COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 15, Number 4, 725–732, 2007

Complete submanifolds of $\mathbb{R}^n$ with finite topology

G. Pacelli Bessa, Luquésio Jorge and J. Fabio Montenegro

We show that a complete m-dimensional immersed submanifold M of $\mathbb{R}^n$ with $a(M) < 1$ is properly immersed and have finite topology, where $a(M) \in [0, \infty]$ is a scaling invariant number that gives the rate that the norm of the second fundamental form decays to zero at infinity. The class of submanifolds $M \subset \mathbb{R}^n$ with $a(M) < 1$ contains all complete minimal surfaces with finite total curvature, all m-dimensional minimal submanifolds with finite total scalar curvature $\int_M |\alpha|^m \, dV < \infty$ and all complete 2-dimensional surfaces with $\int_M |\alpha|^2 \, dV < \infty$ and non-positive curvature with respect to every normal direction.

1. Introduction

Let M be a complete surface minimally immersed in $\mathbb{R}^n$ and let K be its Gaussian curvature. Osserman [6] for $n = 3$ and Chern–Osserman [2] for $n \geq 3$ proved that $\int_M |K| \, dV < \infty$ if and only if M is conformally equivalent to a compact Riemann surface $\overline{M}$ punctured at a finite number of points $\{p_1, \ldots, p_r\}$ and the Gauss map $\Phi : M \to \mathbb{G}_{2,n}$ extends to a holomorphic map $\overline{\Phi} : \overline{M} \to \mathbb{G}_{2,n}$, (see [4] for a clear exposition). Anderson [1] proved a higher-dimension version of Chern–Osserman finite total curvature theorem, i.e., a complete m-dimensional minimally immersed submanifold M of $\mathbb{R}^n$ has finite total scalar curvature $\int_M |\alpha|^m \, dV < \infty$ if and only if M is C^∞-diffeomorphic to a compact smooth Riemannian manifold $\overline{M}$ punctured at a finite number of points $\{p_1, \ldots, p_r\}$ and the Gauss map Φ on M extends to a C^∞-map $\overline{\Phi}$ on $\overline{M}$, where $|\alpha|$ is the norm of the second fundamental form of M.

These results above have appropriate versions in the non-minimal setting. White [7], proved that a complete 2-dimensional surface M immersed in $\mathbb{R}^n$ with $\int_M |\alpha|^2 \, dV < \infty$ and non-positive curvature with respect to every normal direction[1] is homeomorphic to a compact Riemann surface $\overline{M}$ punctured at finite number of points $\{p_1, \ldots, p_r\}$, its Gauss map Φ extends continuously

[1] A submanifold $M \subset \mathbb{R}^n$ is non-positively curved with respect to each normal direction at x if $\det(\eta \cdot \alpha(,)) \leq 0$ for all normals η to M at x; see [7].

to all of $\overline{M}$ and M is properly immersed. It should be observed that the properness of M in White's theorem is a consequence of the first two statements about the immersion, i.e., Jorge and Meeks [3], proved that a complete m-dimensional immersed submanifold M of $\mathbb{R}^n$, homeomorphic to a compact Riemann manifold $\overline{M}$ punctured at finite number of points $\{p_1, \ldots, p_r\}$ and such that the Gauss map Φ extends continuously to all of $\overline{M}$ is properly immersed.

Muller and Sverak [5], answering a question of White, proved that a complete 2-dimensional surface M immersed in $\mathbb{R}^n$ with $\int_M |\alpha|^2 \, dV < \infty$ is properly immersed.

The purpose of this paper is to put another piece on this puzzle showing that a complete m-dimensional submanifold of $\mathbb{R}^n$ with the norm of the second fundamental form uniformly decaying to zero $|\alpha(x)| \to 0$ as $x \to \infty$ in a certain rate is proper and has finite topological type. The decaying rate of $|\alpha(x)| \to 0$ we considered is not fast enough to make $\int_M |\alpha(x)|^m \, dV < \infty$.

To be more precise, let M be a complete m-dimensional submanifold of $\mathbb{R}^n$ and let $K_1 \subset K_2 \subset \cdots$ be an exhaustion sequence of M by compact sets. Fix a point $p \in K_1$ and set $a_i = \sup\{\rho(x) \cdot |\alpha(x)|, \, x \in M \setminus K_i\}$, where $\rho(x) = \mathrm{dist}_M(p, x)$ and $|\alpha(x)|$ is the norm of the second fundamental form of M at x. The a_is form a non-increasing sequence $\infty \geq a_1 \geq a_2 \geq \cdots \geq 0$ with $a_1 = \infty$ if and only if $a_l = \infty$ for all $l \geq 1$. Define the (possibly extended) scaling invariant number $a(M) = \lim_{i \to \infty} a_i \in [0, \infty]$. It can be shown that $a(M)$ does not depend on the exhaustion sequence nor on the point p. It follows from the work of Jorge–Meeks [3] that complete m-dimensional submanifolds M of $\mathbb{R}^n$homeomorphic to a compact Riemannian manifold $\overline{M}$ punctured at finite number of points $\{p_1, \ldots, p_r\}$ and having a well-defined normal vector at infinity have $a(M) = 0$. In particular, complete minimal surfaces in $\mathbb{R}^n$ with finite total curvature, complete 2-dimensional complete surfaces with $\int_M |\alpha|^2 dV < \infty$ and non-positive curvature with respect to every normal direction considered by White or the m-dimensional minimal submanifolds M of $\mathbb{R}^n$ with finite total scalar curvature considered by Anderson have $a(M) = 0$. In our main result, we prove that the larger class of complete m-dimensional immersed submanifolds of $\mathbb{R}^n$ with $a(M) < 1$ share some properties with theses submanifolds with $a(M) = 0$. We prove the following theorem.

Theorem 1.1. *Let M be a complete m-dimensional submanifold of $\mathbb{R}^n$ with $a(M) < 1$. Then M is properly immersed and it is C^∞-diffeomorphic to a compact smooth manifold $\overline{M}$ with boundary.*

Observe that $\int_M |\alpha|^m \, dV < \infty$ is not equivalent to $a(M) < 1$. However, one might ask if Theorem 1.1 holds under finite total scalar curvature $\int_M |\alpha|^m \, dV < \infty$.

For complete m-dimensional minimal submanifolds M of $\mathbb{R}^n$ we define the increasing sequence $b_i = \inf\{\rho^2(x) \cdot \mathrm{Ric}(x)(\nu, \nu), \, |\nu| = 1, \, x \in M \setminus K_i\}$ with $b_1 = -\infty$ if and only if $b_l = -\infty$ for all $l \geq 1$. Define the scaling invariant number $b(M) = \lim_{i\to\infty} b_i \in [-\infty, 0]$. Again, it can be shown that $b(M)$ does not depend on the exhaustion sequence nor on the point p. The proof of Theorem 1.1 can be slightly modified to prove the following version for minimal submanifolds.

Theorem 1.2. *Let M be a complete m-dimensional minimal submanifold of $\mathbb{R}^n$ with $b(M) > -1$. Then M is properly immersed and it is C^∞-diffeomorphic to a compact smooth manifold $\overline{M}$ with boundary.*

2. Proof of Theorem 1.1

2.1. M is properly immersed

Let $\varphi : M^m \hookrightarrow \mathbb{R}^n$ be a complete submanifold with $a(M) < 1$ and let $p \in M$ be a fixed point such that $\varphi(p) = 0 \in \mathbb{R}^n$. There exists a geodesic ball $B_M(p, R_0)$ centered at p with radius R_0 such that for all $x \in M \setminus B_M(p, R_0)$ we have that $\rho(x)|\alpha(x)| \leq c < 1$. Let $f : M^m \to \mathbb{R}$ given by $f(x) = |\varphi(x)|^2$. Fix a point $x \in M \setminus B_M(p, R_0)$ then for $\nu \in T_xM$, $|\nu| = 1$ we have that

$$\begin{aligned} \tfrac{1}{2} \operatorname{Hess} f(x)(\nu, \nu) &= 1 + \langle \varphi(x), \, \alpha(x)(\nu, \nu) \rangle \\ &\geq 1 - |\varphi(x)| \cdot |\alpha(x)| \\ &\geq 1 - \rho(x)|\alpha(x)| \\ &\geq 1 - c. \end{aligned} \tag{2.1}$$

Let $\sigma : [0, \rho(x)] \to M^m$ be a minimal geodesic from p to x. From (2.1) we have for all $t \geq R_0$ that $(f \circ \sigma)''(t)) = \operatorname{Hess} f(\sigma(t))(\sigma', \sigma') \geq 2(1 - c)$ and for $t < R_0$ that $(f \circ \sigma)''(t)) \geq b$, $b = \inf_{x \in B_M(p, R_0)} \{\operatorname{Hess} f(x)(\nu, \nu), \, |\nu| = 1\}$. Thus

$$\begin{aligned} (f \circ \sigma)'(s) &= \int_0^s (f \circ \sigma)''(\tau) \, d\tau \\ &\geq \int_0^{R_0} b \, d\tau + \int_{R_0}^s (1 - c) \, d\tau \\ &\geq b \, R_0 + (1 - c)(s - R_0) \end{aligned} \tag{2.2}$$

$$
\begin{aligned}
f(x) &= \int_0^{\rho(x)} (f\circ\sigma)'(s)\,ds \\
&\geq \int_0^{\rho(x)} b\,R_0 + (1-c)(s-R_0)\,ds \qquad (2.3)\\
&= b\,R_0\,\rho(x) + (1-c)\left(\frac{\rho(x)^2}{2} - R_0\rho(x)\right).
\end{aligned}
$$

Thus $|\varphi(x)|^2 \geq (b-1+c)R_0\rho(x) + (1-c)\rho(x)^2/2$ for all $x \in M \setminus B_M(p,R)$. In fact, this proves that following proposition.

Proposition 2.1. *Let $f : M \to \mathbb{R}$ be a C^2-function defined on a complete Riemannian manifold such that* $\operatorname{Hess} f(x) \geq g(\rho(x))$*, where ρ is the distance function to x_0 and $g : [0,\infty) \to \mathbb{R}$ is a piecewise continuous function. Setting $G(t) = f(x_0) - |\operatorname{grad} f(x_o)|\,t + \int_0^t\int_0^s g(u)\,du\,ds$, $t \in [0,\infty)$ we have that if G is proper and bounded from below then f is proper.*

2.2. M has finite topology

Let $\varphi : M^m \hookrightarrow \mathbb{R}^n$ be a complete immersed submanifold with $a(M) < 1$. To show that M is diffeomorphic to a compact manifold $\overline{M}$ with boundary it suffices to show that R has finitely many critical points. Let $p \in M$ be such that $\varphi(p) = 0 \in \mathbb{R}^n$. We may suppose that $R(x) = |\varphi(x)|$, $x \in M$ is a Morse function. Let $r_0 > 0$ be such that $\Gamma_{r_0} = \varphi(M) \cap \mathbb{S}^{n-1}(r_0)$ is a compact submanifold of $\mathbb{S}^{n-1}(r_0)$ and $\rho(x)\cdot|\alpha(x)| \leq c < 1$ for all $x \in M \setminus \varphi^{-1}(B_{\mathbb{R}^n}(r_0))$. Set $\Lambda_{r_0} = \varphi^{-1}(\Gamma_{r_0})$. For each $x \in \Lambda_{r_0}$ there is an open set $x \in U_x \subset M$ such that $\varphi|U_x$ is an embedding and $\varphi(U_x) \pitchfork \mathbb{S}^{n-1}(r)$, $r \in (r_0-\delta, r_0+\delta)$, $\delta > 0$ small. For each $y \in \Gamma_r \cap \varphi(U_x)$, there is only one unit vector $\nu(y) \in T_{\varphi^{-1}(y)}U_x$ such that $T_y\varphi(U_x) = T_y(\varphi(U_x) \cap \Gamma_r) \oplus [[\varphi_*\nu(y)]]$ and $\langle \varphi_*\nu(y), \eta(y)\rangle > 0$, where $\eta(y) = y/r$ is the unit vector perpendicular to $T_y\mathbb{S}^{n-1}(r)$. Since Λ_{r_0} is compact we find a finite sequence $\{x_1,\ldots,x_k\} \subset \Lambda_{r_0}$ and $\delta = \min\{\delta_1,\ldots,\delta_k\}$ such that using partition of unit we construct by this procedure a smooth vector field ν in $V = \varphi^{-1}(B_{\mathbb{R}^n}(r_0+\delta) \setminus B_{\mathbb{R}^n}(r_0-\delta))$. Identify $\nu(x)$ with $\varphi_*\nu(y)$, $y = \varphi(x)$. Consider the function ψ defined in V given by

$$\psi(x) = \langle \nu(y), \eta(y)\rangle = \cos\theta(y).$$

For each $x \in \Lambda_{r_0}$, let $\xi(t,x)$ be the solution of the following problem on M

$$\begin{aligned} \xi_t &= \frac{1}{\psi}\,\nu(\xi(t,x)), \\ \xi(0,x) &= x. \end{aligned} \tag{2.4}$$

Recall that $R(x) = |\varphi(x)|$. For $X \in TM$ we have that $X(R) = \langle X, \eta\rangle$ and writing $\eta(y) = \sin\theta(y)\,\nu^*(y) + \cos\theta(y)\,\nu(y)$, $\nu^*(y) \perp \nu(y)$, we have that $\operatorname{grad} R = \psi\,\nu$. Set the notation $R(t,x) = |\varphi \circ \xi(t,x)|$. We have that

$$R_t = \left\langle \operatorname{grad} R, \frac{1}{\psi}\,\nu \right\rangle = \left\langle \psi\nu, \frac{1}{\psi}\,\nu \right\rangle = 1 \Longleftrightarrow R = R(t,y) = t + r.$$

We will derive a differential equation that the function $\psi \circ \xi(t,y)$ satisfies.

$$\begin{aligned} \psi_t = \xi_t\langle \nu, \eta\rangle &= \langle D_{(1/\psi)\nu}\nu, \eta\rangle + \langle \nu, D_{\xi_t}\eta\rangle \\ &= \langle \nabla_\nu \nu + \alpha(\nu,\nu), \eta\rangle + \left\langle \nu, D_{\xi_t}\left(\frac{\xi}{R}\right)\right\rangle \end{aligned}$$

But $\langle \nu,\nu\rangle = 1 \;\Rightarrow \langle \nu, \nabla_\nu\nu\rangle = 0$ and $\nabla_\nu\nu \in T_xM \;\Rightarrow\; \nabla_\nu\nu \in (T_xM \cap T_xS^n_R)$ $\Rightarrow \langle \nabla_\nu\nu, \eta\rangle = 0$. On the other hand

$$D_{\xi_t}\left(\frac{\xi}{R}\right) = \frac{(1/\psi)\nu}{R} - \frac{R_t}{R^2}\varphi = \frac{1}{R\psi}\nu - \frac{1}{R}\eta$$

then

$$\psi_t = \frac{1}{\psi}\langle \alpha(\nu,\nu), \eta\rangle + \frac{1}{\psi R} - \frac{\psi}{R} = \frac{\sqrt{1-\psi^2}}{\psi}\langle \alpha(\nu,\nu), \nu^*\rangle + \frac{1-\psi^2}{\psi R} \tag{2.5}$$

To determine a differential equation satisfied by $\sin\theta(t,x) = \sqrt{1-\psi^2}$, we proceed as follows. By (2.5) we have

$$\frac{\psi\psi_t}{\sqrt{1-\psi^2}} = \langle \alpha(\nu,\nu), \nu^*\rangle + \frac{\sqrt{1-\psi^2}}{R} \tag{2.6}$$

Observing that $R(t,x) = t + r$, Equation (2.6) can be written as

$$-(t+r)(\sqrt{1-\psi^2})_t = (t+r)\langle \alpha(\nu,\nu), \nu^*\rangle + \sqrt{1-\psi^2} \tag{2.7}$$

and rewritten as

$$\Big[(t+r)\sqrt{1-\psi^2})\Big]_t + (t+r)\langle \alpha(\nu,\nu), \nu^*\rangle = 0. \tag{2.8}$$

Integrating Equation (2.8), we have the following equation

$$\sqrt{1-\psi^2} = \frac{r}{t+r}\sqrt{1-\psi_0^2} - \frac{1}{t+r}\int_0^t (s+r)\langle \alpha(\nu,\nu),\nu^*\rangle\, ds, \tag{2.9}$$

where $\psi_0 = \psi(\xi(0,x))$. Since $\sin\theta(\xi(t,x)) = \sqrt{1-\psi^2}$ we rewrite (2.9) in the following form

$$\sin\theta(\xi(t,x)) = \frac{r}{t+r}\sin\theta(\xi(0,x)) - \frac{1}{t+r}\int_0^t (s+r)\langle \alpha(\nu,\nu),\nu^*\rangle\, ds \tag{2.10}$$

Now,

$$-\langle \alpha(\nu,\nu),\nu^*\rangle(\xi(s,x)) \le |\alpha|(\xi(s,x)) \le c/\rho(\xi(s,x)) \le c/R(s,x).$$

Substituting in (2.10) and recalling that $R(s,x) = s+r$ we have that

$$\begin{aligned}\sin\theta(\xi(t,x)) &\le \frac{r}{t+r}\sin\theta(\xi(0,y)) + \frac{1}{t+r}\int_0^t (s+r)\frac{c}{s+r}\, ds \\ &= \frac{c\,t + r\sin\theta(\xi(0,x))}{t+r} < 1, \quad \forall t \ge 0.\end{aligned} \tag{2.11}$$

The critical points of R are those x such that $\psi(x) = 0$, or those points where $\sin\theta(x) = 1$. Thus, along the integral curves $\xi(t,y)$, $y \in \Gamma_r$, there is no critical point for the function $R(x) = |\varphi(x)|$. This shows that outside the compact set $M \setminus B_M(p,r_0)$ there are no critical points for R. Since R is a Morse function, its critical points are isolated thus there are finitely many of them. Therefore M has finite topology.

3. Sketch of proof for Theorem 1.2

Let $\varphi : M^m \hookrightarrow \mathbb{R}^n$ be a complete minimal submanifold and let $x \in M$, $\nu \in T_xM$ and $\{e_1,\ldots,e_m = \nu\}$ an orthonormal basis for T_xM. Using the Gauss equation we can compute the Ricci curvature in the direction ν by

$$\begin{aligned} Ric(x)(\nu) &= \left\langle \sum_{i=1}^{m} \alpha_{ii}, \alpha_{mm} \right\rangle - \sum_{i=1}^{m-1} |\alpha_{im}|^2 \\ &= \langle mH - \alpha_{mm}, \alpha_{mm}\rangle - \sum_{i=1}^{m-1} |\alpha_{im}|^2 \\ &= -\sum_{i=1}^{m} |\alpha_{im}|^2, \end{aligned} \tag{3.1}$$

where $\alpha_{ij} = \alpha(e_i, e_j)$. Let $f : M^m \to \mathbb{R}$ given by $f(x) = |\varphi(x)|^2$. The Hessian of f at $x \in M$ and $\nu \in T_xM$, $|\nu| = 1$ satisfies

$$\begin{aligned}\tfrac{1}{2}\mathrm{Hess}\, f(x)(\nu,\nu) &= +\langle \varphi(x),\, \alpha(\nu,\nu)\rangle \\ &\geq [1 - |\varphi(x)| \cdot |\alpha_{mm}|] \\ &\geq \left[1 - |\varphi(x)|\sqrt{-\mathrm{Ric}(x)(\nu)}\right] \qquad (3.2)\\ &\geq \left[1 - \rho(x)\sqrt{-\mathrm{Ric}(x)(\nu)}\right] \\ &\geq 1 - c.\end{aligned}$$

The proof of Theorem 1.1 from Equation (2.1) shows that M is properly immersed in Theorem 1.2. To show that M has finite topological type, observe that

$$|\alpha|(\xi(s,x)) \leq \sqrt{-\mathrm{Ric}(\xi(s,x))(\nu,\nu)} \leq c/\rho(\xi(s,x)) \leq c/R(s,x),$$

and follow the proof of Theorem 1.1 after (2.10) and we still have (2.11).

References

[1] M. Anderson, *The compactification of a minimal subamnifold by the Gauss Map*, Preprint IEHS, 1985.

[2] S.S. Chern and R. Osserman, *Complete minimal surfaces in euclidean n-space*, J. Anal. Math. (19) (1967), 15–34.

[3] Luquésio P. Jorge and W. Meeks III, *The topology of complete minimal surfaces of finite total Gaussian curvature*, Topology **22** (1983), 203–221.

[4] H.B. Lawson, *Lectures on minimal submanifolds*, vol. I, Monografías de Matemática, **14**, Instituto de Matemtica Pura e Aplicada-IMPA, Rio de Janeiro, 1977.

[5] S. Muller and V. Sverak, *On surfaces of finite total curvature*, J. Differential Geom. **42** (1995), 229–258.

[6] R. Osserman, *Global properties of minimal surfaces in $\mathbb{E}^3$ and $\mathbb{E}^n$*, Ann. of Math. **80** (1964), 340–364.

[7] B. White, *Complete surfaces of finite total curvature*, J. Differential Geom. **26** (1987), 315–326.

Departamento de Matemática
Campus do Pici, Bloco 914
Universidad Federal do Cearéäa — UFC
60455-760 Fortaleza-Cearéäa
Brazil
E-mail address: bessa@mat.ufc.br
E-mail address: ljorge@yahoo.com.br
E-mail address: fabio@mat.ufc.br

Received February 7, 2006