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Not all boundary slopes are strongly detected by the character variety

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It has been an open question whether all boundary slopes of hyperbolic knots are strongly detected by the character variety. The main result of this paper produces an infinite family of hyperbolic knots each of which has at least one strict boundary slope that is not strongly detected.

1. Introduction

In 1983 Marc Culler and Peter Shalen introduced a method of constructing essential surfaces in a 3-manifold using representations of the fundamental group into $\mathrm{SL}_2(\mathbb{C})$. In a nutshell, an ideal point of a curve in the character variety gives a non-trivial action of the fundamental group on a Bass–Serre tree, and this action is used to construct embedded essential surfaces in the 3-manifold. Some definitions and facts are recalled below; the reader may consult [2, 5, 6, 10] for details.

Let M be a compact, orientable, irreducible 3-manifold with boundary consisting of a single torus. A *slope* σ on ∂M is a pair $\{\pm s\}$, where $s \in H_1(\partial M)$ is a primitive class. If a basis for $H_1(\partial M)$ is chosen, then a slope is identified with an element of $\mathbb{Q} \cup \{1/0\}$. A properly embedded surface S in M is said to have slope σ if ∂S is a non-empty union of parallel, simple closed curves in ∂M of slope σ. A slope σ is called a *boundary slope of M* if there is an essential surface S in M which has slope σ. (Here, a surface is termed *essential* if it is properly embedded, incompressible, orientable and no component is boundary parallel or a sphere.) A boundary slope of M is *strict* if it is the boundary slope of an essential surface which is not a fiber or a semi-fiber. For instance, any non-zero boundary slope of the exterior of a knot in S^3 (with the standard framing) is strict. The *exterior* of the knot K in S^3 is the closed 3-manifold $S^3 \setminus \nu(K)$, where $\nu(K)$ is a small open tubular neighborhood of K. If σ is a boundary slope of the exterior of K, then σ is also said to be a boundary slope of K as well as of the complement of K, $S^3 \setminus K$.

Let X be an irreducible affine curve in the character variety of M, and let $\widetilde{X}$ be its smooth projective model. Then all but finitely many points of $\widetilde{X}$ correspond to characters in X. The points which do not correspond to characters are termed *ideal points*. Given an ideal point $\tilde{x} \in \widetilde{X}$, the construction by Culler and Shalen associates an essential surface S in M to $\tilde{x}$ in a non-canonical fashion. Some topological information about S, however, can be obtained by evaluating the following elements of the function field $\mathbb{C}(\widetilde{X})$ at $\tilde{x}$. Given a slope $\sigma = \{\pm s\}$, there is a regular function $I_\sigma : X \to \mathbb{C}$ defined by $I_\sigma(\chi) = \chi(\overline{s})$, where $\overline{s}$ is the image of s under the composition of the inverse of the Hurewicz isomorphism with the inclusion $\pi_1(\partial M) \to \pi_1(M)$. The function I_σ lifts to a rational function $I_\sigma : \widetilde{X} \to \mathbb{C} \cup \{\infty\}$. Then either:

(1) there is a unique slope σ such that $I_\sigma(\tilde{x}) \in \mathbb{C}$; or

(2) $I_\sigma(\tilde{x}) \in \mathbb{C}$ for every slope σ.

Accordingly, $\tilde{x}$ is termed of type (1) or type (2). If $\tilde{x}$ is of type (1) and $I_\sigma(\tilde{x}) \in \mathbb{C}$, then every essential surface associated to $\tilde{x}$ has slope σ, and the boundary slope σ of M is said to be *strongly detected by* $\tilde{x}$. If $\tilde{x}$ is of type (2), then the construction can be used to associate a closed essential surface to $\tilde{x}$. Consequently, if there is an essential surface with boundary associated to $\tilde{x}$ and $\tilde{x}$ is of type (2), then the corresponding boundary slope is termed *weakly detected by* $\tilde{x}$. The adverbs "strongly" and "weakly" are often omitted when the distinction is unnecessary. Whilst there is a unique slope detected by an ideal point of type (1), it is not known whether an ideal point of type (2) always detects some slope and whether such a slope is necessarily unique. Note that if M is *small* (i.e., M contains no closed essential surface), then every ideal point is of type (1).

A boundary slope of M is termed *detected* if it is detected by an ideal point of some curve in the character variety of M. A detected boundary slope is *strongly detected* if it is strongly detected by an ideal point of some curve, otherwise it is *weakly detected*. For example, given the complement of a knot in S^3 (with the standard framing), the boundary slope $\lambda = 0/1$ is strongly detected: At the ideal point $\tilde{x}$ of the curve X containing the characters of all abelian representations one has $I_\lambda(\tilde{x}) = 2$ and $I_\mu(\tilde{x}) = \infty$, where $\mu = 1/0$.

It is shown in [2] that the strongly detected boundary slopes of M are precisely the slopes of the sides of the Newton polygon of the A-polynomial

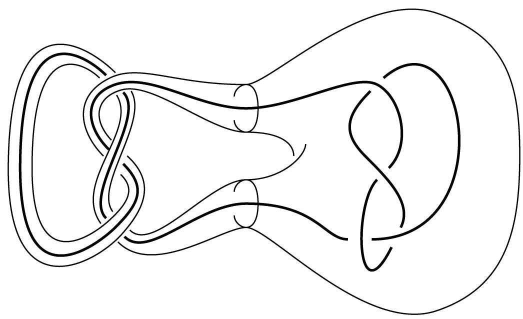

Figure 1: A weakly detected boundary slope.

(if all components of the character variety of M are considered). This result implies that (generally speaking) strongly detected boundary slopes are quite common. The first examples of strict boundary slopes that are not strongly detected were given in 2001 by Schanuel and Zhang [9]. The manifolds involved are a family of graph manifolds, and the slopes in question are weakly detected [Xingru Zhang (personal communication)]. In particular, two interesting questions still remained open.

Question 1.1 (Cooper and Long [3]). *Is every strict boundary slope of the complement of a knot in S^3 strongly detected?*

The following examples of (cylindrical) knot complements containing strict boundary slopes which are weakly detected answer this question negatively. Consider the connected sum K of two knots in S^3 which do not have meridians as boundary slopes: the connected sum of two small knots will suffice (figure 1). The complement of K contains an essential separating annulus with boundary slope $1/0$. Given the decomposition of the fundamental group of $S^3 \setminus K$, it is not difficult to find a curve of characters with ideal points detecting the annulus. However, combining standard arguments involving the limiting representations, any ideal point detecting $1/0$ must be of type (2). In fact, it is not difficult to show that such an ideal point detects a standard swallow-follow torus in $S^3 \setminus K$. Thus, the strict boundary slope $1/0$ of $S^3 \setminus K$ is weakly detected.

Question 1.2 (Schanuel and Zhang [9]). *Let M be an orientable 1-cusped hyperbolic 3-manifold . Is every strict boundary slope of M strongly detected?*

A negative answer to this question is given by the main result of this paper:

Theorem 1.3. *The complement of the pretzel knot* $K_n = (3, 5, 2n+1, 2)$, $n > 1$, *is hyperbolic and has the strict boundary slope* $4(n+4)$ *which is not strongly detected.*

The key idea in the proof of Theorem 1.3 is to use a relationship between certain strongly detected boundary slopes of mutants which was established by Cooper and Long [3]. Recall that a character is termed *irreducible* if it is the character of an irreducible representation; otherwise it is *reducible*. Since a character can be viewed as a function from a group to the complex numbers, its restriction to a subgroup is a character of the subgroup. Let K be a knot in S^3, and assume that K^τ is obtained from K by mutation along the Conway sphere S_4. Let $X(K)$ denote the character variety of $S^3 \setminus K$ and $H = \mathrm{im}(\pi_1(S_4) \to \pi_1(S^3 \setminus K))$. If the boundary slope σ of K is strongly detected by an ideal point of a curve X in $X(K)$ and there is a character on X whose restriction to H is irreducible, then σ is a strongly detected boundary slope of K^τ. Details can be found in [3, 11], where it is not explicitly stated that the respective framings are standard. This can be verified with a direct homology argument; see [1].

Proof of Theorem 1.3. It follows from work by Oertel that K_n is hyperbolic (see [8, Corollary 5]) and that its complement contains two essential 4-punctured spheres with meridional slope (see [8, Proposition 2.13]). The pretzel knot $K_n^\tau = (5, 3, 2n+1, 2)$ is obtained from K_n by performing a mutation along the Conway sphere S_4 separating the first two tangles from the second two. An algorithm due to Hatcher and Oertel [7] is used in

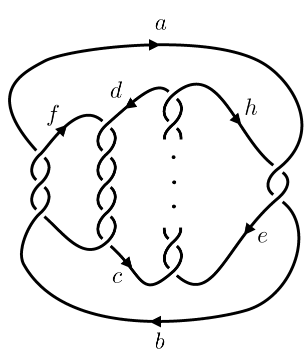

Figure 2: The pretzel knot $K_n = (3, 5, 2n+1, 2)$.

Section 2 to show that K_n has boundary slope $4(n+4)$, but its mutant K_n^τ does not (Proposition 2.4).

Assume that the boundary slope $\sigma = 4(n+4)$ of K_n is strongly detected. Then there is a curve X in $X(K_n)$ with the property that σ is strongly detected by an ideal point $\tilde{x}$ of $\widetilde{X}$. Since σ is not a boundary slope of K_n^τ, it follows from the above discussion of work by Cooper and Long that the restriction to $H = \operatorname{im}(\pi_1(S_4) \to \pi_1(S^3 \setminus K_n))$ of any character on X is reducible. The curve X is therefore contained in the closed algebraic set Y of all characters whose restriction to H is reducible. Moreover, X must contain an irreducible character of $\pi_1(S^3 \setminus K_n)$ since $\sigma \neq 0/1$. The main result of Section 3 (Proposition 3.1) implies that the function I_μ associated to the slope $\mu = 1/0$ is constant on each algebraic component of Y which contains an irreducible character. But then both I_μ and I_σ are finite-valued at $\tilde{x}$, which implies that $\tilde{x}$ is not of type (1) contradicting the assumption that σ is strongly detected by $\tilde{x}$. This completes the proof of the theorem. □

Remark 1.4. The above examples also show that not all boundary slopes are strongly detected by the $\mathrm{PSL}_2(\mathbb{C})$-character variety since each representation into $\mathrm{PSL}_2(\mathbb{C})$ of the fundamental group of the complement of a knot in S^3 lifts to a representation into $\mathrm{SL}_2(\mathbb{C})$.

Question 1.5. *Is the boundary slope* $4(n+4)$ *of* K_n *weakly detected?*

An answer to this question has interesting ramifications. If the slope is not weakly detected, then there is a slope which is not detected by the character variety. If it is, then (using the classification of closed essential surfaces in [8]) one can show that there is an ideal point of the character variety which weakly detects two distinct strict boundary slopes: $4(n+4)$ and $1/0$.

Question 1.6. *Is there a small knot complement with a strict boundary slope that is not strongly* (*and hence also not weakly*) *detected?*

2. Mutants with distinct slope sets

In this section, the algorithm of [7] is used to show that the pretzel knot K_n has the boundary slope $4(n+4)$, whereas K_n^τ does not. For the remainder of this paper, the knots will be denoted in Montesinos' notation by

$$K_n = K\left(\frac{1}{3}, \frac{1}{5}, \frac{1}{2n+1}, \frac{1}{2}\right) \quad \text{and} \quad K_n^\tau = K\left(\frac{1}{5}, \frac{1}{3}, \frac{1}{2n+1}, \frac{1}{2}\right).$$

This notation is used in [7]. The knot K_n is shown in figure 2. The knot K_n^τ is a mutant of K_n by an involution $\tau:\ S_4 \to S_4$, where S_4 is the 4-punctured sphere which separates the $1/3$ and $1/5$ tangles from the other two tangles.

2.1. Setup from [7]

The following notation and conventions are inspired by those in [7] and will be used throughout this subsection.

- K is a 4-tangle pretzel knot in the 3-sphere.
- $M = \mathrm{S}^3 \setminus \nu(K)$, where $\nu(K)$ denotes a regular neighborhood of K.
- We decompose S^3 into the union of four 3-balls $\{B_i\}_{i=0}^3$ such that each B_i contains exactly one of the tangles of K and $\bigcap_0^3 \mathrm{B}_i \cong \mathrm{S}^1$ (the *axis* for K).
- Let $\partial B_i \times [0,1]$ be a collar on ∂B_i inside B_i with $\partial B_i = \partial B_i \times \{1\}$.
- $T_i = K \cap B_i$.
- $1/q_i$ is the rational number corresponding to the tangle T_i.
- P_i denotes the 4-point set $\partial B_i \cap K$.
- S_i is the 4-punctured sphere $\partial B_i \setminus P_i$.

If a surface is properly embedded in M, it may be isotoped to be transverse to $\partial \mathrm{B}_i$ for every i. We always assume that properly embedded surfaces are so arranged.

- If F is a properly embedded surface, then we use F_i to indicate the surface $F \cap B_i$ which is properly embedded in B_i. (Note that ∂F_i is a curve system on the 4-punctured sphere S_i.)

We now review the constructions from [7] which apply to our setting. Curve systems on the 4-punctured sphere are either carried by the train track shown in figure 3(a) or by its mirror image. We usually consider curve systems up to projective class, so rational projective coordinates $[a, b, c] \in \mathbb{Q}P^2$ represent a projective curve system corresponding to the figure. Each projective curve system can also be uniquely represented by an ordered pair (u, v), where $u = b/(a + b)$ and $v = c/(a + b)$. The *slope* of the projective curve system is v. We refer to these pairs as *uv-coordinates.*

A *p/q-tangle*, denoted $\langle \frac{p}{q} \rangle$, is a projective curve system $[1, q-1, p]$, written equivalently as $((q-1)/q,\, p/q)$ in uv-coordinates. A *p/q-circle*,

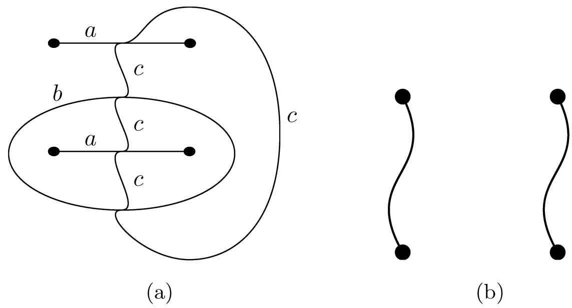

Figure 3: Curve systems and tangles. (a) The traintrack with projective weights a, b and c. (b). The ∞-tangle.

denoted by $\langle\frac{p}{q}\rangle^{\circ}$, is a projective curve system $[0, q, p]$ or, equivalently, $(1, p/q)$. The ∞-*tangle*, denoted by $\langle\infty\rangle$, is the projective class of the pair of vertical arcs shown in figure 3(b), and will be represented by $(-1, 0)$ in uv-coordinates. Note that the tangle T_i is isotopic $(\mathrm{rel}\,\partial B_i)$ to the two component representative for $\langle\frac{1}{q_i}\rangle$ at the level $S_i \times \{0\}$ together with the four arcs $P_i \times [0, 1]$.

We now use the above definitions to define a graph $\mathcal{D}$ in the uv-plane. Our aim is to associate surfaces in $B_i \setminus \nu(T_i)$ to certain paths in this graph. The vertices of $\mathcal{D}$ are the uv-coordinates of the p/q-tangles and p/q-circles for every $p/q \in \mathbb{Q}$ together with the point $\langle\infty\rangle = (-1, 0)$. There are four types of edges in $\mathcal{D}$, *non-horizontal edges, horizontal edges, vertical edges* and *infinity edges*. Two vertices $\langle\frac{p}{q}\rangle$ and $\langle\frac{r}{s}\rangle$ are connected by a non-horizontal edge if $|ps - qr| = 1$ or, equivalently, if $\langle\frac{r}{s}\rangle$ can be obtained from $\langle\frac{p}{q}\rangle$ by surgery on an arc. The horizontal edges connect the vertices $\langle\frac{p}{q}\rangle^{\circ}$ to $\langle\frac{p}{q}\rangle$. The vertical edges connect $\langle m\rangle$ to $\langle m+1\rangle$ for every $m \in \mathbb{Z}$. Finally, the infinity edges connect the integer vertices $\langle m\rangle$ to $\langle\infty\rangle$. If x and y are vertices of $\mathcal{D}$ that are connected by an edge, then we will denote this edge by $[x, y]$. The subgraph $\mathcal{S} \subset \mathcal{D}$ is defined as the portion of $\mathcal{D}$ with u-coordinate in the interval $[0, 1]$. *Rational points* on the graph $\mathcal{D}$ are the points in the set $\mathcal{D} \cap \mathbb{Q}^2$. Rational points on $\mathcal{D}$ correspond to projective curve systems according to the formula $(p/q, r/s) = [s(q-p), sp, rq]$. Figure 4 shows part of the the graph $\mathcal{D}$.

Given an edge $[x, y]$ in $\mathcal{D}$, we subdivide it as follows. For each $m \in \mathbb{Z}^+$ and $k \in \{1, \ldots, m-1\}$, let $\frac{k}{m} \cdot x + \frac{m-k}{m} \cdot y$ denote the point on $[x, y]$ corresponding to the projective curve system represented by k parallel copies of a pair of arcs representing x together with $m-k$ copies of a pair of arcs representing y. It is easy to check that $\frac{k}{m} \cdot x + \frac{m-k}{m} \cdot y$ is represented by a rational point on the edge $[x, y]$.

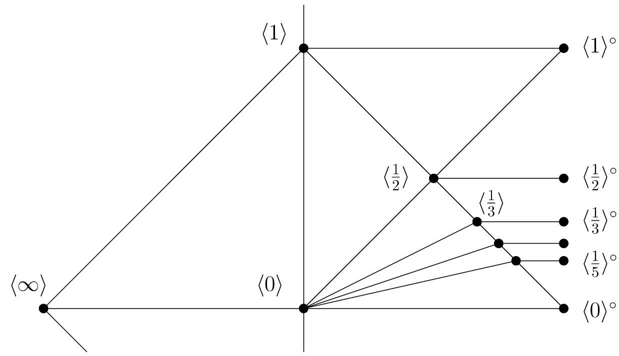

Figure 4: A part of $\mathcal{D}$.

An *edgepath* in $\mathcal{D}$ is a piecewise linear path $[0,1] \to \mathcal{D}$ which starts and ends at rational points of $\mathcal{D}$ which may or may not be vertices of $\mathcal{D}$. An *admissible path system* $\gamma = (\gamma_0, \dots, \gamma_3)$ for K is a 4-tuple of edgepaths in $\mathcal{D}$ with the following four properties.

(E1) For each i, $\gamma_i(0)$ lies on the horizontal edge connecting $\langle \frac{1}{q_i} \rangle^\circ$ to $\langle \frac{1}{q_i} \rangle$ and if $\gamma_i(0) \neq \langle \frac{1}{q_i} \rangle$, then the path γ_i is constant.

(E2) Every γ_i is *minimal*. That is, it never stops and retraces itself and it never travels along two sides of a triangle in $\mathcal{D}$ in succession.

(E3) The points $\gamma_0(1), \dots, \gamma_3(1)$ all have the same u-coordinate and their v-coordinates sum to zero.

(E4) Each γ_i proceeds monotonically from right to left in the sense that traversing vertical edges is permitted. That is, if $0 \leq t_1 < t_2 \leq 1$ then the u-coordinate of $\gamma_i(t_1)$ is at least as big as the u-coordinate of $\gamma_i(t_2)$.

Admissible edgepath systems are divided into the following three types.

- A *type*-I *system* is an admissible edgepath system γ, where each γ_i stays in $\mathcal{S}$ and has no vertical edges.
- A *type*-II *system* is the same as a type-I system except that at least one γ_i has a vertical edge.
- A *type*-III *system* is an admissible edgepath system where the γ_is end to the left of $\mathcal{S}$.

To each edgepath γ_i of an admissible edgepath system γ, Hatcher and Oertel show how to associate a finite number of so-called *candidate surfaces* $\Gamma_i \subset B_i$ with $\partial\Gamma_i \subset T_i \cup \partial B_i$. Γ_i is called *m-sheeted* if the minimum number of intersection points of Γ_i with a small meridian circle of T_i is m. It is shown in [7] that every essential surface in M with non-empty boundary and boundary slope not equal to $1/0$ is isotopic to a union of m-sheeted candidate surfaces $\Gamma = \Gamma_0 \cup \cdots \cup \Gamma_3$. (We refer to Γ as a candidate surface carried by γ.) For the purpose of this paper, it suffices to describe the construction of the candidate surfaces for type-II and -III systems with no constant edgepaths and with the endpoints of the γ_is on vertices of $\mathcal{D}$. Let γ be such an admissible edgepath system. We write γ_i as $[x_r, \dots, x_0]$, where the x_js are the vertices of γ_i and the vertex x_j is followed by the vertex x_{j+1} as we proceed along the edgepath. Likewise, we will interpret $[x_{j+1}, x_j]$ as a directed edge from x_j to x_{j+1}. (Vertices are indexed from left to right to indicate that the edgepath proceeds from right to left.) A complete list of candidate surfaces for γ is described in the following paragraphs.

For any edge $[x_{j+1}, x_j]$ of γ_i and $a < b$, we can build a 1-sheeted candidate surface in $S_i \times [a, b]$ as follows. Let α and β be the curve systems with two arc components in S_i that represent x_j and x_{j+1}, respectively. The surface is

$$\alpha \times \left[a, \frac{a+b}{2}\right) \cup \beta \times \left(\frac{a+b}{2}, b\right] \cup D,$$

where D is a regular neighborhood (saddle) of a surgery arc in $S_i \times \{(a+b)/2\}$ given by the existence of the edge $[x_{j+1}, x_j]$. Up to level preserving isotopy there are two choices for each saddle. One of the two possible surfaces for $[\langle 1\rangle, \langle \frac{1}{2}\rangle]$ is shown in figure 5. To each edgepath γ_i, we can now associate a surface Γ_i which is properly embedded in $\mathrm{B}_i - N(T_i)$ by simply stacking the

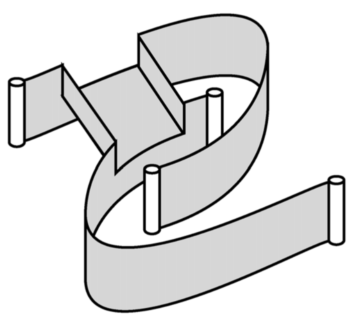

Figure 5: A surface corresponding to the edge $[\langle 1\rangle, \langle \frac{1}{2}\rangle]$.

surfaces associated to the edges of γ_i in the collar $S_i \times [0,1]$. Then $\Gamma_i \cap (S_i \times \{1\})$ is the two component representative of the vertex x_r and $\Gamma_i \cap (S_i \times \{0\})$ is the two component representative of the vertex x_0 and so lies on the tangle T_i. The condition (E3) guarantees that the surfaces $\{\Gamma_i\}$ fit together to give a surface in M.

We have described the construction of 1-sheeted candidate surfaces. If m is a positive integer, then we can construct an m-sheeted surface for γ in a similar way. Again, for every edge $[x_j, x_{j+1}]$ in γ_i, we construct a surface in $S_i \times [a,b]$. This time we further subdivide the edge into m edges separated by the points $\frac{k}{m} \cdot x_j + \frac{m-k}{m} \cdot x_{j+1}$, where $k = 1, \dots, m-1$. Starting with the product surface $\alpha \times \big[a, (a+b)/2m\big)$, we add a saddle at the level $(a+b)/2m$ to pass to the curve system $\frac{1}{m} \cdot x_j + \frac{m-1}{m} \cdot x_{j+1}$. An m-sheeted surface for $[x_j, x_{j+1}]$ is completed by adding saddles to pass between the product surfaces that correspond to each point $\frac{k}{m} \cdot x_j + \frac{m-k}{m} \cdot x_{j+1}$. Just as before, we can build a surface for γ by stacking the surfaces for each edge in each edgepath γ_i and finally gluing everything together to give a surface in M. The meridian of the knot will intersect this surface m times. It is worth noting that we have a choice between two possible saddles every time we pass a point $\frac{k}{m} \cdot x_j + \frac{m-k}{m} \cdot x_{j+1}$. Whence there may be many different m-sheeted candidate surfaces associated to a given admissible path system.

The question whether a candidate surface is essential or not can be studied via the admissible edgepath systems which carry them. This motivates the following terminology. An admissible edgepath system γ is *incompressible* if every candidate surface associated to γ is incompressible, and it is *compressible* if every associated candidate surface is compressible. If there are both compressible and incompressible candidate surfaces associated to γ, then it is said to be *indeterminate.*

For an edgepath γ_i in γ, let e_+ be the number of edges of γ_i which increase slope, and e_- be the number of edges which decrease slope. The numbers e_+ and e_- are independent of the behavior of γ_i to the left of $u = 0$. The *twist number* of γ_i is $t(\gamma_i) = 2(e_- - e_+)$, and the twist number of γ is $t(\gamma) = \sum t(\gamma_i)$. If F is a surface carried by γ, the twist number $t(F)$ of F is defined to be $t(\gamma)$. Hatcher and Oertel show that if S is the Seifert surface for a knot and s an admissible edgepath system which carries S, then the slope of any surface carried by γ is $t(\gamma) - t(s)$.

In order to decide whether an admissible edgepath system is compressible, incompressible or indeterminate, Hatcher and Oertel define the notions of r-values and completely reversible edgepaths. The *r-value* of a leftward directed non-horizontal edge is the denominator of the v-coordinate of the point where the extension of the edge meets the line $u = 1$. The r-value is taken to be positive if the edge travels upwards, and negative if it travels downwards. If γ_i is an edgepath in $\mathcal{D}$, then its *final r-value* is the r-value of the last edge in the path. If γ is an admissible edgepath system, then its *cycle of final r-values* is the 4-tuple of final r-values for the four edgepaths $\{\gamma_i\}$. The cycle of final r-values of γ is defined up to cyclic permutation. An edgepath γ_i is *completely reversible* if each pair of successive edges in γ_i lies in triangles that share a common edge.

2.2. Boundary slopes of K_n and K_n^τ

The aim of this section is to apply the machinery of the previous section to prove that K_n has the boundary slope $4(n+4)$, whereas K_n^τ does not.

Consider the admissible edgepath system $s = (s_i)$ given by

$$\begin{aligned} s_0 &= [\langle\infty\rangle, \langle 1\rangle, \langle 1/2\rangle, \langle 1/3\rangle], \\ s_1 &= [\langle\infty\rangle, \langle 1\rangle, \langle 1/2\rangle, \langle 1/3\rangle, \langle 1/4\rangle, \langle 1/5\rangle], \\ s_2 &= [\langle\infty\rangle, \langle 1\rangle, \langle 1/2\rangle, \ldots, \langle 1/(2n+1)\rangle], \\ s_3 &= [\langle\infty\rangle, \langle 1\rangle, \langle 1/2\rangle]. \end{aligned}$$

Also let $s^\tau = (s_1, s_0, s_2, s_3)$.

Lemma 2.1. *A Seifert surface Σ for K_n is carried by the path system s. Similarly, a Seifert surface Σ^τ for K_n^τ is carried by s^τ. Furthermore, the twist numbers $t(\Sigma)$ and $t(\Sigma^\tau)$ are both $-(14+4n)$.*

Proof. First, we need to show that s and s^τ carry 1-sheeted orientable surfaces. Second, we compute their twist numbers.

Hatcher and Oertel explain how to do the first part on pp. 460–461 of [7]. To use their proceedure, we first make two observations about the admissible path system s. First, the mod 2 reductions of the slopes of all vertices of every s_i are either $1/0$ or $0/1$. Second, there are exactly four odd-integer vertices in the admissible path system s. These observations, together with Hatcher and Oertel's explanation, show that s carries a Seifert surface for K_n. An identical argument works for s^τ.

We calculate the twist number using the formula $t(\gamma) = \sum t(\gamma_i)$. We have

$$\begin{aligned} t(s_0) &= 2(0-2) &= -4, \\ t(s_1) &= 2(0-4) &= -8, \\ t(s_2) &= 2(0-2n) &= -4n, \\ t(s_3) &= 2(0-1) &= -2, \end{aligned}$$

and hence

$$t(s) = \sum t(s_i) = -4 - 8 - 4n - 2 = -(14 + 4n).$$

The same procedure works for K_n^τ. □

Lemma 2.2. $4(n+4)$ *is not a boundary slope of* K_n^τ.

Proof. Since $4(n+4) + (14+4n) = 2$ it suffices to show that every candidate surface with twist number 2 is compressible. Assume that δ is an admissible edgepath system for K_n^τ with $t(\delta) = 2$.

We begin by stating three claims whose proofs are given separately below.

Claim 1. If δ is a type-III system, then

$$\begin{aligned}
\delta_0 &= [\langle\infty\rangle,\ \langle 0\rangle,\ \langle 1/5\rangle]\,,\\
\delta_1 &= [\langle\infty\rangle,\ \langle 1\rangle,\ \langle 1/2\rangle,\ \langle 1/3\rangle]\,,\\
\delta_2 &= [\langle\infty\rangle,\ \langle 0\rangle,\ \langle 1/(2n+1)\rangle]\,,\\
\delta_3 &= [\langle\infty\rangle,\ \langle 0\rangle,\ \langle 1/2\rangle]\,.
\end{aligned}$$

Claim 2. Assume δ is a type-II system and let δ' be the 4-tuple of edgepaths $(\delta_0', \dots, \delta_3')$ obtained by deleting all vertical edges from the edgepaths of δ. Then

$$\begin{aligned}
\delta_0' &= [\langle 0\rangle, \langle 1/5\rangle]\,,\\
\delta_1' &= [\langle 0\rangle, \langle 1\rangle, \langle 1/2\rangle, \langle 1/3\rangle]\,,\\
\delta_2' &= [\langle 0\rangle, \langle 1/(2n+1)\rangle]\,,\\
\delta_3' &= [\langle 0\rangle, \langle 1\rangle, \langle 1/2\rangle]\,.
\end{aligned}$$

Claim 3. δ is not a type-I system.

We now prove the lemma using the claims. By Claim 3, δ is of type III or type II. Assume first that δ is type III. Then using Claim 1, we may assume that δ is the admissible edgepath system listed in Claim 1. Proposition 2.5 of [7] implies that since the sum of integer vertices of δ is 1, if two of the paths δ_i are completely reversible then δ is a compressible path system. Because the triangle $[\langle 1/5\rangle, \langle 0\rangle, \langle 1\rangle]$ shares an edge with the triangle $[\langle 0\rangle, \langle\infty\rangle, \langle 1\rangle]$, δ_0 is completely reversible. Similar arguments apply to δ_2 and δ_3 showing that they are also completely reversible. Hence δ is a compressible path system and there are no incompressible surfaces carried by δ.

Now assume that δ is a type-II system. Using Claim 2, we may assume that δ is of the form mentioned in the claim. In this case, the cycle of final r-values for δ is $(-4, 1, -2n, 1)$. Proposition 2.9 of [7] shows that there is no incompressible surface carried by δ. $\square$

First, we make some definitions. A *basic path* is a minimal path starting at a vertex $\langle \frac{p}{q} \rangle$ that proceeds monotonically to the left (without vertical edges) ending at the left edge of $\mathcal{S}$. A *basic path system* is a path system made up of basic paths. Note that a basic path δ_i starting at $\langle \frac{1}{r} \rangle$ is either

$$[\langle 1 \rangle, \langle 1/2 \rangle, \langle 1/3 \rangle, \ldots, \langle 1/r \rangle] \quad \text{or} \quad [\langle 0 \rangle, \langle 1/r \rangle].$$

Hence any such path ends at either $\langle 0 \rangle$ or $\langle 1 \rangle$. If it ends at $\langle 0 \rangle$, then $t(\delta_i) = 2(1-0) = 2$. If it ends at $\langle 1 \rangle$, then $t(\delta_i) = 2(0 - (r-1)) = 2 - 2r$. If δ is a basic path system for K_n^τ that satisfies (E1), then δ_0 starts at $\langle \frac{1}{5} \rangle$, δ_1 starts at $\langle \frac{1}{3} \rangle$, δ_2 starts at $\langle \frac{1}{2n+1} \rangle$, and δ_3 starts at $\langle \frac{1}{2} \rangle$. So we have

$$t(\delta_0) = \begin{cases} 2 & \text{if } \delta_0 \text{ ends at } \langle 0 \rangle, \\ -8 & \text{if } \delta_0 \text{ ends at } \langle 1 \rangle, \end{cases}$$
$$t(\delta_1) = \begin{cases} 2 & \text{if } \delta_1 \text{ ends at } \langle 0 \rangle, \\ -4 & \text{if } \delta_1 \text{ ends at } \langle 1 \rangle, \end{cases}$$
$$t(\delta_2) = \begin{cases} 2 & \text{if } \delta_2 \text{ ends at } \langle 0 \rangle, \\ -4n & \text{if } \delta_2 \text{ ends at } \langle 1 \rangle, \end{cases}$$
$$t(\delta_3) = \begin{cases} 2 & \text{if } \delta_3 \text{ ends at } \langle 0 \rangle, \\ -2 & \text{if } \delta_3 \text{ ends at } \langle 1 \rangle. \end{cases}$$

Proof of Claim 1. Let δ be a type-III path system with $t(\delta) = 2$. Then δ has no vertical edges, since otherwise some δ_i will travel along a sequence of edges of the form $[\langle \infty \rangle, \langle n \pm 1 \rangle, \langle n \rangle]$ which contradicts the fact that δ_i never travels along two sides of a triangle in $\mathcal{D}$ in succession. Let δ' be the basic path system $\delta \cap \mathcal{S}$. Since extending paths to $\langle \infty \rangle$ does not affect twist number, we have $2 = t(\delta) = t(\delta')$. There are two choices for each of the four paths, and so we have a total of 2^4 possible path systems δ'. The corresponding twist numbers are in the set

$$\{-12, -8, -6, -2, 2, 4, 8, 6-4n, 2-4n, -4n, \\ -4-4n, -8-4n, -10-4n, -14-4n\}.$$

Since n is an integer, $2 \notin \{-4n,\ -4-4n,\ -8-4n\}$ and using our assumption $n \neq 1$ we have $2 \notin \{6-4n,\ 2-4n,\ -10-4n,\ 14-4n\}$. Therefore, we must have $t(\delta') = 2$ for every n. By listing all the possibilities for δ', we see that there is only one for which the equality $t(\delta') = 2$ is independent of n. It extends to the system δ given in the claim. □

Proof of Claim 2. Assume that δ is a type-II path system with $t(\delta) = 2$. Then $t(\delta)$ is determined by the basic path system δ' obtained by deleting all vertical edges of δ. There are infinitely many extensions of δ', which satisfy (E3), formed by adding vertical edges to the ends of the individual paths. However, even if the extensions do not satisfy the minimality condition (E2), any two such extensions will always have the same twist number. Thus, we may choose to work with the extension $\tilde{\delta}$, where all paths end at $\langle 0\rangle$. Then

$$t(\tilde{\delta}_i) = \begin{cases} 2 & \text{if } \delta_i' \text{ ends at } \langle 0\rangle, \\ t(\delta_i') + 2 & \text{if } \delta_i' \text{ ends at } \langle 1\rangle. \end{cases}$$

Again there are 2^4 possibilities. The corresponding twist numbers are in the set

$$\{-6,\ -4,\ -2,\ 0,\ 2,\ 4,\ 6,\ 8,\ 8-4n,\ 6-4n,\ 4-4n,\\ 2-4n,\ -4n,\ -2-4n,\ -4-4n,\ -6-4n\}.$$

As before, we know that $2 \notin \{-4-4n, -4n, 4-4n, 8-4n\}$ since n is an integer. Again using our assumption $n \neq 1$, we have $2 \notin \{-6-4n, -2-4n, 2-4n, 6-4n\}$. The only remaining possibility satisfies the conclusion of the claim. □

Proof of Claim 3. Assume that δ is a type-I path system. Let $(t_0, \ldots, t_3)$ be the 4-tuple of endpoints of the paths δ_i. Since δ is an admissible path system we know that the sum of the vertical coordinates of the t_is is zero. Moreover, every point of every δ_i has vertical coordinate greater than or equal to zero. Therefore $t_i = 0$ for every i. We have

$$\begin{aligned} \delta_0 &= [\langle 0\rangle, \langle 1/3\rangle], \\ \delta_1 &= [\langle 0\rangle, \langle 1/5\rangle], \\ \delta_2 &= [\langle 0\rangle, \langle 1/(2n+1)\rangle], \\ \delta_3 &= [\langle 0\rangle, \langle 1/2\rangle]. \end{aligned}$$

However, for this path system, we have

$$t(\delta) = 2 + 2 + 2 + 2 = 8.$$

We have now established Claim 3. □

Next, we want to show that K_n has boundary slope $4(n+4)$. Consider the path system $\gamma = (\gamma_i)$ in $\mathcal{S}$ given by

$$\begin{aligned}
\gamma_0 &= [\langle 1\rangle, \langle 1/2\rangle, \langle 1/3\rangle]\,,\\
\gamma_1 &= [\langle 0\rangle, \langle 1/5\rangle]\,,\\
\gamma_2 &= [\langle 0\rangle, \langle 1/(2n+1)\rangle]\,,\\
\gamma_3 &= [\langle 1\rangle, \langle 1/2\rangle]\,.
\end{aligned}$$

Note that γ is not an admissible edgepath system, but by adding vertical edges to the ends of the edgepaths, γ extends to an admissible edgepath system.

Lemma 2.3. *For the knot K_n, the path system γ extends to an admissible edgepath system that carries incompressible surfaces all of which have boundary slope $4(n+4)$. Also, every admissible edgepath system that carries this slope is a vertical extension of γ.*

Proof. The cycle of final r-values for the path system γ is $(1, -4, -2n, 1)$. Furthermore, the final slopes of the γ_i have positive sum, γ satisfies (E2), and each γ_i ends on the left edge of $\mathcal{S}$. Therefore, by [7, Proposition 2.9], the γ_is can be extended by vertical edges to form a system that carries an incompressible surface. As in the proof of Claim 2 above, we can calculate the twist number of any such path system by calculating the twist number of the vertical extension γ' given by adding the vertical edge $[\langle 0\rangle, \langle 1\rangle]$ to both γ_0 and γ_3.

We have

$$\begin{aligned}
t(\gamma_0') &= 2(1-2) = -2,\\
t(\gamma_1') &= 2(1-0) = 2,\\
t(\gamma_2') &= 2(1-0) = 2,\\
t(\gamma_3') &= 2(1-1) = 0,
\end{aligned}$$

giving

$$t(\gamma') = \sum t(\gamma_i) = -2 + 2 + 2 = 2.$$

Therefore the slope of any surface carried by a vertical extension of γ is $t(\gamma') - t(s) = 2 + 14 + 4n = 4(n+4)$.

The uniqueness of γ follows exactly as in Lemma 2.2. □

Combining Lemmas 2.2 and 2.3 yields the following result.

Proposition 2.4. *For every $n > 1$, $4(n+4)$ is a boundary slope of K_n but not of K_n^τ. Furthermore, every admissible edgepath system that carries this slope is a vertical extension of γ.*

3. $\mathbf{I}_\mu$ is constant on X

The notation given in the proof of Theorem 1.3 is used throughout this section. Recall that if $4(n+4)$ is a strongly detected boundary slope of K_n, then there is a curve $X \subset X(K_n)$ with the following two properties: (1) X contains an irreducible character, and (2) the restriction of any character on X to H is reducible. Since the set of irreducible characters is dense on X, the fact that I_μ is constant on X follows from the following Proposition.

Proposition 3.1. *There exists a finite set $\Lambda \subset \mathbb{C}$ such that if $\chi \in X(K_n)$ is an irreducible character with the property that $\chi|_H$ is reducible, then $I_\mu(\chi) \in \Lambda$.*

The remainder of this paper is devoted to the proof of this proposition. Three facts that will be used repeatedly and hold in a more general context are established in the next subsection. Subsequently, a lemma pertaining to $\pi_1(S^3 \setminus K_n)$ and Proposition 3.1 are proved.

3.1. Three useful facts

Lemma 3.2. *Assume that a (twist) region in a knot diagram has Wirtinger generators $\{w_0, w_1, \ldots, w_{2k+2}\}$ as shown in figure 6. Then for every $k \geq 0$*

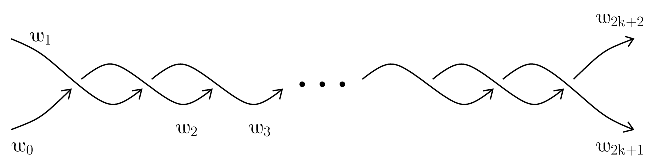

Figure 6: The diagram for Lemma 3.2.

we have

$$w_{2k+2} = (w_0 w_1)^{-k} w_1^{-1} (w_0 w_1)^{k+1}$$

and

$$w_{2k+1} = (w_0 w_1)^{-k} w_1 (w_0 w_1)^{k}.$$

Proof. For $k = 0$, the relations are the Wirtinger relation and the trivial relation, respectively. The conclusion follows by induction. □

Lemma 3.3. *Let G be a group and $x, y \in G$ be elements which have identical images in the abelianization G^{AB} of G. If $\rho :\ G \to \mathrm{SL}_2(\mathbb{C})$ is a homomorphism such that $\rho(G)$ is a group of upper triangular matrices, then $\rho(x)$ and $\rho(y)$ are identical along their diagonals.*

Proof. Let $\Delta < \mathrm{SL}_2(\mathbb{C})$ be the subgroup of upper-triangular matrices and $D < \mathrm{SL}_2(\mathbb{C})$ be the abelian subgroup of diagonal matrices. Then we have an epimorphism $\delta :\ \Delta \to D$ given by

$$\delta \begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix}.$$

Since x and y have the same image in G^{AB} we have $\delta\rho(x) = \delta\rho(y)$. □

Lemma 3.4. *Let $\alpha \in \mathbb{C} \setminus \{0\}$. If $A = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} \alpha & 1 \\ 0 & 1/\alpha \end{smallmatrix}\right)$, then for every $n \in \mathbb{Z}^+$*

$$(AB)^n = \begin{pmatrix} \alpha^{2n} & p_n(\alpha) \\ 0 & \alpha^{-2n} \end{pmatrix},$$

where $p_n(x) \in \mathbb{C}(x)$ and $p_n(x) \neq 1/((1-x^2)x^{2n-1})$.

Proof. First we establish a recursive formula for $p_n(x)$.
Claim 1. $p_1(x) = x$ and $p_n(x) = x^{3-2n} + x^2 p_{n-1}(x)$.
For $n = 1$ we calculate

$$AB = \begin{pmatrix} \alpha^2 & \alpha \\ 0 & \alpha^{-2} \end{pmatrix}$$

to see that $p_1(x) = x$ as claimed. Now assume that

$$(AB)^{n-1} = \begin{pmatrix} \alpha^{2n-2} & p_{n-1}(\alpha) \\ 0 & \alpha^{2-2n} \end{pmatrix}.$$

Multiplying this by AB we see that

$$(AB)^n = \begin{pmatrix} \alpha^{2n} & \alpha^2\, p_{n-1}(\alpha) + \alpha^{3-2n} \\ 0 & \alpha^{-2n} \end{pmatrix}.$$

Hence, Claim 1 is true.
Claim 2. $x^{2n-3}p_n(x) \in \mathbb{C}[x]$.
Again, we establish this by induction. For $n = 1$

$$x^{-1}p_1(x) = x^{-1} \cdot x = 1 \ \in \mathbb{C}[x].$$

Now assume that $x^{2n-5}p_{n-1}(x) \in \mathbb{C}[x]$. Then, using Claim 1, we have

$$\begin{aligned} x^{2n-3}p_n(x) &= x^{2n-3}\big(x^{3-2n} + x^2\, p_{n-1}(x)\big) \\ &= 1 + x^{2n-1}\, p_{n-1}(x). \end{aligned}$$

This is in $\mathbb{C}[x]$ by the inductive assumption.
Finally, we see that $p_n(x) \neq 1/((1-x^2)x^{2n-1})$ because

$$x^{2n-3} \cdot \frac{1}{(1-x^2)x^{2n-1}} = \frac{1}{(1-x^2)x^2} \notin \mathbb{C}[x].$$

The lemma follows. □

3.2. A lemma and a calculation

To simplify notation, let $M = S^3 \setminus K_n$ throughout the remainder of this section. A Wirtinger presentation for $\pi_1(M)$ can be obtained from figure 2 with generating set $\{a, b, c, d, e, f\}$ given by the labels in the figure. We single out the following relations which will be used repeatedly in the arguments that follow:

(R1) $ae = eb$, and

(R2) $d = ab^{-1}c$.

Also, by applying Lemma 3.2 to T_0 and T_1, we get

(R3) $b^{-1} = faf^{-1}a^{-1}f^{-1}$ ($\Leftrightarrow$ $faf = bfa$), and

(R4) $c = (fd)^{-2}d^{-1}(fd)^3$ ($\Leftrightarrow$ $d(fd)^2c = (fd)^3$).

Denote by χ_ρ the character of $\rho \in R(M)$; thus, if $\gamma \in \pi_1(M)$ then $\chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma))$. Since generators in a Wirtinger presentation are conjugate, we have $\chi_\rho(a) = \chi_\rho(b) = \chi_\rho(c) = \chi_\rho(d) = \chi_\rho(e) = \chi_\rho(f)$ for every $\rho \in R(M)$. This will be used implicitly.

Lemma 3.5. *If $\rho \in R(M)$ with $\rho(a) = \rho(b^{-1})$ and $\chi_\rho(a) \neq \pm 2$ then $\chi_\rho(a) = 0$.*

Proof. Since $\chi_\rho(a) \neq \pm 2$ we may conjugate ρ to assume that $\rho(a)$ is diagonal. Let

$$\rho(a) = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \quad \text{and} \quad \rho(e) = \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

The lemma follows from the relation (R1), $ae = eb$. We have

$$\rho(ae) = \begin{pmatrix} \alpha w & \alpha x \\ y/\alpha & z/\alpha \end{pmatrix}$$

and

$$\rho(eb) = \rho(ea^{-1}) = \begin{pmatrix} w/\alpha & \alpha x \\ y/\alpha & \alpha z \end{pmatrix}.$$

Then $\alpha w = w/\alpha$. This implies that either $\alpha^2 - 1 = 0$ or $w = 0$. The first possibility is ruled out by assumption, and hence $w = 0$. Similarly we have $z = 0$. Therefore $\chi_\rho(a) = \chi_\rho(e) = 0$. $\square$

The Conway sphere S_4 separates M into two submanifolds M_1 and M_2. Let M_1 be the piece that contains the tangles T_0 and T_1. Choose a basepoint in S_4 for $\pi_1(M)$ and write $\Gamma_i = \mathrm{im}(\pi_1(M_i) \to \pi_1(M))$. For the convenience of the reader, Proposition 3.1 is stated again:

Proposition 3.1. *There exists a finite set $\Lambda \subset \mathbb{C}$ such that if $\chi \in X(K_n)$ is an irreducible character with the property that $\chi|_H$ is reducible, then $I_\mu(\chi) \in \Lambda$.*

Proof. Since $a \in \pi_1(M)$ is a meridian, it suffices to find a finite set $\Lambda \subset \mathbb{C}$ with the following property. If $\rho \in R(M)$ is irreducible and $\rho|_H$ is reducible, then $\chi_\rho(a) \in \Lambda$. The strategy is to use the relations of the Wirtinger presentation of $\pi_1(M)$ to calculate all conjugacy classes of such representations and to show that the eigenvalues of $\rho(a)$ must satisfy one of finitely many polynomial equations.

Assume that $\rho \in R(M)$ is irreducible and that $\rho|_H$ is reducible. Set $\Lambda = \{0, \pm 2\}$. Finitely many values are added to Λ as we proceed through the proof.

We may replace ρ by a conjugate (which again will be denoted by ρ) with the property that its restriction to H is upper triangular. To simplify notation, given $g \in \pi_1(M)$, $\rho(g)$ will be denoted by $\mathfrak{g}$. Thus, the matrices $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and $\mathfrak{d}$ are all upper triangular. Moreover, we assume that $\mathrm{Tr}(\mathfrak{a}) \notin \Lambda = \{0, \pm 2\}$.

Case 1. $\rho|_H$ is abelian.

Since $\mathrm{Tr}(\mathfrak{a}) \neq \pm 2$, we can conjugate ρ to assume that $\mathfrak{a}$ is diagonal. Since $\rho(H)$ is abelian we have $\mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \{\mathfrak{a}^{\pm 1}\}$. Lemma 3.5 implies $\mathfrak{b} = \mathfrak{a}$, and hence relation (R1) yields $\mathfrak{a}\mathfrak{e} = \mathfrak{e}\mathfrak{a}$. Thus, $\mathfrak{e}$ is diagonal. Since Γ_2 is generated by $\{a, b, c, d, e\}$, $\rho(\Gamma_2)$ is diagonal, and hence $\rho|_{\Gamma_1}$ must be irreducible. Relation (R2) implies that $\mathfrak{d} = \mathfrak{c}$ and so (R3) yields $\mathfrak{f}\mathfrak{a}\mathfrak{f} = \mathfrak{a}\mathfrak{f}\mathfrak{a}$ and (R4) yields $\mathfrak{c}\mathfrak{f}\mathfrak{c}\mathfrak{f}\mathfrak{c} = \mathfrak{f}\mathfrak{c}\mathfrak{f}\mathfrak{c}\mathfrak{f}$. Consider the possibilities $\mathfrak{c} \in \{\mathfrak{a}^{\pm 1}\}$ separately:

Subcase 1(A). $\mathfrak{c} = \mathfrak{a}$.
Relation (R4) yields $\mathfrak{a}\mathfrak{f}\mathfrak{a}\mathfrak{f}\mathfrak{a} = \mathfrak{f}\mathfrak{a}\mathfrak{f}\mathfrak{a}\mathfrak{f}$. Using (R3) we have

$$\begin{aligned} \mathfrak{a}\mathfrak{f}\mathfrak{a}\mathfrak{f}\mathfrak{a} &= \mathfrak{f}\mathfrak{a}(\mathfrak{f}\mathfrak{a}\mathfrak{f}) \\ &= \mathfrak{f}\mathfrak{a}(\mathfrak{a}\mathfrak{f}\mathfrak{a}). \end{aligned}$$

Whence $\mathfrak{a}\mathfrak{f} = \mathfrak{f}\mathfrak{a}$, so $\mathfrak{f}$ is diagonal which contradicts the fact that $\rho|_{\Gamma_1}$ is irreducible.

Subcase 1(B). $\mathfrak{c} = \mathfrak{a}^{-1}$.
Since $\rho|_{\Gamma_1}$ is irreducible, one may conjugate ρ further so that

$$\mathfrak{a} = \begin{pmatrix} \alpha & 1 \\ 0 & 1/\alpha \end{pmatrix} \quad \text{and} \quad \mathfrak{f} = \begin{pmatrix} \beta & 0 \\ r & 1/\beta \end{pmatrix},$$

where $\beta \in \{\alpha^{\pm 1}\}$ and $r \neq 0$.

If $\beta = \alpha$, then (R3) implies $1 = \alpha^2 + r + \frac{1}{\alpha^2}$ and (R4) implies $r^2 - 3r + 1 = 0$. There are at most eight simultaneous solutions to these two equations. The corresponding finite number of values of $\mathrm{Tr}(\mathfrak{a})$ are adjoined to Λ.

If $\beta = 1/\alpha$, then (R3) forces $r = -1$. Together with (R4), this implies that α satisfies the equation $\alpha^8 + \alpha^6 + \alpha^4 + \alpha^2 + 1 = 0$. Again, a finite number of values is added to Λ to account for these representations.

Case 2. $\rho|_H$ is non-abelian and $\rho|_{\Gamma_2}$ is reducible.
It may be assumed that $\mathfrak{a}$ is diagonal and $\mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ and $\mathfrak{e}$ are upper triangular.

Subcase 2(A). $[\mathfrak{a}, \mathfrak{b}] = I$.
Then $\mathfrak{b} \in \{\mathfrak{a}^{\pm 1}\}$, and Lemma 3.3 yields $\mathfrak{b} = \mathfrak{a}$. Relation (R1) gives $[\mathfrak{a}, \mathfrak{e}] = I$ and $\mathfrak{e}$ is diagonal. Also, (R2) implies $\mathfrak{d} = \mathfrak{c}$. Since $\rho(H)$ is non-abelian, $\mathfrak{c}$ cannot be diagonal. Up to conjugation, we may assume that

$$\mathfrak{e} = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \quad \text{and} \quad \mathfrak{c} = \begin{pmatrix} \beta & 1 \\ 0 & 1/\beta \end{pmatrix},$$

where $\beta \in \{\alpha^{\pm 1}\}$. Since c and e have the same image in Γ_2^{AB} and $\rho|_{\Gamma_2}$ is reducible, Lemma 3.3 implies that $\mathfrak{c}$ and $\mathfrak{e}$ are identical along their diagonals. Hence $\beta = \alpha$.
Lemma 3.2 gives $(\mathfrak{e}\mathfrak{c})^n \mathfrak{e} = \mathfrak{c}(\mathfrak{e}\mathfrak{c})^n$ and Lemma 3.4 yields

$$(\mathfrak{e}\mathfrak{c})^n = \begin{pmatrix} \alpha^{2n} & p_n(\alpha) \\ 0 & \alpha^{-2n} \end{pmatrix},$$

where $p_n(x) \in \mathbb{C}(x)$ and $p_n(x) \neq 1/((1 - x^2)x^{2n-1})$.
Equating the upper-right entries of the matrices $(\mathfrak{e}\mathfrak{c})^n \mathfrak{e}$ and $\mathfrak{c}(\mathfrak{e}\mathfrak{c})^n$ gives

$$\alpha\, p_n(\alpha) + \alpha^{-2n} = \frac{1}{\alpha}\, p_n(\alpha).$$

Since $\alpha \notin \{0, \pm 1\}$, we have

$$p_n(\alpha) = \frac{1}{(1 - \alpha^2)\alpha^{2n-1}}.$$

This equation has finitely many solutions because $p_n(x) \neq 1/((1 - x^2)x^{2n-1})$. The corresponding values of $\alpha + \alpha^{-1}$ are added to Λ.

Subcase 2(B). $[\mathfrak{a}, \mathfrak{b}] \neq I$.
Recall that $\mathfrak{a}$ is diagonal and $\rho(\Gamma_2)$ is upper triangular. Since $[\mathfrak{a}, \mathfrak{b}] \neq I$, the upper-right entry of $\mathfrak{b}$ is non-zero, and we can conjugate by diagonal matrices to assume this entry is 1. As above, Lemma 3.3 implies that $\mathfrak{a}$ and $\mathfrak{b}$ are identical along their diagonals, giving:

$$\mathfrak{a} = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \quad \text{and} \quad \mathfrak{b} = \begin{pmatrix} \alpha & 1 \\ 0 & 1/\alpha \end{pmatrix}.$$

Since ρ is irreducible, the lower-left entry of $\mathfrak{f}$ is non-zero, and using $\mathrm{Tr}(\mathfrak{f}) = \mathrm{Tr}(\mathfrak{a}) = \alpha + 1/\alpha$, we obtain

$$\mathfrak{f} = \begin{pmatrix} x & \dfrac{1}{\alpha y}\,(-\alpha x^2 + \alpha^2 x + x - \alpha) \\ y & \dfrac{1}{\alpha}\,(-\alpha x + \alpha^2 + 1) \end{pmatrix},$$

where $y \neq 0$.

Relation (R3) implies $\mathfrak{faf} = \mathfrak{bfa}$. Equating the lower-left entries gives:

$$\alpha x y + \frac{y}{\alpha^2}\,(-\alpha x + \alpha^2 + 1) = 0.$$

Then $y \neq 0$ yields $x = -1/(\alpha(\alpha^2 - 1))$. Making this substitution and equating the upper-left entries of $\mathfrak{faf}$ and $\mathfrak{bfa}$, we arrive at the contradiction $y\alpha = 0$.

Case 3. $\rho|_H$ is non-abelian and $\rho|_{\Gamma_2}$ is irreducible.

Conjugating ρ so that $\mathfrak{a}$ is diagonal and $\mathfrak{b}, \mathfrak{c}$ and $\mathfrak{d}$ are upper triangular, we know that the lower left entry of $\mathfrak{e}$ must be non-zero.

First, note that $[\mathfrak{a}, \mathfrak{b}] \neq I$. Since $\mathfrak{a}$ is diagonal and not parabolic, if $\mathfrak{a}$ and $\mathfrak{b}$ commute, then $\mathfrak{b} = \mathfrak{a}^{-1}$ or $\mathfrak{b} = \mathfrak{a}$. The first possibility is ruled out by Lemma 3.5. If $\mathfrak{a} = \mathfrak{b}$, then (R1) implies that $\mathfrak{a}$ and $\mathfrak{e}$ commute which is not possible because the lower left entry of $\mathfrak{e}$ is non-zero. We may therefore assume that

$$\mathfrak{a} = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \quad \text{and} \quad \mathfrak{b} = \begin{pmatrix} \beta & 1 \\ 0 & 1/\beta \end{pmatrix},$$

where $\beta \in \{\alpha^{\pm 1}\}$. As argued for $\mathfrak{f}$ in Subcase 2(B), there is $y \neq 0$ so that:

$$\mathfrak{e} = \begin{pmatrix} x & \dfrac{1}{\alpha y}\,(-\alpha x^2 + \alpha^2 x + x - \alpha) \\ y & \dfrac{1}{\alpha}\,(-\alpha x + \alpha^2 + 1) \end{pmatrix}.$$

The lower-right entries of $\mathfrak{ae}$ and $\mathfrak{eb}$ are equal due to (R1). If $\beta = \alpha$, this yields:

$$\frac{1}{\alpha^2}\,(-\alpha x + \alpha^2 + 1) = y + \frac{1}{\alpha^2}\,(-\alpha x + \alpha^2 + 1),$$

which implies $y = 0$, a contradiction. We therefore have

$$\mathfrak{b} = \begin{pmatrix} 1/\alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$

Comparing the upper-right entries of $\mathfrak{ae}$ and $\mathfrak{eb}$ gives $x = 0$. Substituting for x, the lower-right entries of $\mathfrak{ae} = \mathfrak{eb}$ imply $y = \alpha^{-2}(1 - \alpha^4)$, and hence

$$\mathfrak{e} = \begin{pmatrix} 0 & \dfrac{\alpha^2}{\alpha^4 - 1} \\ \dfrac{1 - \alpha^4}{\alpha^2} & \dfrac{\alpha^2 + 1}{\alpha} \end{pmatrix}.$$

Since $\mathfrak{c}$ is upper triangular and $\mathrm{Tr}(\mathfrak{c}) = \mathrm{Tr}(\mathfrak{a})$, put

$$\mathfrak{c} = \begin{pmatrix} \gamma & r \\ 0 & 1/\gamma \end{pmatrix},$$

where $\gamma \in \{\alpha^{\pm 1}\}$ and $r \in \mathbb{C}$. Formally assign the matrix $\Theta \in \mathrm{SL}_2(\mathbb{C})$ to represent the product $(\mathfrak{ec})^n$.

Applying Lemma 3.2 to the tangle T_2, we get the relation $h = (ec)^{-n} c(ec)^n$, and from T_3 we have $h = aea^{-1}$, where h is the generator shown in figure 2. We thus conclude that Θ satisfies the relation

$$\Theta^{-1}\mathfrak{c}\,\Theta = \mathfrak{aea}^{-1}. \quad \text{(R5)}$$

Moreover, Θ commutes with $\mathfrak{ec}$, whence

$$\Theta\,\mathfrak{ec} = \mathfrak{ec}\,\Theta. \quad \text{(R6)}$$

Using (R5) and $\det(\Theta) = 1$, Θ can be expressed in terms of α, γ, r, and a new independent variable z. By comparing the matrices in (R6), r is written as a function in α, γ and z. If $\gamma = \alpha$ then the upper-right entries in (R5) give the equality $0 = \alpha^6 - \alpha^4 - \alpha^2 + 1 = (\alpha - 1)^2(\alpha + 1)^2(\alpha^2 + 1)$.[1] But this implies $\mathrm{Tr}(\mathfrak{a}) \in \Lambda$. Thus, $\gamma = 1/\alpha$ and both $\mathfrak{c}$ and Θ are given in terms of α and z:

$$\mathfrak{c} = \begin{pmatrix} 1/\alpha & r \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad \Theta = \begin{pmatrix} \dfrac{\alpha^4 - z^2}{\alpha^4 z} & \dfrac{\alpha z}{\alpha^4 - 1} \\ \dfrac{z(1 - \alpha^4)}{\alpha^5} & z \end{pmatrix},$$

where $r = (\alpha^4 + \alpha^2 z^2 - z^2)/z^2(\alpha^4 - 1)$. Put

$$\mathfrak{f} = \begin{pmatrix} p & q \\ s & t \end{pmatrix}$$

[1]The authors thank one of the referees for factoring this polynomial.

and consider the consequences of (R3). Equating the upper-right entries of $\mathfrak{faf}$ and $\mathfrak{bfa}$ yields $q(\alpha^3 p + \alpha t - 1) - \alpha t = 0$. If $q = 0$, then $t = 0$ and hence $\det(\mathfrak{f}) = 0$; a contradiction. So $q \neq 0$ and solving $\det(\mathfrak{f}) = 1$ for s gives $s = \frac{pt-1}{q}$. Substituting this and looking at the lower-right entries yields an expression for p in terms of α and t. Comparing the lower-left entries gives $(\alpha - t)(\alpha^2 t - t - \alpha) = 0$. There is one subcase for each factor of this polynomial.

Subcase 3(A). $t = \alpha$ ($\rho|_{\Gamma_1}$ is reducible).

We have $\mathfrak{f}$ in terms of α and q. Using (R3), one can solve for q giving:

$$\mathfrak{f} = \begin{pmatrix} 1/\alpha & \dfrac{\alpha^2}{2\alpha^2 - 1} \\ 0 & \alpha \end{pmatrix}.$$

Relation (R4) then yields the polynomial equation

$$(6\alpha^8 - 17\alpha^6 + 13\alpha^4 + 2\alpha^2 - 4)z^2 - 6\alpha^8 + 7\alpha^6 - 2\alpha^4 = 0.$$

By adding $\{\alpha + \alpha^{-1} | 6\alpha^8 - 17\alpha^6 + 13\alpha^4 + 2\alpha^2 - 4 = 0\}$ to Λ, we may solve for z^2 to get

$$z^2 = \frac{\alpha^4\,(6\alpha^4 - 7\alpha^2 + 2)}{6\alpha^8 - 17\alpha^6 + 13\alpha^4 + 2\alpha^2 - 4}.$$

The only remaining relation is $\Theta = (\mathfrak{ec})^n$. Although the expression for Θ in terms of α and z does contain odd powers of z, the resulting expression for Θ^2 only contains even powers of z. Thus, $\mathrm{Tr}(\mathfrak{ec})$ and $\mathrm{Tr}(\Theta^2)$ can be written as rational functions in α:

$$\mathrm{Tr}(\mathfrak{ec}) = \frac{(\alpha^2+1)(10\alpha^4 - 15\alpha^2 + 6)}{\alpha^4\,(2\alpha^2 - 1)(3\alpha^2 - 2)}$$

and

$$\mathrm{Tr}(\Theta^2) = \frac{(\alpha^2+1)(72\alpha^{12} - 288\alpha^{10} + 446\alpha^8 - 278\alpha^6 - 25\alpha^4 + 108\alpha^2 - 36)}{\alpha^4\,(2\alpha^2 - 1)(3\alpha^2 - 2)(6\alpha^6 - 11\alpha^4 + 2\alpha^2 + 4)}.$$

Consider the relation

$$(\star) \qquad \mathrm{Tr}(\Theta^2) = \mathrm{Tr}\big((\mathfrak{ec})^{2n}\big).$$

Using standard trace identities, $\mathrm{Tr}(\Theta^2) = P\big(\mathrm{Tr}(\mathfrak{ec})\big)$ where $P(x) \in \mathbb{Z}[x]$ is a monic polynomial of degree $2n$. It suffices to show that there are at most finitely many α satisfying this equation. This is equivalent to showing that the rational functions $\mathrm{Tr}(\Theta^2)$ and $P\big(\mathrm{Tr}(\mathfrak{ec})\big)$ are not identically equal. To this end, note that $\mathrm{Tr}(\Theta^2)$ has a pole at the roots of the polynomial $6x^6 - 11x^4 + 2x^2 + 4$ whilst $P\big(\mathrm{Tr}(\mathfrak{ec})\big)$ takes a finite value at each of these roots, since any pole of $P\big(\mathrm{Tr}(\mathfrak{ec})\big)$ is a pole of $\mathrm{Tr}(\mathfrak{ec})$. These rational functions are therefore not identically equal and there are only finitely many values for α that satisfy equation $(\star)$. We add the corresponding traces to Λ.

Subcase 3(B). $t = \frac{\alpha}{\alpha^2-1}$ ($\rho|_{\Gamma_1}$ is irreducible).

Just as in Subcase 3(A), relation (R3) yields:

$$\mathfrak{f} = \begin{pmatrix} \dfrac{\alpha^4-\alpha^2-1}{\alpha(\alpha^2-1)} & \dfrac{\alpha^2}{(\alpha^2+1)(\alpha^2-1)^2} \\ \dfrac{\alpha^4-\alpha^2-2}{\alpha^2} & \dfrac{\alpha}{\alpha^2-1} \end{pmatrix}.$$

After adding $\pm\big(2^{1/2} + 2^{-1/2}\big)$ to Λ, relation (R4) implies:

$$\begin{aligned} &(-\alpha^{12} + 8\alpha^{10} - 22\alpha^8 + 23\alpha^6 - 3\alpha^4 - 9\alpha^2 + 4)z^4 \\ &\quad + (2\alpha^{12} - 12\alpha^{10} + 22\alpha^8 - 11\alpha^6 - 3\alpha^4 + 2\alpha^2)z^2 \\ &\quad - \alpha^{12} + 4\alpha^{10} - 4\alpha^8 = 0. \end{aligned} \tag{3.1}$$

Let $\mathcal{C} \subset \mathbb{C}^2$ be the zero set of this polynomial. Setting the variables

$$T = \mathrm{Tr}(\Theta) = \frac{\alpha^4 z^2 + \alpha^4 - z^2}{\alpha^4 z} \quad (\alpha^4 z^2 + \alpha^4 - z^2 - \alpha^4 z T = 0) \tag{3.2}$$

and

$$E = \mathrm{Tr}(\mathfrak{ec}) = \frac{\alpha^4 z^2 - \alpha^4 + z^2}{\alpha^2 z^2} \quad (\alpha^4 z^2 - \alpha^4 + z^2 - \alpha^2 z^2 E = 0), \tag{3.3}$$

we have a corresponding projection

$$\phi : \mathcal{C} \longrightarrow \big\{(T, E) \in \mathbb{C}^2\big\}.$$

Denote the left-hand sides in the polynomial equations in (3.1)–(3.3) by $F_1(\alpha, z)$, $F_2(\alpha, z, T)$ and $F_3(\alpha, z, E)$, respectively. Then successively taking resultants to first eliminate the variable z and then the variable α gives a polynomial $Q(E, T) \in \mathbb{Z}[E, T]$ with $Q = 0$ on the image of ϕ. For details on elimination theory see [4]. The polynomial Q factors as $Q(E,T) = Q_1(E,T)Q_2(E,T)$; the factors are given below. As before, we can express T as a monic polynomial of degree n, $P(E) \in \mathbb{Z}[E]$. The one variable polynomial equation $Q\big(E, P(E)\big) = 0$ holds on $\mathrm{Im}(\phi)$. As long as this polynomial is not identically zero, the image of ϕ is 0-dimensional. We show that this is true by showing that the degree of each factor is bigger than zero. We have:

$$\begin{aligned}
Q_1(E,T) = {} & (-193 + 130E + 17E^2 - 25E^3 + 4E^4)\, T^{10} + (1804 - 918E \\
& - 183E^2 - 21E^3 + 69E^4 + 17E^5 - 15E^6 + 2E^7)\, T^8 \\
& + (-6148 + 1732E + 1019E^2 + 874E^3 - 653E^4 - 21E^5 + 48E^6 \\
& - E^7 - E^8)\, T^6 + (9120 + 96E - 2676E^2 - 2308E^3 + 1805E^4 \\
& - 22E^5 - 253E^6 - 14E^7 + 92E^8 - 35E^9 + 4E^{10})\, T^4 \\
& + (-5440 - 1472E + 3984E^2 + 1088E^3 - 4280E^4 + 992E^5 \\
& + 1960E^6 - 832E^7 - 338E^8 + 270E^9 - 57E^{10} + 4E^{11})\, T^2 \\
& + 1024 - 512E - 4800E^2 + 2496E^3 + 6992E^4 - 4064E^5 \\
& - 3024E^6 + 2464E^7 + 24E^8 - 440E^9 + 148E^{10} - 20E^{11} + E^{12}, \\
Q_2(E,T) = {} & (1024E^4 - 10912E^3 + 37440E^2 - 46944E + 13856)\, T^{10} \\
& + (-512E^7 + 10832E^6 - 108328E^5 + 608826E^4 - 1905305E^3 \\
& + 3168848E^2 - 2476309E + 620777)\, T^8 + (7648E^8 - 175067E^7 \\
& + 1678461E^6 - 8518512E^5 + 23936306E^4 - 35788057E^3 \\
& + 24138626E^2 - 3643367E - 294154)\, T^6 \\
& + (-1024E^{10} + 28320E^9 - 450643E^8 + 4975482E^7 \\
& - 35933695E^6 + 162677046E^5 - 452556067E^4 + 752595598E^3 \\
& - 706177725E^2 + 334806036E - 65721092)\, T^4 \\
& + (-2112E^{11} + 32176E^{10} - 117141E^9 - 107672E^8 \\
& - 5360670E^7 + 92118286E^6 - 551970416E^5 + 1726044320E^4 \\
& - 3064767392E^3 + 3052536872E^2 - 1557218384E \\
& + 318285440)T^2 - 256E^{12} + 7520E^{11} - 61817E^{10} - 151884E^9 \\
& + 4699044E^8 - 20745512E^7 - 22921896E^6 + 499387232E^5 \\
& - 1903989328E^4 + 3615547616E^3 - 3720895056E^2 \\
& + 1944891072E - 400960576.
\end{aligned}$$

Writing Q_j as a polynomial in $\mathbb{Z}[E][T]$, we get

$$Q_j(E,T) \;=\; \sum_{i=0}^{5} q_i(E)\cdot T^{2i}$$

where

$$\begin{aligned}
&\deg\big(q_0(E)\big) = 12 \Longrightarrow \deg\big(q_0(E)\cdot P(E)^0\big) = 12\\
&\deg\big(q_1(E)\big) = 11 \Longrightarrow \deg\big(q_1(E)\cdot P(E)^2\big) = 11+2n\\
&\deg\big(q_2(E)\big) = 10 \Longrightarrow \deg\big(q_2(E)\cdot P(E)^4\big) = 10+4n\\
&\deg\big(q_3(E)\big) = 8 \Longrightarrow \deg\big(q_3(E)\cdot P(E)^6\big) = 8+6n\\
&\deg\big(q_4(E)\big) = 7 \Longrightarrow \deg\big(q_4(E)\cdot P(E)^8\big) = 7+8n\\
&\deg\big(q_5(E)\big) = 4 \Longrightarrow \deg\big(q_5(E)\cdot P(E)^{10}\big) = 4+10n.
\end{aligned}$$

Since $n > 1$, the degree $4+10n$ term in $q_5(E)\cdot P(E)^{10}$ is the unique term of highest degree in $Q_j\big(E,P(E)\big)$. Therefore,

$$\deg\big(Q_j\big(E,P(E)\big) = 4+10n > 0.$$

We have now established that $\mathrm{Im}(\phi)$ is a finite set of points. The proof will be complete if we show that the fibers of ϕ are finite. Using Equations (3.2) and (3.3),

$$z \;=\; \frac{\alpha^2 T}{2\alpha^2 - E}.$$

This, together with Equation (3.3), implies that

$$0 \;=\; (T^2-4)\alpha^4 + (4E - T^2E)\alpha^2 - E^2 + T^2.$$

Therefore, the fibers are finite over every (E,T) unless $T^2 = 4$ and $E^2 = 4$. These points are not in the image of ϕ, since the system of equations

$$F_1(\alpha,z) = F_2(\alpha,z,T) = F_3(\alpha,z,E) = 0$$

has no solution when $T^2 = 4$ and $E^2 = 4$.

Since in every case at most finitely many possibilities for $\chi_\rho(a)$ are encounted, it follows that there is a finite set Λ with the property that if ρ is an irreducible representation in $R(M)$ such that $\rho|_H$ is reducible, then $\chi_\rho(a) \in \Lambda$. $\square$

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