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Computing linear Hodge integrals

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All Hodge integrals with at most one λ-class can be expressed as polynomials in terms of lower-dimensional Hodge integrals with at most one λ-class. Algorithm to compute any given Hodge integral with at most one λ-class is discussed and some examples are presented.

1. Introduction

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne–Mumford moduli stack of stable curves of genus g with n marked points. A Hodge integral is an integral of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \dots \psi_n^{j_n} \lambda_1^{k_1} \dots \lambda_g^{k_g},$$

where ψ_i is the first Chern class of the cotangent line bundle at the ith marked point, and $\lambda_1, \dots, \lambda_g$ are the Chern classes of the Hodge bundle. Hodge integrals arise naturally in the calculations of Gromov–Witten invariants by localization techniques [21, 22, 23, 24]. Their explicit evaluations are difficult problems. The famous *Witten's conjecture/Kontsevich's theorem* [20, 11, 25] gives a recursive relation of Hodge integrals involving ψ classes only;

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \dots \psi_n^{j_n} \tag{1.1}$$

and some of them can be computed recursively through *string equation* and *KdV hierarchy.* In [2], Faber developed an algorithm to compute intersection numbers of type (1.1). Also, Getzler [4] obtained recursion relations for the case of $g = 2$, one of which is given as;

$$\begin{aligned}\langle\langle\tau_{k+2}(x)\rangle\rangle_2 = &\langle\langle\tau_{k+1}(x)\gamma_a\rangle\rangle_0\langle\langle\gamma^a\rangle\rangle_2 + \langle\langle\tau_k(x)\gamma_a\rangle\rangle_0\langle\langle\tau_1(\gamma^a)\rangle\rangle_2\\ &- \langle\langle\tau_k(x)\gamma_a\rangle\rangle_0\langle\langle\gamma^a\gamma_b\rangle\rangle_0\langle\langle\gamma^b\rangle\rangle_2\\ &+ \frac{7}{10}\langle\langle\tau_k(x)\gamma_a\gamma_b\rangle\rangle_0\langle\langle\gamma^a\rangle\rangle_1\langle\langle\gamma^b\rangle\rangle_1\\ &+ \frac{1}{10}\langle\langle\tau_k(x)\gamma_a\gamma_b\rangle\rangle_0\langle\langle\gamma^a\gamma^b\rangle\rangle_1 - \frac{1}{240}\langle\langle\tau_k(x)\gamma_a\rangle\rangle_1\langle\langle\gamma^a\gamma_b\gamma^b\rangle\rangle_0\\ &+ \frac{13}{240}\langle\langle\tau_k(x)\gamma_a\gamma^a\gamma_b\rangle\rangle_0\langle\langle\gamma^b\rangle\rangle_1\\ &+ \frac{1}{960}\langle\langle\tau_k(x)\gamma_a\gamma^a\gamma_b\gamma^b\rangle\rangle_0 \text{ for } k \geq 0.\end{aligned}$$

When the λ-classes are involved, the computation of Hodge integrals is not easy. There was λ_g-conjecture which computes for the case of one top-degree λ-class as [3];

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g = \binom{2g+n-3}{k_1,\ldots,k_n}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!}, \tag{1.2}$$

where B_{2g} are Bernoulli numbers and $k_1 + \cdots + k_n = 2g - 3 + n$. By using Mariño–Vafa formula [15, 18], the case of λ_{g-1} with one marked point can be computed as;

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-1}\lambda_{g-1} = b_g\sum_{i=1}^{2g-1}\frac{1}{i} - \frac{1}{2}\sum_{g_1+g_2=g}\frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!}b_{g_1}b_{g_2} \tag{1.3}$$

and the case of more than one marked points can be computed by repeatedly applying the cut-and-join equation:

(1.4)
$$\frac{\partial\Omega}{\partial\tau} = \frac{\sqrt{-1}\lambda}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial^2\Omega}{\partial p_i\partial p_j} + ijp_{i+j}\frac{\partial\Omega}{\partial p_i}\frac{\partial\Omega}{\partial p_j} + (i+j)p_ip_j\frac{\partial\Omega}{\partial p_{i+j}}\right)$$

which was used in the proof of Mariño–Vafa formula [15, 16, 18], and proven to be an effective tool in studying Hodge integrals.

The moduli space of relative stable morphisms admits a natural S^1-action induced from the S^1-action on the target space. And as a result of the localization formula applied to it, the following convolution formula is obtained;

Theorem 6.1. *For any partition μ and e with $|e| < |\mu| + l(\mu) - \chi$, we have*

$$\left[\lambda^{l(\mu)-\chi}\right] \sum_{|\nu|=|\mu|} \Phi^{\bullet}_{\mu,\nu}(-\lambda) z_\nu \mathcal{D}^{\bullet}_{\nu,e}(\lambda) = 0$$

where the sum is taken over all partitions ν of the same size as μ.

Here χ is the prescribed Euler number of domain curves, $\left[\lambda^a\right]$ means taking the coefficient of λ^a, $\Phi^{\bullet}(\lambda)$ is a generating series of double Hurwitz numbers, and $\mathcal{D}^{\bullet}(\lambda)$ is a certain generating series of Hodge integrals. This formula gives many relations between *linear Hodge integrals*, i.e., Hodge integrals with at most one λ-class, and it is enough to consider the special case of $\mu = (d)$ for some positive integers d to compute all Hodge integrals with at most one λ-class. More precisely, the following theorem is proved in Section 7;

Theorem 7.2. *Any given Hodge integral with one λ-class:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \lambda_j,$$

where $k_1, \dots, k_n \in \mathbb{N} \cup \{0\}$, $j \in \{0, 1, 2, \dots, g\}$, is explicitly expressed as a polynomial in terms of lower-dimensional Hodge integrals with one λ-class. Therefore it computes all linear Hodge integrals.

The rest of the paper is organized as follows: In Section 2, we summarize various versions of localization formulas which will be used in computing Hodge integrals. In Section 3, the relative moduli space is defined and the natural S^1-action on it is introduced. Also the fixed locus of the S^1-action and their corresponding description in terms of graphs is discussed. In Section 4, we compute the Euler class of the normal bundle of the fixed locus of the S^1-action in the relative moduli space. In Section 5, the double Hurwitz numbers and its description in terms of Hodge integrals over a certain moduli space is discussed. In Section 6, the recursion formula which gives relations between Hodge integrals with at most one λ-class is proved. In Section 7, the recursion formula is used prove that all Hodge integrals with at most one λ-class is explicitly expressed as a polynomial in terms of lower-dimensional Hodge integrals with at most one λ-class. In Section 8, an algorithm to implement the recursion formula and to compute each Hodge integral with at most one λ-class is discussed. In Section 9, some examples of the algorithm in Section 8 are presented.

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2. Localization formula

In this section, I will summarize various versions of localization formulas.

2.1. Equivariant cohomology

Let G be a compact Lie group acting on M. The equivariant cohomology of M is defined as the ordinary cohomology of the space M_G obtained from a fixed universal G-bundle EG, by the mixing construction

$$M_G = EG \times_G M$$

Here, G acts on the right of EG and on the left of M, and the notation means that we identify $(pg, q) \sim (p, gq)$ for $p \in EG$, $q \in M$, $g \in G$. Hence M_G is the bundle with fiber M over the classifying space BG associated to the universal bundle $EG \to BG$. We have natural projection map $\pi : M_G \to BG$ and $\sigma : M_G \to M/G$, which fits into the mixing diagram of Cartan and Borel:

$$\begin{tikzcd}
EG \arrow[d] & EG \times M \arrow[l] \arrow[r] \arrow[d] & M \arrow[d] \\
BG & E \times_G M \arrow[l, "\pi"'] \arrow[r, "\sigma"] & M/G
\end{tikzcd}$$

If G acts smoothly on M, then we have $M_G \simeq M/G$. This is not true in general but it turns out that M_G is a better functorial construction and the proper homotopy theoretic quotient of M by G. In any case, the equivariant cohomology, denoted by $H^*_G(M)$, is defined by

$$H^*_G(M) = H^*(M_G)$$

and constitutes a contravariant functor from G-spaces to modules over the base ring $H^*_G := H^*_G(\mathrm{pt}) = H^*(BG)$. The map σ defines a natural map $\sigma^* : H^*(M/G) \to H^*_G(M)$ which is an isomorphism if G acts freely. The inclusion $i : M \to M_G$ induces a natural map $i^* : H^*_G(M) \to H^*(M)$.

2.2. Atiyah–Bott localization formula

Following [1], let $i : V \hookrightarrow M$ be a map of compact manifolds [1]. The tubular neighborhood of V inside M can be identified with the normal bundle of V. On the total space of the normal bundle, there is the Thom form Φ_V which has compact support in the fibers and integrates to one in each fiber. Extending this form by zero gives a form in M, and multiplying by Φ_V provides a map $H^*(V) \cong H^{*+k}(M, M\backslash V) \to H^*(M)$. In particular, the cohomology class $1 \in H^0(V)$ is sent to the Thom class and this class restricts to be the Euler class of the normal bundle of V in M, $\mathcal{N}_{V/M}$. Hence, we see that

$$i^* i_* 1 = e(\mathcal{N}_{V/M}).$$

This also holds in equivariant cohomology by same argument applied to V_G, M_G. The theorem of Atiyah and Bott says that an inverse of the Euler class of the normal bundle always exists along the fixed locus of a group action. Precisely, $i^*/e(\mathcal{N}_{V/M})$ is the inverse of i_* in equivariant cohomology, i.e., for any equivariant class ϕ,

$$\phi = \sum_F \frac{i_* i^* \phi}{e(\mathcal{N}_{F/M})}$$

holds, where F runs over the fixed locus of the group action. In the integrated form, we have

$$\int_M \phi = \sum_F \int_F \frac{i^* \phi}{e(\mathcal{N}_{F/M})}.$$

2.3. Functorial localization formula

Let X and Y be T-manifolds. Assume $f : X \to Y$ is a T-equivariant map, $j_E : E \hookrightarrow Y$ is a fixed component in Y, and $i_F : F \hookrightarrow f^{-1}(E)$ is a fixed component in X. For any equivariant class $\omega \in H_T^*(X)$, we have the diagrams;

$$\begin{tikzcd}
F \arrow[r, "i_F"] \arrow[d, "g=f|_F"'] & X \arrow[d, "f"] \\
E \arrow[r, "j_E"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\frac{i_F^*(\omega)}{e_T(F/X)} \arrow[d, "g_!"] & \omega \arrow[l, "i_F^*"'] \arrow[d, "f_!"] \\
g_!\left[\frac{i_F^*(\omega)}{e_T(F/X)}\right] & f_!(\omega) \arrow[l, "j_E^*"']
\end{tikzcd}$$

Applying the Atiyah–Bott localizatio formula with the naturality relation $f_!(\omega \cdot f^*\alpha) = f_!\omega \cdot \alpha$, we obtain the functorial localization formula:

$$g_!\Big[\frac{i_F^*(\omega)}{e_T(F/X)}\Big] = \frac{j_E^* f_!(\omega)}{e_T(E/Y)}. \tag{2.1}$$

2.4. Virtual functorial localization formula

The above functorial localization formula is also valid in the case where X and F are virtual fundamental classes [7, 8]. In this paper, we will use $\big[\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1,\mu)\big]^{\mathrm{vir}}$ for X, and $\big[F_\Gamma\big]^{\mathrm{vir}}$ for F. Hence for any equivariant class ω, we have:

$$\int_{\left[\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1,\mu)\right]^{\mathrm{vir}}} \omega = \sum_{F_\Gamma} \int_{\left[F_\Gamma\right]^{\mathrm{vir}}} \frac{i_\Gamma^*(\omega)}{e_T(F_\Gamma/\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu))}. \tag{2.2}$$

3. Relative moduli space and S^1-action

3.1. Moduli space of relative morphisms

For any non-negative integer m, let

$$\mathbb{P}^1[m] = \mathbb{P}^1_{(0)} \cup \mathbb{P}^1_{(1)} \cup \ldots \cup \mathbb{P}^1_{(m)}$$

be a chain of $(m+1)$ copies of $\mathbb{P}^1$ such that $\mathbb{P}^1_{(l)}$ is glued to $\mathbb{P}^1_{(l+1)}$ at $p_1^{(l)}$ for $0 \le l \le m-1$. The irreducible component $\mathbb{P}^1_{(0)}$ is referred to as *the root component*, and other irreducible components are called *bubble components.* Two points $p_1^{(l)} \ne p_1^{(l+1)}$ in $\mathbb{P}^1_{(l)}$ are fixed. Denote by $\pi[m] : \mathbb{P}^1[m] \to \mathbb{P}^1$ the map which is identity on the root component and contracts all bubble components to $\mathbb{P}^{(0)}_1$. Also denote by $\mathbb{P}^1(m) = \mathbb{P}^1_{(1)} \cup \ldots \cup \mathbb{P}^1_{(m)}$ the union of bubble components of $\mathbb{P}^1[m]$.

For a fixed partition μ of a positive integer d, let $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1,\mu)$ be *the moduli space of relative morphisms* $f : \big(C; x_1, \ldots, x_{l(\mu)}, z_1, \ldots, z_n\big) \to \big(\mathbb{P}^1[m], p_1^{(m)}\big)$ such that

(1) $\big(C; x_1, \ldots, x_{l(\mu)}, z_1, \ldots, z_n\big)$ is a possibly disconnected prestable curve of Euler number χ with $l(\mu) + n$ marked points. Here, the marked points are *unordered.*

(2) $f^{-1}(p_1^{(m)}) = \sum_{i=1}^{l(\mu)} \mu_i x_i$ as Cartier divisors and $\deg(\pi[m] \circ f) = |\mu|$.

(3) The preimage of each node in $\mathbb{P}^1[m]$ consists of nodes of C. If $f(y)=p_1^{(l)}$ and C_1 and C_2 are two irreducible components of C which intersects at y, then $f|_{C_1}$ and $f|_{C_2}$ have the same contact order to $p_1^{(l)}$ at y.

(4) The automorphism group of f is finite. Here, an automorphism of f consists of an automorphism of the domain curve and and automorphism of the pointed curve $\big(\mathbb{P}^1(m), p_1^{(0)}, p_1^{(m)}\big)$.

Following [12,13], $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1,\mu)$ is a separated, proper Deligne–Mumford stack with a perfect obstruction theory of virtual dimension $r=-\chi+|\mu|+l(\mu)+n$, and hence has a virtual fundamental class of degree r.

3.2. Torus action

Consider the $\mathbb{C}^*$-action $t\cdot[z^0:z^1]=[tz^0:z^1]$ on $\mathbb{P}^1$. There are two fixed points $p_0=[0:1]$ and $p_1=[1:0]$. Extend this action to the action on $\mathbb{P}^1[m]$ by identifying the root component with $\mathbb{P}^1$ and giving trivial actions on bubble components. Then this extended action on $\mathbb{P}^1[m]$ induces an action on $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1,\mu)$.

3.3. Fixed locus

The connected components of the fixed points set of $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1,\mu)$ under the induced torus action can be parametrized by labeled graphs. For any $f:\big(C;x_1,\dots,x_{l(\mu)},z_1,\dots,z_n\big)\to\mathbb{P}^1[m]$ representing a fixed point of the $\mathbb{C}^*$-action on $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1,\mu)$, the restriction of $\hat{f}:=\pi[m]\circ f:C\to\mathbb{P}^1$ to an irreducible component of C is either a constant map to one of the $\mathbb{C}^*$-fixed points p_0,p_1, or a covering of $\mathbb{P}^1$ fully ramified over p_0 and p_1. Associate a labeled graph Γ to the $\mathbb{C}^*$-fixed point $\big[f:\big(C;x_1,\dots,x_{l(\mu)},z_1,\dots,z_n\big)\to\mathbb{P}^1[m]\big]$ as follows:

(1) For each connected component C_v of $\hat{f}^{-1}(\{p_0,p_1\})$, assign a *vertex* v, a label $g(v)$ which is the arithmetic genus of C_v, and a label $i(v)=\begin{cases}0 & \text{if } \hat{f}(C_v)=p_0,\\ 1 & \text{if } \hat{f}(C_v)=p_1.\end{cases}$ Denote by $V(\Gamma)^{(k)}$ the set of vertices v with $i(v)=k$, for $k=0,1$. The set $V(\Gamma)$ of vertices of the graph Γ will then be the disjoint union of $V(\Gamma)^{(0)}$ and $V(\Gamma)^{(1)}$.

(2) Assign an *edge* e to each rational irreducible component C_e of C such that $\hat{f}|_{C_e}$ is not a constant map. Then $\hat{f}|_{C_e}$ is fully ramified over p_0 and p_1 with degree $d(e)$. Let $E(\Gamma)$ denote the set of edges of Γ.

(3) The set of flags is given by $F(\Gamma) = \{(v,e) : v \in V(\Gamma), e \in E(\Gamma), C_v \cap C_e \neq \emptyset\}$

(4) For each $v \in V(\Gamma)$, define $d(v) = \sum_{(v,e)\in F(\Gamma)} d(e)$ and let $\nu(v)$ be the partition of $d(v)$ determined by $\{d(e) : (v,e) \in F(\Gamma)\}$. In case $m > 0$, we assign an additional label for each $v \in V(\Gamma)^{(1)}$: let $\mu(v)$ be the partition of $d(v)$ determined by the ramification of $f\,|_{C_v} : C_v \to \mathbb{P}^1(m)$ over $p_1^{(m)}$.

Let $G_\chi(\mathbb{P}^1, \mu)$ be the set of all the graphs associated to the $\mathbb{C}^*$-fixed points in $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1, \mu)$. We now describe the set of fixed points associated to a given graph $\Gamma \in G_\chi(\mathbb{P}^1, \mu)$.

- *Case* $m = 0$: Any $\mathbb{C}^*$-fixed point in $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1, \mu)$ which is represented by a morphism to $\mathbb{P}^1[0] = \mathbb{P}^1$ is associated to a graph $\Gamma_0 \in G^0_\chi(\mathbb{P}^1, \mu)$ such that

$$\begin{aligned}
V(\Gamma_0)^{(0)} &= \{v_1^0, \ldots, v_k^0\}, \quad g(v_i^0) = g_i, \quad k \in \mathbb{N}, \quad \sum_i (2 - 2g_i) = \chi \\
V(\Gamma_0)^{(1)} &= \{v_1^\infty, \ldots, v_{l(\mu)}^\infty\}, \quad g(v_1^\infty) = \ldots = g(v_{l(\mu)}^\infty) = 0, \\
E(\Gamma_0) &= \{e_1, \ldots, e_{l(\mu)}\}, \quad d(e_i) = \mu_i \quad \text{for } i = 1, \ldots, l(\mu) \\
j(v) &= \text{number of } z_i\text{'s mapped to } v, \quad \sum_{V(\Gamma_0)^{(0)}} j(v) = n
\end{aligned}$$

The two end-points of the edge e_i are v_j^0 and v_i^∞ for some $1 \leq j \leq k$. Let $\mu(v_i^0) = \{\mu_j | e_j \text{ has } v_i^0 \text{ as an endpoint}\}$. Define

$$\overline{\mathcal{M}}_{\Gamma_0} = \prod_{1 \leq i \leq k} \overline{\mathcal{M}}_{g_i, l(\mu(v_i^0)) + j(v_i^0)},$$

where we take $\overline{\mathcal{M}}_{0,1} = \overline{\mathcal{M}}_{0,2} = \overline{\mathcal{M}}_{1,0} = \{\mathrm{pt}\}$, then there is a morphism $i_{\Gamma_0} : \overline{\mathcal{M}}_{\Gamma_0} \to \overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1, \mu)$ whose image is the fixed locus F_{Γ_0} associated to Γ_0. Hence i_{Γ_0} induces an isomorphism $\overline{\mathcal{M}}_{\Gamma_0}/A_{\Gamma_0} \cong F_{\Gamma_0}$, where A_{Γ_0} is the automorphism group of any morphism associated to the graph Γ_0, which can be obtained from the short exact sequence

$$1 \longrightarrow \prod_{i=1}^{l(\mu)} \mathbb{Z}_{\mu_i} \longrightarrow A_{\Gamma^0} \longrightarrow \mathrm{Aut}(\Gamma_0) \longrightarrow 1.$$

The virtual dimension of F_{Γ_0} with only stable vertices is

$$d_{\Gamma_0} = -\frac{3}{2}\chi + l(\mu) + n.$$

For any vertex $v \in V(\Gamma_0)^{(0)}$, introduce *multiplicity of* v, $m(v)$, as follows:

$$m(v) = |\{w \in V(\Gamma_0)^{(0)} \mid g(v) = g(w),$$
$$\mu(v) = \mu(w),\, j(v) = j(w)\}|.$$

Pick one representatives from each group of identical vertices $\{v_1, \ldots, v_l\}$ so that $\sum m(v_k) = |V(\Gamma_0)^{(0)}|$. Denote by $m(v_k) = m_k$ and now we can find the order of automorphism group of any $\Gamma_0 \in G^0_{\chi,n}(\mathbb{P}^1, \mu)$ to be:

$$|\mathrm{Aut}\,\Gamma_0| = \prod_k m_k!\,|\,j(v_k)!\ \mathrm{Aut}\,\mu(v_k)\,|^{m_k} \tag{3.1}$$

- *Case* $m > 0$: For any given $f : (C; x_1, \ldots, x_{l(\mu)}, z_1, \ldots, z_n) \to \mathbb{P}^1[m]$, consider f_0 and f_∞ which are defined as follows:
  - f_0: Let $C_0 = f^{-1}(p_0^{(0)})$, $\{y_1, \ldots, y_{l(\nu)}\} = f^{-1}(p_1^{(0)})$, and $\{z_1, \ldots, z_{n_0}\}$ mapped to C_0. Then $f_0 : (C_0; y_1, \ldots, y_{l(\nu)}, z_1, \ldots, z_{n_0}) \to \mathbb{P}^1$ corresponds to the case of $m = 0$.
  - f_∞: Let $C_\infty = f^{-1}(\mathbb{P}^1(m))$ and $f_\infty = f\,|_{C_\infty}$, then f_∞ corresponds to an element of $\overline{\mathcal{M}}^{(1)}_\Gamma$ corresponding to the graph Γ described below.

  Classify the vertices of the graph Γ as follows:

$$V^I(\Gamma)^{(0)} = \{v \in V(\Gamma)^{(0)} : r_0(v) = -1\},$$
$$V^{II}(\Gamma)^{(i)} = \{v \in V(\Gamma)^{(i)} : r_i(v) = 0\},\ \text{ for } i = 0, 1,$$
$$V^S(\Gamma)^{(i)} = \{v \in V(\Gamma)^{(i)} : r_i(v) > 0\},\ \text{ for } i = 0, 1,$$

where $r_0(v) = 2g(v) - 2 + \mathrm{val}(v) + j(v)$, for $v \in V(\Gamma)^{(0)}$,

$$r_1(v) = 2g(v) - 2 + l(\mu(v)) + l(\nu(v)) + j(v),\ \text{ for } v \in V(\Gamma)^{(1)}.$$

Define $\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}^{(0)}_\Gamma \times \overline{\mathcal{M}}^{(1)}_\Gamma$, where $\overline{\mathcal{M}}^{(0)}_\Gamma = \prod_{v \in V^S(\Gamma)^{(0)}} \overline{\mathcal{M}}_{g(v),\mathrm{val}(v)+j(v)}$ and $\overline{\mathcal{M}}^{(1)}_\Gamma$ is the moduli space of morphisms $\hat{f} : \hat{C} \to \mathbb{P}^1(m)$ such that

(1) $\hat{C} = \bigsqcup_{v \in V(\Gamma)^{(1)}} C_v$

(2) For each $v \in V(\Gamma)^{(1)}$, $\big(C_v; x_{v,1}, \ldots, x_{v,l(\mu(v))}, y_{v,1}, \ldots, y_{v,l(\nu(v))}, z_1, \ldots, z_{j(v)}\big)$ is a prestable curve of genus $g(v)$ with $l(\mu(v)) + l(\nu(v)) + j(v)$ ordered marked points.

(3) As Cartier divisors, $(\hat{f}\mid_{C_v})^{-1}(p_1^{(0)}) = \sum_{j=1}^{l(\nu(v))} \nu(v)_j y_{v,j}$, $(\hat{f}\mid_{C_v})^{-1}(p_1^{(m)}) = \sum_{i=1}^{l(\mu(v))} \mu(v)_i x_{v,i}$. The morphism $(\hat{f}\mid_{C_v})^{-1}(E) \to E$ is of degree $d(v)$ for each irreducible component E of $\mathbb{P}^1(m)$.

(4) The automorphism group of $\hat{f}$ is finite. Here, an automorphism of $\hat{f}$ consists of an automorphism of the domain curve $\hat{C}$ and an automorphism of the pointed curve $(\mathbb{P}^1(m), p_1^{(0)}, p_1^{(m)})$, which is an element of $(\mathbb{C}^*)^m$.

There is a morphism $i_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1, \mu)$ whose image is the fixed locus F_Γ associated to the graph Γ. Hence i_Γ induces an isomorphism $\overline{\mathcal{M}}_\Gamma / A_\Gamma \cong F_\Gamma$, where A_Γ is the automorphism group of any morphism associated to the graph Γ, which can be obtained from the short exact sequence

$$1 \longrightarrow \prod_{e \in E(\Gamma)} \mathbb{Z}_{d(e)} \longrightarrow A_\Gamma \longrightarrow \mathrm{Aut}(\Gamma) \longrightarrow 1.$$

The virtual dimension of $\overline{\mathcal{M}}_\Gamma^{(1)}$ is given by $d_\Gamma^{(1)} = \Big(\sum_{v \in V(\Gamma)^{(1)}} r_1(v)\Big) - 1$. Use the identities

$$-\chi = -\chi_0 - \chi_\infty + 2l(\nu), \quad g = \sum_{v \in V(\Gamma)} g(v) - \mid V(\Gamma) \mid + \mid E(\Gamma) \mid + 1$$

to get the virtual dimension of F_Γ:

$$\begin{aligned}
d_\Gamma = d_\Gamma^{(0)} + d_\Gamma^{(1)} &= -\frac{3}{2}\chi_0 + l(\nu) - \chi_\infty + l(\mu) + l(\nu) - 1 + n_\infty \\
&= \sum_{v \in V^S(\Gamma)^{(0)}} \big(3g(v) - 3 + \mathrm{val}(v)\big) + \left(\sum_{v \in V(\Gamma)^{(1)}} r_1(v)\right) - 1 + n_\infty \\
&= 2g - 3 + l(\mu) + \sum_{v \in V^S(\Gamma)^{(0)}} \big(g(v) - 1\big) + \mid V^I(\Gamma)^{(0)} \mid + n_\infty \\
&= -\frac{1}{2}\chi_0 - \chi + l(\mu) - 1 + n_\infty.
\end{aligned}$$

By similar observation as in the case of $m = 0$, we find the order of automorphism group of any given $\Gamma \in G^{\infty}_{\chi}(\mathbb{P}^1, \mu)$ to be:

(3.2)

$$|\mathrm{Aut}\,\Gamma| = |\mathrm{Aut}\,\Gamma_0|\ |\mathrm{Aut}\,\mu|\Big(\prod n_k!\Big)\left(\prod_{V(\Gamma)^{(1)}} j(v)!\ |\mathrm{Aut}\,\mu(v)|\ |\mathrm{Aut}\,\nu(v)|\right).$$

In this case, n_k is the multiplicity of vertices in $V(\Gamma)^{(1)}$ with identical $\mu(v)$, $\nu(v)$, $j(v)$ and $g(v)$. The presence of automorphism group of μ is due to the fact that we can exchange two marked points with same ramification type μ_i without changing the type of corresponding graph.

4. Computation of $e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma})$

In this section, I will summarize the computations of $e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma})$ in [15] which will be needed for localization computation. Denote by (ω) the 1-dimensional representation of $\mathbb{C}^*$ given by $\lambda \cdot z = \lambda^{\omega} z$ for $\lambda \in \mathbb{C}^*$, $z \in \mathbb{C}$. For a given graph $\Gamma \in G_{g,n}(\mathbb{P}^1, \mu)$, let

$$[f : (C, x_1, \ldots, x_{l(\mu)}, z_1, \ldots, z_n) \longrightarrow \mathbb{P}^1[m]] \tag{4.1}$$

be a fixed point of the $\mathbb{C}^*$-action on $\overline{\mathcal{M}}^{\bullet}_{\chi,n}(\mathbb{P}^1, \mu)$ associated to Γ. In order to apply the virtual functorial localization formula (2.2), we need to compute the Euler class of the virtual normal bundle $\mathcal{N}^{vir}_{\Gamma}$ of $[f]$. Given a flag $(v, e) \in F(\Gamma)$, denote by $q_{(v,e)} \in C$ the node at which C_v and C_e intersect. Also let $\psi_{(v,e)}$ denote the first Chern class of the cotangent line bundles over $\overline{\mathcal{M}}_{\Gamma}$, i.e., the fiber at the fixed point (4.1) is given by $T^*_{q(v,e)} C_v$. The Euler class $e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma})$ is given by;

$$\frac{1}{e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma})} = \frac{e_T(\hat{T}^2)}{e_T(\hat{T}^1)},$$

where T^1, T^2 are the tangent space and the obstruction space of $\overline{\mathcal{M}}^{\bullet}_{\chi,n}(\mathbb{P}^1, \mu)$, respectively. The $\hat{\cdot}$-notation denotes the moving part, i.e. $\hat{T}^1$ and $\hat{T}^2$ are the moving parts of the vector bundles T^1 and T^2, respectively. Here, T^1 and T^2 can be computed through the following two exact sequences [13]:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^0(\Omega_C(D), \mathcal{O}_C) \longrightarrow H^0(\mathbf{D}^{\bullet}) \longrightarrow T^1 \longrightarrow \mathrm{Ext}^1(\Omega_C(D), \mathcal{O}_C) \\ &\longrightarrow H^1(\mathbf{D}^{\bullet}) \longrightarrow T^2 \longrightarrow 0, \end{aligned}$$

$$0 \longrightarrow H^0(C, f^*(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)}))^\vee) \longrightarrow H^0(\mathbf{D}^\bullet) \longrightarrow \bigoplus_{l=0}^{m-1} H^0_{\mathrm{et}}(\mathbf{R}^\bullet_l)$$
$$\longrightarrow H^1(C, f^*(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)}))^\vee) \longrightarrow H^1(\mathbf{D}^\bullet) \longrightarrow \bigoplus_{l=0}^{m-1} H^1_{\mathrm{et}}(\mathbf{R}^\bullet) \longrightarrow 0,$$

where $\omega_{\mathbb{P}^1[m]}$ is the dualizing sheaf of $\mathbb{P}^1[m]$, $D = x_1 + \cdots + x_{l(\mu)}$ is the branch divisor, and for $n_l =$ the number of nodes over $p_1^{(l)}$;

$$H^0_{\mathrm{et}}(\mathbf{R}^\bullet_l) \cong \bigoplus_{q \in f^{-1}(p_1^{(l)})} T_q\big(f^{-1}(\mathbb{P}^1_{(l)})\big) \cong \mathbb{C}^{\oplus n_l}$$
$$H^1_{\mathrm{et}}(\mathbf{R}^\bullet_l) \cong \big(T_{p_1^{(l)}}\mathbb{P}^1_{(l)} \otimes T_{p_1^{(l)}}\mathbb{P}^1_{(l+1)}\big)^{\oplus(n_l-1)}.$$

Recall the map $\pi[m] : \mathbb{P}^1[m] \to \mathbb{P}^1$, and observe that for $\hat{f} = \pi[m] \circ f$ we have

$$f^*\big(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)})\big)^\vee \cong \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1).$$

Let F_Γ be the set of fixed points associated to $\Gamma \in G_{\chi,n}(\mathbb{P}^1, \mu)$ and assume that

$$\big[f : (C, x_1, \dots, x_{l(\mu)}, z_1, \dots, z_n) \longrightarrow \mathbb{P}^1[m]\big] \in F_\Gamma \subset \overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1, \mu)$$

The $\mathbb{C}^*$-action on $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1, \mu)$ induces $\mathbb{C}^*$-actions on

$$\mathrm{Ext}^0(\Omega_C(D), \mathcal{O}_C), \;\; H^0(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)), \;\; \bigoplus_{l=0}^{m-1} H^0_{\mathrm{et}}(\mathbf{R}^\bullet_l),$$
$$\mathrm{Ext}^1(\Omega_C(D), \mathcal{O}_C), \;\; H^1(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)), \;\; \bigoplus_{l=0}^{m-1} H^1_{\mathrm{et}}(\mathbf{R}^\bullet_l).$$

The moving part of each of these groups form vector bundles over $\overline{\mathcal{M}}_\Gamma$. We will use the same notation $\hat{}$ to denote the induced vector bundles. In particular,

$$\widehat{\bigoplus_{l=0}^{m-1} H^0_{\mathrm{et}}(\mathbf{R}^\bullet_l)} = 0, \qquad \text{and}$$
$$\widehat{\bigoplus_{l=0}^{m-1} H^1_{\mathrm{et}}(\mathbf{R}^\bullet_l)} = \begin{cases} 0 & \text{if } m = 0, \\ H^1_{\mathrm{et}}(\mathbf{R}^\bullet_0) = \big(T_{p_1^{(0)}}\mathbb{P}^1_{(0)} \otimes T_{p_1^{(0)}}\mathbb{P}^1_{(1)}\big)^{\oplus(n_0-1)} & \text{if } m > 0. \end{cases}$$

Hence we have

$$\frac{1}{e_T(\mathcal{N}_\Gamma^{\mathrm{vir}})} = \frac{e_T(\hat{T}^2)}{e_T(\hat{T}^1)} = \frac{e_T\big(\widehat{\mathrm{Ext}^0(\Omega_C(D), \mathcal{O}_C)}\big)e_T\big(\widehat{H^1(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1))}\big)e_T\big(\widehat{\bigoplus_{l=0}^{m-1} H^1_{\mathrm{et}}(\mathbf{R}_l^\bullet)}\big)}{e_T\big(\widehat{H^0(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1))}\big)e_T\big(\widehat{\mathrm{Ext}^1(\Omega_C(D), \mathcal{O}_C)}\big)}$$

Case $m = 0$: For each $v \in V(\Gamma_0)$ and $\mu_{v,1}, \dots, \mu_{v,l(\mu(v))}$ the ramification type in the vertex v, we have under the convention to write $\mu_{v,2} = \infty$ when $g(v) = 0, l(\mu(v)) = l(e(v)) = 1$;

$$\widehat{\bigoplus_{l=0}^{m-1} H^1_{\mathrm{et}}(\mathbf{R}_l^\bullet)_v} = 0.$$

$$\mathrm{Ext}^0(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v = \begin{cases} \left(\dfrac{1}{\mu_{v,1}}\right) & \text{if } v \in \mathrm{I}, \\ 0 & \text{if } v \in \mathrm{II} \text{ or S}. \end{cases}$$

$$\mathrm{Ext}^1(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v = \begin{cases} 0 & \text{if } v \in \mathrm{I}, \\ \left(\dfrac{1}{\mu_{v,1}} + \dfrac{1}{\mu_{v,2}}\right) & \text{if } v \in \mathrm{II}, \\ \displaystyle\bigoplus_{i=1}^{l(\mu(v))} T_{q_{(v,e_{v,i})}} C_v \otimes T_{q_{(v,e_{v,i})}} C_{e_{v,i}} & \text{if } v \in \mathrm{S}. \end{cases}$$

Hence we can compute their contributions to be;

$$e_T\left(\widehat{\bigoplus_{l=0}^{m-1} H^1_{\mathrm{et}}(\mathbf{R}_l^\bullet)_v}\right) = 1.$$

$$e_T\big(\mathrm{Ext}^0(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v\big) = \begin{cases} \dfrac{u}{\mu_{v,1}} & \text{if } v \in \mathrm{I}, \\ 1 & \text{if } v \in \mathrm{II} \text{ or S}. \end{cases}$$

$$e_T\big(\mathrm{Ext}^1(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v\big) = \begin{cases} 1 & \text{if } v \in \mathrm{I}, \\ \dfrac{u}{\mu_{v,1}} + \dfrac{u}{\mu_{v,2}} & \text{if } v \in \mathrm{II}, \\ \displaystyle\prod_{i=1}^{l(\mu(v))} \left(\tfrac{u}{\mu_{v,i}} - \psi_{v,i}\right) & \text{if } v \in \mathrm{S}. \end{cases}$$

For the contributions from the rest, consider the normalization sequence when $v \in \mathrm{S}$;

$$
\begin{aligned}
0 \longrightarrow \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \big(\hat{f}\mid_{C_v}\big)^*\mathcal{O}_{\mathbb{P}^1}(1) \oplus \bigoplus_{i=1}^{l(\mu(v))} \big(\hat{f}\mid_{C_{e_{v,i}}}\big)^*\mathcal{O}_{\mathbb{P}^1}(1) \\
\longrightarrow \bigoplus_{i=1}^{l(\mu(v))} \mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \longrightarrow 0.
\end{aligned}
$$

The corresponding long exact sequence becomes

$$
\begin{aligned}
0 &\longrightarrow H^0\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \\
&\longrightarrow H^0\big(C_v, (\hat{f}\mid_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \oplus \bigoplus_{i=1}^{l(\mu(v))} H^0\big(C_{e_{v,i}}, (\hat{f}\mid_{C_{e_{v,i}}})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \\
&\longrightarrow \bigoplus_{i=1}^{l(\mu(v))} \mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \longrightarrow H^1\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \\
&\longrightarrow H^1\big(C_v, (\hat{f}\mid_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \oplus \bigoplus_{i=1}^{l(\mu(v))} H^1\big(C_{e_{v,i}}, (\hat{f}\mid_{C_{e_{v,i}}})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow 0
\end{aligned}
$$

and the representations of $\mathbb{C}^*$ are given by

$$
\begin{aligned}
0 \longrightarrow H^0\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow H^0\big(C_v, \mathcal{O}_{C_v}\big) \otimes (1) \oplus \bigoplus_{i=1}^{l(\mu(v))} \left(\bigoplus_{a=1}^{\mu_{v,i}} \left(\frac{a}{\mu_{v,i}} \right) \right) \\
\longrightarrow \bigoplus_{i=1}^{l(\mu(v))} (1) \longrightarrow H^1\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow H^1\big(C_v, \mathcal{O}_{C_v}\big) \otimes (1) \longrightarrow 0.
\end{aligned}
$$

Hence their ratio can computed as;

$$
\begin{aligned}
&\frac{e_T\big(H^1(\widehat{C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)})\big)}{e_T\big(H^0(\widehat{C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)})\big)} \\
&\qquad = \prod_v \left[\Lambda^\vee_{g(v)}(u) u^{l(\mu(v))-1} \prod_{i=1}^{l(\mu(v))} \left(\frac{\mu_{v,i}^{\mu_{v,i}}}{\mu_{v,i}!} u^{-\mu_{v,i}} \right) \right]
\end{aligned}
$$

which also works for the case of $v \in \mathrm{I}$ or $v \in \mathrm{II}$.

Case $m > 0$: Let ψ^t be the first Chern class of the line bundle over $\overline{\mathcal{M}}_\Gamma^{(1)}$ whose fiber at $\big[\hat{f} : \hat{C} \to \mathbb{P}^1(m)\big]$ is $T^*_{p_1^{(0)}}\mathbb{P}^1(m)$. So $\psi^t = \nu_{v,i}\psi_{(v,e_{v,i})}$ for $v \in V(\Gamma)^{(1)}$, $(v, e_{v,i}) \in F$. By similar observation as in the case of $m = 0$, we can find that

$$\widehat{\bigoplus_{l=0}^{m-1} H^1_{\mathrm{et}}(\mathbf{R}_l^\bullet)} = \Big(T_{p_1^{(0)}}\mathbb{P}^1_{(0)} \otimes T_{p_1^{(0)}}\mathbb{P}^1_{(1)}\Big)^{|E(\Gamma)|-1},$$

$$\mathrm{Ext}^0(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v = \begin{cases} \left(\dfrac{1}{\nu_{v,1}}\right) & \text{if } v \in \mathrm{I}, \\ 0 & \text{if } v \in \mathrm{II} \text{ or } \mathrm{S}, \end{cases}$$

$$\mathrm{Ext}^1(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v = \begin{cases} 0 & \text{if } v \in \mathrm{I}, \\ \left(\dfrac{1}{\nu_{v,1}} + \dfrac{1}{\nu_{v,2}}\right) & \text{if } v \in \mathrm{II}, \\ \displaystyle\bigoplus_{i=1}^{l(\nu(v))} T_{q_{(v,e_{v,i})}} C_v \otimes T_{q_{(v,e_{v,i})}} C_{e_{v,i}} & \text{if } v \in \mathrm{S} \text{ or } \mathrm{T}. \end{cases}$$

Hence we can compute their contributions to be;

$$e_T\left(\widehat{\bigoplus_{l=0}^{m-1} H^1_{\mathrm{et}}(\mathbf{R}_l^\bullet)}\right) = \big(-u - \psi^t\big)^{|E(\Gamma)|-1},$$

$$e_T\big(\mathrm{Ext}^0(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v\big) = \begin{cases} \dfrac{u}{\nu_{v,1}} & \text{if } v \in \mathrm{I}, \\ 1 & \text{if } v \in \mathrm{II} \text{ or } \mathrm{S}, \end{cases}$$

$$e_T\big(\mathrm{Ext}^1(\widehat{\Omega_C(D)}, \mathcal{O}_C)_v\big) = \begin{cases} 1 & \text{if } v \in \mathrm{I}, \\ \dfrac{u}{\nu_{v,1}} + \dfrac{u}{\nu_{v,2}} & \text{if } v \in \mathrm{II}, \\ \displaystyle\prod_{i=1}^{l(\nu(v))} \left(\tfrac{u}{\nu_{v,i}} - \psi_{v,i}\right) & \text{if } v \in \mathrm{S}, \\ \displaystyle\prod_{i=1}^{l(\nu(v))} \left(\tfrac{-u}{\nu_{v,i}} - \psi_{v,i}\right) & \text{if } v \in \mathrm{T}. \end{cases}$$

Also consider the following normalization sequence

$$0 \longrightarrow \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \bigoplus_{S\cup T}\big(\hat{f}\,|_{C_v}\big)^*\mathcal{O}_{\mathbb{P}^1}(1) \oplus \bigoplus_{e\in E(\Gamma)}\big(\hat{f}\,|_{C_e}\big)^*\mathcal{O}_{\mathbb{P}^1}(1)$$
$$\longrightarrow \bigoplus_{II}\mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \oplus \bigoplus_{S}\left(\bigoplus_{(v,e)\in F}\mathcal{O}_{\mathbb{P}^1}(1)_{p_0}\right)$$
$$\oplus \bigoplus_{T}\left(\bigoplus_{(v,e)\in F}\mathcal{O}_{\mathbb{P}^1}(1)_{p_1}\right) \longrightarrow 0$$

and the corresponding long exact sequence

$$0 \longrightarrow H^0\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow \bigoplus_{S\cup T} H^0\big(C_v, \big(\hat{f}\,|_{C_v}\big)^*\mathcal{O}_{\mathbb{P}^1}(1)\big)$$
$$\oplus \bigoplus_{e\in E(\Gamma)} H^0\big(C_e, \big(\hat{f}\,|_{C_e}\big)^*\mathcal{O}_{\mathbb{P}^1}(1)\big)$$
$$\longrightarrow \bigoplus_{II}\mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \oplus \bigoplus_{S}\left(\bigoplus_{(v,e)\in F}\mathcal{O}_{\mathbb{P}^1}(1)_{p_0}\right) \oplus \bigoplus_{T}\left(\bigoplus_{(v,e)\in F}\mathcal{O}_{\mathbb{P}^1}(1)_{p_1}\right)$$
$$\longrightarrow H^1\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow \bigoplus_{ST} H^1\big(C_v, (\hat{f}\,|_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1)\big)$$
$$\oplus \bigoplus_{e\in E(\Gamma)} H^1\big(C_e, (\hat{f}\,|_{C_e})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow 0$$

The representations of $\mathbb{C}^*$ are given by

$$0 \longrightarrow H^0\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow \bigoplus_{S} H^0\big(C_v, \mathcal{O}_{C_v}\big)\otimes(1)$$
$$\oplus \bigoplus_{T} H^0\big(C_v, \mathcal{O}_{C_v}\big)\otimes(0) \oplus \bigoplus_{e\in E(\Gamma)}\left(\bigoplus_{a=1}^{d(e)}\left(\frac{a}{d(e)}\right)\right)$$
$$\longrightarrow \bigoplus_{II}(1) \oplus \bigoplus_{S}\left(\bigoplus_{(v,e)\in F}(1)\right) \oplus \bigoplus_{T}\left(\bigoplus_{(v,e)\in F}(0)\right) \longrightarrow H^1\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big)$$
$$\longrightarrow \bigoplus_{S} H^1\big(C_v, \mathcal{O}_{C_v}\big)\otimes(1) \oplus \bigoplus_{T} H^1\big(C_v, \mathcal{O}_{C_v}\big)\otimes(0) \longrightarrow 0$$

from which we can compute their ratio to be;

$$\frac{e_T\big(H^1(\widehat{C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)})\big)}{e_T\big(H^0(\widehat{C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)})\big)} = \prod_{V(\Gamma)^{(0)}} \left[\Lambda^\vee_{g(v)}(u)u^{l(\nu(v))-1}\right] \prod_{i=1}^{l(\nu)} \left[\frac{\nu_i^{\nu_i}}{\nu_i!}u^{-\nu_i}\right].$$

After combining all the contributions, we find the following Feynman rules;

$$\begin{aligned}
\frac{1}{e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma_0})} &= \left[\prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!}u^{-\mu_i}\right] \left[\prod_I \frac{u}{\mu_{v,1}}\right] \left[\prod_{II} \frac{u}{(u/\mu_{v,1}) + (u/\mu_{v,2})}\right] \\
&\quad \times \left[\prod_S \frac{\Lambda^\vee_{g(v)}(u)}{u} \left(\prod_{i=1}^{l(\mu(v))} \frac{u}{(u/\mu_{v,i}) - \psi_{v,i}}\right)\right] \\
\frac{1}{e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma})} &= \frac{-\prod \nu_i}{u+\psi^t} \left[\prod_{i=1}^{l(\nu)} \frac{\nu_i^{\nu_i}}{\nu_i!}u^{-\nu_i}\right] \left[\prod_I \frac{u}{\nu_{v,1}}\right] \left[\prod_{II} \frac{u}{(u/\nu_{n,1}) + (u/\nu_{v,2})}\right] \\
&\quad \times \left[\prod_S \frac{\Lambda^\vee_{g(v)}(u)}{u} \left(\prod_{i=1}^{l(\nu(v))} \frac{u}{(u/\nu_{v,i}) - \psi_{v,i}}\right)\right].
\end{aligned}$$

5. Double Hurwitz numbers

In this section, I will summarize some properties of double Hurwitz numbers and their relation to Hodge integrals [15, 21].

5.1. General result of Hurwitz numbers

Let X be a Riemann surface of genus h. Given n partitions $\eta^1, \dots, \eta^n$ of d, denote by $H^X_d(\eta^1, \dots, \eta^n)^\bullet$ and $H^X_d(\eta^1, \dots, \eta^n)^\circ$ the weighted counts of *possibly disconnected* and *connected* Hurwitz covers of type $(\eta^1, \dots, \eta^n)$, respectively. The following *Burnside formula* is well known:

$$H^X_d(\eta^1, \dots, \eta^n)^\bullet = \sum_{|\rho|=d} \left(\frac{\dim R_\rho}{d!}\right)^{2-2h} \prod_{i=1}^{n} \mid C_{\eta^i} \mid \frac{\chi_\rho(C_{\eta^i})}{\dim R_\rho}$$

5.2. Double Hurwitz numbers

Consider a cover $C \to \mathbb{P}^1$ of genus g, ramification type ν, μ at two points p_0 and p_1, respectively, and ramification type (2) at r other points. By

Riemann–Hurwitz formula, we have $r = 2g - 2 + l(\nu) + l(\mu)$. Let $\eta^1 = \cdots = \eta^r = (2)$ and introduce notations

$$H_g^\circ(\nu,\mu) = H_d^{\mathbb{P}^1}(\nu,\mu,\eta^1,\ldots,\eta^r)^\circ, \quad H_\chi^\bullet(\nu,\mu) = H_d^{\mathbb{P}^1}(\nu,\mu,\eta^1,\ldots,\eta^r)^\bullet$$

By applying Burnside formula, we obtain

$$\begin{aligned} H_\chi^\bullet(\nu,\mu) &= \sum_{|\xi|=d}\left(\mid C_{(2)}\mid \frac{\chi_\xi(C_{(2)})}{\dim R_\xi}\right)^r \frac{\chi_\xi(C_\nu)}{z_\nu}\frac{\chi_\xi(C_\mu)}{z_\mu} \\ &= \sum_{|\xi|=d} f_\xi(2)^r \frac{\chi_\xi(C_\nu)}{z_\nu}\frac{\chi_\xi(C_\mu)}{z_\mu}. \end{aligned}$$

Define generating series of double Hurwitz numbers as follows:

$$\begin{aligned} \Phi^\circ_{\nu,\mu}(\lambda) &= \sum_{g\geq 0} H_g^\circ(\nu,\mu)\frac{\lambda^{2g-2+l(\nu)+l(\mu)}}{(2g-2+l(\nu)+l(\mu))!} \\ \Phi^\circ(\lambda;p^0,p^\infty) &= \sum_{\nu,\mu}\Phi^\circ_{\nu,\mu}(\lambda)p^0_\nu p^\infty_\mu, \\ \Phi^\bullet_{\nu,\mu}(\lambda) &= \sum_{\chi} H_\chi^\bullet(\nu,\mu)\frac{\lambda^{-\chi+l(\nu)+l(\mu)}}{(-\chi+l(\nu)+l(\mu))!} \\ \Phi^\bullet(\lambda;p^0,p^\infty) &= 1+\sum_{\nu,\mu}\Phi^\bullet_{\nu,\mu}(\lambda)p^0_\nu p^\infty_\mu. \end{aligned}$$

We will need the following initial value formula of double Hurwitz numbers;

Lemma 5.1.

$$\Phi^\bullet_{\nu,\mu}(0) = \frac{1}{z_\nu}\delta_{\nu,\mu}$$

Proof. This is a direct consequence of the orthogonal relations for characters of S_d

$$\sum_\xi \frac{\chi_\nu(C_\xi)\chi_\mu(C_\xi)}{z_\xi} = \delta_{\nu,\mu}, \quad \text{and} \quad \sum_{|\xi|=d}\chi_\xi(C_\nu)\chi_\xi(C_\mu) = z_\nu\delta_{\nu,\mu} \qquad \square$$

5.3. Relating double Hurwitz numbers with Hodge integrals

We can extend the notion of $\mathbb{P}^1[m]$ to have bubble components on both directions. Denote by

$$\mathbb{P}^1[m_0,m_\infty] = \mathbb{P}^1_{(-m_0)}\cup\cdots\cup\mathbb{P}^1_{(-1)}\cup\mathbb{P}^1_{(0)}\cup\mathbb{P}^1_{(1)}\cup\cdots\cup\mathbb{P}^1_{(m_\infty)}.$$

As before, we call $\mathbb{P}^1_{(0)}$ the *root component*, $\mathbb{P}^1_0(m_0) = \mathbb{P}^1_{(-m_0)} \cup \cdots \cup \mathbb{P}^1_{(-1)}$ the *bubble component at* 0, and $\mathbb{P}^1_\infty(m_\infty) = \mathbb{P}^1_{(1)} \cup \cdots \cup \mathbb{P}^1_{(m_\infty)}$ the *bubble component at* ∞. Define $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mu^0, \mu^\infty)$ as the moduli space of morphisms $f : C \to \mathbb{P}^1[m_0, m_\infty]$ such that

(1) $\big(C; x_1, \ldots, x_{l(\mu^\infty)}, y_1, \ldots, y_{l(\mu^0)}\big)$ is a possibly-disconnected prestable curve with $l(\mu^0) + l(\mu^\infty)$ unordered marked points.

(2) $\chi = \sum_i (2 - 2g_i)$ where g_i is the genus of each connected component of C.

(3) As Cartier divisors, $f^{-1}(p_0^{(-m_0)}) = \sum_{j=1}^{l(\mu^0)} \mu^0_j y_j$, $f^{-1}(p_1^{(m_\infty)}) = \sum_{i=1}^{l(\mu^\infty)}$ $\mu^\infty_i x_i$.

(4) The automorphism group of f is finite. Here, an automorphism of f consists of an automorphism of the domain curve C and automorphisms of the pointed curves $(\mathbb{P}^1_0(m_0), p_0^{(-m_0)}, p_1^{(-1)})$ and $(\mathbb{P}^1_\infty(m_\infty),$ $p_0^{(1)}, p_1^{(m_\infty)})$, which is an element of $(\mathbb{C}^*)^{m_0}$ and $(\mathbb{C}^*)^{m_\infty}$, respectively.

We can extend the standard action $t \cdot [z, w] = [tz, w]$ on $\mathbb{P}^1$ to $\mathbb{P}^1[m_0, m_\infty]$ by trivial action on the bubble components at 0 and ∞. Then this action induces $\mathbb{C}^*$-action on $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^0, \mu^\infty)$. Let $\pi[m_0, m_\infty] : \mathbb{P}^1[m_0, m_\infty] \to \mathbb{P}^1$ be the projection which contracts both bubble components and $f_r = \pi[m_0,$ $m_\infty] \circ f$. Denote by ν the partition of $d = |\mu^0| = |\mu^\infty|$ by the degrees of f_r on each rational irreducible components. For any morphism $f : \big(C, x_i, y_j\big) \to$ $\mathbb{P}^1[m_0, m_\infty]$ which represents a fixed point of $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^0, \mu^\infty)$ under this action, one of the following four cases must hold:

- $m_0 = m_\infty = 0$: We have $f = f_r$, $\mu^0 = \mu^\infty = \nu$.
- $m_0 = 0, m_\infty > 0$: Let $\chi_\infty = \sum(2 - 2g_i^\infty)$, where g_i^∞ is the genus of each connected component of $f^{-1}(\mathbb{P}^1_\infty(m_\infty))$. In this case, we have $\mu^0 = \nu$, $\chi_\infty = \chi$, $\chi_0 = 2l(\nu)$.
- $m_0 > 0, m_\infty = 0$: Let $\chi_0 = \sum(2 - 2g_j^0)$, where g_j^0 is the genus of each connected component of $f^{-1}(\mathbb{P}^1_0(m_0))$. In this case, we have $\mu^\infty = \nu$, $\chi_0 = \chi$, $\chi_\infty = 2l(\nu)$.
- $m_0 > 0, m_\infty > 0$: We have $\chi = \chi_0 + \chi_\infty - 2l(\nu)$.

Hence, we can see that χ_0, χ_∞, ν determines each connected component of $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^0, \mu^\infty)^{\mathbb{C}^*}$. Consider the branch morphism

$$\mathrm{Br} : \overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^0, \mu^\infty) \longrightarrow \mathrm{Sym}^{-\chi + l(\mu^0) + l(\mu^\infty)}\mathbb{P}^1 \;\cong\; \mathbb{P}^{-\chi + l(\mu^0) + l(\mu^\infty)}$$

The double Hurwitz numbers for possibly disconnected covers of $\mathbb{P}^1$ can be defined by

$$H^\bullet_\chi(\mu^0,\mu^\infty)=\frac{1}{|\operatorname{Aut}(\mu^0)||\operatorname{Aut}(\mu^\infty)|}\int_{\left[\overline{\mathcal{M}}^\bullet_\chi(\mu^0,\mu^\infty)\right]^{\mathrm{vir}}}\mathrm{Br}^*\big(H^{-\chi+l(\mu^0)+l(\mu^\infty)}\big)$$

under the assumption $-\chi+l(\mu^0)+l(\mu^\infty)>0$ where $H\in H^2(\mathbb{P}^{-\chi+l(\mu^0)+l(\mu^\infty)})$ is the hyperplane class. We want to compute this integration by virtual localization. The connected components of $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^0,\mu^\infty)^{\mathbb{C}^*}$ can be described as follows:

- $m_0=0, m_\infty>0: \mathcal{F}(\nu;2l(\nu),\chi)\cong\Big(\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\nu,\mu^\infty)//\mathbb{C}^*\Big)/\prod\mathbb{Z}_{\mu_i^\infty}$.
- $m_0>0, m_\infty=0: \mathcal{F}(\nu;\chi,2l(\nu))\cong\Big(\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^0,\nu)//\mathbb{C}^*\Big)/\prod\mathbb{Z}_{\mu_j^0}$.
- $m_0>0, m_\infty>0: \mathcal{F}(\nu;\chi_0,\chi_\infty)\cong\Big(\big(\overline{\mathcal{M}}^\bullet_{\chi_0}(\mathbb{P}^1,\mu^0,\nu)//\mathbb{C}^*\big)$
$$\times\big(\overline{\mathcal{M}}^\bullet_{\chi_\infty}(\mathbb{P}^1,\nu,\mu^\infty)//\mathbb{C}^*\big)\Big)\Big/\left(\operatorname{Aut}(\nu)\prod_{i=1}^{l(\nu)}\mathbb{Z}_{\nu_i}\right).$$

Let $\mathcal{N}^{\mathrm{vir}}_{\nu;\chi_0,\chi_\infty}$ be the pull-back of the virtual normal bundle of $\mathcal{F}(\nu,\chi_0,\chi_\infty)$ in $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^0,\mu^\infty)$. By computations similar to those $e_T(\mathcal{N}^{\mathrm{vir}}_\Gamma)$, we obtain

$$\frac{1}{e_T\big(\mathcal{N}^{\mathrm{vir}}_{\nu;2l(\nu),\chi}\big)}=\frac{-a_\nu}{u+\psi^0},\quad \frac{1}{e_T\big(\mathcal{N}^{\mathrm{vir}}_{\nu;\chi,2l(\nu)}\big)}=\frac{a_\nu}{u-\psi^\infty},$$
$$\frac{1}{e_T\big(\mathcal{N}^{\mathrm{vir}}_{\nu;\chi_0,\chi_\infty}\big)}=\frac{-A_\nu}{u+\psi^0}\times\frac{a_\nu}{u-\psi^\infty},$$

where ψ^0 and ψ^∞ are the first Chern classes of the cotangent line bundle $T^*_{p_0^{(0)}}\mathbb{P}^1[m_0,m_\infty]$ and $T^*_{p_1^{(0)}}\mathbb{P}^1[m_0,m_\infty]$, respectively. Let $r=-\chi+l(\mu^0)+l(\mu^\infty)$ and observe that

$$\begin{aligned}\mathrm{Br}(\mathcal{F}(\nu;\chi_0,\chi_\infty))&=\big(-\chi_\infty+l(\nu)+l(\mu^\infty)\big)p_1\\&\quad+\big(-\chi_0+l(\mu^0)+l(\nu)\big)p_0\in\mathbb{P}^r i^*_{\nu,\chi_0,\chi_\infty}\end{aligned}$$
$$\mathrm{Br}^*\left(\prod_{k=1}^r(H-w_k)\right)=\left(\prod_{k=1}^r\big(-\chi_0+l(\mu^0)+l(\nu)-w_k\big)\right)u^r$$

By taking special values for w:

$$\begin{aligned}&w=(0,1,\dots,-\chi+l(\mu^0)+l(\mu^\infty)-1)\quad\text{and}\\&w=(1,2,\dots,-\chi+l(\mu^0)+l(\mu^\infty)),\end{aligned}$$

we can compute to obtain

$$
\begin{aligned}
&\frac{H^\bullet_\chi(\mu^0,\mu^\infty)}{(-\chi+l(\mu^0)+l(\mu^\infty))!}\\
&\quad=\frac{1}{|\operatorname{Aut}(\mu^0)||\operatorname{Aut}(\mu^\infty)|}\int_{\left[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^0,\mu^\infty)//\mathbb{C}^*\right]^{\mathrm{vir}}}\left(\psi^0\right)^{-\chi+l(\mu^0)+l(\mu^\infty)-1}\\
&\quad=\frac{1}{|\operatorname{Aut}(\mu^0)||\operatorname{Aut}(\mu^\infty)|}\int_{\left[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^0,\mu^\infty)//\mathbb{C}^*\right]^{\mathrm{vir}}}\left(\psi^\infty\right)^{-\chi+l(\mu^0)+l(\mu^\infty)-1}.
\end{aligned}
$$

6. Recursion formula

In this section, I will summarize the results from previous sections to prove the following recursion formula on Hodge integrals. Let $e=(k_1,\dots,k_n)$ be a partition, where k_i's are allowed to be zero.

Theorem 6.1. *For any partition μ and e with $|e|<|\mu|+l(\mu)-\chi$, we have*

$$
\left[\lambda^{l(\mu)-\chi}\right]\sum_{|\nu|=|\mu|}\Phi^\bullet_{\mu,\nu}(-\lambda)z_\nu\mathcal{D}^\bullet_{\nu,e}(\lambda)=0, \tag{6.1}
$$

where the sum is taken over all partitions ν of the same size as μ.

Here $\left[\lambda^a\right]$ means taking the coefficient of λ^a. Let me first introduce some notations;

$$
\mathcal{D}_{g,\nu,e}=\begin{cases}
\dfrac{\nu_1^{\nu_1-2}}{\nu_1!} & \text{if } \big(g,l(\nu)+l(e)\big)=\big(0,1\big),\\[2ex]
\dfrac{1}{|\operatorname{Aut}\nu|}\dfrac{\nu_1^{\nu_1}\nu_2^{\nu_2}}{\nu_1!\nu_2!}\dfrac{1}{\nu_1+\nu_2} & \text{if } \big(g,l(\nu),l(e)\big)=\big(0,2,0\big),\\[2ex]
\dfrac{\nu_1^{\nu_1}}{\nu_1!}\displaystyle\sum_{k=0}^{e_1}\dfrac{1}{\nu_1^{1+k}}\binom{e_1}{k} & \text{if } \big(g,l(\nu),l(e)\big)=\big(0,1,1\big),\\[2ex]
\dfrac{1}{l(e)!\,|\operatorname{Aut}\nu|}\left[\displaystyle\prod_{i=1}^{l(\nu)}\dfrac{\nu_i^{\nu_i}}{\nu_i!}\right] & \\
\quad\times\displaystyle\int_{\overline{\mathcal{M}}_{g,l(\nu)+l(e)}}\dfrac{\Lambda_g^\vee(1)\displaystyle\prod_{j=1}^{l(e)}(1-\psi_j)^{e_j}}{\displaystyle\prod_{i=1}^{l(\nu)}\left(1-\nu_i\psi_i\right)} & \text{otherwise.}
\end{cases}
$$

$$\mathcal{D}(\lambda,p,q) = \sum_{|\nu|\geq 1}\sum_{g\geq 0} \lambda^{2g-2+l(\nu)} p_\nu q_e \mathcal{D}_{g,\nu,e}$$

$$\mathcal{D}^\bullet(\lambda,p,q) = \exp\big(\mathcal{D}(\lambda,p,q)\big) =: \sum_{|\nu|\geq 0} \lambda^{-\chi+l(\nu)} p_\nu q_e \mathcal{D}^\bullet_{\chi,\nu,e}$$

$$= \sum_{|\nu|\geq 0} p_\nu q_e \mathcal{D}^\bullet_{\nu,e}(\lambda),$$

where p_i, q_j's are formal variables with $p_\nu = p_{\nu_1}\times\cdots\times p_{\nu_{l(\nu)}}$ and $q_e = q_{e_1}\times\cdots\times q_{e_{l(e)}}$.

Proof. For any given μ and χ such that $|\mu|+l(\mu)>\chi$, applying the localization formula to the class $\prod_{j=1}^n \psi_j^{k_j}\mathrm{ev}_j^* H$, where H is the hyperplane class of $\mathbb{P}^1$ yields;

$$0 = \int_{\overline{\mathcal{M}}_{\chi,n}(\mathbb{P}^1,\mu)} \prod_{j=1}^n \psi_j^{k_j}\mathrm{ev}_j^* H \quad \text{since} \quad \deg \prod_{j=1}^n \psi_j^{k_j}\mathrm{ev}_j^* H < \dim \overline{\mathcal{M}}_{\chi,n}(\mathbb{P}^1,\mu)$$

Applying localization formula on the equivariant-lift of the above integral yields

$$0 = \sum_{\Gamma_0\in G^0_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_{\Gamma_0}|}\int_{\overline{\mathcal{M}}_{\Gamma_0}} \frac{\prod_{j=1}^n (u-\psi_j)^{k_j}\mathrm{ev}_j^* H_T}{e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma_0})}$$

$$+ \sum_{\Gamma\in G^\infty_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_{\Gamma}|}\int_{\overline{\mathcal{M}}_{\Gamma}} \frac{\prod_{j=1}^n (u-\psi_j)^{k_j}\mathrm{ev}_j^* H_T}{e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma})}.$$

Here H_T is the lift of H to the equivariant hyperplane class. Choose H in such a way that $H(0)=0$ and $H(\infty)=u$, then we have $H_T(0)=u$ and $H_T(\infty)=0$. Also let $u=1$ for simplicity, then the formula reduces to:

$$\sum_{\Gamma_0\in \overline{G}^0_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_{\Gamma_0}|}\int_{\overline{\mathcal{M}}_{\Gamma_0}} \frac{\prod_{j=1}^n (1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma_0})}$$

$$+ \sum_{\Gamma\in \overline{G}^\infty_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_{\Gamma}|}\int_{\overline{\mathcal{M}}_{\Gamma}} \frac{\prod_{j=1}^n (1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma})} = 0,$$

where $\overline{G}^0_{\chi,n}$ and $\overline{G}_{\chi,n}$ are the set of graphs corresponding to fixed locus with $m=0$ and $m>0$ with all the marked points $z_1,\ldots,z_n$ concentrated on the vertices in $V(\Gamma_0)^{(0)}$ and $V(\Gamma)^{(0)}$, respectively. We can compute the summand for Γ_0 as follows:

$$
\begin{aligned}
&\int_{\overline{\mathcal{M}}_{\Gamma_0}} \frac{\prod_{j=1}^{n}(1-\psi_j)^{k_j}}{e_T(\mathcal{N}_{\Gamma_0}^{\mathrm{vir}})} \\
&\quad = \left[\prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!}\right] \left[\prod_{I} \int_{\{\mathrm{pt}\}} \frac{1}{\mu_{v,1}}\right] \left[\prod_{II} \int_{\{\mathrm{pt}\}} \frac{1}{(1/\mu_{v,1})+(1/\mu_{v,2})}\right] \\
&\qquad \times \left[\prod_{S} \int_{\overline{\mathcal{M}}_{g(v),l(\mu(v))+l(e(v))}} \frac{\Lambda_{g(v)}^{\vee}(1)\prod_{j=1}^{j(v)}(1-\psi_{v,j})^{k_{v,j}}}{\prod_{i=1}^{l(\mu(v))}\big((1/\mu_{v,i})-\psi_{v,i}\big)}\right] \\
&\quad = \prod_{V(\Gamma_0)} z_{\mu(v)} j(v)! \mathcal{D}_{g(v),\mu(v)}
\end{aligned}
$$

Similarly, we can compute the summand for Γ as follows:

$$
\begin{aligned}
&\int_{\overline{\mathcal{M}}_{\Gamma}} \frac{\prod_{j=1}^{n}(1-\psi_j)^{k_j}}{e_T(\mathcal{N}_{\Gamma}^{\mathrm{vir}})} \\
&\quad = \left[\prod_{i=1}^{l(\nu)} \frac{\nu_i^{\nu_i}}{\nu_i!}\right] \left[\prod_{I} \int_{\{\mathrm{pt}\}} \frac{1}{\nu_{v,1}}\right] \left[\prod_{II} \int_{\{\mathrm{pt}\}} \frac{1}{(1/\nu_{n,1})+(1/\nu_{v,2})}\right] \\
&\qquad \times \left[\prod_{S} \int_{\overline{\mathcal{M}}_{g(v),l(\nu(v))+l(e(v))}} \frac{\Lambda_{g(v)}^{\vee}(1)\prod_{j=1}^{j(v)}(1-\psi_{v,j})^{k_{v,j}}}{\prod_{i=1}^{l(\nu(v))}\big((1/\nu_{v,i})-\psi_{v,i}\big)}\right] \left[\int_{\overline{\mathcal{M}}_{\Gamma}^{(1)}} \frac{-\prod \nu_i}{1+\psi^t}\right] \\
&\quad = \left[\prod_{V(\Gamma_0)} z_{\nu(v)}\, j(v)!\, \mathcal{D}_{g(v),\nu(v),e(v)}\right] \\
&\qquad \times \left[(-1)^{-\chi+l(\mu)+l(\nu)} \prod_{i=1}^{l(\nu)} \nu_i\right] \int_{\left[\overline{\mathcal{M}}_{\Gamma}^{(1)}\right]^{\mathrm{vir}}} \left(\psi^t\right)^{-\chi_\infty+l(\mu)+l(\nu)-1}.
\end{aligned}
$$

And the integration over $\overline{\mathcal{M}}_\Gamma^{(1)}$ can be related to double Hurwitz numbers as follows:

$$\begin{aligned}|\mathrm{Aut}\,\mu|\,|\mathrm{Aut}\,\nu|\;H^\bullet_{\chi_\infty}(\mu,\nu) &= \big(-\chi_\infty + l(\mu) + l(\nu)\big)!\\ &\quad\times\int_{[\overline{\mathcal{M}}^\bullet_{\chi_\infty}(\mathbb{P}^1,\mu,\nu)//\mathbb{C}^*]^{\mathrm{vir}}} (\psi^0)^{-\chi_\infty+l(\mu)+l(\nu)-1}\\ &= \frac{\big(-\chi_\infty + l(\mu) + l(\nu)\big)!}{\Big(\prod n_k!\Big)\Big(\prod_{V(\Gamma)^{(1)}} |\mathrm{Aut}\,\mu(v)|\;|\mathrm{Aut}\,\nu(v)|\Big)}\\ &\quad\times\int_{\left[\overline{\mathcal{M}}_\Gamma^{(1)}\right]^{\mathrm{vir}}} \big(\psi^t\big)^{-\chi_\infty+l(\mu)+l(\nu)-1},\end{aligned}$$

since marked points in $\overline{\mathcal{M}}_\Gamma^{(1)}$ are ordered. Recall that $|A_{\Gamma_0}|$ and $|A_\Gamma|$ are given by:

$$\begin{aligned}|A_{\Gamma_0}| &= \left(\prod_{i=1}^{l(\mu)}\mu_i\right)\prod_k m_k!\,\big(j(v_k)!\,|\,\mathrm{Aut}\,\mu(v_k)\,|\big)^{m_k}\\ |A_\Gamma| &= \left(\prod_{i=1}^{l(\nu)}\nu_i\right)\left(\prod_k m_k!\,\big(j(v_k)!\,|\,\mathrm{Aut}\,\nu(v_k)\,|\big)^{m_k}\right)|\mathrm{Aut}\,\mu\,|\\ &\quad\times\Big(\prod n_k!\Big)\left(\prod_{V(\Gamma)^{(1)}} j(v)!\,|\,\mathrm{Aut}\,\mu(v)|\;|\mathrm{Aut}\,\nu(v)|\right).\end{aligned}$$

Also observe that

$$\begin{aligned}&\frac{\prod_{V(\Gamma_0)} z_{\mu(v)}\,j(v)!\,\mathcal{D}_{g(v),\mu(v),e(v)}}{\Big(\prod_{i=1}^{l(\mu)}\mu_i\Big)\prod_k m_k!\,\big(\,j(v)!\,|\,\mathrm{Aut}\,\mu(v_k)\,|\big)^{m_k}}\\ &= \frac{1}{|V(\Gamma_0)|!}\binom{|V(\Gamma_0)|}{m_1,\ldots,m_l}\prod_k \mathcal{D}_{g(v_k),\mu(v_k),e(v_k)}\end{aligned}$$

which is the coefficient of $\lambda^{-\chi+l(\mu)}p_\mu q_e$ in the expansion of $\mathcal{D}^\bullet(\lambda,p,q)$. Now the original equation can be simplified as follows;

$$\begin{aligned}
0 &= \sum_{\Gamma_0\in\overline{G}^0_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_{\Gamma_0}|}\int_{\overline{\mathcal{M}}_{\Gamma_0}} \frac{\prod_{j=1}^n(1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma_0})}\\
&\quad+\sum_{\Gamma\in\overline{G}^\infty_{\chi,n}(\mathbb{P}^1,\mu)} \frac{1}{|A_{\Gamma}|}\int_{\overline{\mathcal{M}}_{\Gamma}} \frac{\prod_{j=1}^n(1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{\mathrm{vir}}_{\Gamma})}\\
&= \mathcal{D}^\bullet_{\chi,\mu,e}+\sum_\nu\sum_{-\chi_\infty+l(\mu)+l(\nu)\neq 0}\frac{(-1)^{-\chi_\infty+l(\mu)+l(\nu)}H^\bullet_{\chi_\infty}(\mu,\nu)}{(-\chi_\infty+l(\mu)+l(\nu))!}z_\nu\mathcal{D}^\bullet_{\chi_0,\nu,e}\\
&= \sum_\nu\sum_{\chi_0,\chi_\infty}\frac{(-1)^{-\chi_\infty+l(\mu)+l(\nu)}H^\bullet_{\chi_\infty}(\mu,\nu)}{(-\chi_\infty+l(\mu)+l(\nu))!}z_\nu\mathcal{D}^\bullet_{\chi_0,\nu,e},
\end{aligned}$$

where $\chi=\chi_0+\chi_\infty-2\,l(\nu)$ and the initial value formula for double Hurwitz numbers is used at the last equality. Summing over χ yields that we have for all $|e|+\chi<|\mu|+l(\mu)$

$$\left[\lambda^{l(\mu)-\chi}\right]\sum_{|\nu|=|\mu|}\Phi^\bullet_{\mu,\nu}(-\lambda)z_\nu\mathcal{D}^\bullet_{\nu,e}(\lambda)=0. \qquad \square$$

As a remark, the above recursion formula contains the single Hurwitz number formula and the ELSV formula which can be observed by comparing the recursion type and the initial values [5, 9, 10, 14, 17, 19].

7. Computing linear Hodge integrals

It is enough to consider the case of $\mu=(d)$ for some positive integers d to compute all linear Hodge integrals. And in this case, we have a closed formula for the double Hurwitz numbers as follows;

Theorem 7.1. ([6], Theorem 3.1.)*Let $r=r^g_{(d),\beta}$. For $g\geq 0$, and $\beta\vdash d$ with n parts,*

$$H^g_{(d),\beta}=r!d^{r-1}\left[t^{2g}\right]\prod_{k\geq 1}\left(\frac{\sinh(kt/2)}{kt/2}\right)^{c_k}=\frac{r!d^{r-1}}{2^{2g}}\sum_{\lambda\vdash g}\frac{\xi_{2\lambda}S_{2\lambda}}{|\mathrm{Aut}\,\lambda|},$$

where $r=2g-1+l(\beta)$ and $c_1=($number of 1's in $\beta)-1$, $c_k=($number of k's in $\beta)$ for $k>1$. $\left[t^{2g}\right]$ means taking the coefficient of t^{2g}.

Here the double Hurwitz number is counted with multiplicity, hence in our notation it will read as follows

$$\frac{H^{\bullet}_{\chi_\infty}((d),\nu)}{(-\chi_\infty+1+l(\nu))!}=\frac{d^{-\chi_\infty+l(\nu)}}{|\mathrm{Aut}\,\nu|}[t^{2g}]\prod_{k\geq 1}\left(\frac{\sinh(kt/2)}{kt/2}\right)^{c_k}. \tag{7.1}$$

And in this case, (6.1) can be written as

(7.2)
$$\sum_{|\nu|=d}\sum_{\chi_0,\chi_\infty}\mathcal{D}^{\bullet}_{\chi_0,\nu,e}z_\nu(-1)^{-\chi_\infty+1+l(\nu)}\frac{d^{-\chi_\infty+l(\nu)}}{|\mathrm{Aut}\,\nu|}[t^{2g}]\prod_{k\geq 1}\left(\frac{\sinh(kt/2)}{kt/2}\right)^{c_k}=0.$$

Now fix ν and consider the case where there are m vertices in $V(\Gamma)^{(0)}$. Then we have splitting of χ_0, ν, and e into $\{g_1,\ldots,g_m\}$, $\{\nu(v_1),\ldots,\nu(v_m)\}$, and $\{e(v_1),\ldots,e(v_m)\}$ such that $e(v_i)$'s are allowed to be empty and

$$\sum_{i=1}^{m}(2-2g_i)=\chi_0,\qquad \bigcup_{i=1,\ldots,m}\nu(v_i)=\nu,\qquad \bigcup_{i=1,\ldots,m}e(v_i)=e.$$

Each vertex will correspond to a certain Hodge integral on $\overline{\mathcal{M}}_{g(v),l(\nu(v))+l(e(v))}$ with dimension $3g(v)-3+l(\nu(v))+l(e(v))$. There are conditions on m, $\chi,\chi_0,\chi_\infty,l(e(v))$, and $l(\nu(v))$: let $\overline{g}(v)$ denote $\sum_{w\neq v}g(w)$,

$$m\leq l(\nu),\quad l(\nu(v))\leq l(\nu)-m+1,\quad \chi=\chi_0+\chi_\infty-2l(\nu),\quad l(e(v))\leq l(e),$$

$$\chi_\infty\leq 2\min\{l((d)),l(\nu)\}=2,\quad \chi_0=\sum_{i=1}^{m}(2-2g(v_i))=2m-2g(v)-2\overline{g}(v).$$

From these conditions, we can deduce that

$$\begin{aligned}3g(v)&=3m-3\overline{g}(v)-\frac{3}{2}\chi_0=3m-3\overline{g}(v)-\frac{3}{2}\chi+\frac{3}{2}\chi_\infty-3l(\nu)\\&\leq-\frac{3}{2}\chi+3m-\frac{3}{2}\big(2l(\nu)-\chi_\infty\big).\end{aligned}$$

where equality holds if and only if $\overline{g}(v) = \sum_{w\neq v} g(w) = 0$. Now we can find the upper bound for the dimension of Hodge integral as follows.

$$\begin{aligned}
&3g(v) - 3 + l(\nu(v)) + l(e(v)) \\
&\quad \leq -\frac{3}{2}\chi + 3m - 3 - \frac{3}{2}\big(2l(\nu) - \chi_\infty\big) + l(\nu(v)) + l(e) \\
&\quad \leq \Big[3g - 3 + l(e) + 1\Big] + \Big[3m - 3 - 3l(\nu) + \frac{3}{2}\chi_\infty + l(\nu) - m\Big] \\
&\quad \leq \Big[3g - 3 + l(e) + 1\Big] + \Big[2\big(m - l(\nu)\big) + \frac{3}{2}\big(\chi_\infty - 2\big)\Big] \leq 3g - 2 + l(e)
\end{aligned}$$

and the equality holds if and only if

$$m = l(\nu), \quad \chi_\infty = 2, \quad e(w) = \emptyset \text{ for all } w \neq v, \quad g(w) = 0 \text{ for all } w \neq v,$$

i.e., when each part of ν is splitted into separate vertices, and all the marked points other than the ramification divisor as well as all genuses are concentrated on one vertex on the 0th side. Now we can compute any linear Hodge integral as follows: Say we want to compute Hodge integrals of the form

$$\int_{\overline{M}_{g,n+1}} \psi_0^{k_0} \ldots \psi_n^{k_n} \lambda_j$$

where $j + \sum_{i=0}^{n} k_i = 3g - 2 + n$. Assume $0 \leq k_0 \leq \ldots \leq k_n$ and let $e = (k_1, \ldots, k_n)$ and $\chi = 2 - 2g$. Then for any positive integer d such that $d > \chi + |e| - 1$, the recursion formula (7.2) expresses the top-dimensional Hodge integrals in terms of lower-dimensional Hodge integrals as follows:

$$\begin{aligned}
&\sum_{|\nu|=d} \left[\left(\prod_{i=1}^{l(\nu)} \frac{\nu_i^{\nu_i - 1}}{\nu_i!} \right) \frac{(-1)^{l(\nu)-1} d^{l(\nu)-2}}{n!} \right. \\
&\quad \left. \times \sum_{i=1}^{l(\nu)} \frac{\nu_i^2}{|\mathrm{Aut}\, \hat{\nu}_i|} \int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\Lambda_g^\vee(1) \prod_{j=1}^{n} (1 - \psi_j)^{k_j}}{1 - \nu_i \psi_0} \right] \\
&\qquad = \text{terms consisting of lower-dimensional Hodge integrals only,}
\end{aligned}$$

where $\operatorname{Aut}\hat{\nu_i}$ is the automorphism group of the partition $\hat{\nu_i} = (\nu_1, \ldots, \hat{\nu_i}, \ldots, \nu_{l(\nu)})$. In this expression, the Hodge integral term expands to:

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\Lambda_g^\vee(1)\prod_{j=1}^n (1-\psi_j)^{k_j}}{1-\nu_i\psi_0} \\
&\quad = \int_{\overline{\mathcal{M}}_{g,n+1}} \left(1-\lambda_1+\lambda_2+\cdots+(-1)^g\lambda_g\right)\left(\sum_{a_0=0}^{\infty} \nu_i^{a_0}\psi_0^{a_0}\right) \\
&\quad\quad \times \prod_{j=1}^n \left[\sum_{a_j=0}^{k_j} (-1)^{a_j}\binom{k_j}{a_j}\psi_j^{a_j}\right] \\
&\quad = \sum_{k+a_j=3g-2+n} (-1)^{3g-2+n}\left[\nu_i^{a_0}\prod_{j=1}^n \binom{k_j}{a_j}\right]\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_0^{a_0}\times\cdots\times\psi_n^{a_n}\lambda_k.
\end{aligned}$$

Hence the previous expression can be written as

$$\begin{aligned}
&\sum_{k+a_j=3g-2+n} C_d(k,(a_j))\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_0^{a_0}\times\cdots\times\psi_n^{a_n}\lambda_k \\
&\quad = \text{lower-dimensional terms,}
\end{aligned} \tag{7.3}$$

where $C_d(k,(a_j))$ are constants defined as follows:

$$\begin{aligned}
C_d\big(k,(a_j)\big) = \sum_{|\nu|=d} \Bigg[&\left(\prod_{i=1}^{l(\nu)} \frac{\nu_i^{\nu_i-1}}{\nu_i!}\right)\frac{(-1)^{l(\nu)-1}d^{l(\nu)-2}}{n!} \\
&\times \sum_{i=1}^{l(\nu)} \frac{\nu_i^2}{|\operatorname{Aut}\hat{\nu_i}|}\left(\nu_i^{a_0}\prod_{j=1}^n\binom{k_j}{a_j}\right)\Bigg].
\end{aligned}$$

Now we have infinitely many linear relations of finitely many linear Hodge integrals of fixed dimension $3g-2+n$ since equation (7.3) holds for all positive integers d such that $d > 1-2g+\sum k_j$. Moreover the coefficients $C_d(k,(a_j))$ form Vandermonde-type matrices and it can be proved that one can always find a set of positive integers $\{d_1,\ldots,d_l\}$ which will give linearly independent relations to solve for all the linear Hodge integrals of given dimension $3g-2+n$ in terms of the values of lower-dimensional Hodge integrals with at most one λ-class. So we just proved the following theorem;

Theorem 7.2. *Any given linear Hodge integral*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \lambda_j, \tag{7.4}$$

where $k_1, \dots, k_n \in \mathbb{N} \cup \{0\}$, $j \in \{0, 1, 2, \dots, g\}$, *is explicitly expressed as a polynomial in terms of lower-dimensional Hodge integrals with one* λ*-class. Therefore it computes all Hodge integrals with one* λ*-class.*

8. Algorithm to compute Hodge integrals with one $\boldsymbol{\lambda}$-class

In this section, I will derive explicit formula to compute all linear Hodge integrals that can be implemented through computer algorithm. It is clear that Theorem 7.2 can be implemented using computer: We can use formula (7.1) to compute double Hurwitz numbers. For error-free computational purpose, there are well-developed C++ libraries for multi-precision computing, for example GNU MP.

For any given linear Hodge integral

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \lambda_j.$$

Let $e = (k_1, \dots, k_{n-1})$ and $\chi = 2 - 2g$. Start with $d = \chi - 1 + |e|, \dots$ and find linear relations (6.1) as follows: Run over partitions ν of size d. For a fixed ν, run over pairs (χ_0, χ_∞) which satisfies $\chi = \chi_0 + \chi_\infty - 2\,l(\nu)$, $\chi_\infty \leq 2$, $\chi_0, \chi_\infty \in 2\,\mathbb{Z}$. Now for a fixed pair (χ_0, χ_∞), we can compute double Hurwitz number $\mathcal{D}^\bullet_{\chi_\infty}(\,(d)\,, \nu)$ as follows: When $c_1 \geq 0$, i.e., when there are one or more 1's in ν, we have

$$[t^{2g}] \prod_{k\geq 1} \Big(\frac{\sinh(kt/2)}{kt/2}\Big)^{c_k} = \sum_{(b_k)} \prod_{b_k \neq 0} \sum_{(a_j^k)} \frac{k^{2b_k}}{2^{2b_k} \prod_j (2a_j^k + 1)!},$$

where $b_k = 0$ if $c_k = 0$ and $\chi_\infty = 2 - 2g_\infty$, $\sum_k b_k = g_\infty$, $\sum_{j=1,\dots,c_k} a_j^k = b_k$ with $b_k, a_j^k \in \mathbb{N} \cup \{0\}$. When $c_1 = -1$, i.e., when there is no 1 in ν, let h be the minimum numbered part of ν. Then we have $c_h \geq 1$ and

$$\left(\frac{\sinh(t/2)}{t/2}\right)^{-1}\left(\frac{\sinh(ht/2)}{ht/2}\right) = \frac{1}{h}\left(e^{(h-1)t/2} + e^{(h-3)t/2} + \cdots + e^{(-h+1)t/2}\right)$$

$$= \begin{cases} \sum\limits_{m=0}^{\infty}\left[\frac{1}{h\,2^{2m-1}(2m)!}\left(\sum\limits_{l} l^{2m}\right)\right]t^{2m} \\ \quad \text{when } h \text{ is even, } l = 1, 3, \ldots, h-1, \\ \frac{1}{h} + \sum\limits_{m=0}^{\infty}\left[\frac{1}{h\,2^{2m-1}(2m)!}\left(\sum\limits_{l} l^{2m}\right)\right]t^{2m} \\ \quad \text{when } h \text{ is odd, } l = 2, 4, \ldots, h-1. \end{cases}$$

Hence we have

$$\prod_{k\geq 1}\left(\frac{\sinh(kt/2)}{kt/2}\right)^{c_k}$$

$$= \left(\frac{\sinh(t/2)}{t/2}\right)^{-1}\left(\frac{\sinh(ht/2)}{ht/2}\right)\left(\frac{\sinh(ht/2)}{ht/2}\right)^{c_h-1}\prod_{k>h}\left(\frac{\sinh(kt/2)}{kt/2}\right)^{c_k}$$

$$= \left(\frac{1}{h}\delta_{h,\mathrm{odd}} + \sum_{m=0}^{\infty}\left[\frac{1}{h\,2^{2m-1}(2m)!}\left(\sum_{l} l^{2m}\right)\right]t^{2m}\right)\left(\frac{\sinh(ht/2)}{ht/2}\right)^{c_h-1}$$

$$\times \prod_{k>h}\left(\frac{\sinh(kt/2)}{kt/2}\right)^{c_k}$$

and in this case, for $g > 0$;

$$[t^{2g}]\prod_{k\geq 1}\left(\frac{\sinh(kt/2)}{kt/2}\right)^{c_k} = \sum_{(b_k)}\left[\frac{1}{h\,2^{2b_1-1}(2b_1)!}\left(\sum_{l} l^{2b_1}\right)\right]$$

$$\times \prod_{b_k\neq 0,\, k\geq h}\ \sum_{(a_j^k)} \frac{k^{2b_k}}{2^{2b_k}\prod_j (2a_j^k+1)!},$$

where $b_h = 0$ if $c_h = 1$ and $b_k = 0$ if $c_k = 0$ for $k \neq h$. Using these formulas and (7.1), we can compute double Hurwitz numbers. In order to compute $\left[\lambda^{l(\nu)-\chi_0}\right]\mathcal{D}^{\bullet}_{\nu,e}(\lambda)$, first run over the number of vertices $m = 1, 2, \ldots, l(\nu)$. For each m, find all possible groupings of ν, e, and all possible splittings of χ_0. Note that e admits groupings with empty components. Now find all triples $(\nu(v), e(v), g(v))$ according to the equivalence condition of vertices discussed in Section 3. Then the contribution of $\left[\lambda^{l(\nu)-\chi_0}\right]\mathcal{D}^{\bullet}_{\nu,e}(\lambda)$ will be the product of the combination factor $1/\prod m_i!$ and the expansions of $\mathcal{D}_{g(v),\nu(v),e(v)}$ which can be obtained by

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_{g,l(\nu)+l(e)}} \frac{\Lambda_g^{\vee}(1)\prod_{j=1}^{l(e)}\left(1-\psi_j\right)^{e_j}}{\prod_{i=1}^{l(\nu)}\left(1-\nu_i\psi_i\right)} \\
&\quad = \int_{\overline{\mathcal{M}}_{g,l(\nu)+l(e)}} \left[1-\lambda_1+\cdots+(-1)^g\lambda_g\right]\prod_{j=1}^{l(e)}\left[\sum_{(\tilde{l}_j)}(-1)^{\tilde{l}_j}\binom{e_j}{\tilde{l}_j}\psi_j^{\tilde{l}_j}\right] \\
&\qquad \times \prod_{i=1}^{l(\nu)}\left[\sum_{l_i}\nu_i^{l_i}\psi_i^{l_i}\right] \\
&\quad = \sum_{k,(l_i),(\tilde{l}_j)}(-1)^{k+\sum\tilde{l}_j}\left[\prod_{j=1}^{l(e)}\binom{e_j}{\tilde{l}_j}\right]\left[\prod_{i=1}^{l(\nu)}\nu_i^{l_i}\right]\int_{\overline{\mathcal{M}}_{g,l(\nu)+l(e)}}\lambda_k\prod_{i=1}^{l(\nu)}\psi_i^{l_i}\prod_{j=1}^{l(e)}\psi_j^{\tilde{l}_j},
\end{aligned}$$

where $l_i \geq 0$, $0 \leq \tilde{l}_j \leq e_j$, $0 \leq k \leq g$, and $k+\sum_i l_i+\sum_j \tilde{l}_j = 3g-3+l(\nu)+l(e)$. Some of them will have maximum dimension $3g-3+n$ for the situations described in Section 7. Those are treated as unknowns and all others are lower-dimensional Hodge integrals or initial values which are already computed. Summing over all pairs (χ_0,χ_∞) and ν will give a linear relation between Hodge integrals of the dimension $3g-3+n$. Now we can follow same step as above for other values of d and obtain more linear relations. Observe that the number of unknowns are independent of d and actually bounded by the number of partitions of $3g-3+n$, and hence we will have a system of linear relations which can be solved by simple Gaussian elimination method. Thus in each dimension, it amounts to solve a matrix equation of size $N \times N$ when N is, at worst case, the number of partitions of dimension.

9. Examples

In this section, I will illustrate how the algorithm developed in this paper works and show that the results match with previously known methods. There are 16 possible combinations of linear Hodge integrals in moduli space dimension 4; namely

$$\begin{array}{llll}
\int_{\overline{\mathcal{M}}_{0,7}} \psi^4, & \int_{\overline{\mathcal{M}}_{0,7}} \psi^3\psi^1, & \int_{\overline{\mathcal{M}}_{0,7}} \psi^2\psi^2, & \int_{\overline{\mathcal{M}}_{0,7}} \psi^2\psi^1\psi^1, \\
\int_{\overline{\mathcal{M}}_{0,7}} \psi^1\psi^1\psi^1\psi^1, & \int_{\overline{\mathcal{M}}_{1,4}} \psi^3\lambda_1, & \int_{\overline{\mathcal{M}}_{1,4}} \psi^4, & \int_{\overline{\mathcal{M}}_{1,4}} \psi^2\psi^1\lambda_1, \\
\int_{\overline{\mathcal{M}}_{1,4}} \psi^3\psi^1, & \int_{\overline{\mathcal{M}}_{1,4}} \psi^1\psi^1\psi^1\lambda_1, & \int_{\overline{\mathcal{M}}_{1,4}} \psi^2\psi^1\psi^1, & \int_{\overline{\mathcal{M}}_{1,4}} \psi^2\psi^2, \\
\int_{\overline{\mathcal{M}}_{1,4}} \psi^1\psi^1\psi^1\psi^1, & \int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2, & \int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1, & \int_{\overline{\mathcal{M}}_{2,1}} \psi^4.
\end{array}$$

The first 13 integrals can be computed by applying the algorithm to the triples:

$$\begin{aligned}
(d,g,e) = {} & (3,0,(0,0,0,0,0,0)), (4,0,(1,0,0,0,0,0)), (5,0,(2,0,0,0,0,0)), \\
& (2,1,(1,0,0)), (5,0,(1,1,0,0,0,0)), (6,0,(1,1,1,0,0,0)), \\
& (1,1,(0,0,0)), (2,1,(0,0,0)), (3,1,(1,0,0)), (3,1,(1,1,0)), \\
& (4,1,(1,1,0)), (3,1,(2,0,0)), (4,1,(1,1,1)).
\end{aligned}$$

It is straightforward to solve the resulting linear systems and to show that they match with known values. The last 3 integrals can be computed using $(d,g) = (1,2), (2,2)$, and $(3,2)$ with $e = \emptyset$. The resulting linear system is

$$\begin{aligned}
\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - \int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + \int_{\overline{\mathcal{M}}_{2,1}} \psi^4 &= 0, \\
7\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - 15\int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + 31\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 &= \frac{1}{240}, \\
25\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - 90\int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + 301\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 &= \frac{5}{48}.
\end{aligned}$$

One can solve this linear system to obtain the solution set

$$\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 = \frac{7}{5760}, \quad \int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 = \frac{1}{480}, \quad \int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = \frac{1}{1152}.$$

The first value matches with λ_g-formula (1.2), since we have

$$\sum_{k=0}^{m}\binom{m+1}{k}B_k=0,\ \text{ for } m>0 \implies B_4=-\frac{1}{30}$$
$$\binom{2*2+1-3}{2}\frac{2^{2*2-1}-1}{2^{2*2-1}}\frac{|B_{2*2}|}{(2*2)!}=\frac{7}{5760}.$$

The second value matches with (1.3), since we have

$$b_g=\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!},\ \text{ for } g>0 \implies b_1=\frac{1}{24},\quad b_2=\frac{7}{5760}$$
$$b_g\sum_{i=1}^{2g-1}\frac{1}{i}-\frac{1}{2}\sum_{g_1+g_2=g}\frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!}b_{g_1}b_{g_2}=\frac{7}{5760}\Big(1+\frac{1}{2}+\frac{1}{3}\Big)$$
$$-\frac{1}{2}\frac{1}{3!}\Big(\frac{1}{24}\Big)^2=\frac{1}{480}.$$

And the last value matches with the result in [26, p. 36].

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