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Images of real submanifolds under finite holomorphic mappings

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We give some results concerning the smoothness of the image of a real-analytic submanifold in complex space under the action of a finite holomorphic mapping. For instance, if the submanifold is not contained in a proper complex subvariety, we give a necessary and sufficient condition guaranteeing that its image is smooth and the mapping is transversal to the image.

1. Introduction and main results

In this paper we study finite holomorphic mappings of real-analytic submanifolds in $\mathbb{C}^N$. Recall that a germ of a holomorphic mapping $H : (\mathbb{C}^N, p_0) \to (\mathbb{C}^N, \tilde{p}_0)$ is *finite* at p_0 if $H^{-1}(\tilde{p}_0) \cap U = \{p_0\}$ for a sufficiently small open neighborhood U of p_0 in $\mathbb{C}^N$. If $V \subset \mathbb{C}^N$ is a germ at p_0 of a real-analytic subvariety and H is a finite holomorphic mapping, then its image, $H(V)$, is contained in a germ at $\tilde{p}_0$ of a real-analytic subvariety of the same dimension. We consider here the case where V is a (germ at p_0 of a) real-analytic submanifold and ask for geometric conditions guaranteeing that the image $\tilde{V} := H(V)$ is again a (germ at $\tilde{p}_0$ of a) submanifold and H is transversal to $\tilde{V}$ at p_0. Our main result (Theorem 1.1) generalizes to higher codimension earlier work of Baouendi and the second author (see [3]). This study is partly motivated by the recent interest in the structure of nondegenerate mappings (e.g., finite holomorphic mappings) taking one real-analytic submanifold in $\mathbb{C}^N$ into another. We mention here only the papers [4, 5, 9, 10, 12, 13], and refer the reader to these papers for precise results and a more extensive bibliography.

A standard method of studying real-analytic submanifolds involves the complexification of such manifolds. In this paper we are led to the study of questions involving images of complex manifolds under finite holomorphic mappings. The key step in our proof of the main result concerning images of real-analytic submanifolds (Theorem 1.1 below) is a result for complex submanifolds (Theorem 2.1 below). A natural question arises concerning the hypotheses in Theorem 2.1. This open question is formulated precisely in Section 4, and some partial results are given there.

Before stating our main result, we must first introduce some notation. Let M be a real-analytic submanifold of codimension d in $\mathbb{C}^N$ with $p_0 \in M$. We let $\mathcal{M}$ be the usual complexification of M in some neighborhood of $(p_0, \bar{p}_0)$ in $\mathbb{C}^N \times \mathbb{C}^N$; i.e., $\mathcal{M}$ is defined near $(p_0, \bar{p}_0)$ in $\mathbb{C}^N \times \mathbb{C}^N$ by $\rho_1(Z, \zeta) = \cdots = \rho_d(Z, \zeta) = 0$ if M is defined near p_0 by

$$\rho_1(Z, \bar{Z}) = \cdots = \rho_d(Z, \bar{Z}) = 0. \tag{1.1}$$

We shall also associate to a holomorphic mapping $H : (\mathbb{C}^N, p_0) \to (\mathbb{C}^N, \tilde{p}_0)$ its complexification $\mathcal{H}\colon (\mathbb{C}^N \times \mathbb{C}^N, (p_0, \bar{p}_0)) \to (\mathbb{C}^N \times \mathbb{C}^N, (\tilde{p}_0, \bar{\tilde{p}}_0))$ defined by $\mathcal{H}(Z, \zeta) = (H(Z), \bar{H}(\zeta))$, where $\bar{H}(\zeta) := \overline{H(\bar{\zeta})}$. Observe that the mapping H sends M into another real-analytic submanifold $\widetilde{M}$ if and only if the complexified mapping $\mathcal{H}$ sends $\mathcal{M}$ to $\widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}}$ is the complexification of $\widetilde{M}$. Also, observe that the mapping H is finite if and only if $\mathcal{H}$ is finite. It is easy to check that a necessary condition for $H(M)$ to be smooth is that $\mathcal{H}(\mathcal{M})$ is smooth, but the converse is not true in general. (See Remark 3.1.)

A real-analytic submanifold M is called *generic* if $T_pM + J(T_pM) = T_p\mathbb{C}^N$ for every $p \in M$. Here, T_pY denotes the (real) tangent space at p to a manifold Y, and $J : T\mathbb{C}^N \to T\mathbb{C}^N$ is the complex structure on $\mathbb{C}^N$. An equivalent definition can be given in terms of local defining equations (1.1) for M near p_0, namely $\partial\rho_1 \wedge \cdots \wedge \partial\rho_d \neq 0$ on M. A generic submanifold M is said to be of *finite type* at p_0 (in the sense of Kohn and Bloom–Graham) if the (complex) Lie algebra $\mathfrak{g}_M$ generated by all smooth $(1,0)$ and $(0,1)$ vector fields tangent to M satisfies $\mathfrak{g}_M(p_0) = \mathbb{C}T_{p_0}M$, where $\mathbb{C}T_{p_0}M$ is the complexified tangent space to M. Recall that a germ of a smooth mapping $g : (\mathbb{R}^k, x) \to (\mathbb{R}^\ell, y)$ is said to be *transversal* to a smooth submanifold $Y \subset \mathbb{R}^\ell$ at y if

$$T_yY + dg(T_x(\mathbb{R}^k)) = T_y(\mathbb{R}^\ell). \tag{1.2}$$

We shall say that a holomorphic mapping $H : (\mathbb{C}^N, p_0) \to (\mathbb{C}^N, \tilde{p}_0)$ is transversal to a real-analytic submanifold $\widetilde{M} \subset \mathbb{C}^N$ at $\tilde{p}_0$ if it is transversal to M at 0 as a real mapping $H : (\mathbb{R}^{2N}, p_0) \to (\mathbb{R}^{2N}, \tilde{p}_0)$. Finally, the holomorphic mapping H is said to be CR *transversal* to a generic submanifold $\widetilde{M}$ at $\tilde{p}_0$ if

$$T^{1,0}_{\tilde{p}_0}\widetilde{M} + dH(T^{1,0}_{p_0}\mathbb{C}^N) = T^{1,0}_{\tilde{p}_0}\mathbb{C}^N. \tag{1.3}$$

Here $T^{1,0}_{\tilde{p}_0}\widetilde{M}$ denotes the the smooth $(1,0)$ vectors in $T_{\tilde{p}_0}\mathbb{C}^N$ that are tangent to $\widetilde{M}$ at $\tilde{p}_0$.

The following theorem is the main result of this paper.

Theorem 1.1. *Let M be a (germ of a) real-analytic submanifold through p_0 in $\mathbb{C}^N$ and $H : (\mathbb{C}^N, p_0) \to (\mathbb{C}^N, \tilde{p}_0)$ a germ of a finite holomorphic mapping. Consider the two properties:*

(i) *The image $H(M)$ is a germ at $\tilde{p}_0$ of a real-analytic submanifold.*

(ii) *The complexified germ $\mathcal{H}$ satisfies $\mathcal{H}^{-1}(\mathcal{H}(\mathcal{M})) = \mathcal{M}$, where $\mathcal{M}$ denotes the complexification of M.*

If M is not contained in any proper complex analytic subvariety through p_0, then

$$\text{(i) with } H \text{ is transversal to } H(M) \text{ at } \tilde{p}_0. \iff \text{(ii)} \tag{1.4}$$

If M is generic and of finite type at p_0, then

$$\text{(i) with } H(M) \text{ generic} \iff \text{(ii)} . \tag{1.5}$$

Moreover, in the latter case, if either (i) *or* (ii) *is satisfied, then the image $H(M)$ is of finite type at $\tilde{p}_0$, and H is* CR *transversal to $H(M)$ at $\tilde{p}_0$.*

Remark 1.2. It is well known that if $\widetilde{M}$ is a real-analytic submanifold and H is transversal to $\widetilde{M}$ at $\tilde{p}_0$, then $H^{-1}(\widetilde{M})$ is necessarily a real-analytic submanifold. Moreover, if $\widetilde{M}$ is generic and H is CR transversal to $\widetilde{M}$, then $H^{-1}(\widetilde{M})$ is generic. Theorem 1.1 can be viewed as providing partial converses to these statements.

Since a smooth real hypersurface in $\mathbb{C}^N$ is necessarily a generic submanifold, we have the following corollary, which shows that condition (iii) of Theorem 1 in [3] is extraneous.

Corollary 1.3. *If M is a real-analytic hypersurface of finite type at p_0 in $\mathbb{C}^N$ and $H : (\mathbb{C}^N, p_0) \to (\mathbb{C}^N, \tilde{p}_0)$ a germ of a finite holomorphic mapping, then $H(M)$ is a real-analytic, real submanifold if and only if $\mathcal{H}^{-1}(\mathcal{H}(\mathcal{M})) = \mathcal{M}$, where $\mathcal{M}$ denotes the complexification of M.*

Remark 1.4. By using Theorem 4 in [3], we may replace (i) by the condition (i′): the image $H(M)$ is a germ at $\tilde{p}_0$ of a smooth submanifold.

Remark 1.5. If (ii) in Theorem 1.1 is satisfied, then $H^{-1}(H(M)) = M$. However, even in the case of a hypersurface, the latter condition does not imply (i) or (ii) (see Remark 1.11 in [3] for an example).

The following example shows that the image of a generic manifold of finite type under a finite holomorphic mapping may not be a CR manifold at 0. (Recall that $M \subset \mathbb{C}^N$ is CR *at* 0 if the mapping $p \mapsto \dim T^{0,1}M$ is constant for p in a neighborhood of 0. A generic manifold through 0 is necessarily CR at 0.) This example therefore shows that the condition that $H(M)$ is generic cannot be omitted in (1.5).

Example 1.6. Let $M \subset \mathbb{C}^3$ be the generic hypersurface given by

$$M := \left\{ (z, w_1, w_2) \in \mathbb{C}^3 : \operatorname{Im} w_1 = \frac{|z|^2}{2},\ \operatorname{Im} w_2 = \frac{|z|^4}{2} \right\} \tag{1.6}$$

and $H = (F_1, F_2, G) : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be the finite mapping given by

$$F_1(z, w) = z, \quad F_2(z, w) = w_1 + iw_2, \quad G(z, w) = (w_1 - iw_2)^2 \tag{1.7}$$

Let $M \subset \mathbb{C}^3$ be the real submanifold given by

$$\widetilde{M} := \{(\tilde{z}_1, \tilde{z}_2, \tilde{w}) \in \mathbb{C}^3 : \tilde{w} = (\overline{\tilde{z}}_2 + i|\tilde{z}_1|^2 + |\tilde{z}_1|^4)^2\}. \tag{1.8}$$

It is easily checked that $\widetilde{M}$ is not CR at 0. One can check by direct calculation that $H(M) \subset \widetilde{M}$. To see that H maps M onto $\widetilde{M}$, let $(\tilde{z}_1^0, \tilde{z}_2^0, \tilde{w}^0) \in \widetilde{M}$. Taking $z^0 := \tilde{z}_1^0$, $\operatorname{Re} w_1^0 = \operatorname{Re} \tilde{z}_2^0 + |\tilde{z}_1^0|^4/2$, $\operatorname{Re} w_2^0 = \operatorname{Im} \tilde{z}_2^0 - |\tilde{z}_1^0|^2/2$, we have $F_1(z^0, w^0) = \tilde{z}_1^0$ and $F_1(z^0, w^0) = \tilde{z}_2^0$, which proves the desired surjectivity.

Note that $\mathbb{C}^3$ is the lowest dimensional complex space in which one can find an example of the above type. Indeed, for a generic submanifold in $\mathbb{C}^2$ to be of finite type at a point, it must be a real hypersurface, so this case is covered by Corollary 1.3. However, a totally real generic submanifold in $\mathbb{C}^2$ can be mapped onto a nongeneric submanifold in $\mathbb{C}^2$, as is shown by the following example.

Example 1.7. Let

$$M := \{(w_1, w_2) \in \mathbb{C}^2 : \operatorname{Im} w_1 = \operatorname{Im} w_2 = 0\}$$

and let $H(w_1, w_2) := (w_1 + iw_2, (w_1 - iw_2)^2)$. Then H is finite, and H maps M onto the surface

$$\widetilde{M} := \{\tilde{z}, \tilde{w}) \in \mathbb{C}^2 : \tilde{w} = \overline{\tilde{z}}^2\},$$

which again is not CR at 0 and hence not generic.

Example 1.8. If M is contained in a complex analytic subvariety, then the implication $\Leftarrow$ in (1.4) does not hold in general. Consider $M \subset \mathbb{C}^2$ given by $z_2 = 0$ and the mapping $H(z_1, z_2) = (z_1, z_2^2)$. Observe that the complexification $\mathcal{M} \subset \mathbb{C}^2 \times \mathbb{C}^2$ of M is the submanifold of points $(z_1, z_2, \zeta_1, \zeta_2)$ such that $z_2 = \zeta_2 = 0$, and the complexified map is given by $\mathcal{H}(z,\zeta) = (z_1, z_2^2, \zeta_1, \zeta_2^2)$. Clearly, we have $\mathcal{H}^{-1}(\mathcal{H}(\mathcal{M})) = \mathcal{M}$. The image $\tilde{M} := H(M)$ is a submanifold at 0 (i.e., (i)), but H is not transverse to $\tilde{M}$ at 0. However, we do not know of any examples where (ii) holds, but (i) does not. For further discussion about this point, see Section 4 of this paper.

As mentioned above, Theorem 1.1 generalizes and sharpens a theorem of Baouendi and the second author (see [3], Theorem 1, part (B)) for the case, where M is an essentially finite real-analytic hypersurface. We should also point out that the conclusion in Theorem 1.1 that H is CR transversal to $H(M)$, when M is generic and of finite type, provided that $H(M)$ is a generic manifold, was proved in a recent paper [5] (see Theorem 1.1) by the authors. We conclude the introduction by mentioning two corollaries concerning ranks of finite holomorphic mappings that follow from Theorem 1.1.

Corollary 1.9. *Let M be a real-analytic generic submanifold of finite type through p_0 in $\mathbb{C}^N$, and $H : (\mathbb{C}^N, p_0) \to (\mathbb{C}^N, \tilde{p}_0)$ a germ of a finite holomorphic mapping. Let $\mathcal{M}$ and $\mathcal{H}$ denote the complexifications of M and H, respectively. If $\mathcal{H}^{-1}(\mathcal{H}(\mathcal{M})) = \mathcal{M}$, then*

$$\operatorname{rk} \frac{\partial H}{\partial Z}(p_0) \geq \operatorname{codim} M, \tag{1.9}$$

where rk *denotes the rank of a matrix and* codim M *is the real codimension of M in $\mathbb{C}^N$.*

By combining Theorem 1.1 above with a theorem from [5], we obtain, as a corollary, a sufficient geometric condition for a finite mapping to be a local biholomorphism at a given point. For this recall that M is said to be *finitely nondegenerate* at p_0 if

$$\operatorname{span}_{\mathbb{C}} \left\{ L^\alpha \left(\frac{\partial \rho^j}{\partial Z} \right)(p_0) : j = 1, \ldots, d,\ \alpha \in \mathbb{N}_+^n \right\} = \mathbb{C}^N, \tag{1.10}$$

where $\operatorname{span}_{\mathbb{C}}$ denotes the vector space spanned over $\mathbb{C}$ and $L^\alpha := L_1^{\alpha_1}, \ldots, L_n^{\alpha_n}$. Here, $L_1, \ldots, L_n$ is a basis for the smooth $(0,1)$ (or CR) vector fields tangent to M near p_0, and M is defined locally near p_0 by (1.1). A direct consequence of Theorem 1.1 above and Theorem 1.2 in [5] is the following result.

Corollary 1.10. *Let M, $\mathcal{M}$, H, and $\mathcal{H}$ be as in Corollary* 1.9. *Assume, in addition, that M is finitely nondegenerate at p_0. If $\mathcal{H}^{-1}(\mathcal{H}(\mathcal{M})) = \mathcal{M}$, then H is a local biholomorphism at p_0.*

2. Images of complex manifolds under finite mappings

The study of images of complex analytic manifolds and varieties under finite holomorphic mappings has a long history (see e.g., [7, 8, 14, 15]). The proof of Theorem 1.1 is mainly based on the following result concerning images of complex manifolds.

Theorem 2.1. *Let X be a complex submanifold through 0 in $\mathbb{C}^k$ and $f : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ a germ of a finite holomorphic mapping such that*

$$\det \frac{\partial f}{\partial z}\bigg|_X \not\equiv 0, \tag{2.1}$$

where $z = (z_1, \ldots, z_k)$ are coordinates in $\mathbb{C}^k$. Then, the following are equivalent:

(a) *$f(X)$ is a germ of a manifold at 0 and f is transversal to $f(X)$,*

(b) *$f^{-1}(f(X)) = X$ as germs at 0.*

Remark 2.2. Without the assumption (2.1), condition (b) in Theorem 2.1 does not imply that f is transversal to $f(X)$, as is shown by the example given in Remark 1.8 above. We do not know if condition (b) implies that $f(X)$ is a manifold without assuming (2.1). If X is of dimension one, then it is shown in Theorem 4.1 below that in fact (b) does imply that $f(X)$ is a manifold even without assuming (2.1).

Remark 2.3. We note that without the condition of transversality in (a), condition (b) need not hold. For example, consider $X = \{(z, w) : w = 0\}$ and the mapping $f(z, w) = (z^2 + w^2, zw)$. It can be easily checked that $f(X) = X$, but $f^{-1}(f(X)) \neq X$.

Proof of Theorem 2.1. The proof of (a)$\Longrightarrow$(b) is immediate by the transversality assumption (without using (2.1)). We shall prove (b) $\Longrightarrow$ (a). We first observe that, by the proper mapping theorem, $f(X)$ is a complex analytic (irreducible) subvariety, of the same dimension as X, through 0 in $\mathbb{C}^k$. Let q be the codimension of X in $\mathbb{C}^k$. We choose local coordinates

$(x, y) \in \mathbb{C}^p \times \mathbb{C}^q$, with $p + q = k$, vanishing at the origin in $\mathbb{C}^k$ such that X is given locally by $y = 0$.

Our first claim is that the coordinate functions y_l, $l = 1, \ldots, q$, belong to the ideal $I(f(x, y))$. If m denotes the multiplicity at the origin of the finite mapping $f : \mathbb{C}^k \to \mathbb{C}^k$, then assertion (b) implies that the m points (counted with their multiplicities) in $f^{-1}(w)$, for an arbitrary $w \in f(X)$ sufficiently close to the origin, are all contained in $X = \{(x, 0)\}$. Moreover, it follows from the condition (2.1) that these m preimages will all be distinct (multiplicity one) for a set of w which is open and dense in the variety $f(X)$. (The set of points w in $f(X)$ for which $f^{-1}(\{w\})$ consists of fewer than m points is contained in the image of $\{z : \det(\partial f/\partial z)(z) = 0\}$, which by the assumption (2.1) does not contain X.) Thus, if we let $h : (\mathbb{C}^p, 0) \to (\mathbb{C}^k, 0)$ denote the holomorphic mapping defined by $h(x) = f(x, 0)$, then for an open and dense set of $w \in f(X)$ there are m preimages of w under h. It follows (see e.g., Proposition 1 on p. 94 in [1]; see also Proposition 2.4 on p. 168 of [6]) that the multiplicity of h at 0 is at least m, i.e.,

$$m \leq \dim_{\mathbb{C}} \mathbb{C}\{x\}/I(f(x, 0)), \tag{2.2}$$

where $\mathbb{C}\{x\}$ denotes the ring of convergent power series in x (or ring of germs at 0 of holomorphic functions in $\mathbb{C}^p$). Consider the homomorphism $\phi : \mathbb{C}\{x, y\} \to \mathbb{C}\{x\}$ defined by $\phi(g)(x) = g(x, 0)$. Clearly, ϕ is surjective and sends $I(f(x, y))$ into $I(f(x, 0))$. Hence, ϕ induces a surjective homomorphism $\phi^* : \mathbb{C}\{x, y\}/I(f(x, y)) \to \mathbb{C}\{x\}/I(f(x, 0))$, so that $\dim_{\mathbb{C}} \mathbb{C}\{x\}/I(f(x, 0)) \leq m$. On the other hand, m is the multiplicity of the finite mapping f, i.e.,

$$m = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/I(f(x, y)), \tag{2.3}$$

and hence, by (2.2), we must have that

$$\dim_{\mathbb{C}} \mathbb{C}\{x\}/I(f(x, 0)) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/I(f(x, y)) \tag{2.4}$$

and ϕ^* is an isomorphism. Since $\phi^*(y_l) = 0$, for $l = 1, \ldots, q$, we conclude that

$$y_l \in I(f(x, y)), \quad l = 1, \ldots, q, \tag{2.5}$$

as claimed above.

Notice that as a consequence of (2.5), $\partial f/\partial y(0)$ has rank q. After a linear invertible transformation in the target space $\mathbb{C}^k$ (if necessary), we can decompose its coordinates as $w = (\xi, \eta) \in \mathbb{C}^p \times \mathbb{C}^q$ and write the mapping $f(x, y)$ as $f(x, y) = (R(x, y), S(x, y))^t$, where $R = (R_1, \dots, R_p)^t$, $S = (S_1 \dots, S_q)^t$, and

$$\frac{\partial R}{\partial y}(0) = 0_{p\times q}, \quad \frac{\partial S}{\partial y}(0) = I_{q\times q}, \tag{2.6}$$

where $0_{p\times q}$ denotes the $(p \times q)$-matrix whose entries are all 0 and $I_{q\times q}$ the $(q \times q)$ identity matrix. Hence, we can further write the components of the mapping as

$$R(x, y) = R_0(x) + R_1(x, y)y, \quad S(x, y) = y + S_0(x) + S_1(x, y)y, \tag{2.7}$$

where $R_1(x, y)$ and $S_1(x, y)$ are $(p \times q)$-matrix and $(q \times q)$-matrix valued functions, respectively, with $R_1(0) = 0$ and $S_1(0) = 0$. Observe that the restriction to $y = 0$ is given by $h(x) = f(x, 0) = (R_0(x), S_0(x))^t$.

Lemma 2.4. *With the notation introduced above, the germ at* 0 *of the holomorphic mapping* $R_0 : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ *is finite with multiplicity* m *and*

$$S_0(x) = g(R_0(x)), \tag{2.8}$$

where $g : (\mathbb{C}^p, 0) \to (\mathbb{C}^q, 0)$ *is the germ of the holomorphic mapping that satisfies*

$$g(\xi) := \frac{1}{m}\sum_{\nu=1}^{m} S_0(x^\nu(\xi)) \tag{2.9}$$

for a generic point $\xi \in \mathbb{C}^p$*; here,* $x^\nu(\xi)$ *denote the* m *distinct preimages of* ξ *under the mapping* R_0*.*

Proof of Lemma 2.4. In view of (2.5), we have

$$y = A(x, y)R(x, y) + B(x, y)S(x, y), \tag{2.10}$$

for some matrix valued functions A, B. If we expand $A(x, y)$ and $B(x, y)$ in y in Taylor form, writing $A(x, y) = A_0(x) + \mathrm{O}(y)$ and $B(x, y) = B_0(x) + \mathrm{O}(y)$, we conclude by substituting (2.7) into (2.10) and setting $x = 0$ that

$$B_0(0) = I_{q\times q}. \tag{2.11}$$

Similarly, by setting $y = 0$ we obtain

$$A_0(x)R_0(x) + B_0(x)S_0(x) = 0. \tag{2.12}$$

It follows from (2.11) that $B_0(x)$ is invertible near 0 and therefore, by (2.12), the components of $S_0(x)$ are in the ideal $I(R_0(x))$. In other words, we have $I(f(x,0)) = I(R_0(x))$. Thus, by (2.4) the number m of preimages of a generic $w \in f(X)$ under $h : (\mathbb{C}^p, 0) \to (\mathbb{C}^k, 0)$ is also the multiplicity of the mapping $R_0 : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ as claimed in the lemma. If we write $w = (\xi, \eta) \in \mathbb{C}^p \times \mathbb{C}^q$ for a point in $f(X)$, then for generic ξ we have

$$\eta = S_0(x^\nu(\xi)), \quad \nu = 1, \dots, m. \tag{2.13}$$

If we define g by (2.9), then g extends to a holomorphic function near 0, since it is a symmetric function of the roots $x^1, \dots, x^m$, and (2.8) can be verified directly from (2.13). □

We may now complete the proof of Theorem 2.1. Let g be as in Lemma 2.4. If we make the change of variables

$$\xi' = \xi, \quad \eta' = \eta - g(\xi) \tag{2.14}$$

in the target space $\mathbb{C}^k$ then by writing

$$g(R_0(x) + R_1(x,y)y) = g(R_0(x)) + \mathrm{O}(|x| + |y|)y$$

and using Lemma 2.4, the mapping $f(x,y)$ takes the form

$$f(x,y) = (R_0(x) + R_1(x,y)y, y + \tilde{S}_1(x,y)y)^t, \tag{2.15}$$

where $\tilde{S}_1(x,y)$ is a $(q \times q)$-matrix-valued function with $\tilde{S}_1(0) = 0$. By (2.15), the q-dimensional complex subvariety $f(X)$, where $X = \{y = 0\}$, is contained in the q-dimensional plane $\{\eta' = 0\}$ in $\mathbb{C}^k$. This proves that $f(X) = \{\eta' = 0\}$ and hence is a submanifold at 0. It is obvious from the form (2.15) of the mapping that f is transversal to $f(X)$. This completes the proof of Theorem 2.1. □

Remark 2.5. The condition (2.1) in Theorem 2.1 is only used to deduce that the generic number of preimages of the mapping $f|_X : (X,0) \to (f(X),0)$ equals the multiplicity of the mapping $f\colon (\mathbb{C}^k,0) \to (\mathbb{C}^k,0)$. In fact, these two properties are equivalent as the reader can verify.

3. Proof of Theorem 1.1

Proof. Without loss of generality, we may take $p_0 = \tilde{p}_0 = 0$. We assume first that M is a real-analytic submanifold that is not contained in any proper complex subvariety of $\mathbb{C}^N$. To prove the implication $\Longrightarrow$ of (1.4), we suppose that $\widetilde{M} := H(M)$ is a germ at 0 of a real-analytic submanifold and that H is transversal to $H(M)$ at 0. Observe that $\widetilde{M}$ is of the same dimension as M, since H is a finite mapping. If $\tilde\rho(Z,\bar Z)$, where $\tilde\rho = (\tilde\rho_1,\ldots,\tilde\rho_d)$, is a defining function for $\widetilde{M}$, then the fact that H is transversal implies that $\rho(Z,\bar Z) := \tilde\rho(H(Z),\bar H(\bar Z))$ is a defining function for M. By simply replacing $\bar Z$ by ζ in the above, we conclude that $\mathcal{H}^{-1}(\widetilde{\mathcal{M}}) = \mathcal{M}$ which is the assertion (ii).

To prove the implication $\Longleftarrow$ in (1.4), we shall need the observation that

$$\det \frac{\partial \mathcal{H}}{\partial (Z,\zeta)}\bigg|_{\mathcal{M}} \not\equiv 0.$$

Indeed, if $\det(\partial\mathcal{H}/\partial(Z,\zeta))|_{\mathcal{M}} \equiv 0$, then by the specific form of $\mathcal{H}$ it would follow that $|\det(\partial H/\partial Z)|_M|^2 \equiv 0$ contradicting the assumptions that H is finite and M is not contained in a proper complex-analytic subvariety. We may now apply Theorem 2.1 with $X := \mathcal{M}$ and $f := \mathcal{H}$ to conclude that $\widetilde{\mathcal{M}} := \mathcal{H}(\mathcal{M})$ is a germ at 0 of a manifold with $\mathcal{H}$ transversal to $\widetilde{\mathcal{M}}$ at 0. Since $\mathcal{H}$ is finite, $\widetilde{\mathcal{M}}$ has the same dimension as $\mathcal{M}$. Moreover, $\widetilde{\mathcal{M}}$ satisfies the "reality" symmetry: if $(Z,\zeta)\in\widetilde{\mathcal{M}}$, then $(\bar\zeta,\bar Z)\in\widetilde{\mathcal{M}}$. The latter is easily verified (and the verification is left to the reader) from the fact that $\mathcal{M}$ has this symmetry, by using the specific form of $\mathcal{H}$. The symmetry implies that one can find defining equations for $\widetilde{\mathcal{M}}$ near 0 of the form $\tilde\rho(Z,\zeta) = 0$, where $\tilde\rho = (\tilde\rho_1,\ldots,\tilde\rho_d)$, and for each $1\le j\le d$ we have $\tilde\rho_j(Z,\bar Z)$ is real-valued. It follows that the real-valued equation $\tilde\rho(Z,\bar Z)=0$ defines a real-analytic submanifold $\widetilde{M}\subset\mathbb{C}^N$ through 0 of codimension d. By construction, H sends M into $\widetilde{M}$. We must show that H sends M onto $\widetilde{M}$ in the sense of germs at 0. The fact that $\mathcal{H}$ is transversal to $\widetilde{\mathcal{M}}$ at 0 means that $\rho(Z,\zeta) := \tilde\rho(H(Z),\bar H(\zeta))$ is a defining function for $\mathcal{M}$ at 0. Clearly, this also means that $\rho(Z,\bar Z)$ is a defining function for M at 0 and, hence, $H^{-1}(\widetilde{M}) = M$ as germs at 0. Since any representative of the germ H near 0 is an open mapping, we conclude that $H(M) = \widetilde{M}$ as germs at 0. This completes the proof of the implication $\Longleftarrow$ in (1.4).

To complete the proof of Theorem 1.1, assume that M is a generic submanifold of finite type at 0. We shall show that (1.5) holds. Note first that M generic implies that M is not contained in any proper complex submanifold of $\mathbb{C}^N$. It follows from (1.4) that (ii) $\Longrightarrow$ (i) and H is transversal to $H(M)$ at 0. As above, we note that $\rho(Z,\bar Z) = \tilde\rho(H(Z),\overline{H(Z)})$, where $\tilde\rho(Z,\bar Z)$ is a defining function for $H(M)$ near 0, is a defining function for M

near 0. Now, by the chain rule

$$\frac{\partial \rho}{\partial Z}(0) = \frac{\partial \tilde{\rho}}{\partial Z}(0)\frac{\partial H}{\partial Z}(0)$$

and, hence, the rank of $(\partial\tilde{\rho}/\partial Z)(0)$ must be d since M is generic. Consequently, $H(M)$ is generic.

For the implication $\Longrightarrow$ in (1.5), let $\widetilde{M} := H(M)$ be a generic submanifold. It follows from Proposition 2.3 in [5] that $\widetilde{M}$ is of finite type at 0, since H is finite. Finally, the mapping H is CR transversal (and hence transversal) to $\widetilde{M}$ at 0, by Theorem 1.1 of [5]. The rest of the proof of (1.5) now follows from (1.4). $\square$

Remark 3.1. To prove that $H(M)$ is smooth in the proof above, we have used not only that $\mathcal{H}(\mathcal{M})$ is smooth but also that $\mathcal{H}$ is transversal to $\mathcal{H}(\mathcal{M})$. In general, $\mathcal{H}(\mathcal{M})$ smooth does not imply that $H(M)$ is smooth. For instance, consider $M = \mathbb{R}$ in $\mathbb{C}$ and the mapping $z \mapsto z^2$. However, the reverse implication does hold, i.e., if $H(M)$ is smooth, then $\mathcal{H}(\mathcal{M})$ is also smooth.

4. Further results on images of curves under finite mappings

In this section, we shall address the following question, which was alluded to above.

Question: *Let X be a complex submanifold through 0 in $\mathbb{C}^k$, and $f : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ a germ of a finite holomorphic mapping. Does the identity $f^{-1}(f(X)) = X$, as germs at 0, imply that $f(X)$ is a submanifold at 0?*

An equivalent formulation can be given as follows.

Question′: *Let $\tilde{X}$ be a complex subvariety through 0 in $\mathbb{C}^k$, and $f : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ a germ of a finite holomorphic mapping. Assume that $X := f^{-1}(\tilde{X})$ is a submanifold at 0. Does this imply that $\tilde{X}$ is a submanifold at 0?*

As mentioned above, we do not know the answer in general. However, the answer for one-dimensional submanifolds is affirmative in view of the following result.

Theorem 4.1. *Let X be a complex submanifold of dimension one (i.e., a smooth complex curve) through 0 in $\mathbb{C}^k$ and $f : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ a germ* *of a finite holomorphic mapping. If* $f^{-1}(f(X)) = X$ *as germs at* 0*, then* $\tilde{X} := f(X)$ *is a germ at* 0 *of a submanifold.*

An equivalent formulation of this result in the spirit of the second formulation of the question above is the following.

Theorem 4.2. *Let* $\tilde{X}$ *be a complex subvariety of dimension one* (*i.e., a complex curve*) *through* 0 *in* $\mathbb{C}^k$ *and* $f : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ *a germ of a finite holomorphic mapping. If* $X := f^{-1}(\tilde{X})$ *is a germ at* 0 *of a submanifold* (*i.e., a smooth curve*), *then* $\tilde{X}$ *is also a submanifold* (*i.e., a smooth curve*) *at* 0.

Proof of Theorem 4.1. Let $Z = (z, w) \in \mathbb{C} \times \mathbb{C}^{k-1}$ be local coordinates at 0 in which $X = \{(z, w) : w = 0\}$. We also choose coordinates $\tilde{Z}$ at 0 in the target copy of $\mathbb{C}^k$ such that, after possibly another change of coordinates in the z variable,

$$g(z) := f(z, 0) = (z^m, f_2(z, 0), \ldots, f_k(z, 0)), \tag{4.1}$$

where $f_j(z, 0) = \mathrm{O}(z^m)$ for $j = 2, \ldots, k$. We expand the $f_j(z, 0)$, $j = 2, \ldots, k$, in their Taylor series

$$f_j(z, 0) = \sum_{l=1}^{\infty} a_{jl} z^l, \quad j = 2, \ldots, k. \tag{4.2}$$

We define

$$q := \gcd(m, \{l : a_{1l} \neq 0\}, \ldots, \{l : a_{kl} \neq 0\}), \tag{4.3}$$

where $\gcd(n_1, n_2, \ldots)$ denotes the greatest common divisor of the numbers $n_1, n_2, \ldots$. We observe that there are holomorphic functions $h_j(z)$, for $j = 2, \ldots, k$, such that

$$f_j(z, 0) = h_j(z^q), \quad j = 2, \ldots, k. \tag{4.4}$$

We claim that, for any $t \neq 0$, the preimages of $g(t)$ are $\{\epsilon_0 t, \ldots, \epsilon_{q-1} t\}$, where $\epsilon_0, \ldots, \epsilon_{q-1}$ are the qth roots of unity. The fact that all the points $\epsilon_j t$, $j = 0, \ldots, q-1$, are preimages is clear from the definition of q and (4.4). Conversely, if z_0 is a preimage of $g(t_0)$ for some $t_0 \neq 0$, then $z_0 = \epsilon t_0$ for some mth root of unity ϵ. Since the only possible preimages of a point $g(t)$ with t close to t_0 are of the form $z = \delta t$ for some mth root of unity δ, we conclude that $z = \epsilon t$ is a preimage of $g(t)$ for all t near t_0. Hence, we have

$f_j(\epsilon t, 0) = f_j(t, 0)$ for $j = 2, \ldots, k$, which implies

$$\sum_{l=1}^{\infty} a_{jl} \epsilon^l t^l = \sum_{l=1}^{\infty} a_{jl} t^l, \quad j = 2, \ldots, k, \tag{4.5}$$

for t near t_0. Consequently, $\epsilon^l = 1$ for all l such that $a_{jl} \neq 0$ for some $j = 2, \ldots, k$. Since $\epsilon^m = 1$ as well, we conclude that $\epsilon^q = 1$, where q is as defined in (4.3). This proves the claim above.

If $q = m$, then it is clear that $\tilde{X}$ is smooth at 0. Thus, to complete the proof of Theorem 4.1 it suffices to assume that $1 \leq q < m$ and show that $f^{-1}(f(X))$ contains but is not equal to X. This is an immediate consequence of the following lemma, since $f(t, 0)$ is $\mathrm{O}(t^m)$ and $q < m$. □

Lemma 4.3. *Let f and X be as in Theorem* 4.1 *and q the multiplicity of the mapping $f|_X : X \to \tilde{X} := f(X)$. If $f^{-1}(f(X)) = X$ as germs at* 0, *then there is a germ at* 0 *of a holomorphic function $F : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ such that $F(f(t, 0)) = ct^j$, where $c \neq 0$ and $1 \leq j \leq q$.*

Proof. We retain the normalizations as in the beginning of the proof of Theorem 4.1. Let p be the multiplicity of the finite mapping $f : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ and, for generic $\tilde{Z}$, let $Z^1 := Z^1(\tilde{Z}), \ldots, Z^p := Z^p(\tilde{Z})$ be the preimages of $\tilde{Z}$ under f. For $j = 1, \ldots, p$, we form the jth symmetric combination of these preimages, i.e., $F^j(\tilde{Z}) = (F_1^j(\tilde{Z}), \ldots F_k^j(\tilde{Z}))$, where

$$F_i^j(\tilde{Z}) = (-1)^j \sum_{1 \leq l_1 < \cdots < l_j \leq p} Z_i^{l_1} \ldots Z_i^{l_j}. \tag{4.6}$$

As is well known, the mappings F^j, originally defined only for generic $\tilde{Z}$, extend as holomorphic mappings $(\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$. Since all the preimages of $f(t, 0)$, for $(t, 0) \in X$, are assumed to lie on X and, hence are of the form $(\epsilon_i t, 0)$ with $\epsilon_i^q = 1$ for $i = 0, 1, \ldots, q - 1$, we conclude that

$$F^j(f(t, 0)) = (c_j t^j, 0), \quad j = 1, \ldots, p, \tag{4.7}$$

for some constants c_j. Thus, to prove Lemma 4.3, it suffices to show that $c_j \neq 0$ for some $j \leq q$. To this end, we introduce the Weierstrass polynomial

(4.8)

$$P(\tilde{Z}, x) := x^p + F_1^1(\tilde{Z}) x^{p-1} + \cdots + F_1^{p-1}(\tilde{Z}) x^1 + F_1^p(\tilde{Z}) = \prod_{l=1}^{p} (x - Z_1^l),$$

where the last identity only holds for generic $\tilde{Z}$. We let $R(t,x)$ denote the polynomial $P(f(t,0),x)$ and observe that $R(t,x)$ has the form

$$R(t,x) = x^p + c_1 t x^{p-1} + \cdots + c_{p-1} t^{p-1} x + c_p t^p. \tag{4.9}$$

Moreover, by construction, the distinct roots of $R(t,x)$ are precisely $x_i = \epsilon_i t$, $i = 0, 1, \ldots, q$.

Lemma 4.4. *Let $Q(y)$ be a monic polynomial of degree p*

$$Q(y) = y^p + e_1 y^{p-1} + \cdots + e_{p-1} y + e_p. \tag{4.10}$$

If all the roots of $Q(y)$ are q-roots of unity with $q \leq p$, then there is $1 \leq j \leq q$ such that $e_j \neq 0$.

Proof. Let $S(w_1, \ldots, w_p, y)$ be the monic polynomial in y with polynomial coefficients $c_i(w)$ given by

$$S(w,y) = \prod_{l=1}^{p} (y - w_i) = y^p + c_1(w) y^{p-1} + \cdots + c_{p-1}(w) y + c_p(w). \tag{4.11}$$

Note that the polynomial $c_i(w)$ is homogeneous of total degree i in the complex variables $w_1, \ldots, w_p$ and is invariant under all permutations of the w_i. Furthermore, by the well-known theorem of elementary invariant theory (see e.g., [2], Theorem 5.3.4), if $d(w)$ is any polynomial invariant under all permutations of w, then there is a polynomial $a(b_1, \ldots, b_p)$ such that

$$d(w) = a(c_1(w), \ldots, c_p(w)) \tag{4.12}$$

In view of the homogeneity of the $c_i(w)$, it follows from (4.12) that if $d_q(w)$ is homogeneous of degree q, then there is a polynomial $a_q(b_1, \ldots, b_q)$ (necessarily of degree $\leq q$ and with no constant term) such that

$$d_q(w) = a_q(c_1(w), \ldots, c_q(w)) \tag{4.13}$$

Now take $d_q(w) := \sum_{i=1}^{p} w_i^q$. If $w_i = \epsilon_i$ is a q-root of unity for $i = 1, \ldots p$, then $d_q(\epsilon_1, \ldots, \epsilon_p) = p$, and, in particular, is not zero. It follows from (4.13) that $c_{j_0}((\epsilon_1, \ldots, \epsilon_p) \neq 0$ for some j_0 with $1 \leq j_0 \leq q$. Therefore, since all the roots of $Q(y)$ given by (4.10) are assumed to be q-roots of unity, it follows that the coefficient e_{j_0} of $Q(y)$ is not zero. This proves Lemma 4.4. $\square$

To complete the proof of Lemma 4.3, we set $y = x/t$ in (4.9) and define $Q(y) := R(t, ty)/t^p$, i.e.,

$$Q(y) = y^p + c_1 y^{p-1} + \cdots + c_{p-1} y + c_p. \tag{4.14}$$

All the roots of $Q(y)$ are q-roots of unity by construction. By Lemma 4.4, there is $1 \leq j \leq q$ such that $c_j \neq 0$. This proves Lemma 4.3. $\square$

We conclude this paper by giving an equivalent algebraic reformulation of the question posed in the beginning of this section. Let I be an ideal in $\mathbb{C}\{Z\}$. Recall that I is the ideal of a complex analytic subvariety X at 0 if and only if $I = \sqrt{I}$, i.e., I is radical. The subvariety X is a submanifold at 0 if and only if the ring $\mathbb{C}\{Z\}/I$ is regular, i.e., isomorphic to a power series ring $\mathbb{C}\{t\}$.

Question″: *Let $\phi : \mathbb{C}\{x_1, \ldots, x_k\} \to \mathbb{C}\{z_1, \ldots, z_k\}$ be an injective $\mathbb{C}$-algebra homomorphism such that $\mathbb{C}\{z_1, \ldots, z_k\}$ is integral over $\phi(\mathbb{C}\{x_1, \ldots, x_k\})$. Let I be a radical ideal in $\mathbb{C}\{x_1, \ldots, x_k\}$ and J the ideal in $\mathbb{C}\{z_1, \ldots, z_k\}$ generated by $\phi(I)$. Assume that the ring $\mathbb{C}\{z_1, \ldots, z_k\}/\sqrt{J}$ is regular. Does this imply that $\mathbb{C}\{x_1, \ldots, x_k\}/I$ is regular?*

In this formulation, the answer is negative if the field of complex numbers is replaced by any field of characteristic $0 < p < \infty$ in view of the following example, communicated to us by Joseph Lipman, who attributed it to Melvin Hochster. Let K be a field of characteristic p and consider the homomorphism $\phi : K\{u, v, w\} \to K\{x, y, z\}$ given by $u \mapsto x^p$, $v \mapsto y^p$, and $w \mapsto z$. If we let I be the (prime) ideal generated by $w^p + uv$, then J is the ideal generated by $z^p + x^p y^p$, which in characteristic p is equal to $(z + xy)^p$. The radical $\sqrt{J}$ is then generated by $z + xy$, and $\mathbb{C}\{z_1, \ldots, z_k\}/\sqrt{J}$ is regular. However, $\mathbb{C}\{x_1, \ldots, x_k\}/I$ is not regular. Another counterexample (in characteristic 3) was attributed to Bill Heinzer. Lipman also informed us that he has an algebraic proof [11] showing that if "regular" in Question″ above is replaced by "normal", then the answer is affirmative (indeed, he proved this statement in a more general context). For an ideal I for which $\dim \mathbb{C}\{z_1, \ldots, z_k\}/I = 1$, normal is the same as regular and, hence, Lipman's arguments yield another proof of Theorem 4.2 above. We would like to take this opportunity to thank the above mentioned people for their help and interest in our question.

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