

Uniqueness and pseudolocality theorems of the mean curvature flow

BING-LONG CHEN AND LE YIN

Mean curvature flow evolves isometrically immersed base manifolds M in the direction of their mean curvatures in an ambient manifold \bar{M} . If the base manifold M is compact, the short-time existence and uniqueness of the mean curvature flow are well known. For complete isometrically immersed submanifolds of arbitrary codimensions, the existence and uniqueness are still unsettled even in the Euclidean space. In this paper, we solve the uniqueness problem affirmatively for the mean curvature flow of general codimensions and general ambient manifolds. In the second part of the paper, inspired by the Ricci flow, we prove a pseudolocality theorem of mean curvature flow. As a consequence, we obtain a strong uniqueness theorem, which removes the assumption on the boundedness of the second fundamental form of the solution.

1. Introduction

Let $(\bar{M}^{\bar{n}}, \bar{g})$ be a complete Riemannian (compact or noncompact) manifold, and $X_0 : (M^n, g) \rightarrow \bar{M}^{\bar{n}}$ be an isometrically immersed Riemannian manifold. For any fixed point $x_0 \in M^n$, $X, Y \in T_{x_0} M^n$, the second fundamental form Π at x_0 is defined by $\Pi(X, Y) = \bar{\nabla}_{\tilde{X}} \tilde{Y} - \nabla_{\tilde{X}} \tilde{Y} = (\bar{\nabla}_{\tilde{X}} \tilde{Y})^\perp$, where M^n is regarded as a submanifold of \bar{M} locally by the isometry X_0 , $\bar{\nabla}$ and ∇ are the covariant derivatives of \bar{g} and g , respectively, \tilde{X}, \tilde{Y} are any smooth extensions of X and Y on $\bar{M}^{\bar{n}}$. In a local coordinate system $\{x^1, x^2, \dots, x^n\}$ on M^n , denote the second fundamental form by $h_{ij} = \Pi((\partial/\partial x^i), (\partial/\partial x^j))$ and the mean curvature by $H = g^{ij} h_{ij}$. The mean curvature flow (MCF) is a deformation $X_t : M^n \rightarrow \bar{M}^{\bar{n}}$ of X_0 in the direction of the mean curvature H

$$(1.1) \quad \frac{\partial}{\partial t} X(x, t) = H(x, t), \quad \text{for } x \in M^n \text{ and } t \geq 0,$$

with $X(x, 0) = X_0(x)$, where M^n is equipped with the induced metric from $X(\cdot, t) : M^n \rightarrow \bar{M}^{\bar{n}}$ and $H(x, t)$ is the corresponding mean curvature.

We can write (1.1) in another form

$$(1.2) \quad \frac{\partial}{\partial t} X(x, t) = \Delta X(x, t), \text{ for } x \in M^n \text{ and } t \geq 0,$$

where $\Delta X^\alpha(x, t) = g^{ij}(x, t)((\partial^2 X^\alpha / \partial x^i \partial x^j) - \Gamma_{ij}^k(\partial X^\alpha / \partial x^k) + \bar{\Gamma}_{\beta\gamma}^\alpha(\partial X^\beta / \partial x^i)(\partial X^\gamma / \partial x^j))$ is the harmonic map Laplacian from the manifold $(M^n, g_{ij}(\cdot, t))$ to $(\bar{M}^{\bar{n}}, \bar{g})$, and $g_{ij}(\cdot, t)$ is the induced metric from the inclusion map $X(\cdot, t)$.

Various weak solutions to the MCF have been studied in the past 30 years by many mathematicians with different approaches, e.g., Brakke solutions, the level set solutions, etc. The existence, uniqueness and nonuniqueness of weak solutions for Euclidean (non)smooth hypersurface have been extensively studied. In this paper, motivated by geometric applications, we consider the classical solutions in general ambient Riemannian manifolds.

When M^n is compact, the MCF (1.1) has a unique short-time solution, since (1.2) is a (degenerate) quasi-linear parabolic equation. For codimensional one complete immersed local Lipschitz hypersurfaces in the Euclidean space, we refer the readers to see [11, 12]. For submanifolds of arbitrary codimensions in a general ambient Riemannian manifold, the short-time existence and the uniqueness of (1.1) have not been established in the literature. In this paper, we deal with the uniqueness problem of the mean curvature flow and derive the pseudolocality estimate.

The first main theorem of this paper is the following.

Theorem 1.1. *Let $(\bar{M}^{\bar{n}}, \bar{g})$ be a complete Riemannian manifold of dimension \bar{n} such that the curvature and its covariant derivatives up to order 2 are bounded and the injectivity radius is bounded from below by a positive constant, i.e., there are constants \bar{C} and $\bar{\delta}$ such that*

$$|\bar{\text{Rm}}| + |\bar{\nabla} \bar{\text{Rm}}| + |\bar{\nabla}^2 \bar{\text{Rm}}|(x) \leq \bar{C}, \quad \text{inj}(\bar{M}^{\bar{n}}, x) > \bar{\delta} > 0,$$

for all $x \in \bar{M}^{\bar{n}}$. Let $X_0 : M^n \rightarrow \bar{M}^{\bar{n}}$ be an isometrically immersed Riemannian manifold with bounded second fundamental form in $\bar{M}^{\bar{n}}$. Suppose $X_1(x, t)$ and $X_2(x, t)$ are two solutions to the mean curvature flow (1.1) on $M^n \times [0, T]$ with the same X_0 as initial data and with bounded second fundamental forms on $[0, T]$. Then $X_1(x, t) = X_2(x, t)$ for all $(x, t) \in M^n \times [0, T]$.

We remark that the uniqueness of the Ricci flow has been established by Zhu and the first author in [5]. More precisely, it was proved in [5] that the solutions of the Ricci flow in the class of bounded curvature with the same

initial data are unique. We refer the readers to see an interesting application of this uniqueness theorem to the theory of the Ricci flow with surgery in dimension 3 and 4 [2, 4, 21]. We hope this MCF uniqueness theorem will also play roles in the theory of the mean curvature flow with surgery.

Since the MCF is degenerate in tangent directions, it is not a strictly parabolic system. In order to apply the standard theory of strict parabolic equations, we use the De Turck trick [7]. The idea is to pull back the MCF through a family of diffeomorphisms of the base manifold M^n generated by solving a harmonic map flow coupled with the MCF, this gives us the MCF in harmonic map flow gauge, which is a strict parabolic system. Then we apply the uniqueness of the strict parabolic system. The issue is not quite straight forward as it seems. Because before applying the uniqueness theorem of a strict parabolic system on a noncompact manifold, we encounter two analytic difficulties. The first one is that we need to establish a short-time existence for the harmonic map flow between complete manifolds. The second one is to get a priori estimates for the harmonic map flow so that after pulling back, the solutions to the strictly parabolic system still satisfy suitable smooth or growth conditions.

In the classical theory of the harmonic map flow, people usually would like to impose certain convexity conditions to ensure the existence (e.g., the negative curvature condition [13] or convex condition [8]). We observed that in [5] the condition of injectivity radius bounded from below by a positive constant ensures certain uniform (local) convexity and this is sufficient to give the short-time existence and a priori estimates for the harmonic map flow. Note that the MCF is a kind of harmonic map flow with varying base metrics. In order to deal with the a priori estimates for MCF and harmonic map flow coupled with MCF, we have to consider the general harmonic map flow. These estimates have been dealt with systematically in this paper (Sections 2, 3 and 4).

Note that the injectivity radius of a Riemannian manifold with bounded curvature may decay exponentially. In the Ricci flow case [5], since we only have the curvature bound, we need make more effort to overcome this difficulty.

The difference of Theorem 1.1 with [5] is between the extrinsic and intrinsic geometries. In the present case, instead of the metrics as in the Ricci flow, we consider the equation of the position function.

As a direct consequence of Theorem 1.1, we have

Corollary 1.2. *Let $(\bar{M}^{\bar{n}}, \bar{g})$ be assumed as in Theorem 1.1 and $X_t : M^n \rightarrow \bar{M}^{\bar{n}}$ be a solution to the mean curvature flow (1.1) on $M^n \times [0, T]$ with*

bounded second fundamental forms on $[0, T]$, and with complete isometric immersed $X_0 : M \rightarrow M$ initial data. Let $\bar{\sigma}$ be an isometry of (\bar{M}^n, \bar{g}) such that there is an isometry σ of (M^n, g) to itself satisfying

$$(1.3) \quad (\bar{\sigma} \circ X_0)(x) = (X_0 \circ \sigma)(x)$$

for all $x \in M^n$. Then we have

$$(1.4) \quad (\bar{\sigma} \circ X_t)(x) = (X_t \circ \sigma)(x)$$

for all $(x, t) \in M^n \times [0, T]$. In particular, the isometry subgroup of (M^n, g) induced by an isometry subgroup of (\bar{M}^n, \bar{g}) at initial time remains to be an isometry subgroup of (M^n, g_t) for any $t \in [0, T]$.

From the PDE point of view, it is a natural condition in Theorem 1.1 to assume that the second fundamental form of the solution is bounded. In the last part of the paper, we try to remove this condition. We remark that in [6], Chou and Zhu had obtained the strong uniqueness of the curve shortening flow for the locally Lipschitz continuous properly embedded curve whose two ends are presentable as graphs over semi-infinite line. Our strong uniqueness theorem is the following.

Theorem 1.3. *Let \bar{M} be an \bar{n} -dimensional complete Riemannian manifold satisfying $\sum_{i=0}^3 |\bar{\nabla}^i \bar{R}m| \leq c_0^2$ and $\text{inj}(\bar{M}) \geq i_0 > 0$. Let $X_0 : M \rightarrow \bar{M}$ be an n -dimensional isometrically properly embedded submanifold with bounded second fundamental form in \bar{M} . We assume $X_0(M)$ is uniform graphic with some radius $r > 0$. Suppose $X_1(x, t)$ and $X_2(x, t)$ are two smooth solutions to the mean curvature flow (1.1) on $M \times [0, T_0]$ properly embedded in \bar{M} with the same X_0 as initial data. Then there is $0 < T_1 \leq T_0$ such that $X_1(x, t) = X_2(x, t)$ for all $(x, t) \in M \times [0, T_1]$.*

Here roughly speaking, uniform graphic with radius $r (> 0)$ means that for any $x_0 \in X_0(M)$, $X_0(M) \cap B_{\bar{M}}(x_0, r)$ is a graph. We say a submanifold $M \subset \bar{M}$ is properly embedded in a ball $B_{\bar{M}}(x_0, r_0)$ if either M is closed or ∂M has distance $\geq r_0$ from x_0 . A submanifold $M \subset \bar{M}$ is said to be properly embedded in (complete manifold) \bar{M} if either M is closed or there is an $x_0 \in \bar{M}$ such that M is properly embedded in $B_{\bar{M}}(x_0, r_0)$ for any $r_0 > 0$.

The strong uniqueness theorem was proved as a consequence of Theorem 1.1 and pseudolocality theorem.

The pseudolocality theorem says that the behavior of the solution at a point can be controlled by the initial data of nearby points, no matter the

solution or initial data outside the neighborhood behaviors like. Precisely the following theorem is proved in this paper.

Theorem 1.4. *Let \bar{M} be an \bar{n} -dimensional manifold satisfying $\sum_{i=0}^3 |\bar{\nabla}^i \bar{R}_m| \leq c_0^2$ and $\text{inj}(\bar{M}) \geq i_0 > 0$. Then for every $\alpha > 0$ there exist $\varepsilon > 0$, $\delta > 0$ depending only on the constants \bar{n} , c_0 and i_0 with the following property. Suppose we have a smooth solution to the mean curvature flow $M_t \subset \bar{M}$ properly embedded in $B_{\bar{M}}(x_0, r_0)$ for $t \in [0, T]$, where $0 < T \leq \varepsilon^2 r_0^2$, and assume that at time zero, M_0 is a local δ -Lipschitz graph of radius r_0 at $x_0 \in M$ with $r_0 \leq (i_0/2)$. Then we have an estimate of the second fundamental form*

$$|A|(x, t)^2 \leq \frac{\alpha}{t} + (\varepsilon r_0)^{-2}$$

on $B_{\bar{M}}(x_0, \varepsilon r_0) \cap M_t$, for any $t \in [0, T]$.

We refer the reader to see the precise definition of δ -Lipschitz graph in Section 7. The third covariant derivative of the curvature is a technical assumption which could be improved, we assume it only for simplicity. For most of interesting cases, we have all covariant derivative bounds.

We remark that for codimension one uniformly local Lipschitz hypersurface in Euclidean space, the estimate was firstly derived by Ecker and Huisken [11, 12]. For higher-codimension case, under an additional condition which assumes that the submanifold is compact, the estimate was proved by Wang [24]. In codimension, one case, the constant δ in Theorem 1.4 does not need to be small; however, in higher-codimension case, as noted by [24], the smallness assumption is necessary in view of the example of Lawson and Osserman [18]. The strategy of the proofs of [11, 12, 24] is to find a suitable gradient function. The philosophy is that this gradient function will serve as the lower order quantity as in the Bernstein trick, and the second fundamental form is the higher order quantity, then apply the maximum principle. This method has some difficulties in higher-codimensional case in current stage, treating pure local estimates for noncompact submanifolds (see [24]).

Our approach is completely different. This approach can be regarded as an integral version of Bernstein trick. It is a mean curvature flow analog of the corresponding estimate in Ricci flow given by Perelman [20]. The idea of localizing monotonicity formulas (see [9], or formula (7.8) in Section 7 of the present paper) plays a crucial role in our argument. We refer the readers to see [1, 9, 10, 16] (Euclidean case) for relevant details and references.

As a nontrivial corollary of Theorem 1.4, we have

Corollary 1.5. *Let \bar{M} be an \bar{n} -dimensional complete manifold satisfying $\sum_{i=0}^3 |\bar{\nabla}^i \bar{\text{Rm}}| \leq c_0^2$ and $\text{inj}(\bar{M}) \geq i_0 > 0$. Let $X_0 : M \rightarrow \bar{M}$ be an n -dimensional isometrically properly embedded submanifold with bounded second fundamental form $|A| \leq c_0$ in \bar{M} . We assume $M_0 = X_0(M)$ is uniform graphic with some radius $r > 0$. Suppose $X(x, t)$ is a smooth solution to the mean curvature flow (1.1) on $M \times [0, T_0]$ properly embedded in \bar{M} with X_0 as initial data. Then there is $T_1 > 0$ depending upon c_0, i_0, r and the dimension \bar{n} such that*

$$|A|(x, t) \leq 2c_0$$

for all $x \in M, 0 \leq t \leq \min\{T_0, T_1\}$.

This paper is organized as follows. In Section 2, we derive the injectivity radius estimate of an immersed manifold and some preliminary estimates for a general harmonic map flow. In Section 3, the higher-derivative estimates for the MCF are derived. In Section 4, we study the harmonic map flow coupled with the MCF. In Section 5, we deal with the uniqueness theorem of the mean curvature De Turck flow (or MCF in harmonic map flow gauge). In Section 6, we prove the uniqueness Theorem 1.1 and Corollary 1.2. In Section 7, we establish the pseudolocality Theorems 1.4 and 1.5 and prove the strong uniqueness Theorem 1.3.

2. Preliminary estimates

In the first part of this section, we will derive the injectivity radius estimate for isometrically immersed manifold M^n .

Theorem 2.1. *Let $(\bar{M}^{\bar{n}}, \bar{g})$ be a complete Riemannian manifold of dimension \bar{n} with bounded curvature and the injectivity radius is bounded from below by a positive constant, i.e., there are constants \bar{C} and $\bar{\delta}$ such that*

$$(2.1) \quad |\bar{\text{Rm}}|(x) \leq \bar{C} \quad \text{and} \quad \text{inj}(\bar{M}^{\bar{n}}, x) \geq \bar{\delta} > 0, \quad \text{for all } x \in \bar{M}^{\bar{n}}.$$

Let $X : M^n \rightarrow \bar{M}^{\bar{n}}$ be a complete isometrically immersed manifold with bounded second fundamental form $|h_{ij}^\alpha| \leq C$ in $\bar{M}^{\bar{n}}$, then there is a positive constant $\delta = \delta(\bar{C}, \bar{\delta}, C, \bar{n})$ such that the injectivity radius of M^n satisfies

$$(2.2) \quad \text{inj}(M^n, x) \geq \delta > 0, \quad \text{for all } x \in M^n.$$

Proof. Fix $x_0 \in M^n$, let $\{y^1, y^2, \dots, y^{\bar{n}}\}$ and $\{x^1, x^2, \dots, x^n\}$ be any two local coordinates of $\bar{M}^{\bar{n}}$ and M^n at $y_0 (= X(x_0))$ and x_0 , respectively, recall that the second fundamental form can be written in these local coordinates in the following form

$$\begin{aligned} h_{ij}^\alpha &= \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial y^\alpha}{\partial x^k} + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j} \\ (2.3) \quad &= \nabla_i \nabla_j (y^\alpha) + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j}, \text{ for } \alpha = 1, 2, \dots, \bar{n}, \end{aligned}$$

where $\nabla_i \nabla_j (y^\alpha)$ is the Hessian of y^α , which is viewed as a function of M^n near x_0 . In the following argument, we denote by \bar{C}_1 various constants depending only on \bar{C} , C and $\bar{\delta}$.

Define $f(x) = \bar{d}^2(y_0, X(x))$ on $M^n \cap X^{-1}(\bar{B}(y_0, \bar{C}_1))$ for some $\bar{C}_1 \leq \bar{\delta}$, then $\nabla_j f = (\partial f / \partial y^\alpha)(\partial y^\alpha / \partial x^j)$ and the Hessian of f with respect to the metric g on $M^n \cap X^{-1}(\bar{B}(y_0, \bar{C}_1))$ can be computed as follows

$$\begin{aligned} \nabla_i \nabla_j f &= \frac{\partial}{\partial x^i} \nabla_j f - \Gamma_{ij}^k \nabla_k f \\ &= \left(\frac{\partial^2 f}{\partial y^\alpha \partial y^\beta} - \bar{\Gamma}_{\alpha\beta}^\gamma \frac{\partial f}{\partial y^\gamma} \right) \frac{\partial y^\alpha}{\partial x^j} \frac{\partial y^\beta}{\partial x^i} \\ &\quad + \frac{\partial f}{\partial y^\alpha} \left(\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial y^\alpha}{\partial x^k} + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j} \right) \\ (2.4) \quad &= \bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{d}^2 \frac{\partial y^\alpha}{\partial x^j} \frac{\partial y^\beta}{\partial x^i} + 2 \bar{d} \bar{\nabla}_\alpha \bar{d} \cdot h_{ij}^\alpha. \end{aligned}$$

Using Hessian comparison theorem on $\bar{M}^{\bar{n}}$ and choosing \bar{C}_1 suitable small so that \bar{d} is suitable small, we get

$$(2.5) \quad \nabla_i \nabla_j f \geq \frac{1}{2} g_{ij}$$

on $M^n \cap X^{-1}(\bar{B}(y_0, \bar{C}_1))$. Now we claim that any closed geodesic starting and ending at x_0 on (M^n, g) must have length $\geq 2\bar{C}_1$.

We argue by contradiction. Indeed, suppose we have a closed geodesic $\gamma : [0, L] \rightarrow M^n$ of length $L < 2\bar{C}_1$, $X \circ \gamma$ must be contained in $\bar{B}(y_0, \bar{C}_1)$, then by (2.5), we have

$$(2.6) \quad \frac{d^2}{ds^2} f \circ \gamma(s) = \nabla^2 f(\dot{\gamma}, \dot{\gamma}) \geq \frac{1}{2}, \quad s \in [0, L].$$

By the maximum principle, we have

$$\sup_{s \in [0, L]} f \circ \gamma(s) \leq f \circ \gamma(0),$$

this implies that γ is just a point $\gamma(0)$. The contradiction proves the claim.

On the other hand, by the Gauss equation,

$$R_{ijkl} = \bar{R}_{ijkl} + (h_{ik}^\alpha h_{jl}^\beta - h_{il}^\alpha h_{kj}^\beta) \bar{g}_{\alpha\beta}(\cdot, 0),$$

we see that

$$(2.7) \quad |\text{Rm}| \leq \bar{C} + 2C^2.$$

Finally, by Klingenberg lemma [3], the injectivity radius of (M^n, g) at x_0 is given by

$$\begin{aligned} \text{inj}(M^n, g, x_0) &= \min \left\{ \begin{array}{l} \text{the conjugate radius at } x_0, \\ \frac{1}{2} \text{ the length of the shortest closed geodesic at } x_0 \end{array} \right\} \\ &\geq \min \left\{ \frac{\pi}{\sqrt{\bar{C} + 2C^2}}, \bar{C}_1 \right\}. \end{aligned}$$

The proof of the theorem is completed. □

Let N be a Riemannian manifold, the distance function $d(y_1, y_2)$ can be regarded as a function on $N \times N$. In the next theorem, we will estimate the Hessian of the distance function, which is viewed as the function of two variables. The crucial computation of the Hessian was carried out in [23].

Theorem 2.2. *Let N^n be a complete Riemannian manifold of dimension n satisfying*

$$(2.8) \quad |\text{Rm}| \leq K_0, \quad \text{inj}(N^n) \geq i_0 > 0.$$

Let $d(y_1, y_2)$ be the distance function regarded as a function on $N \times N$, then there is a positive constant $C = C(K_0, i_0)$ such that when $d(y_1, y_2) \leq \min\{(i_0/2), (1/4\sqrt{K_0})\}$, we have

$$(2.9) \quad \begin{aligned} &\text{(i)} \quad |\nabla^2 d^2|(y_1, y_2) \leq C, \\ &\text{(ii)} \quad (\nabla^2 d^2)(X, X) \geq 2|X_1 - P_\gamma^{-1} X_2|^2 - C|X|^2 d^2 \\ &\text{for all } X \in T_{(y_1, y_2)} N^n \times N^n, \end{aligned}$$

where $X = X_1 + X_2$, $X_1 \in T_{y_1}N^n$, $X_2 \in T_{y_2}N^n$, ∇ is the covariant derivative of $N \times N$, γ is the unique geodesic connecting y_1 and y_2 in N^n , and P_γ is the parallel translation of N^n along γ .

Proof. Set $\psi(y_1, y_2) = d_{N^n}^2(y_1, y_2)$. Then ψ is a smooth function of (y_1, y_2) when $d(y_1, y_2) \leq \min\{(i_0/2), (1/4\sqrt{K_0})\}$. Now we recall the computation of $\text{Hess}(\psi)$ in [23]. For any $(u, v) \in D = \{(u, v) : (u, v) \in N^n \times N^n, d_{N^n}(u, v) \leq \min\{(i_0/2), (1/4\sqrt{K_0})\}\} \setminus \{(u, u) : u \in N^n\}$, let γ_{uv} be the minimal geodesic from u to v and $e_1 \in T_uN^n$ be the tangent vector to γ_{uv} at u . Then $e_1(u, v)$ defines a smooth vector field on D . Let $\{e_i\}$ be an orthonormal basis for T_uN^n which depends on u smoothly. By parallel translation of $\{e_i\}$ along γ , we define $\{\bar{e}_i\}$ an orthonormal basis for T_vN^n . Thus $\{e_1, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n\}$ is a local frame on D . Then for any $X = X_1 + X_2 \in T_{(u,v)}D$ with

$$X_1 = \sum_{i=1}^n \xi_i e_i \quad \text{and} \quad X_2 = \sum_{i=1}^n \eta_i \bar{e}_i,$$

by the formula (16) in [23],

$$\begin{aligned} \frac{1}{2}\text{Hess}(\psi)(X, X) &= \sum_{i=1}^n (\xi_i - \eta_i)^2 + \int_0^r t \langle \nabla_{e_1} V, \nabla_{e_1} V \rangle \\ &\quad + \int_0^r t \langle \nabla_{\bar{e}_1} V, \nabla_{\bar{e}_1} V \rangle - \int_0^r t \langle R(e_1, V)V, e_1 \rangle \\ &\quad - \int_0^r t \langle R(\bar{e}_1, V)V, \bar{e}_1 \rangle, \end{aligned} \tag{2.10}$$

where V is a Jacobi field on geodesic σ (connecting (v, v) to (u, v)) and $\bar{\sigma}$ (connecting (u, u) to (u, v)) of length $r = \sqrt{\psi}$ with X as the boundary values, where X is extended to be a local vector field by letting its coefficients with respect to $\{e_1, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n\}$ be constant (see [23]). By the Jacobi equation, we have the estimates

$$|V| \leq C(K_0, i_0)|X|, \quad r|\nabla_{e_1} V| \leq C(K_0, i_0)|X|, \quad r|\nabla_{\bar{e}_1} V| \leq C(K_0, i_0)|X|$$

under the assumption $d(y_1, y_2) \leq \min\{(i_0/2), (1/4\sqrt{K_0})\}$. Thus by (2.10) we have

$$|\text{Hess}(\psi)| \leq C(K_0, i_0),$$

this proves (i). Similarly, when $d(y_1, y_2) \leq \min\{(i_0/2), (1/4\sqrt{K_0})\}$, by (2.10), we have

$$\begin{aligned} \frac{1}{2} \text{Hess}(\psi)(X, X) &\geq \sum_{i=1}^n (\xi_i - \eta_i)^2 - \int_0^r t \langle R(e_1, V)V, e_1 \rangle \\ &\quad - \int_0^r t \langle R(\bar{e}_1, V)V, \bar{e}_1 \rangle \\ &\geq \sum_{i=1}^n (\xi_i - \eta_i)^2 - C(K_0, i_0) |X|^2 r^2. \end{aligned}$$

This proves (ii). The Theorem is proved. □

For future applications, in the next part of this section, we will calculate the equations of derivatives of general harmonic map flow. Since the MCF is a kind of harmonic map flow with varying base metrics evolved by MCF, the formulas computed here are very useful in deriving the higher derivative-estimates in Section 3 and 4. The formulas are of interest in their own rights. First we fix some notations.

Let F be a map from a Riemannian manifold (M, g_{ij}) to another Riemannian manifold $(N, \bar{g}_{\alpha\beta})$, let $F^{-1}TN$ be the pull back of the tangent bundle of N , we equip the bundle $(T^*M)^{\otimes p} \otimes F^{-1}TN$ the connection and metric induced from the connections and metrics of M and N . Let u be a section of $(T^*M)^{\otimes(p-1)} \otimes F^{-1}TN$. In local coordinates $\{x^i\}$ and $\{y^\alpha\}$ of M and N with $y = F(x)$, we have $|u|^2 = u_{i_1 i_2 \dots i_{p-1}}^\alpha u_{j_1 j_2 \dots j_{p-1}}^\beta g^{i_1 j_1} \dots g^{i_{p-1} j_{p-1}} \bar{g}_{\alpha\beta}$. The coefficients of the covariant derivative ∇u can be computed by the formula

$$(\nabla u)_{i_1 i_2 \dots i_{p-1} i_p}^\alpha = \frac{\partial u_{i_1 i_2 \dots i_{p-1}}^\alpha}{\partial x^{i_p}} - \Gamma_{i_p i_j}^l u_{i_1 i_2 \dots i_{j-1} l i_{j+1} \dots i_{p-1}}^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial F^\beta}{\partial x^{i_p}} u_{i_1 i_2 \dots i_{p-1}}^\gamma,$$

where Γ and $\bar{\Gamma}$ are connection coefficients of M and N , respectively. We can define the Laplacian of u by $\Delta u = \text{tr}_g \nabla^2 u = g^{ij} (\nabla^2 u)_{\dots ij}$. Recall the Ricci identity

$$\begin{aligned} (\nabla^2 u)_{\dots ij}^\alpha - (\nabla^2 u)_{\dots ji}^\alpha &= -R_{ij i_m l} u_{\dots i_{m-1} l i_{m+1} \dots}^\alpha g^{kl} \\ (2.11) \quad &\quad + \bar{R}_{\beta\gamma\delta\zeta} \frac{\partial F^\beta}{\partial x^j} \frac{\partial F^\gamma}{\partial x^i} g^{\alpha\delta} u_{\dots}^\zeta. \end{aligned}$$

Note that the derivative ∇F ($\nabla_i F^\alpha = (\partial F^\alpha / \partial x^i)$) is a section of the bundle $T^*M \otimes F^{-1}TN$, the higher-derivative $\nabla^p F$ is a section of $(T^*M)^{\otimes p} \otimes F^{-1}TN$.

If we have a family of metrics $g_{ij}(\cdot, t)$ on M and a family of maps $F(\cdot, t)$ from M to N , then for each time t , we can still define the bundle $(T^*M)^{\otimes p} \otimes F^{-1}TN$ and define the covariant derivative ∇ . It is a useful observation that the natural time derivative $(\partial/\partial t)$ is not covariant with the metrics. We define a covariant time derivative D_t as follows. For any section $u_{i_1 \dots i_p}^\alpha$ of $(T^*M)^{\otimes p} \otimes F^{-1}TN$, we define

$$D_t u_{i_1 \dots i_p}^\alpha = \frac{\partial}{\partial t} u_{i_1 \dots i_p}^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial F^\beta}{\partial t} u_{i_1 \dots i_p}^\gamma.$$

It is a routine computation which shows that the operator D_t is covariant.

Proposition 2.3. *Let M be a manifold with a family of metrics $g_{ij}(x, t)$, (N, \bar{g}) a Riemannian manifold. Let $F(\cdot, t)$ be a solution to the harmonic map flow with respect to the evolving metrics g_t and \bar{g}*

$$(2.12) \quad \frac{\partial}{\partial t} F(x, t) = \Delta F(x, t), \text{ for } x \in M^n \text{ and } t \geq 0,$$

where $\Delta F(x, t)$ is the harmonic map Laplacian of F defined by metrics $g_{ij}(x, t)$ and \bar{g} . Then we have

$$(2.13) \quad \begin{aligned} (D_t - \Delta)\nabla^k F &= \sum_{l=0}^{k-1} \nabla^l [(R_M * g^{-2} + \bar{R}_N * (\nabla F)^2 * g^{-1} * \bar{g}^{-1})] * \nabla^{k-l} F \\ &\quad + \sum_{l=1}^{k-1} g^{-1} * \nabla^l \frac{\partial g}{\partial t} * \nabla^{k-l} F, \end{aligned}$$

where $\nabla^l(A * B)$ represents the linear combinations of $\nabla^l A * B$, $\nabla^{l-1} A * \nabla B, \dots, A * \nabla^l B$ with universal coefficients.

Proof. For $k = 1$, by direct computation and Ricci identity, we have

$$\begin{aligned} \frac{\partial}{\partial t} \nabla_i F^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha F_i^\beta (\Delta F)^\gamma &= \nabla_i \Delta F^\alpha \\ &= \Delta \nabla_i F^\alpha - R_i^l \nabla_l F^\alpha \\ &\quad + \bar{R}_{\beta\delta\gamma}^\alpha \nabla_i F^\beta \nabla_k F^\delta \nabla_l F^\gamma g^{kl}. \end{aligned}$$

For $k \geq 2$, we prove by induction. Since

$$\begin{aligned} \frac{\partial}{\partial t}(\nabla^k F)_{i_1 \dots i_k}^\alpha &= \frac{\partial}{\partial x^{i_k}} \frac{\partial}{\partial t}(\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\alpha - \Gamma_{i_k i_l}^p \frac{\partial}{\partial t}(\nabla^{k-1} F)_{i_1 \dots p \dots i_{k-1}}^\alpha \\ &\quad + \bar{\Gamma}_{\beta\gamma}^\alpha F_{i_k}^\beta \frac{\partial}{\partial t}(\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\gamma \\ &\quad + \left(g^{-1} * \nabla \frac{\partial g}{\partial t} * \nabla^{k-1} F \right)_{i_1 \dots i_k}^\alpha \\ &\quad + \frac{\partial}{\partial y^\delta} \bar{\Gamma}_{\beta\gamma}^\alpha (\Delta F)^\delta F_{i_k}^\beta (\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\gamma \\ &\quad + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial}{\partial t} F_{i_k}^\beta (\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\gamma, \end{aligned}$$

we have

$$\begin{aligned} D_t(\nabla^k F)_{i_1 \dots i_k}^\alpha &= \frac{\partial}{\partial x^{i_k}} D_t(\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\alpha - \Gamma_{i_k i_l}^p D_t(\nabla^{k-1} F)_{i_1 \dots p \dots i_{k-1}}^\alpha \\ &\quad + \bar{\Gamma}_{\beta\gamma}^\alpha F_{i_k}^\beta D_t(\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\gamma \\ &\quad + \left(g^{-1} * \nabla \frac{\partial g}{\partial t} * \nabla^{k-1} F \right)_{i_1 \dots i_k}^\alpha \\ &\quad + \frac{\partial}{\partial y^\delta} \bar{\Gamma}_{\beta\gamma}^\alpha (\Delta F)^\delta F_{i_k}^\beta (\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\gamma \\ &\quad + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial}{\partial t} F_{i_k}^\beta (\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\gamma \\ &\quad - \frac{\partial}{\partial x^{i_k}} \left[\bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial F^\beta}{\partial t} (\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\gamma \right] \\ &\quad + \Gamma_{i_k i_l}^p \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial F^\beta}{\partial t} (\nabla^{k-1} F)_{i_1 \dots p \dots i_{k-1}}^\gamma \\ &\quad - \bar{\Gamma}_{\beta\gamma}^\alpha \bar{\Gamma}_{\delta\xi}^\gamma F_{i_k}^\beta \frac{\partial F^\delta}{\partial t} (\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\xi + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial F^\beta}{\partial t} (\nabla^k F)_{i_1 \dots i_k}^\gamma. \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial}{\partial x^{i_k}} \left[\bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial F^\beta}{\partial t} (\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\gamma \right] &= \frac{\partial}{\partial y^\beta} \bar{\Gamma}_{\delta\gamma}^\alpha F_{i_k}^\beta \frac{\partial F^\delta}{\partial t} (\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\gamma \\ &\quad + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial}{\partial x^{i_k}} \frac{\partial F^\beta}{\partial t} (\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\gamma + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial F^\beta}{\partial t} (\nabla^k F)_{i_1 \dots i_k}^\gamma \\ &\quad + \Gamma_{i_k i_l}^p \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial F^\beta}{\partial t} (\nabla^{k-1} F)_{i_1 \dots p \dots i_{k-1}}^\gamma - \bar{\Gamma}_{\delta\gamma}^\alpha \bar{\Gamma}_{\beta\xi}^\gamma F_{i_k}^\beta \frac{\partial F^\delta}{\partial t} (\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\xi, \end{aligned}$$

we have

$$D_t(\nabla^k F)_{i_1 \dots i_k}^\alpha = [\nabla D_t(\nabla^{k-1} F)]_{i_1 \dots i_k}^\alpha + \left(g^{-1} * \nabla \frac{\partial g}{\partial t} * \nabla^{k-1} F \right)_{i_1 \dots i_k}^\alpha \\ + \bar{R}_{\delta\beta\gamma}^\alpha (\Delta F)^\delta F_{i_k}^\beta (\nabla^{k-1} F)_{i_1 \dots i_{k-1}}^\gamma.$$

Combining with Ricci identity

$$\nabla \Delta \nabla^{k-1} F = \Delta \nabla^k F + \nabla [(R_M * g^{-2} + \bar{R}_N * (\nabla F)^2 * g^{-1} * \bar{g}^{-1}) * \nabla^{k-1} F]$$

and induction on k , we have

$$(D_t - \Delta)(\nabla^k F) = g^{-1} * \nabla \frac{\partial g}{\partial t} * \nabla^{k-1} F \\ + \bar{R}_N * \nabla F * \nabla^2 F * \nabla^{k-1} F * g^{-1} * \bar{g}^{-1} \\ + \nabla [(D_t - \Delta)\nabla^{k-1} F] \\ + \nabla [(R_M * g^{-2} + \bar{R}_N * (\nabla F)^2 * g^{-1} * \bar{g}^{-1}) * \nabla^{k-1} F] \\ = \nabla [(D_t - \Delta)\nabla^{k-1} F] \\ + \nabla [(R_M * g^{-2} + \bar{R}_N * (\nabla F)^2 * g^{-1} * \bar{g}^{-1}) * \nabla^{k-1} F] \\ + g^{-1} * \nabla \frac{\partial g}{\partial t} * \nabla^{k-1} F \\ = \sum_{l=0}^{k-1} \nabla^l [(R_M * g^{-2} + \bar{R}_N * (\nabla F)^2 * g^{-1} * \bar{g}^{-1}) * \nabla^{k-l} F] \\ + \sum_{l=1}^{k-1} g^{-1} * \nabla^l \frac{\partial g}{\partial t} * \nabla^{k-l} F.$$

We finish the proof of the proposition. \square

Corollary 2.4. *Let $F(\cdot, t)$ be assumed as in Proposition 2.3. Then we have*

$$(2.14) \quad \left(\frac{\partial}{\partial t} - \Delta \right) |\nabla^k F|^2 \leq -2|\nabla^{k+1} F|^2 + \left\langle \sum_{l=0}^{k-1} \left\{ \nabla^l [(R_M * g^{-2} \\ + \bar{R}_N * (\nabla F)^2 * g^{-1} * \bar{g}^{-1})] \right. \right. \\ \left. \left. + g^{-1} * \nabla^l \frac{\partial g}{\partial t} \right\} * \nabla^{k-l} F, \nabla^k F \right\rangle \\ + g^{-(k+1)} \frac{\partial g}{\partial t} * (\nabla^k F)^2 * \bar{g}.$$

Proof. Since $|\nabla^k F|^2 = (\nabla^k F)^\alpha_{i_1 \dots i_k} (\nabla^k F)^\beta_{j_1 \dots j_k} g^{i_1 j_1} \dots g^{i_k j_k} \bar{g}_{\alpha\beta}$, and

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla^k F|^2 &= 2 \frac{\partial}{\partial t} (\nabla^k F)^\alpha_{i_1 \dots i_k} (\nabla^k F)^\beta_{j_1 \dots j_k} g^{i_1 j_1} \dots g^{i_k j_k} \bar{g}_{\alpha\beta} \\ &\quad + \frac{\partial \bar{g}_{\alpha\beta}}{\partial y^\delta} \frac{\partial F^\delta}{\partial t} (\nabla^k F)^\alpha_{i_1 \dots i_k} (\nabla^k F)^\beta_{j_1 \dots j_k} g^{i_1 j_1} \dots g^{i_k j_k} \\ &\quad + g^{-(k+1)} * \frac{\partial g}{\partial t} * (\nabla^k F)^2 * \bar{g} \\ &= 2D_t (\nabla^k F)^\alpha_{i_1 \dots i_k} (\nabla^k F)^\beta_{j_1 \dots j_k} g^{i_1 j_1} \dots g^{i_k j_k} \bar{g}_{\alpha\beta} \\ &\quad + g^{-(k+1)} * \frac{\partial g}{\partial t} * (\nabla^k F)^2 * \bar{g}, \end{aligned}$$

then (2.14) follows from Proposition 2.3. □

3. Higher-derivative estimates for the mean curvature flow

Now we come back to MCF, suppose $X(\cdot, t)$ is a solution to MCF equation (1.2), $g(\cdot, t)$ is the family of the induced metrics on M^n from $(\bar{M}^{\bar{n}}, \bar{g})$ by $X(\cdot, t)$, then

$$(3.1) \quad \frac{\partial}{\partial t} g_{ij} = -2H^\alpha h_{ij}^\beta \bar{g}_{\alpha\beta}.$$

Note that $(\partial g / \partial t) = (\nabla^2 X)^2 * \bar{g} * g^{-1}$ and $R_M = \bar{R}_{\bar{M}} * (\nabla X)^4 + (\nabla^2 X)^2 * \bar{g}$. Combining with corollary 2.4, we have

Proposition 3.1. *Let $(\bar{M}^{\bar{n}}, \bar{g})$ be a Riemannian manifold of dimension \bar{n} . Let $X_0 : M^n \rightarrow \bar{M}^{\bar{n}}$ be an isometrically immersed manifold in $\bar{M}^{\bar{n}}$. Suppose $X(x, t)$ is a solution of MCF on $M^n \times [0, T]$ with X_0 as initial data. Then*

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) |\nabla^k X|^2 &\leq -2|\nabla^{k+1} X|^2 + \left\langle \sum_{l=0}^{k-1} \nabla^l [(\nabla^2 X)^2 * \bar{g} * g^{-2} \right. \\ &\quad \left. + \bar{R}_{\bar{M}} * (\nabla X)^4 * g^{-2} \right. \\ &\quad \left. * \bar{g} * \bar{g}^{-1}] * \nabla^{k-l} X, \nabla^k X \right\rangle \\ (3.2) \quad &+ g^{-(k+2)} * \bar{g}^2 * (\nabla^2 X)^2 * (\nabla^k X)^2. \end{aligned}$$

Now we are ready to derive the higher-derivative estimates of the second fundamental form of MCF provided that we have bounded the second fundamental form. Before deriving the higher-derivative estimates, we need to construct a family of cut-off functions ξ_k , which are used also in the next section. For each integer $k > 0$, let ξ_k be a smooth nonincreasing function from $(-\infty, +\infty)$ to $[0, 1]$ so that $\xi_k(s) = 1$ for $s \in (-\infty, (1/2) + (1/2^{k+1})]$, and $\xi_k(s) = 0$ for $s \in [(1/2) + (1/2^k), +\infty)$; moreover for any $\epsilon > 0$ there exists a universal $C_{k,\epsilon} > 0$ such that

$$(3.3) \quad |\xi'_k(s)| + |\xi''_k(s)| \leq C_{k,\epsilon} \xi_k(s)^{1-\epsilon}.$$

Theorem 3.2. (*local estimates*) *Let $(\bar{M}^{\bar{n}}, \bar{g})$ be a complete Riemannian manifold of dimension \bar{n} . Let $X_0 : M^n \rightarrow \bar{M}^{\bar{n}}$ be an isometrically immersed complete manifold in $\bar{M}^{\bar{n}}$. Suppose $X(x, t)$ is a solution to the mean curvature flow (1.1) on $M^n \times [0, T]$ with X_0 as initial data and with bounded second fundamental forms $|h^\alpha_{ij}| \leq \bar{C}$ on $[0, T]$. Then for any fixed $x_0 \in M^n$ and any geodesic ball $B_0(x_0, a)$ of radius $a > 0$ of initial metric g_{ij} , for any $k \geq 3$, we have*

$$(3.4) \quad |\nabla^k X|(x, t) \leq \frac{C_k}{t^{(k-2)/2}}, \quad \text{for all } (x, t) \in B_0\left(x_0, \frac{a}{2}\right) \times [0, T],$$

where the constant C_k depends on \bar{C} , T , \bar{n} , a and the bounds of the curvature and its covariant derivatives up to order $k - 1$ of the ambient manifold \bar{M} on its geodesic ball $B_{\bar{M}}(X_0(x_0), a + 1 + \sqrt{\bar{n}\bar{C}T})$.

Proof. Since $|(\partial/\partial t)X| = |H| \leq \sqrt{\bar{n}\bar{C}}$, it is not hard to see that under the evolution of MCF, at any time $t \in [0, T]$, $X_t(B_0(x_0, a))$ is contained in $B_{\bar{M}}(X_0(x_0), a + 1 + \sqrt{\bar{n}\bar{C}T})$. For any fixed $a > 0$, $k > 0$, we denote by C_k various constants depending only on a , \bar{C} , T , \bar{n} and the bounds of the curvature and its covariant derivatives up to order $k - 1$ of the ambient manifold \bar{M} on its ball $B_{\bar{M}}(X_0(x_0), a + 1 + \sqrt{\bar{n}\bar{C}T})$.

By Proposition 3.1, we have

$$(3.5) \quad \begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) |\nabla^2 X|^2 &\leq -2|\nabla^3 X|^2 + C_2 + C_2|\nabla^3 X| \\ &\leq -|\nabla^3 X|^2 + C_2 \end{aligned}$$

and

$$\begin{aligned}
 \left(\frac{\partial}{\partial t} - \Delta\right) |\nabla^3 X|^2 &\leq -2|\nabla^4 X|^2 + C_3(|\nabla^3 X|^3 + |\nabla^3 X|^2 + |\nabla^3 X| \\
 &\quad + |\nabla^4 X||\nabla^3 X|) \\
 (3.6) \qquad \qquad \qquad &\leq -|\nabla^4 X|^2 + C_3|\nabla^3 X|^3 + C_3.
 \end{aligned}$$

Combining (3.5) and (3.6), for any constant $A > 0$ we have

$$\begin{aligned}
 \left(\frac{\partial}{\partial t} - \Delta\right) ((A + |\nabla^2 X|^2)|\nabla^3 X|^2) &\leq (-|\nabla^3 X|^2 + C_3)|\nabla^3 X|^2 \\
 &\quad + 8|\nabla^3 X|^2|\nabla^4 X||\nabla^2 X| + [-|\nabla^4 X|^2 \\
 (3.7) \qquad \qquad \qquad &\quad + C_3|\nabla^3 X|^3 + C_3](A + |\nabla^2 X|^2).
 \end{aligned}$$

Since $|\nabla^2 X|^2$ is bounded by assumption, by choosing A suitable large, let $u = (A + |\nabla^2 X|^2)|\nabla^3 X|^2$ and $v = tu$, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) u \leq -\frac{1}{C_3}u^2 + C_3$$

and

$$(3.8) \qquad \qquad \qquad \left(\frac{\partial}{\partial t} - \Delta\right) v \leq \frac{1}{t} \left(-\frac{1}{C_3}v^2 + C_3\right).$$

Now we need a cut-off function technique as in [5]. Let $\xi(x) = \xi_3(d_0(x, x_0)/a)$, where ξ_3 is the cut-off function satisfying (3.3) for $k = 3$. Then the function $\xi(x)$ satisfies

$$\begin{aligned}
 \xi(x) &= 1, \quad \text{for } x \in B_0\left(x_0, \left(\frac{1}{2} + \frac{1}{2^4}\right)a\right), \\
 \xi(x) &= 0, \quad \text{for } x \in M \setminus B_0(x_0, a), \\
 |\nabla_0 \xi|^2 &\leq C_3 \xi, \\
 (\nabla_0^2 \xi)_{ij} &\geq -C_3 \xi^{1/2} g_{ij}(\cdot, 0),
 \end{aligned}$$

(3.9)

where we used the Hessian comparison theorem. Since by Gauss equation, the curvature of the initial metric is bounded from below by a constant, which depends on \bar{C} and the curvature bound on the ball $B_{\bar{M}}(X_0(x_0))$,

$a + 1 + \sqrt{n}\bar{C}T$) of the ambient manifold. The last formula holds in the sense of support functions. Define $\phi(x, t) = \xi(x)v(x, t)$. Then we have

$$(3.10) \quad \left(\frac{\partial}{\partial t} - \Delta\right)\phi \leq \frac{1}{t} \left(-\frac{1}{C_3}\xi v^2 - tv\Delta\xi - 2t\nabla\xi \cdot \nabla v + C_3\xi\right).$$

Suppose $\phi(x, t)$ achieves its maximum value over $M^n \times [0, T]$ at some point $(x_1, t_1) \in B(x_0, a) \times (0, T]$, i.e.,

$$\phi(x_1, t_1) = \max_{M \times [0, T]} \phi(x, t).$$

Suppose the point x_1 does not lie in the cut locus of x_0 , then

$$(3.11) \quad \frac{\partial\phi}{\partial t}(x_1, t_1) \geq 0, \quad \nabla v(x_1, t_1) = -\frac{\nabla\xi}{\xi}v, \quad \Delta\phi(x_1, t_1) \leq 0.$$

By (3.10) and (3.11), at (x_1, t_1) we have

$$(3.12) \quad 0 \leq -\frac{1}{C_3}\xi v^2 - t_1v\Delta\xi + 2t_1\frac{|\nabla\xi|^2}{\xi}v + C_3\xi.$$

Note that the second fundamental form is bounded in $M^n \times [0, T]$, the metrics $g_{ij}(\cdot, t)$ are equivalent. Since

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = \left(g^{-1} * \nabla \frac{\partial g}{\partial t}\right)_{ij}^k = g^{-2} * \bar{g} * \nabla^2 X * \nabla^3 X,$$

we have

$$\begin{aligned} |\Gamma_{ij}^k(x_1, t_1) - \Gamma_{0ij}^k(x_1)| &\leq C(\bar{n})\bar{C} \int_0^{t_1} |\nabla^3 X| dt \\ &\leq C(\bar{n})\bar{C} \int_0^{t_1} \left(\frac{\phi}{\xi t}\right)^{1/2}(x_1, t) dt \\ &\leq C_3 \frac{\phi(x_1, t_1)^{1/2}}{\xi(x_1)^{1/2}}, \end{aligned}$$

where we used the fact that ϕ achieves its maximum at (x_1, t_1) . Thus at (x_1, t_1) , we have

$$\begin{aligned} -\Delta\xi &= -g^{ij}\nabla_i\nabla_j\xi \\ &= -g^{ij}(\nabla_{0i}\nabla_{0j}\xi + (\Gamma_{0ij}^k - \Gamma_{ij}^k)\nabla_{0k}\xi) \\ &\leq C_3 + C_3\frac{\phi(x_1, t_1)^{1/2}}{\xi(x_1)^{1/2}}|\nabla\xi|. \end{aligned}$$

Substituting into (3.12), multiplying by $\xi(x_1)$ and combining with (3.9), we have at (x_1, t_1)

$$\begin{aligned} 0 &\leq -\frac{1}{C_3}\xi^2v^2 + \left(C_3 + C_3\phi(x_1, t_1)^{1/2}\frac{|\nabla\xi|}{\xi^{1/2}}\right)\xi v + 2\frac{|\nabla\xi|^2}{\xi}\xi v + C_3\xi^2 \\ &\leq -\frac{1}{C_3}\phi^2 + C_3\phi^{3/2} + C_3\phi + C_3. \end{aligned}$$

This implies

$$\phi(x_1, t_1) \leq C_3,$$

hence we have

$$|\nabla^3 X| \leq \frac{C_3}{t^{1/2}}$$

on $B_0(x_0, ((1/2) + (1/2^4))a) \times [0, T]$. If x_1 lies on the cut locus of x_0 , then by applying a standard support function technique as in [22], the same estimate is still valid.

For higher-derivatives, we prove by induction. Fix $x_0 \in M^n$, $a > 0$, suppose

$$(3.13) \quad |\nabla^k X| \leq \frac{C_k}{t^{k-2/2}}, \quad k = 3, \dots, m - 1,$$

on $B_0(x_0, ((1/2) + (1/2^{k+1}))a) \times [0, T]$. Now we prove the estimate for $k = m$.

By induction hypothesis and Proposition 3.1, we have

$$\begin{aligned}
 \left(\frac{\partial}{\partial t} - \Delta\right) |\nabla^m X|^2 &\leq -2|\nabla^{m+1} X|^2 + \left\langle \sum_{l=0}^{m-1} \nabla^l [(\nabla^2 X)^2 * \bar{g} * g^{-2} \right. \\
 &\quad \left. + \bar{R}_{\bar{M}} * (\nabla X)^4 * g^{-2} * \bar{g} * \bar{g}^{-1}] * \nabla^{m-l} X, \nabla^m X \right\rangle \\
 &\quad + g^{-(m+2)} * \bar{g}^2 * (\nabla^2 X)^2 * (\nabla^m X)^2 \\
 &\leq -2|\nabla^{m+1} X|^2 + C_m \sum_{l=0}^{m-1} \left\{ \sum_{l_1+l_2=l} |\nabla^{2+l_1} X| |\nabla^{2+l_2} X| \right. \\
 &\quad \left. + \sum_{l_1+\dots+l_4=l} |\nabla^{l_1+1} X| |\nabla^{l_2+1} X| |\nabla^{l_3+1} X| |\nabla^{l_4+1} X| \right\} \\
 &\quad \times |\nabla^{m-l} X| |\nabla^m X| \\
 &\leq -2|\nabla^{m+1} X|^2 + C_m [|\nabla^{m+1} X| |\nabla^m X| \\
 &\quad + (|\nabla^3 X| + 1) |\nabla^m X|^2 + t^{-(m-2/2)} |\nabla^m X|] \\
 (3.14) \quad &\leq -|\nabla^{m+1} X|^2 + \frac{C_m}{t^{1/2}} |\nabla^m X|^2 + C_m t^{-(m-2/2)} |\nabla^m X|
 \end{aligned}$$

and

$$\begin{aligned}
 \left(\frac{\partial}{\partial t} - \Delta\right) |\nabla^{m-1} X|^2 &\leq -|\nabla^m X|^2 + \frac{C_{m-1}}{t^{1/2}} |\nabla^{m-1} X|^2 \\
 &\quad + C_{m-1} t^{-(m-3/2)} |\nabla^{m-1} X| \\
 (3.15) \quad &\leq -|\nabla^m X|^2 + \frac{C_{m-1}}{t^{m-3+(1/2)}}
 \end{aligned}$$

on $B_0(x_0, ((1/2) + (1/2^m))a) \times [0, T]$.

Define

$$\psi(x, t) = (A + t^{m-3} |\nabla^{m-1} X|^2) |\nabla^m X|^2 t^{m-2}$$

for A to be determined later. Combining (3.14) and (3.15), we have for suitable large A as before

$$\begin{aligned}
 \left(\frac{\partial}{\partial t} - \Delta\right) \psi &\leq \frac{2m-5}{t} \psi + t^{m-3} |\nabla^m X|^2 t^{m-2} \left(-|\nabla^m X|^2 + \frac{C_{m-1}}{t^{m-3+(1/2)}} \right) \\
 &\quad + t^{m-2} (A + t^{m-3} |\nabla^{m-1} X|^2) \left(-|\nabla^{m+1} X|^2 \right. \\
 &\quad \left. + \frac{C_m}{t^{1/2}} |\nabla^m X|^2 + C_m t^{-(m-2/2)} |\nabla^m X| \right) \\
 &\quad + 8t^{2m-5} |\nabla^{m-1} X| |\nabla^m X|^2 |\nabla^{m+1} X|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{2m-5}{t}\psi - \frac{1}{2t}[t^{m-2}|\nabla^m X|^2]^2 \\
 &\quad + \frac{C_m}{t^{1/2}}[t^{m-2}|\nabla^m X|^2] + C_m[t^{m-2}|\nabla^m X|^2]^{1/2} \\
 &\leq \frac{1}{t}\left[-\frac{1}{C_m}\psi^2 + C_m\psi + C_m\psi^{1/2}\right] \\
 (3.16) \quad &\leq \frac{1}{t}\left[-\frac{1}{C_m}\psi^2 + C_m\right]
 \end{aligned}$$

on $B_0(x_0, ((1/2) + (1/2^m))a) \times [0, T]$. To apply the cut-off function technique to (3.16) as before, we note that by the estimate for $k = 3$, we know that

$$|\Gamma - \Gamma_0| \leq C(\bar{n})\bar{C} \int_0^T |\nabla^3 X| dt \leq C_3 \int_0^T \frac{1}{\sqrt{t}} dt \leq C_3.$$

By calculating the equation of $\xi_m(d_0(x_0, \cdot)/a)\psi$ using (3.16), and repeating the same procedure of applying maximum principle as before, we can prove that

$$\xi_m\left(\frac{d_0(x_0, \cdot)}{a}\right)\psi \leq C_m \text{ on } B_0(x_0, a) \times [0, T],$$

which implies

$$|\nabla^m X|(x, t) \leq \frac{C_m}{t^{m-2/2}}, \text{ for all } (x, t) \in B_0\left(x_0, \left(\frac{1}{2} + \frac{1}{2^{m+1}}\right)a\right) \times [0, T].$$

We complete the induction step and the theorem is proved. □

Corollary 3.3. *Let $(\bar{M}^{\bar{n}}, \bar{g})$ be a complete Riemannian manifold satisfying*

$$|\bar{\nabla}^k \bar{Rm}|(\cdot) \leq \bar{C}, \text{ for } k \leq 2.$$

Let $X_0 : M^n \rightarrow \bar{M}^{\bar{n}}$ be an isometrically immersed complete manifold in $\bar{M}^{\bar{n}}$. Suppose $X(\cdot, t)$ is a solution of MCF on $M^n \times [0, T]$ with X_0 as initial data and with bounded second fundamental forms $|h_{ij}^\alpha| \leq \bar{C}$ on $[0, T]$. Then there is a constant C_1 depending only on \bar{C} , \bar{n} and T such that

$$(3.17) \quad |\nabla Rm|(x, t) \leq \frac{C_1}{t^{1/2}}, \text{ for all } (x, t) \in M^n \times [0, T].$$

Moreover, for any fixed $x_0 \in M^n$ and any ball $B_0(x_0, a)$ of radius $a > 0$ of initial metric g_{ij} , and for any $k \geq 2$, there is a constant C_k depending only

on a, \bar{C}, \bar{n}, T and the bounds of the curvature and its derivatives up to order $k + 1$ of the ambient manifold on its geodesic ball $B_{\bar{M}}(X_0(x_0), a + 1 + \sqrt{\bar{n}\bar{C}T})$, such that

$$(3.18) \quad |\nabla^k Rm|(x, t) \leq \frac{C_k}{t^{k/2}}, \quad \text{for all } (x, t) \in B_0\left(x_0, \frac{a}{2}\right) \times [0, T].$$

Proof. This follows from Gauss equation and Theorem 3.2. □

4. Harmonic map flow coupled with mean curvature flow

Let X_t be the solution of MCF as in Theorem 1.1, $g_{ij}(x, t)$ the induced Riemannian metrics. Let $f : M^n \rightarrow N^m$ be a map from M^n to a fixed Riemannian manifold $(N^m, \hat{g}_{\alpha\beta})$. Then the harmonic map flow coupled with MCF is the following evolution equation of maps

$$\begin{aligned} \frac{\partial}{\partial t} f(x, t) &= \Delta f(x, t), \quad \text{for } x \in M^n, t > 0, \\ f(x, 0) &= f(x), \quad \text{for } x \in M^n, \end{aligned}$$

where the Harmonic map Laplacian Δ is defined by using the metric $g_{ij}(x, t)$ and $\hat{g}_{\alpha\beta}(y)$, i.e.,

$$\Delta f^\alpha(x, t) = g^{ij}(x, t) \nabla_i \nabla_j f^\alpha(x, t),$$

and

$$\nabla_i \nabla_j f^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f^\alpha}{\partial x^k} + \hat{\Gamma}_{\beta\gamma}^\alpha \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j}.$$

Here we use $\{x^i\}$ and $\{y^\alpha\}$ to denote the local coordinates of M^n and N^m respectively, Γ_{ij}^k and $\hat{\Gamma}_{\beta\gamma}^\alpha$ the corresponding Christoffel symbols of g_{ij} and $\hat{g}_{\alpha\beta}$.

Now we fix a metric $\hat{g} = g(\cdot, T)$ on M^n , and let $(N^m, \hat{g}) = (M^n, \hat{g})$. Note that the ambient manifold (\bar{M}, \bar{g}) in Theorem 1.1 satisfies the assumption of Corollary 3.3. By Corollary 3.3 and Theorem 2.1, we know that there are positive constants $\hat{C}_1, \hat{\delta}$ depending only on \bar{C}, T, \bar{n} and $\bar{\delta}$ such that

$$(4.1) \quad \begin{aligned} |\hat{R}_N| + |\hat{\nabla} \hat{R}_N| &\leq \hat{C}_1, \\ \text{inj}(N, \hat{g}) &\geq \hat{\delta} > 0. \end{aligned}$$

Moreover, by (3.18) of Corollary 3.3, for any fixed $y_0 \in N$, for any $k \geq 2$, there is a constant \hat{C}_k depending only on \bar{C}, \bar{n}, T and the bounds of the

curvature and its derivatives up to order $k + 1$ of the ambient manifold on its ball $B_{\bar{M}}(X_0(y_0), 2e^{\sqrt{\bar{n}\bar{C}^2T}} + 1 + \sqrt{\bar{n}\bar{C}T})$, such that

$$(4.2) \quad |\hat{\nabla}^k \hat{R}_N|(y) \leq \hat{C}_k, \text{ for all } y \in \hat{B}(y_0, 1).$$

In this section, we will establish the existence theorem for the above harmonic map flow coupled with MCF. More precisely, we will prove

Theorem 4.1. *There exists $0 < T_0 < T$, depending only on $\bar{C}, T, \bar{n}, \bar{\delta}$, such that the harmonic map flow coupled with mean curvature flow*

$$(4.3) \quad \begin{aligned} \frac{\partial}{\partial t} F(x, t) &= \Delta F(x, t), \quad x \in M^n, t > 0, \\ F(\cdot, 0) &= \text{Identity}, \quad x \in M^n \end{aligned}$$

has a solution on $M^n \times [0, T_0]$ such that the following estimates hold. There is a constant C_2 depending only on $\bar{C}, \bar{\delta}, \bar{n}$ and T such that

$$(4.4) \quad |\nabla F| + |\nabla^2 F| \leq C_2.$$

For any $k \geq 3$, $B_0(x_1, 1) \subset M^n$, there is a constant C_k depending only on $\bar{C}, \bar{\delta}, T, \bar{n}$ and x_1 such that

$$(4.5) \quad |\nabla^k F| \leq C_k t^{-(k-2/2)}, \text{ on } B_0(x_1, 1) \times [0, T_0].$$

We will adapt the strategy of [5] by solving the corresponding initial-boundary value problem on a sequence of exhausted bounded domains $D_1 \subseteq D_2 \subseteq \dots$ with smooth boundaries and $D_j \supseteq B_0(x_0, j + 1)$,

$$(4.6) \quad \begin{aligned} \frac{\partial}{\partial t} F^j(x, t) &= \Delta F^j(x, t) \\ F^j(x, 0) &= x \text{ for all } x \in D_j, \\ F^j(x, t) &= x \text{ for all } x \in \partial D_j, \end{aligned}$$

and taking a convergent subsequence of F^j as $j \rightarrow \infty$, where x_0 is a fixed point in M^n .

First we need the zero order estimate for the Dirichlet problem (4.6).

Lemma 4.2. *There exist positive constants $T_1 > 0$ and $C > 0$ such that for any j , if F^j solves problem(4.6) on $\bar{D}_j \times [0, T'_1]$ with $T'_1 \leq T_1$, then we have*

$$\hat{d}(x, F^j(x, t)) \leq C\sqrt{t}$$

for any $(x, t) \in D_j \times [0, T'_1]$, where \hat{d} is the distance with respect to the metric \hat{g} .

Proof. For simplicity, we drop the superscript j . In the following argument, we denote by C various positive constants depending only on the constants \bar{C} , $\bar{\delta}$, T , and \bar{n} in Theorem 1.1. Note that $\hat{d}(y_1, y_2)$ is the distance function on the target (M^n, \hat{g}) , which can be regarded as a function on $M^n \times M^n$ with the product metric. Let $\varphi(y_1, y_2) = (1/2)\hat{d}^2(y_1, y_2)$ and $\rho(x, t) = \varphi(x, F(x, t))$. We compute

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)\rho &= \hat{d}(x, F(x, t)) \left(-\frac{\partial \hat{d}}{\partial y_1^\alpha} \Delta \text{Id}^\alpha\right) \\ &\quad - g^{ij} \left\{ \frac{\partial^2 \varphi}{\partial y_1^\alpha \partial y_1^\beta} - (\hat{\Gamma}_{\alpha\beta}^\gamma \circ \text{Id}) \frac{\partial \varphi}{\partial y_1^\gamma} \right\} \frac{\partial \text{Id}^\alpha}{\partial x^i} \frac{\partial \text{Id}^\beta}{\partial x^j} \\ &\quad - 2g^{ij} \frac{\partial^2 \varphi}{\partial y_1^\alpha \partial y_2^\beta} \frac{\partial \text{Id}^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} \\ &\quad - g^{ij} \left\{ \frac{\partial^2 \varphi}{\partial y_2^\alpha \partial y_2^\beta} - (\hat{\Gamma}_{\alpha\beta}^\gamma \circ F) \frac{\partial \varphi}{\partial y_2^\gamma} \right\} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} \\ &= -\hat{d} \frac{\partial \hat{d}}{\partial y_1^\alpha} \Delta \text{Id}^\alpha - g^{ij} \text{Hess}(\varphi)(V_i, V_j), \end{aligned}$$

where

$$V_i = \frac{\partial \text{Id}^\alpha}{\partial x^i} \frac{\partial}{\partial y_1^\alpha} + \frac{\partial F^\alpha}{\partial x^i} \frac{\partial}{\partial y_2^\alpha}.$$

By Theorem 3.2, there is a constant C depending only on \bar{C} , T and \bar{n} such that

$$(4.7) \quad \left| \frac{\partial \Gamma}{\partial t} \right| \leq C |\nabla^3 X| \leq \frac{C}{\sqrt{t}}.$$

Since

$$\Delta \text{Id} = g^{-1} * (\hat{\Gamma} \circ \text{Id} - \Gamma) = g^{-1} * (\Gamma(\cdot, T) - \Gamma(\cdot, t))$$

then we have $|\Delta \text{Id}| \leq C$ by (4.7), this implies

$$\left(\frac{\partial}{\partial t} - \Delta\right)\rho \leq C\hat{d} - g^{ij}\text{Hess}(\varphi)(V_i, V_j).$$

By (4.1), the curvature of \hat{g} is bounded by some constant \hat{K} , the injectivity radius of \hat{g} has a uniform positive lower bound $\hat{\delta}$. We claim that if $\hat{d}(x, F(x, t)) \leq \min\{\hat{\delta}/2, 1/4\sqrt{\hat{K}}\}$, then

$$g^{ij}\text{Hess}(\varphi)(V_i, V_j) \geq \frac{1}{2}|\nabla F|^2 - C.$$

Firstly, by Theorem 2.2(i), we have $|\text{Hess}(\varphi)| \leq C$ under the assumption of the claim. On the other hand, the Hessian comparison theorem at those points not lying on the cut locus shows that

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial y_2^\alpha \partial y_2^\beta} - (\hat{\Gamma}_{\alpha\beta}^\gamma \circ F) \frac{\partial \varphi}{\partial y_2^\gamma} &\geq \frac{\pi}{4} \hat{g}_{\alpha\beta}, \\ \frac{\partial^2 \varphi}{\partial y_1^\alpha \partial y_1^\beta} - (\hat{\Gamma}_{\alpha\beta}^\gamma \circ \text{Id}) \frac{\partial \varphi}{\partial y_1^\gamma} &\geq \frac{\pi}{4} \hat{g}_{\alpha\beta}. \end{aligned}$$

Combining the above inequalities, we have

$$\begin{aligned} g^{ij}\text{Hess}(\varphi)(V_i, V_j) &\geq \frac{\pi}{4}|\nabla F|^2 - C|\nabla F| - C \\ &\geq \frac{1}{2}|\nabla F|^2 - C, \end{aligned}$$

which proves the claim. Hence when $\hat{d}(x, F(x, t)) \leq \min\left\{\left(\hat{\delta}/2\right), \left(1/4\sqrt{\hat{K}}\right)\right\}$, we have

$$(4.8) \quad \left(\frac{\partial}{\partial t} - \Delta\right)\rho \leq C\hat{d} - \frac{1}{2}|\nabla F|^2 + C.$$

By maximum principle we have

$$\hat{d}(x, F(x, t)) \leq C\sqrt{t} \text{ whenever } \hat{d}(x, F(x, t)) \leq \min\left\{\frac{\hat{\delta}}{2}, \frac{1}{4\sqrt{\hat{K}}}\right\}.$$

Therefore there exists $T_1 \leq (1/C^2) \min^2\left\{\left(\hat{\delta}/2\right), \left(1/4\sqrt{\hat{K}}\right)\right\}$ such that

$$\hat{d}(x, F(x, t)) \leq C\sqrt{t}, \text{ for } t \leq T_1' (\leq T_1),$$

we have proved the lemma. □

After proving the above lemma, we can apply the standard parabolic equation theory to get a local existence for the initial-boundary value problem (4.6) as follows. This is similar to [5], we include the proof here for completeness.

Lemma 4.3. *There exists a positive constant $T_2(\leq T_1)$ depending only on the dimension n , the constants T_1 and C obtained in the previous lemma such that for each j , the initial-boundary value problem (4.6) has a smooth solution F^j on $\bar{D}_j \times [0, T_2]$.*

Proof. For an arbitrarily fixed point \bar{x} in M^n , we consider the normal coordinates $\{x^i\}$ and $\{y^\alpha\}$ of the metric g_{0ij} and the metric $\hat{g}_{\alpha\beta}$, respectively, around \bar{x} . Locally the equation (4.6) is written as a system of equations

$$(4.9) \quad \begin{aligned} \frac{\partial y^\alpha}{\partial t}(x, t) = g^{ij}(x, t) \left[\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k(x, t) \frac{\partial y^\alpha}{\partial x^k} \right. \\ \left. + \hat{\Gamma}_{\beta\gamma}^\alpha(y^1(x, t), \dots, y^n(x, t)) \frac{\partial y^\beta}{\partial x^i} \frac{\partial y^\gamma}{\partial x^j} \right]. \end{aligned}$$

Note that $\hat{\Gamma}_{\beta\gamma}^\alpha(\bar{x}) = 0$. Since by (4.1) the curvature of metric \hat{g} and its first covariant derivative are bounded on the whole target manifold, by applying Corollary 4.12 in [14], we know that there is some uniform constant \hat{C} such that if $\hat{d}(y, \bar{x}) \leq (1/\hat{C})$, then $|\hat{\Gamma}_{\beta\gamma}^\alpha(y)| \leq \hat{C}\hat{d}(y, \bar{x})$. (This fact is proved essentially in [14], although it is not explicitly stated.) By Lemma 4.2, $\hat{d}(x, F(x, t)) \leq C\sqrt{t}$, we conclude that the coefficients of the quadratic terms on the RHS of (4.9) can be as small as we like, provided $T_2 > 0$ is sufficiently small (independent of \bar{x} and j).

Now for fixed j , we consider the corresponding parabolic system of the difference of the map F^j and the identity map. Clearly the coefficients of the quadratic terms of the gradients are also very small. Thus, whenever (4.9) has a solution on a time interval $[0, T'_2]$ with $T'_2 \leq T_2$, we can argue exactly as in the proof of Theorem 6.1 in Chapter VII of the book [19] to bound the norm of ∇F^j on the time interval $[0, T'_2]$ by a positive constant depending only on g_{0ij} , and $\hat{g}_{\alpha\beta}$ over the domain D_{j+1} , the L^∞ bound of F^j obtained in the previous lemma, and the boundary ∂D_j . Hence by the same argument as in the proof of Theorem 7.1 in Chapter VII of the book [19], we deduce that the initial-boundary value problem (4.9) has a smooth solution F^j on $\bar{D}_j \times [0, T_2]$. \square

To get a convergent sequence of F^j , we need the following uniform estimates.

Lemma 4.4. *There exists a positive constant T_3 , $0 < T_3 \leq T_2$, independent of j , such that if F^j solves*

$$\begin{aligned} \frac{\partial}{\partial t} F^j(x, t) &= \Delta F^j(x, t) \text{ on } D_j \times [0, T_3], \\ F^j(x, 0) &= x \text{ on } D_j. \end{aligned}$$

Then for any $B_0(x_1, 1) \subset D_j$, there is a positive constant $C = C(\bar{C}, \bar{\delta}, \bar{n}, T)$ such that

$$|\nabla F^j| + |\nabla^2 F^j| \leq C$$

on $B_0(x_1, (1/2)) \times [0, T_3]$, and for any $k \geq 3$ there exist constants $C_k = C(k, \bar{C}, \bar{\delta}, T, \bar{n}, x_1)$ satisfying

$$|\nabla^k F^j| \leq C_k t^{-(k-2/2)}$$

on $B_0(x_1, (1/2)) \times [0, T_3]$.

Proof. We drop the superscript j . We denote by C various constants depending only on $\bar{C}, \bar{\delta}, T, \bar{n}$. We first estimate $|\nabla F|$. By Corollary 2.4, we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) |\nabla F|^2 &\leq -2|\nabla^2 F|^2 + \left\langle \left(\left[R_M * g^{-2} + \hat{R}_N * (\nabla F)^2 * g^{-1} * \hat{g}^{-1} \right] \right. \right. \\ &\quad \left. \left. + g^{-1} * \frac{\partial g}{\partial t} \right) * \nabla F, \nabla F \right\rangle + g^{-2} \frac{\partial g}{\partial t} * (\nabla F)^2 * \hat{g}. \end{aligned}$$

Note that $(\partial g / \partial t) = (\nabla^2 X)^2 * \bar{g} * g^{-1}$, $R_M = \bar{R}_{\bar{M}} * (\nabla X)^4 + (\nabla^2 X)^2 * \bar{g}$, the second fundamental form $\nabla^2 X$ and curvature $\bar{R}_{\bar{M}}$ are bounded by assumption, we know that $|(\partial g / \partial t)|$ and $|R_M|$ are bounded. The above formula gives

$$(4.10) \quad \frac{\partial}{\partial t} |\nabla F|^2 \leq \Delta |\nabla F|^2 - 2|\nabla^2 F|^2 + C|\nabla F|^2 + C|\nabla F|^4.$$

On the other hand, we know from (4.8) that

$$\frac{\partial}{\partial t} \rho \leq \Delta \rho - \frac{1}{2} |\nabla F|^2 + C,$$

where $\rho(x, t) = (1/2)\hat{d}^2(x, F(x, t))$. For any $a > 0$ to be determined later, we compute

$$\begin{aligned} \frac{\partial}{\partial t}[(a + \rho)|\nabla F|^2] &\leq \Delta[(a + \rho)|\nabla F|^2] - 2\nabla\rho \cdot \nabla|\nabla F|^2 \\ &\quad - 2(a + \rho)|\nabla^2 F|^2 + C(a + \rho)|\nabla F|^2 + C(a + \rho)|\nabla F|^4 \\ &\quad - \frac{1}{2}|\nabla F|^4 + C|\nabla F|^2. \end{aligned}$$

Since

$$\begin{aligned} -2\nabla\rho \cdot \nabla|\nabla F|^2 &\leq C\hat{d}(|\nabla F| + |\nabla F|^2)|\nabla^2 F| \\ &\leq C(|\nabla F|^2 + |\nabla F|^4)\hat{d} + C\hat{d}|\nabla^2 F|^2 \end{aligned}$$

and $\hat{d}(\cdot, F(\cdot, t)) \leq C\sqrt{t}$, by taking $a = (1/8C)$ and T_3 suitable small, we have

$$\frac{\partial}{\partial t}[(a + \rho)|\nabla F|^2] \leq \Delta[(a + \rho)|\nabla F|^2] - \frac{1}{8C}|\nabla^2 F|^2 - \frac{1}{4}|\nabla F|^4 + C$$

for $t \leq T_3$. Let $u = (a + \rho)|\nabla F|^2$, then

$$(4.11) \quad \frac{\partial u}{\partial t} \leq \Delta u - \frac{1}{C}u^2 + C$$

for $t \leq T_3$. Let $\xi(x) = \xi_1(d_0(x_1, x))$ be a cut-off function, where ξ_1 is the nonincreasing smooth function in (3.3) supported in $[0, 1]$ and equal to 1 in $[0, \frac{3}{4}]$. Note that at $t = 0$, $u = ag^{ij}(\cdot, 0)g_{ij}(\cdot, T) \leq C$. Then by computing the equation of ξu and applying the maximum principle as before, we have

$$\xi u(x, t) \leq C \text{ on } M^n \times [0, T_3],$$

this implies

$$|\nabla F| \leq C \text{ on } B_0\left(x_1, \frac{3}{4}\right) \times [0, T_3].$$

We now estimate $|\nabla^2 F|$. By Corollary 2.4 again

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) |\nabla^2 F|^2 &\leq -2|\nabla^3 F|^2 + \left\langle \sum_{l=0}^1 \left\{ \nabla^l [(R_M * g^{-2} \right. \right. \\ &\quad \left. \left. + \hat{R}_N * (\nabla F)^2 * g^{-1} * \hat{g}^{-1})] \right. \right. \\ &\quad \left. \left. + g^{-1} * \nabla^l \frac{\partial g}{\partial t} \right\} * \nabla^{2-l} F, \nabla^2 F \right\rangle \\ &\quad + g^{-(3)} \frac{\partial g}{\partial t} * (\nabla^2 F)^2 * \hat{g}, \end{aligned}$$

and by (3.4), (3.17) and (4.1), we know $\sqrt{t}|\nabla \frac{\partial g}{\partial t}| + \sqrt{t}|\nabla R_M| + |\hat{\nabla} \hat{R}_N| \leq C$, and

$$(4.12) \quad \frac{\partial}{\partial t} |\nabla^2 F|^2 \leq \Delta |\nabla^2 F|^2 - 2|\nabla^3 F|^2 + C|\nabla^2 F|^2 + \frac{C}{\sqrt{t}} |\nabla^2 F|$$

on $B_0(x_1, \frac{3}{4}) \times [0, T_3]$. This implies

$$(4.13) \quad \frac{\partial}{\partial t} |\nabla^2 F| \leq \Delta |\nabla^2 F| + C|\nabla^2 F| + \frac{C}{\sqrt{t}}.$$

By (4.10) we have

$$\frac{\partial}{\partial t} |\nabla F|^2 \leq \Delta |\nabla F|^2 - 2|\nabla^2 F|^2 + C.$$

Let

$$u = |\nabla^2 F| + |\nabla F|^2 - 2C\sqrt{t} + 2C\sqrt{T},$$

then

$$(4.14) \quad \frac{\partial}{\partial t} u \leq \Delta u - u^2 + C \text{ on } B_0\left(x_1, \frac{3}{4}\right) \times [0, T_3].$$

Define the cut-off function $\xi(x) = \xi_2(d_0(x_1, x))$. Note that at $t = 0$, $|\nabla^2 F| = |\Gamma_0 - \hat{\Gamma}| \leq C$, then $u|_{t=0} \leq C$. Using the similar maximum principle argument as before, we get

$$\xi u \leq C \text{ on } B_0\left(x_1, \frac{1}{2} + \frac{1}{2^2}\right) \times [0, T_3],$$

which implies

$$|\nabla^2 F| \leq C \text{ on } B_0\left(x_1, \frac{1}{2} + \frac{1}{2^3}\right) \times [0, T_3].$$

To derive the higher-derivative estimates we prove by induction on k . We denote by C_k various constants, depending only on \bar{C} , T , $\bar{\delta}$, \bar{n} , and the bounds of the ambient manifold \bar{M} curvature and its covariant derivatives up to order k on its ball $B_{\bar{M}}(X_0(x_1), C)$ for suitable C .

Now suppose we have proved

$$(4.15) \quad |\nabla^l F| \leq \frac{C_l}{t^{(l-2)/2}}, \quad l = 2, \dots, k - 1$$

on $B_0(x_1, ((1/2) + (1/2^k))) \times [0, T_3]$. By Corollaries 2.4 and 3.3, Theorem 3.2 and using (4.15), we get

$$(4.16) \quad \frac{\partial}{\partial t} |\nabla^k F|^2 \leq \Delta |\nabla^k F|^2 - 2 |\nabla^{k+1} F|^2 + C_k |\nabla^k F|^2 + \frac{C_k}{t^{(k-1)/2}} |\nabla^k F|,$$

which implies

$$(4.17) \quad \frac{\partial}{\partial t} |\nabla^k F| \leq \Delta |\nabla^k F| + C_k |\nabla^k F| + \frac{C_k}{t^{(k-1)/2}}.$$

We also have

$$(4.18) \quad \frac{\partial}{\partial t} |\nabla^{k-1} F|^2 \leq \Delta |\nabla^{k-1} F|^2 - 2 |\nabla^k F|^2 + \frac{C_{k-1}}{t^{k-(5/2)}}.$$

Let

$$u = t^{(k-2)/2} |\nabla^k F| + t^{k-3} |\nabla^{k-1} F|^2.$$

By combining (4.17) and (4.18), we obtain

$$(4.19) \quad \frac{\partial}{\partial t} u \leq \Delta u - \frac{1}{t} (u^2 + C_k)$$

on $B_0(x_1, ((1/2) + (1/2^k))) \times [0, T_3]$. Using the cut-off function $\xi(x) = \xi_k(d_0(x_1, x))$, (4.19) and applying maximum principle as before, we conclude with

$$|\nabla^k F| \leq \frac{C_k}{t^{(k-2)/2}} \quad \text{on } B_0 \left(x_1, \left(\frac{1}{2} + \frac{1}{2^{k+1}} \right) \right) \times [0, T_3].$$

Therefore we complete the proof of Lemma 4.4. □

Proof of Theorem 4.1. Now we combine the above three lemmas to prove Theorem 4.1. We have known that there is a $T_3 > 0$ such that for each j ,

the equation

$$\begin{aligned} \frac{\partial}{\partial t} F^j(x, t) &= \Delta F^j(x, t) \\ F^j(x, 0) &= x \text{ for all } x \in D_j, \\ F^j(x, t) &= x \text{ for all } x \in \partial D_j \end{aligned}$$

has a smooth solution F^j on $\bar{D}_j \times [0, T_3]$. Since $D_j \supset B_0(x_0, j + 1)$, by choosing any $x_1 \in B_0(x_0, j)$ in Lemma 4.4 we have

$$|\nabla F^j| + |\nabla^2 F^j| \leq C$$

on $B_0(x_0, j) \times [0, T_3]$, where C depends only on $\bar{C}, \bar{n}, \bar{\delta}, T$. Moreover for any $x_1 \in B_0(x_0, j), k \geq 3$, there is a C_k depending on $\bar{C}, \bar{\delta}, T, \bar{n}$ and x_1 such that

$$|\nabla^k F^j|(x_1, t) \leq C_k t^{-((k-2)/2)}.$$

Then we can take a convergent subsequence of F^j (as $j \rightarrow \infty$) to get the desired F with the desired estimates. So the proof of Theorem 4.1 is completed. \square

For later purpose, now we need to derive some preliminary estimate of $g_{ij}(x, t)$ with respect to $F^* \hat{g}$. Let $\hat{g}_{ij} = (F^* \hat{g})_{ij}$.

Proposition 4.5. *Under the assumption of Theorem 4.1, there exist $0 < T_4 \leq T_3$ and $C > 0$ depending only on $\bar{C}, \bar{n}, \bar{\delta}$ and T such that for all $(x, t) \in M^n \times [0, T_4]$, we have*

$$(4.20) \quad \frac{1}{C} \hat{g}_{ij}(x, t) \leq g_{ij}(x, t) \leq C \hat{g}_{ij}(x, t).$$

Proof. Note that $|\nabla F|^2 = \hat{g}_{ij} g^{ij} \leq C$, which implies $\hat{g}_{ij}(x, t) \leq C g_{ij}(x, t)$. For the reverse inequality, since the curvature of $g_{ij}(\cdot, t)$ is bounded, we compute the equation of $\hat{g}_{ij}(x, t)$ on the domain,

$$\begin{aligned} \frac{\partial}{\partial t} \hat{g}_{ij} &= \Delta \hat{g}_{ij} - R_{ik} F_l^\alpha F_j^\beta \hat{g}_{\alpha\beta} g^{kl} - R_{jk} F_l^\alpha F_i^\beta \hat{g}_{\alpha\beta} g^{kl} \\ &\quad + 2 \hat{R}_{\alpha\beta\gamma\delta} F_i^\alpha F_k^\beta F_j^\gamma F_l^\delta g^{kl} - 2 \hat{g}_{\alpha\beta} F_{ki}^\alpha F_{lj}^\beta g^{kl} \\ &\geq \Delta \hat{g}_{ij} - R_{ik} \hat{g}_{jl} g^{kl} - R_{jk} \hat{g}_{il} g^{kl} - C |\nabla F|^2 \hat{g}_{ij} - 2 |\nabla^2 F|^2 g_{ij} \\ (4.21) \quad &\geq \Delta \hat{g}_{ij} - C g_{ij}. \end{aligned}$$

Note that for suitable large constant C , we have

$$\frac{\partial}{\partial t} g_{ij} \leq C g_{ij}, \quad 0 < t < T,$$

and $\hat{g}_{ij} \geq \frac{1}{C} g_{ij}$ at time 0. Thus for $t \leq 1/C^3$, we have

(4.22)

$$\left(\frac{\partial}{\partial t} - \Delta\right) \left(\hat{g}_{ij} + \left(C^2 t - \frac{1}{C}\right) g_{ij}\right) \geq \left[-C + C^2 + C \left(C^2 t - \frac{1}{C}\right)\right] g_{ij} \geq 0.$$

Note that

$$\left(\hat{g}_{ij} + \left(C^2 t - \frac{1}{C}\right) g_{ij}\right) |_{t=0} \geq 0.$$

Since $|\nabla^2 X| + \sqrt{t} |\nabla^3 X| \leq C$ and the curvature is bounded, then there is a smooth proper function φ with $\varphi(x) \geq 1 + d_0(x_0, x)$, $|\nabla\varphi| + |\nabla^2\varphi| \leq C$. So Hamilton’s maximum principle for tensors on complete manifolds is applicable, we get

$$\hat{g}_{ij} + \left(C^2 t - \frac{1}{C}\right) g_{ij} \geq 0 \quad \text{for } t \leq \min\{T_3, C^{-3}\},$$

which implies

$$g_{ij} \leq 2C \hat{g}_{ij}$$

for $t \leq T_4 = \min\{T_3, 1/2C^3\}$.

The proof of the proposition is completed. □

As a consequence, we know that the solution of the harmonic map flow coupled with the MCF is a family of diffeomorphisms.

Corollary 4.6. *Let $F(x, t)$ be assumed as in the previous proposition. Then $F(\cdot, t)$ are diffeomorphisms from M to N for all $t \in [0, T_4]$.*

Proof. Note that (4.20) implies that F are local diffeomorphisms. For any $x_1 \neq x_2$, we claim that $F(x_1, t) \neq F(x_2, t)$ for all $t \in [0, T_4]$. Suppose not; then there is the first time $t_0 > 0$ such that $F(x_1, t_0) = F(x_2, t_0)$. Choose small $\sigma > 0$ so that there exist a neighborhood \hat{O} of $F(x_1, t_0)$ and a neighborhood O of x_1 such that $F^{-1}(\cdot, t)$ is a diffeomorphism from \hat{O} to O for each $t \in [t_0 - \sigma, t_0]$, and let $\hat{\gamma}$ be a shortest geodesic (parametrized by arc

length) on the target (with respect to the metric \hat{g}) with $\hat{\gamma}(0) = F(x_1, t)$, $\hat{\gamma}(l) = F(x_2, t)$ and $\hat{\gamma} \subset \hat{O}$. We compute

$$(4.23) \quad \frac{\partial}{\partial t} \hat{d}(F(x_1, t), F(x_2, t)) = \langle V(F(x_2, t)), \hat{\gamma}'(l) \rangle_{\hat{g}} - \langle V(F(x_1, t)), \hat{\gamma}'(0) \rangle_{\hat{g}},$$

where $V(F(x, t)) = (\partial/\partial t)F(x, t)$. Now we pull back everything by F^{-1} to O ,

$$\begin{aligned} \frac{\partial}{\partial t} \hat{d}(F(x_1, t), F(x_2, t)) &= \langle P_{-\hat{\gamma}}V - V, \hat{\gamma}'(0) \rangle_{F^*\hat{g}} \\ &\geq - \sup_{x \in F^{-1}\hat{\gamma}} |\hat{\nabla}V|(x, t) \hat{d}(F(x_1, t), F(x_2, t)), \end{aligned}$$

where $P_{\hat{\gamma}}$ is the parallel translation along $F^{-1}\hat{\gamma}$ using the connection defined by $F^*\hat{g}$. Since

$$\hat{\nabla}_k V^l = \nabla_k V^\alpha \frac{\partial x^l}{\partial y^\alpha},$$

where $\nabla_k V^\alpha$ is the covariant derivative of the section V^α of the bundle $F^{-1}TN$. Thus by (4.20) in proposition 4.5, we have

$$|\hat{\nabla}_k V^l| = [\nabla_k V^\alpha \nabla_l V^\beta \hat{g}_{\alpha\beta} \hat{g}^{kl}]^{1/2} \leq C |\nabla^3 F| \leq \frac{C}{\sqrt{t}},$$

where the constant C depends on the x_1 and x_2 and is independent of t by (4.5) of Theorem 4.1. Therefore, for $t \in [t_0 - \sigma, t_0]$, we have

$$\hat{d}(F(x_1, t), F(x_2, t)) \leq e^{C(\sqrt{t_0} - \sqrt{t_0 - \sigma})} \hat{d}(F(x_1, t_0), F(x_2, t_0)) = 0,$$

which contradicts with the choice of t_0 . The corollary is proved. □

5. Mean curvature De Turck flow

From the previous section, we know that the harmonic map flow coupled with MCF with identity as initial data has a short-time solution $F(x, t)$ which maintains being a diffeomorphism with good estimates. Let $\bar{X} = X \circ F^{-1}$ be a family of maps defined from $(N, \hat{g}_{\alpha\beta})$ to $\bar{M}^{\bar{n}}$, then \bar{X} satisfies the following mean curvature De Turck flow

$$(5.1) \quad \frac{\partial}{\partial t} \bar{X} = g^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X} \quad \text{for } y \in N,$$

where $g^{\alpha\beta}$ is the inverse matrix of $g_{\alpha\beta}(\cdot, t) = ((F^{-1})^*g(\cdot, t))_{\alpha\beta}$, $\hat{\nabla}$ is the covariant derivative with respect to $\hat{g}_{\alpha\beta}$. We denote the local coordinates of \bar{M} by $\{\bar{z}^{\bar{\alpha}}\}$. It is not hard to see

$$\begin{aligned}
 g_{\alpha\beta}(y, t) &= g_{ij}(x, t) \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \\
 (5.2) \qquad &= \bar{g}_{\bar{\alpha}\bar{\beta}} \frac{\partial X^{\bar{\alpha}}}{\partial x^i} \frac{\partial X^{\bar{\beta}}}{\partial x^j} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} = \frac{\partial \bar{X}^{\bar{\gamma}}}{\partial y^\alpha} \cdot \frac{\partial \bar{X}^{\bar{\delta}}}{\partial y^\beta} \bar{g}_{\bar{\gamma}\bar{\delta}}(\bar{X}(y, t)),
 \end{aligned}$$

this implies that the metric $g_{\alpha\beta}(y, t)$ is just the induced metric from the ambient space by the map \bar{X} . Since

$$\hat{\Gamma}_{\alpha\beta}^\gamma(y) - \Gamma_{\alpha\beta}^\gamma(y, t) = (\nabla^2 F)_{ij}^\gamma \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta},$$

we have

$$\begin{aligned}
 (5.3) \qquad &\frac{1}{C} \hat{g}_{\alpha\beta}(y) \leq g_{\alpha\beta}(y, t) \leq C \hat{g}_{\alpha\beta}(y), \\
 &|\hat{\Gamma}_{\alpha\beta}^\gamma(y) - \Gamma_{\alpha\beta}^\gamma(y, t)| \leq C,
 \end{aligned}$$

by Theorem 4.1 and Proposition 4.5.

Let X_1 and X_2 be two solutions of MCF with bounded second fundamental form and with the same initial value X_0 assumed as in the Theorem 1.1. Let $g_{ij}^1(x, t)$ and $g_{ij}^2(x, t)$ be the corresponding induced metrics. As in Section 4, we solve the harmonic map flows coupled with MCF with the same target $(M^n, \hat{g}_{\alpha\beta})$, where $\hat{g} = g^1(T)$, respectively,

$$\begin{aligned}
 (5.4) \qquad &\frac{\partial}{\partial t} F_1 = \Delta_{g^1, \hat{g}} F_1 \\
 &F_1|_{t=0} = \text{Identity on } M^n,
 \end{aligned}$$

and

$$\begin{aligned}
 (5.5) \qquad &\frac{\partial}{\partial t} F_2 = \Delta_{g^2, \hat{g}} F_2 \\
 &F_2|_{t=0} = \text{Identity on } M^n,
 \end{aligned}$$

where $\Delta_{g^k, \hat{g}}$ is the harmonic map Laplacian defined by the metric $g_{ij}^k(x, t)$, and $\hat{g}_{\alpha\beta}$ for $k = 1, 2$, respectively. By Section 4, we obtain two solutions $F_1(x, t)$ and $F_2(x, t)$ such that Theorem 4.1 holds with $F = F_1$ and $F = F_2$. Corollary 4.6 says that $F_1(x, t)$ and $F_2(x, t)$ are diffeomorphisms for any $t \in [0, T_4]$. Let $g_{1\alpha\beta}(y, t) = ((F_1^{-1})^*g^1)_{\alpha\beta}(y, t)$ and $g_{2\alpha\beta}(y, t) = ((F_2^{-1})^*g^2)_{\alpha\beta}(y, t)$. Then $\bar{X}_1 = X_1 \circ F_1^{-1}$ and $\bar{X}_2 = X_2 \circ F_2^{-1}$ are two solutions to the mean curvature De Turck flow (5.1) with the same initial value X_0 ,

$$(5.6) \quad \begin{aligned} \frac{\partial}{\partial t} \bar{X}_1 &= g_1^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X}_1, \quad \text{on } M^n \times [0, T_4], \\ \bar{X}_1|_{t=0} &= X_0, \quad \text{on } M^n, \end{aligned}$$

$$(5.7) \quad \begin{aligned} \frac{\partial}{\partial t} \bar{X}_2 &= g_2^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X}_2, \quad \text{on } M^n \times [0, T_4], \\ \bar{X}_2|_{t=0} &= X_0, \quad \text{on } M^n, \end{aligned}$$

where $g_{1\alpha\beta}$ and $g_{2\alpha\beta}$ are the corresponding induced metrics from the target $(\bar{M}^{\bar{n}}, \bar{g}_{\alpha\bar{\beta}})$ by the maps \bar{X}_1 and \bar{X}_2 by (5.2).

Proposition 5.1. *Under the assumptions of Theorem 1.1, there is some $T_5 > 0$ depending only on $\bar{C}, \bar{\delta}, T$ and \bar{n} such that*

$$\bar{X}_1(y, t) = \bar{X}_2(y, t) \quad \text{on } M^n \times [0, T_5]$$

for the two solutions of mean-De Turck flow constructed above.

Proof. Let $\psi(\bar{z}_1, \bar{z}_2) = d_{\bar{M}}^2(\bar{z}_1, \bar{z}_2)$ be the square of the distance function on \bar{M} which is viewed as a function of $(\bar{z}_1, \bar{z}_2) \in \bar{M} \times \bar{M}$. Set $u(y, t) = d_{\bar{M}}^2(\bar{X}_1(y, t), \bar{X}_2(y, t))$. Let $\Delta_k = g_k^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta$ for $k = 1, 2$. By direct computation, we have

$$\begin{aligned} \frac{\partial}{\partial t} u(y, t) &= 2d_{\bar{M}}(\bar{X}_1, \bar{X}_2) \frac{\partial d}{\partial \bar{z}_1^{\bar{\xi}}} \Delta_1 \bar{X}_1^{\bar{\xi}} + 2d_{\bar{M}}(\bar{X}_1, \bar{X}_2) \frac{\partial d}{\partial \bar{z}_2^{\bar{\zeta}}} \Delta_2 \bar{X}_2^{\bar{\zeta}}, \\ g_1^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta u(y, t) &= 2d_{\bar{M}}(\bar{X}_1, \bar{X}_2) \left[\frac{\partial d}{\partial \bar{z}_1^{\bar{\xi}}} \Delta_1 \bar{X}_1^{\bar{\xi}} + \frac{\partial d}{\partial \bar{z}_2^{\bar{\zeta}}} \Delta_1 \bar{X}_2^{\bar{\zeta}} \right] \\ &\quad + \text{Hess}(\psi)(Z_\alpha, Z_\beta) g_1^{\alpha\beta}, \end{aligned}$$

where $Z_\alpha = (\partial \bar{X}_1^{\bar{\xi}} / \partial y^\alpha)(\partial / \partial \bar{z}_1^{\bar{\xi}}) + (\partial \bar{X}_2^{\bar{\zeta}} / \partial y^\alpha)(\partial / \partial \bar{z}_2^{\bar{\zeta}}) \in T_{(\bar{X}_1, \bar{X}_2)} \bar{M} \times \bar{M}$, $\alpha = 1, 2, \dots, n$ are vector fields on $\bar{M} \times \bar{M}$. Combining these two formulas,

we have

$$(5.8) \quad \left[\frac{\partial}{\partial t} - g_1^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \right] u(y, t) = -2d_{\bar{M}}(\bar{X}_1, \bar{X}_2) \frac{\partial d}{\partial \bar{z}_2^\zeta} ((\Delta_1 - \Delta_2) \bar{X}_2)^\zeta - \text{Hess}(\psi)(Z_\alpha, Z_\beta) g_1^{\alpha\beta}.$$

Note that

$$(5.9) \quad \begin{aligned} (\Delta_1 - \Delta_2) \bar{X}_2 &= g_1^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X}_2 - g_2^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X}_2 \\ &= g_1^{\alpha\gamma} g_2^{\beta\delta} (g_{2\delta\gamma} - g_{1\delta\gamma}) \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X}_2, \\ \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X}_2 &= \nabla_{2\alpha} \nabla_{2\beta} \bar{X}_2 + (\hat{\Gamma} - \Gamma_2) * \nabla \bar{X}_2, \end{aligned}$$

where Γ_2 and ∇_2 are the Christoffel symbol and the covariant derivative of the metric $g_{2\alpha\beta}(y, t)$.

For each $y \in M^n$ and $t \in [0, T]$, if $\bar{X}_1(y, t) \neq \bar{X}_2(y, t)$, denote the minimal geodesic on \bar{M} from $\bar{X}_1(y, t)$ to $\bar{X}_2(y, t)$ by σ , and denote the parallel translation of \bar{M} along σ by P_σ , then we have

$$(5.10) \quad \begin{aligned} g_{1\delta\gamma}(y, t) - g_{2\delta\gamma}(y, t) &= \left\langle \bar{X}_{1*} \left(\frac{\partial}{\partial y^\delta} \right), \bar{X}_{1*} \left(\frac{\partial}{\partial y^\gamma} \right) \right\rangle_{\bar{g}} \\ &\quad - \left\langle \bar{X}_{2*} \left(\frac{\partial}{\partial y^\delta} \right), \bar{X}_{2*} \left(\frac{\partial}{\partial y^\gamma} \right) \right\rangle_{\bar{g}} \\ &= \left\langle \bar{X}_{1*} \left(\frac{\partial}{\partial y^\delta} \right), \bar{X}_{1*} \left(\frac{\partial}{\partial y^\gamma} \right) \right\rangle_{\bar{g}} \\ &\quad - \left\langle P_\sigma^{-1} \left(\bar{X}_{2*} \left(\frac{\partial}{\partial y^\delta} \right) \right), P_\sigma^{-1} \left(\bar{X}_{2*} \left(\frac{\partial}{\partial y^\gamma} \right) \right) \right\rangle_{\bar{g}} \\ &= \left\langle \bar{X}_{1*} \left(\frac{\partial}{\partial y^\delta} \right) - P_\sigma^{-1} \left(\bar{X}_{2*} \left(\frac{\partial}{\partial y^\delta} \right) \right), \bar{X}_{1*} \left(\frac{\partial}{\partial y^\gamma} \right) \right\rangle_{\bar{g}} \\ &\quad + \left\langle P_\sigma^{-1} \left(\bar{X}_{2*} \left(\frac{\partial}{\partial y^\delta} \right) \right), \bar{X}_{1*} \left(\frac{\partial}{\partial y^\gamma} \right) \right. \\ &\quad \left. - P_\sigma^{-1} \left(\bar{X}_{2*} \left(\frac{\partial}{\partial y^\gamma} \right) \right) \right\rangle_{\bar{g}}. \end{aligned}$$

If $\bar{X}_1(y, t) = \bar{X}_2(y, t)$, $P_\sigma = \text{Identity}$, the above formula still holds.

In the following argument, we compute norms by using the metrics g_1 and \bar{g} . For example,

$$|\hat{\Gamma} - \Gamma_2|^2 = (\hat{\Gamma} - \Gamma_2)_{\alpha\beta}^\gamma (\hat{\Gamma} - \Gamma_2)_{\alpha'\beta'}^{\gamma'} g_1^{\alpha\alpha'} g_1^{\beta\beta'}$$

and

$$|\nabla_2^2 \bar{X}_2|^2 = \bar{g}_{\bar{\xi}\bar{\zeta}} g_1^{\alpha\alpha'} g_1^{\beta\beta'} \nabla_{2\alpha} \nabla_{2\beta} \bar{X}_2^{\bar{\xi}} \nabla_{2\alpha'} \nabla_{2\beta'} \bar{X}_2^{\bar{\zeta}}.$$

We denote by C various constants depending only on the constants \bar{C} , T , \bar{n} and $\bar{\delta}$ in the main Theorem 1.1. Then by (5.3), we have

$$\begin{aligned} |\hat{\Gamma} - \Gamma_2| &\leq C, \\ |\hat{\nabla}^2 \bar{X}_2| &\leq C|\hat{\Gamma} - \Gamma_2| + C|\nabla_2^2 \bar{X}_2| \leq C, \\ (5.11) \quad |g_2| + |g_2^{-1}| &\leq C, \end{aligned}$$

where $|\nabla_2^2 \bar{X}_2|$ is just the norm of the second fundamental form of $X_2 : M^n \rightarrow \bar{M}^{\bar{n}}$ which is bounded by \bar{C} . Combining (5.9) (5.10) and (5.11), we have

$$\begin{aligned} |(\Delta_1 - \Delta_2)\bar{X}_2|^2 &\leq Cg_1^{\delta\gamma} \left\langle X_{1*} \left(\frac{\partial}{\partial y^\delta} \right) - P_\sigma^{-1} \left(X_{2*} \left(\frac{\partial}{\partial y^\delta} \right) \right), X_{1*} \left(\frac{\partial}{\partial y^\gamma} \right) \right. \\ (5.12) \quad &\quad \left. - P_\sigma^{-1} \left(X_{2*} \left(\frac{\partial}{\partial y^\gamma} \right) \right) \right\rangle_{\bar{g}}. \end{aligned}$$

By choosing an orthonormal frame at y so that $g_{1\alpha\beta} = \delta_{\alpha\beta}$, then we have

$$\text{Hess}(\psi)(Z_\alpha, Z_\beta)g_1^{\alpha\beta} = \sum_{\alpha=1}^n \text{Hess}(\psi)(Z_\alpha, Z_\alpha).$$

Note that

$$Z_\alpha = Z_{\alpha 1} + Z_{\alpha 2}, \text{ for } \alpha = 1, 2, \dots, n,$$

where $Z_{\alpha 1} = (\partial \bar{X}_1^{\bar{\xi}} / \partial y^\alpha)(\partial / \partial \bar{z}_1^{\bar{\xi}}) = X_{1*}(\partial / \partial y^\alpha)$ and $Z_{\alpha 2} = (\partial \bar{X}_2^{\bar{\zeta}} / \partial y^\alpha)(\partial / \partial \bar{z}_2^{\bar{\zeta}}) = X_{2*}(\partial / \partial y^\alpha)$.

Recall that by Theorem 2.2(ii), there is a constant C such that if $d_{\bar{M}}(\bar{z}_1, \bar{z}_2) \leq \min \{ (1/4\sqrt{\bar{C}}), (\bar{\delta}/2) \}$, we have

$$(\nabla^2 d^2)(Z, Z) \geq 2|Z_1 - P_\sigma^{-1}Z_2|^2 - C|Z|^2 d^2 \text{ for all } Z \in T_{(\bar{z}_1, \bar{z}_2)}\bar{M}^{\bar{n}} \times \bar{M}^{\bar{n}},$$

where $Z = Z_1 + Z_2$, $Z_1 \in T_{\bar{z}_1} \bar{M}^{\bar{n}}$, $Z_2 \in T_{\bar{z}_2} \bar{M}^{\bar{n}}$. Hence if $d_{\bar{M}}(\bar{X}_1, \bar{X}_2) \leq \min \{ (1/4\sqrt{\bar{C}}), (\bar{\delta}/2) \}$, then

$$(5.13) \quad \sum_{\alpha=1}^n \text{Hess}(\psi)(Z_\alpha, Z_\alpha) \geq \sum_{\alpha=1}^n 2 \left| \bar{X}_{1*} \left(\frac{\partial}{\partial y^\alpha} \right) - P_\sigma^{-1} \bar{X}_{2*} \left(\frac{\partial}{\partial y^\alpha} \right) \right|^2 - Cd_{\bar{M}}(\bar{X}_1, \bar{X}_2)^2$$

since $|Z_\alpha| \leq \bar{C}$.

Combining (5.8), (5.12) and (5.13), if $u^{1/2} \leq \min\{ (1/4\sqrt{\bar{C}}), (\bar{\delta}/2) \}$, then we have

$$(5.14) \quad \begin{aligned} \left(\frac{\partial}{\partial t} - g_1^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \right) u(y, t) &\leq Cd_{\bar{M}}(\bar{X}_1, \bar{X}_2) \sum_{\alpha=1}^n 2 \left| \bar{X}_{1*} \left(\frac{\partial}{\partial y^\alpha} \right) \right. \\ &\quad \left. - P_\sigma^{-1} \bar{X}_{2*} \left(\frac{\partial}{\partial y^\alpha} \right) \right|^2 \\ &\quad - 2 \sum_{\alpha=1}^n \left| \bar{X}_{1*} \left(\frac{\partial}{\partial y^\alpha} \right) - P_\sigma^{-1} \bar{X}_{2*} \left(\frac{\partial}{\partial y^\alpha} \right) \right|^2 \\ &\quad + Cd_{\bar{M}}^2(\bar{X}_1, \bar{X}_2) \\ &\leq Cu. \end{aligned}$$

Now we show that $u^{1/2} \leq \min \{ (1/4\sqrt{\bar{C}}), (\bar{\delta}/2) \}$ on some time interval $[0, T_5]$.

For any $(y, t) \in \hat{M} \times [0, T_4]$, we have

$$(5.15) \quad \begin{aligned} u^{1/2}(y, t) &\leq d_{\bar{M}}(X_1 \circ F_1^{-1}(y, t), X_1 \circ F_1^{-1}(y, 0)) \\ &\quad + d_{\bar{M}}(X_1 \circ F_1^{-1}(y, 0), X_2 \circ F_2^{-1}(y, 0)) \\ &\quad + d_{\bar{M}}(X_2 \circ F_2^{-1}(y, t), X_2 \circ F_2^{-1}(y, 0)) \\ &\triangleq I_1 + I_2 + I_3. \end{aligned}$$

By the mean curvature flow equation (1.1), we know

$$I_2 \leq d_{\bar{M}}(X_1(y, t), X_1(y, 0)) + d_{\bar{M}}(X_2(y, t), X_2(y, 0)) \leq 2\sqrt{n}\bar{C}t.$$

By (4.4) and (4.23), for any $x_1, x_2 \in M^n$, we get

$$\frac{\partial}{\partial t} \hat{d}(F_1(x_1, t), F_1(x_2, t)) \geq -C,$$

this implies

$$(5.16) \quad \hat{d}(x_1, x_2) \leq \hat{d}(F_1(x_1, t), F_1(x_2, t)) + Ct.$$

By (5.16) and Lemma 4.2, it follows

$$\begin{aligned}
 I_1 &= d_{\bar{M}}(X_1 \circ F_1^{-1}(y, t), X_1 \circ F_1^{-1}(y, 0)) \\
 &\leq d_{(M, g^1(\cdot, t))}(F_1^{-1}(y, t), y) \\
 &\leq C\hat{d}(F_1^{-1}(y, t), y) \\
 &\leq Ct + C\hat{d}(y, F_1(y, t)) \\
 &\leq C\sqrt{t}.
 \end{aligned}$$

The estimate of I_3 is similar. Therefore, we have

$$(5.17) \quad u^{1/2}(y, t) \leq C\sqrt{t}$$

for some constant C depending only on $\bar{C}, \bar{\delta}, T$ and \bar{n} .

Although $g_1^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta$ is not the standard Laplacian, the maximum principle is still applicable. For completeness, we include the proof in the following.

Since the curvature of (M, \hat{g}) is bounded, it is well-known that there is a function φ such that

$$\begin{aligned}
 \frac{1}{C}(1 + d_{\hat{g}}(y_0, y)) \leq \varphi(y) \leq C(1 + d_{\hat{g}}(y_0, y)) \\
 |\hat{\nabla}\varphi| + |\hat{\nabla}^2\varphi| \leq C.
 \end{aligned}$$

Note g_1 is equivalent to \hat{g} . For any small $\varepsilon > 0$ and large $A > 0$, we have

$$\left(\frac{\partial}{\partial t} - g_1^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta \right) (e^{-Ct}u(y, t) - \varepsilon e^{At}\varphi) \leq -\frac{\varepsilon A}{2}e^{At}\varphi < 0.$$

Then the classical maximum principle implies that for any fixed t_0 the maximal value of $(e^{-Ct}u(y, t) - \varepsilon e^{At}\varphi)$ on $M \times [0, t_0]$ cannot be achieved for any point (y, t) with $0 < t \leq t_0$. Hence $e^{-Ct}u(y, t) - \varepsilon e^{At}\varphi \leq 0$ for any $t \in [0, T_5]$ for some $T_5 > 0$. Let $\varepsilon \rightarrow 0$, we conclude that $u \equiv 0$ on $[0, T_5]$. This implies $\bar{X}_1 = \bar{X}_2$, on $M \times [0, T_5]$. We complete the proof of Proposition 5.1. \square

6. Proof of the uniqueness Theorem 1.1

Now we are ready to prove Theorem 1.1. Let $X_1(x, t)$ and $X_2(x, t)$ be two solutions of MCF with bounded second fundamental form and with the same initial data. We solve the corresponding harmonic map flow (5.4) and (5.5) (with the same target (M, \hat{g}) , $\hat{g} = g_1(T)$), respectively, to obtain

two solutions $F_1(x, t)$ and $F_2(x, t)$ on some common time interval. Then $\bar{X}_1 = X_1 \circ F_1^{-1}$ and $\bar{X}_2 = X_2 \circ F_2^{-1}$ are two solutions to the mean curvature De Turck flow with the same initial value. By Proposition 5.1 we know $\bar{X}_1 \equiv \bar{X}_2$ on $[0, T_5]$. So in order to prove $X_1(x, t) \equiv X_2(x, t)$, we only need to show $F_1 \equiv F_2$.

We know

$$\begin{aligned} \Delta_1 F_1^\alpha &= g_1^{\beta\gamma} (\hat{\Gamma}_{\beta\gamma}^\alpha - \Gamma_{1\beta\gamma}^\alpha) \circ F_1, \\ \Delta_2 F_2^\alpha &= g_2^{\beta\gamma} (\hat{\Gamma}_{\beta\gamma}^\alpha - \Gamma_{2\beta\gamma}^\alpha) \circ F_2. \end{aligned}$$

Since $\bar{X}_1 \equiv \bar{X}_2$, we know $g_{1\alpha\beta}(y, t) = g_{2\alpha\beta}(y, t)$ on $[0, T_5]$, and the vector fields $V_1 \equiv V_2$ on the target, where

$$\begin{aligned} V_1^\alpha &= g_1^{\beta\gamma} (\hat{\Gamma}_{\beta\gamma}^\alpha - \Gamma_{1\beta\gamma}^\alpha), \\ V_2^\alpha &= g_2^{\beta\gamma} (\hat{\Gamma}_{\beta\gamma}^\alpha - \Gamma_{2\beta\gamma}^\alpha). \end{aligned}$$

Therefore, the two families of maps F_1 and F_2 satisfy the same ODE with the same initial value:

$$\begin{aligned} \frac{\partial}{\partial t} F_1 &= V \circ F_1 \\ F_1(\cdot, 0) &= \text{Identity}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} F_2 &= V \circ F_2 \\ F_2(\cdot, 0) &= \text{Identity}. \end{aligned}$$

So for any $x \in M^n$, letting γ be a shortest geodesic (parametrized by arc length) on the target with $\gamma(0) = F_1(x, t)$ and $\gamma(l) = F_2(x, t)$, we have

$$\begin{aligned} \frac{\partial}{\partial t} \hat{d}(F_1(x, t), F_2(x, t)) &= \langle V, \gamma'(l) \rangle - \langle V, \gamma'(0) \rangle \\ &= \langle P_\gamma^{-1} V - V, \gamma'(0) \rangle \\ &\leq \sup_{y \in \gamma} |\hat{\nabla} V|(y, t) \hat{d}(F_1(x, t), F_2(x, t)), \end{aligned}$$

where $P_\gamma^{-1} V$ is the parallel transport of $V(F_2(x, t), t)$ along the geodesic γ back to the tangent space of the point $F_1(x, t)$. We have seen in the proof of Corollary 4.6 that $\sup_{y \in \gamma} |\hat{\nabla} V|(y, t) \leq \frac{C}{\sqrt{t}}$ for some C depending on x but

independent of t . Since $\hat{d}(F_1(x, 0), F_2(x, 0)) \equiv 0$, we conclude that

$$F_1(x, t) \equiv F_2(x, t).$$

So we have proved $X_1(x, t) = X_2(x, t)$, for all $x \in M$ and $t \in [0, T_5]$. Clearly, we can extend the interval $[0, T_5]$ to the whole $[0, T]$ by applying the same argument on $[T_5, T]$.

The proof of Theorem 1.1 is completed. □

Corollary 1.2 is a direct consequence of Theorem 1.1. Indeed, let $\bar{\sigma}$ and σ be two isometries of $(\bar{M}^{\bar{n}}, \bar{g})$ and (M^n, g) , respectively, such that $(\bar{\sigma} \circ X_0)(x) = (X_0 \circ \sigma)(x)$ for any $x \in M^n$. Since $\bar{\sigma} \circ X_t$ and $X_t \circ \sigma$ are two solutions to the MCF (1.1) with bounded second fundamental form on $M^n \times [0, T]$ and with the same initial value, then by Theorem 1.1, we have

$$(\bar{\sigma} \circ X_t)(x) = (X_t \circ \sigma)(x)$$

for any $x \in M^n$ and $t \in [0, T]$. The proof of the Corollary 1.2 is completed. □

7. Pseudolocality theorem

We begin with a few terminologies for the sake of convenience. An n -dimensional submanifold $M \subset \bar{M}$ is said to be a local δ -Lipschitz graph of radius r_0 at $P \in M$, if there is a normal coordinate system $(y^1, \dots, y^{\bar{n}})$ of \bar{M} around P with $T_P M = \text{span}\{(\partial/\partial y^1), \dots, (\partial/\partial y^n)\}$, a vector valued function $F : \{y' = (y^1, \dots, y^n) \mid (y^1)^2 + \dots + (y^n)^2 < r_0^2\} \rightarrow \mathbb{R}^{\bar{n}-n}$ with $F(0) = 0$, $|DF|(0) = 0$ such that $M \cap \{|y'| < r_0\} = \{(y', F(y')) \mid |y'| < r_0\}$ and $|DF|^2(y') = \sum_{i,\beta} (\partial F^\beta / \partial y^i)(\partial F^\beta / \partial y^i) < \delta^2$. The submanifold M_0 is said to be graphic in the ball $B_{\bar{M}}(x_0, r_0)$, if the above holds for $\delta = \infty$.

We say a submanifold $M \subset \bar{M}$ is properly embedded in a ball $B_{\bar{M}}(x_0, r_0)$ if either M is closed or ∂M has distance $\geq r_0$ from x_0 . We say a submanifold $M \subset \bar{M}$ is properly embedded in \bar{M} if either M is closed or there is an $x_0 \in \bar{M}$ such that M is properly embedded in $B_{\bar{M}}(x_0, r_0)$ for any $r_0 > 0$. It is clear that if \bar{M} is complete and M is properly embedded in \bar{M} , then M is complete. A properly embedded submanifold M is said to be uniform graphic with radius r_0 if for any $x_0 \in M$ it is graphic in the ball $B_{\bar{M}}(x_0, r_0)$.

The following lemma says that if the second fundamental form is controlled, then (a piece of) the submanifold is a local δ -Lipschitz graph of suitable radius.

Lemma 7.1. *Let \bar{M} be an \bar{n} -dimensional complete Riemannian manifold satisfying*

$$|\bar{\text{Rm}}| + |\bar{\nabla}\bar{\text{Rm}}|(x) \leq \bar{C}, \quad \text{inj}(\bar{M}) \geq i_0 > 0.$$

There exists a constant $C_1 > 0$ with the following property. Let $\{x^1, \dots, x^{\bar{n}}\}$ be normal coordinates of \bar{M} of radius r_0 around x_0 with $T_{x_0}\bar{M} = \text{span}\{(\partial/\partial x^1), \dots, (\partial/\partial x^{\bar{n}})\}$, where M is an n -dimensional submanifold properly embedded in $B_{\bar{M}}(x_0, r_0)$, $x_0 \in M$, $r_0 \leq (1/C_1)$, and the second fundamental form $|A| \leq (1/r_0)$. Then there exists a map $F : \{(x^1, \dots, x^n) \mid (x^{1^2} + \dots + x^{n^2})^{1/2} < (r_0/96)\} \rightarrow \mathbb{R}^{\bar{n}-n}$ with $F(0) = 0$, $|DF|(0) = 0$ such that the connected component containing x_0 of $M \cap \{(x^1, \dots, x^{\bar{n}}) \mid (x^{1^2} + \dots + x^{n^2})^{1/2} < (r_0/96)\}$ can be written as a graph $\{(x', F(x')) \mid |x'| = (x^{1^2} + \dots + x^{n^2})^{1/2} < (r_0/96)\}$ and

$$(7.1) \quad |DF|(x') \leq \frac{36}{r_0}|x'|,$$

$x' = (x^1, \dots, x^n) \in B_{\mathbb{R}^n}(0, (r_0/96))$, where $|DF|(x')^2 = \sum_{i=1}^n \sum_{\alpha=n+1}^{\bar{n}} (\partial F^\alpha / \partial x^i)(\partial F^\alpha / \partial x^i)(x')$.

Proof. Let $X = (X^1, \dots, X^{\bar{n}}) = (x', F(x'))$, $x' = (x^1, \dots, x^n)$, be a graph representation of the local isometric embedding of the connected component containing x_0 of $M \cap \{(x^1, \dots, x^{\bar{n}}) \mid (x^{1^2} + \dots + x^{n^2})^{1/2} < r_1\}$ (for some $r_1 \leq (r_0/96)$) into \bar{M} under the exponential map.

Define

$$|\nabla F|^2 = \sum_{i,j=1}^n \sum_{\alpha=n+1}^{\bar{n}} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\alpha}{\partial x^j} g^{ij}, \quad |DF|^2 = \sum_{i=1}^n \sum_{\alpha=n+1}^{\bar{n}} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\alpha}{\partial x^i}.$$

By choosing C_1 large, we have

$$\frac{1}{2}\delta_{\alpha\beta} \leq \bar{g}_{\alpha\beta} \leq 2\delta_{\alpha\beta}, \quad |\bar{\Gamma}_{\alpha\beta}^\gamma| \leq 1, \quad \frac{1}{2}\delta_{ij} \leq g_{ij} \leq 2(1 + |DF|^2)\delta_{ij}.$$

For $\alpha \geq n + 1$, $i, j \leq n$, recall the coefficients of the second fundamental form is given by

$$A_{ij}^\alpha = \frac{\partial X^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial X^\alpha}{\partial x^k} + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial X^\beta}{\partial x^i} \frac{\partial X^\gamma}{\partial x^j} = \nabla_{ij}^2 F^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial X^\beta}{\partial x^i} \frac{\partial X^\gamma}{\partial x^j}.$$

Note that

$$\begin{aligned} \left| \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial X^\beta}{\partial x^i} \frac{\partial X^\gamma}{\partial x^j} \right|^2 &= \bar{\Gamma}_{\beta'\gamma'}^{\alpha'} \frac{\partial X^{\beta'}}{\partial x^i} \frac{\partial X^{\gamma'}}{\partial x^j} \bar{\Gamma}_{\beta\gamma}^\alpha \frac{\partial X^\beta}{\partial x^k} \frac{\partial X^\gamma}{\partial x^l} g^{ik} g^{jl} \bar{g}_{\alpha\alpha'} \leq C(\bar{n}), \\ |\nabla^2 F|^2 &= \sum_{\alpha, \beta \geq n+1; i, j, k, l \leq n} \nabla_{ij}^2 F^\alpha \nabla_{kl}^2 F^\beta \delta_{\alpha\beta} g^{ik} g^{jl} \\ &\leq 4(|A|^2 + C(\bar{n})) \\ &\leq 4r_0^{-2} + C(\bar{n}), \end{aligned}$$

and

$$|\nabla|\nabla F|| \leq |\nabla^2 F|.$$

This implies

$$(7.2) \quad |\nabla F|(\cdot) \leq 3r_0^{-1} d_M(x_0, \cdot).$$

Since $g_{ij} \leq 2(\delta_{ij} + \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\alpha}{\partial x^j}) \leq 2(1 + |DF|^2)\delta_{ij}$, it follows that

$$|\nabla F|^2 \geq \frac{1}{4} \frac{|DF|^2}{1 + |DF|^2}$$

and

$$(7.3) \quad |DF|^2 \leq \frac{4|\nabla F|^2}{1 - 4|\nabla F|^2}.$$

Combining (7.2) and (7.3), it follows that

$$|DF|(\cdot) \leq 9r_0^{-1} d_M(x_0, \cdot) \text{ on } B_M\left(x_0, \frac{r_0}{24}\right).$$

Since $d_M(x_0, \cdot) \leq 2d_{\bar{M}}(x_0, \cdot)$ by (2.5), we have

$$|DF|(\cdot) \leq 18r_0^{-1} \sup_{B_M(0, (r_0/24))} (1 + |DF|)|x'| \leq 36r_0^{-1}|x'|,$$

and we conclude that

$$|DF|(x') \leq 36r_0^{-1}|x'|, \text{ whenever } |x'| \leq \frac{r_0}{96}.$$

The above argument shows that there is $C_1 > 0$ such that under the exponential map, once the connected component of M can be expressed as a

graph $(x', F(x'))$ on $B_{\mathbb{R}^n}(0, r_1)$, for $r_1 \leq (r_0/96)$, then the estimate (7.1) holds. Hence the connected component of M can be expressed as a graph on the ball $B_{\mathbb{R}^n}(0, (r_0/96))$. \square

For future applications in pseudolocality theorem, we need a local graph representation for mean curvature flow.

Lemma 7.2. *Fix $k \geq 1$. Let \bar{M} be an \bar{n} -dimensional complete manifold satisfying*

$$\sum_{i=0}^{k+1} |\bar{\nabla}^i \bar{\text{Rm}}|(x) \leq \bar{C}, \quad \text{inj}(\bar{M}) \geq i_0 > 0.$$

There exists a constant $C_1 > 0$ with the following property. Suppose M_s , $s \in [-r_0^2, 0]$ is a solution of MCF properly embedded in $B_{\bar{M}}(x_0, r_0)$, $x_0 \in M_0$, $r_0 \leq (1/C_1)$, with $\sum_{i=0}^k |\nabla^i A| r_0^{i+1} \leq 1$ on $B_{\bar{M}}(x_0, r_0)$. Denote by $x_0^s \in M_s$ the orbit of x_0 . Let $\{x^1, \dots, x^{\bar{n}}\}$ be normal coordinates of \bar{M} of radius r_0 around x_0 with $T_{x_0} M_0 = \text{span}\{(\partial/\partial x^1), \dots, (\partial/\partial x^{\bar{n}})\}$. Then there exist a family of smooth maps $F_s : \{(x^1, \dots, x^{\bar{n}}) \mid (x^{1^2} + \dots + x^{\bar{n}^2})^{1/2} < (r_0/C_1)\} \rightarrow \mathbb{R}^{\bar{n}-n}$ with $F_0(0) = 0$, $|D_0 F|(0) = 0$, $e\bar{x}p_{x_0}((0, F_s(0))) = x_0^s$ such that the connected component of $M_s \cap \{(x^1, \dots, x^{\bar{n}}) \mid (x^{1^2} + \dots + x^{\bar{n}^2})^{1/2} < (r_0/C_1)\}$ (under the exponential map $e\bar{x}p_{x_0}$) containing x_0^s can be written as a graph $\{(x', F_s(x')) \mid |x'| = (x^{1^2} + \dots + x^{\bar{n}^2})^{1/2} < (r_0/C_1)\}$; moreover we have $\sum_{i=1}^{k+2} r_0^{i+1} |D^i F_s| \leq C_1$.

Actually, by the MCF equation $(\partial/\partial s)X = \Delta X$, where $X = (x', F_s(x'))$ is the graph representations on $B(0, r_1)$ for some $r_1 < (r_0/C_1)$, we have information on $|(\partial/\partial s)F_s| r_0 + |(\partial/\partial s)DF_s| r_0^2 \leq C_1$. This gives $|F_s(0)| \leq C s r_0^{-1}$ and $|DF_s|(0) \leq C s r_0^{-2}$. Similarly, by integrating $|\nabla|\nabla F|| \leq |\nabla^2 F|$, we know the graph representation holds in a ball of uniform radius (r_1/C_1) . The higher-derivative $D^i F$ can be estimated by $\sum_{j \leq i} |\nabla^j F|$ by definitions. \square

Now we state the pseudolocality theorem for the MCF.

Theorem 7.3. *Let \bar{M} be an \bar{n} -dimensional complete manifold satisfying $\sum_{i=0}^3 |\bar{\nabla}^i \bar{\text{Rm}}| \leq c_0^2$ and $\text{inj}(\bar{M}) \geq i_0 > 0$. Then for every $\alpha > 0$ there exist $\varepsilon > 0$, $\delta > 0$ with the following property. Suppose we have a smooth solution to the mean curvature flow $M_t \subset \bar{M}$ properly embedded in $B_{\bar{M}}(x_0, r_0)$ for $t \in [0, T]$ with $0 < T \leq \varepsilon^2 r_0^2$, and assume that at time zero, M_0 is a local δ -Lipschitz graph of radius r_0 at $x_0 \in M_0$ with $r_0 \leq (i_0/2)$. Then we have*

an estimate of the second fundamental form

$$(7.4) \quad |A|(x, t)^2 \leq \frac{\alpha}{t} + (\varepsilon r_0)^{-2}$$

on $B_{\bar{M}}(x_0, \varepsilon r_0) \cap M_t$, for any $t \in [0, T]$.

Proof. We argue by contradiction. By scaling we may assume $r_0 = 1$. Suppose there exist fixed $c_0 > 0$, $i_0 > 0$, $\alpha > 0$, and a sequence of $\varepsilon, \delta \rightarrow 0$ and smooth solutions to the mean curvature flow $M_t \subset \bar{M}$ for $t \in [0, T] \subseteq [0, \varepsilon^2]$ such that at time zero, M_0 is a local δ -Lipschitz graph of radius 1 at $x_0 \in M$. But there is some (x_1, t_1) satisfying $0 \leq t_1 \leq T$ and $x_1 \in B_{\bar{M}}(x_0, \varepsilon)$ such that

$$|A|(x_1, t_1)^2 > \frac{\alpha}{t_1} + \varepsilon^{-2}.$$

Denote by E_α the set of points (x, t) satisfying $|A|(x, t)^2 \geq (\alpha/t)$. Now we use the Perelman's point-picking technique [20] to choose another point which controls nearby points in its scale.

Lemma 7.4. *For any $K > 0$ with $K\varepsilon < (1/100n)$, let M_t be assumed as in the theorem, suppose $|A|(x_1, t_1)^2 \geq (\alpha/t_1) + \varepsilon^{-2}$ for some (x_1, t_1) satisfying $0 \leq t_1 \leq T \leq \varepsilon^2$ and $x_1 \in B_{\bar{M}}(x_0, \varepsilon)$, then one can find $(\bar{x}, \bar{t}) \in E_\alpha$ with $0 < \bar{t} \leq T$, $d_{\bar{M}}(x_0, \bar{x}) \leq (2K + 1)\varepsilon$ such that*

$$(7.5) \quad |A|(x, t) \leq 4Q$$

whenever $\bar{t} - (3/4)\alpha Q^{-2} \leq t \leq \bar{t}$, $d_{\bar{M}}(x, \bar{x}) \leq KQ^{-1}$, where $Q = |A|(\bar{x}, \bar{t})$.

Firstly, we claim that there exists $(\bar{x}, \bar{t}) \in E_\alpha$ with $0 < \bar{t} \leq T$, $d_{\bar{M}}(x_0, \bar{x}) \leq (2K + 1)\varepsilon$ such that

$$|A|(x, t) \leq 4|A|(\bar{x}, \bar{t})$$

whenever $(x, t) \in E_\alpha$, $0 \leq t \leq \bar{t}$, $d_{\bar{M}}(x_0, x) \leq d_{\bar{M}}(x_0, \bar{x}) + K|A|(\bar{x}, \bar{t})^{-1}$.

The argument is by contradiction. If (x_1, t_1) can not be chosen for (\bar{x}, \bar{t}) , one can find $(x_2, t_2) \in E_\alpha$ with $0 \leq t_2 \leq t_1$, $d_{\bar{M}}(x_0, x_2) \leq d_{\bar{M}}(x_0, x_1) + K|A|(x_1, t_1)^{-1}$, $|A|(x_2, t_2) > 4|A|(x_1, t_1)$. Inductively, we have a sequence of $(x_k, t_k) \in E_\alpha$ with $0 \leq t_k \leq t_{k-1}$, $d_{\bar{M}}(x_0, x_k) \leq d_{\bar{M}}(x_0, x_{k-1}) + K|A|(x_{k-1}, t_{k-1})^{-1}$, $|A|(x_k, t_k) > 4|A|(x_{k-1}, t_{k-1})$. Therefore, we have

$$|A|(x_k, t_k) > 4^{k-1}|A|(x_1, t_1) \geq 4^{k-1}\varepsilon^{-1}$$

and $d_{\bar{M}}(x_0, x_k) \leq d_{\bar{M}}(x_0, x_1) + K \sum_{i=1}^{\infty} (4^{i-1}|A|(x_1, t_1))^{-1} \leq (2K + 1)\varepsilon < 1/2$. Since the solution is smooth, we get a contradiction as k large enough.

For the chosen (\bar{x}, \bar{t}) , if $(x, t) \notin E_\alpha$, $\bar{t} - (3/4)\alpha Q^{-2} \leq t \leq \bar{t}$, then

$$|A|^2(x, t) \leq \frac{\alpha}{t} \leq \frac{\alpha}{\bar{t} - (3/4)\alpha Q^{-2}} \leq 4Q^2.$$

If $(x, t) \in E_\alpha$ and $d_{\bar{M}}(x, \bar{x}) \leq K|A|(\bar{x}, \bar{t})^{-1}$, by above claim we still have the estimate. The lemma is proved.

Continuing the proof of Theorem 7.3.

Choose $K = (1/\sqrt{\varepsilon})$. Let (\bar{x}, \bar{t}) be the point obtained in Lemma 7.4. Consider the auxiliary functions

$$\varphi(x, t) = (4\pi(\bar{t} - t))^{-(n/2)} \exp \left[- \left(1 + \frac{1}{\varepsilon}(t - \bar{t}) \right) \frac{d_{\bar{M}}^2(\bar{x}, x)}{4(t - \bar{t})} - \frac{n}{2\varepsilon}t \right],$$

and

$$\psi(x, t) = \left(1 - \frac{d_{\bar{M}}(\bar{x}, x)^2 + 3nt}{\rho^2} \right)_+^3$$

on $\bar{M} \times [0, \bar{t}]$, where $\rho = \min\{(1/2), (1/c_0\sqrt{e}), i_0, \sqrt{\varepsilon}\}$. They are also functions on M by composing the inclusion maps. We will compute their equations on M . Since the sectional curvature of \bar{M} satisfies $-c_0^2 \leq \sec \leq c_0^2$, by comparison theorem and mean curvature flow equation, we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta \right) d_{\bar{M}}(\bar{x}, \cdot)^2 &= 4d_{\bar{M}}\bar{\nabla}d_{\bar{M}} \cdot H + \text{tr}(\text{Hess}(d_{\bar{M}}^2(\bar{x}, \cdot)) |_{TM}) \\ &\geq 4d_{\bar{M}}\bar{\nabla}d_{\bar{M}} \cdot H + 2n \frac{c_0 d_{\bar{M}}(\bar{x}, \cdot)}{\tan c_0 d_{\bar{M}}(\bar{x}, \cdot)} \\ &\geq 4d_{\bar{M}}\bar{\nabla}d_{\bar{M}} \cdot H + 2n \left(1 - \frac{1}{2}c_0^2 d_{\bar{M}}^2(\bar{x}, \cdot) \right), \\ \left(\frac{\partial}{\partial t} - \Delta \right) d_{\bar{M}}(\bar{x}, \cdot)^2 &= -\text{tr}(\text{Hess}(d_{\bar{M}}^2(\bar{x}, \cdot)) |_{TM}) \\ &\geq -2nc_0 d_{\bar{M}}(\bar{x}, \cdot) \coth(c_0 d_{\bar{M}}(\bar{x}, \cdot)) \geq -3n \end{aligned}$$

whenever $d_{\bar{M}}(\bar{x}, \cdot)^2 < \min\{(1/c_0^2)e, i_0^2\}$, $t \in [0, \bar{t}]$. Hence we have

$$(7.6) \quad \left(\frac{\partial}{\partial t} - \Delta \right) \psi \leq 0$$

and

$$\begin{aligned}
 \left(\frac{\partial}{\partial t} + \Delta - |H|^2\right) \varphi &= \varphi \left[\frac{n}{2(\bar{t} - t)} - \frac{1 + (1/\varepsilon)(t - \bar{t})}{4(\bar{t} - t)} \left(\frac{\partial}{\partial t} + \Delta\right) d_{\bar{M}}(\bar{x}, \cdot)^2 \right. \\
 &\quad - \frac{(1 + (1/\varepsilon)(t - \bar{t}))d_{\bar{M}}(\bar{x}, \cdot)^2}{4(\bar{t} - t)^2} \\
 &\quad + \frac{(1 + (1/\varepsilon)(t - \bar{t}))^2 |\nabla d_{\bar{M}}(\bar{x}, \cdot)|^2}{16(\bar{t} - t)^2} \\
 &\quad \left. - \frac{(1/\varepsilon)d_{\bar{M}}(\bar{x}, \cdot)^2}{4(\bar{t} - t)} - \frac{n}{2\varepsilon} - |H|^2 \right] \\
 &\leq \varphi \left[-\frac{1 + (1/\varepsilon)(t - \bar{t})}{(\bar{t} - t)} d_{\bar{M}} \bar{\nabla} d_{\bar{M}} \cdot H \right. \\
 &\quad - \frac{(1 + (1/\varepsilon)(t - \bar{t}))d_{\bar{M}}(\bar{x}, \cdot)^2}{4(\bar{t} - t)^2} \\
 &\quad + \frac{(1 + (1/\varepsilon)(t - \bar{t}))^2 |\nabla d_{\bar{M}}(\bar{x}, \cdot)|^2}{16(\bar{t} - t)^2} \\
 &\quad \left. - \frac{[(1/\varepsilon) - (1 + (1/\varepsilon)(t - \bar{t}))nc_0^2] d_{\bar{M}}(\bar{x}, \cdot)^2 - |H|^2}{4(\bar{t} - t)} \right] \\
 (7.7) \quad &\leq - \left| H + \left(1 + \frac{1}{\varepsilon}(t - \bar{t})\right) \frac{d_{\bar{M}}(\bar{x}, \cdot) \bar{\nabla}^\perp d_{\bar{M}}(\bar{x}, \cdot)}{2(\bar{t} - t)} \right|^2 \varphi
 \end{aligned}$$

whenever $d_{\bar{M}}(\bar{x}, \cdot) < \rho$, $t \in [0, \bar{t}]$. We used $0 < 1 + (1/\varepsilon)(t - \bar{t}) \leq 1$. In the above and following argument, we regard the mean curvature flow M_t is a smooth family of $F_t : M \rightarrow \bar{M}$, $(\varphi\psi) \circ F_t$ is a C^2 function on $M \times [0, \bar{t}]$ with compact support in M . So $\int_{M_t} \varphi\psi = \int_M \varphi\psi dv_t$ is a C^2 function in t . Combining (7.6) and (7.7), we get the monotonicity formula

$$(7.8) \quad \frac{d}{dt} \int_{M_t} \varphi\psi \leq - \int_{M_t} \left| H + \left(1 + \frac{1}{\varepsilon}(t - \bar{t})\right) \frac{d_{\bar{M}}(\bar{x}, \cdot) \bar{\nabla}^\perp d_{\bar{M}}(\bar{x}, \cdot)}{2(\bar{t} - t)} \right|^2 \varphi\psi$$

on $[0, \bar{t}]$. This implies

$$\begin{aligned}
 &\int_{\bar{t} - (1/2)\alpha Q^{-2}}^{\bar{t}} \left[\int_{M_t} \left| H + \left(1 + \frac{1}{\varepsilon}(t - \bar{t})\right) \frac{d_{\bar{M}}(\bar{x}, \cdot) \bar{\nabla}^\perp d_{\bar{M}}(\bar{x}, \cdot)}{2(\bar{t} - t)} \right|^2 \varphi\psi \right] dt \\
 (7.9) \quad &\leq \int_{M_{\bar{t} - (1/2)\alpha Q^{-2}}} \varphi\psi - \int_{M_{\bar{t}}} \varphi\psi.
 \end{aligned}$$

Since the solution is smooth and properly embedded, ψ is compactly supported, we have $\lim_{t \rightarrow \bar{t}-} \int_{M_t} \varphi \psi = e^{-(n/2\varepsilon)\bar{t}} (1 - (3n\bar{t}/\rho^2))^3$. Now we claim that there is $\beta > 0$ such that as $\varepsilon, \delta \rightarrow 0$, we have

$$(7.10) \quad \int_{M_{\bar{t}-(1/2)\alpha Q^{-2}}} \varphi \psi \geq (1 + \beta) e^{-(n/2\varepsilon)\bar{t}} \left(1 - \frac{3n\bar{t}}{\rho^2}\right)^3.$$

We still argue by contradiction. Suppose not, then there is a subsequence of $\varepsilon, \delta \rightarrow 0$ and

$$(7.11) \quad \int_{\bar{t}-(1/2)\alpha Q^{-2}}^{\bar{t}} \left[\int_{M_t} \left| H + \left(1 + \frac{1}{\varepsilon}(t - \bar{t})\right) \frac{d_{\bar{M}}(\bar{x}, \cdot) \bar{\nabla}^\perp d_{\bar{M}}(\bar{x}, \cdot)}{2(\bar{t} - t)} \right|^2 \varphi \psi dv \right] dt \leq \beta \rightarrow 0.$$

Parabolic scaling the solution around (\bar{x}, \bar{t}) with the factor Q and shifting the \bar{t} to 0 and \bar{x} to origin O , i.e., let $(\tilde{M}, \tilde{g}) = (\bar{M}, Q^2 \bar{g})$ be the new target manifold, $\tilde{M}_s = M_{\bar{t}+Q^{-2}s}$, $-(3/4)\alpha \leq s \leq 0$ be the new family of submanifolds, which is still solution of MCF. By (7.5), the normalized second fundamental form satisfies $|\tilde{A}| \leq 4$ on $B_{\tilde{M}}(\bar{x}, K)$, $-(3/4)\alpha \leq s \leq 0$. By Theorem 3.2, we have $|\tilde{\nabla} \tilde{A}| + |\partial^2 \tilde{g}_{\alpha\beta}| \leq \text{Const.}$ on $B_{\tilde{M}}(\bar{x}, (K/2))$, $-(5/8)\alpha \leq s \leq 0$. Note that $K \rightarrow \infty$.

Now we are going to consider the convergence of the MCF on changing target manifolds. We clarify the meaning of the convergence in the following.

Denote the orbit of \bar{x} under MCF by $\bar{x}^s \in \tilde{M}_s$ such that $\bar{x}^0 = \bar{x}$. Note the injectivity radius of the new target (\tilde{M}, \tilde{g}) tends to infinity as $\varepsilon \rightarrow 0$. Let $\{x^1, \dots, x^{\bar{n}}\}$ be normal coordinates of \tilde{M} of radius $\gg 1$ around \bar{x} with $T_{\bar{x}} \tilde{M}_0 = \text{span}\{(\partial/\partial x^1), \dots, (\partial/\partial x^{\bar{n}})\}$, and $\tilde{g}_{\alpha\beta}$ be the metric coefficients of \tilde{M} in this coordinates. By [14], we have $|\tilde{g}_{\alpha\beta} - \delta_{\alpha\beta}|(x) \leq CQ^{-2}|x|^2$ and $|\partial \tilde{g}_{\alpha\beta}| + |\partial^2 \tilde{g}_{\alpha\beta}| \leq C$. By Arzela–Ascoli theorem, after taking a subsequence of $\varepsilon \rightarrow 0$, $\tilde{g}_{\alpha\beta}$ tends to $\delta_{\alpha\beta}$ in $C^{2-\gamma}$ topology for any $0 < \gamma < 1$.

By Lemma 7.2, there exist a family of maps $F_s : \{(x^1, \dots, x^{\bar{n}}) \mid (x^{1^2} + \dots + x^{\bar{n}^2})^{1/2} < 1\} \rightarrow \mathbb{R}^{\bar{n}-n}$ with $F_0(0) = 0$, $|DF_0|(0) = 0$, such that the connected component containing \bar{x}^s of $\tilde{M}_s \cap \{(x^1, \dots, x^{\bar{n}}) \mid (x^{1^2} + \dots + x^{\bar{n}^2})^{1/2} < 1\}$ can be written as a graph $\{(x', F_s(x')) \mid |x'| = (x^{1^2} + \dots + x^{\bar{n}^2})^{1/2} < 1\}$. Moreover, we can show

$$\sum_{i=1}^4 |D^i F| + \sum_{i=1}^2 \left(\left| \frac{\partial^i}{\partial s^i} F \right| + \left| D^i \frac{\partial F}{\partial s} \right| \right) \leq C,$$

where D and the norm are the natural differential and norm in Euclidean ordinates of $N \subset \mathbb{R}^n$ and the garget $\mathbb{R}^{\bar{n}}$. By Arzela–Ascoli theorem, $F(x', s)$ will converge to $F^\infty(x', s)$ in the topology of $\mathcal{C}^{3/2}(\overline{B(0, (1/2))}) \times [-(5\alpha/8), 0]$, $\mathbb{R}^{\bar{n}} \cap \mathcal{C}^3(\overline{B(0, (1/2))}, \mathbb{R}^{\bar{n}})$.

If we set $X = (x', F(x'))$ being the map from $N := B(0, 1)$ to \tilde{M} , then the MCF equation can be written as

$$\frac{\partial X}{\partial s} = \Delta X,$$

where Δ is the harmonic Laplacian defined by using the induced metric $X^*\tilde{g}$ and the target metric \tilde{g} . Since $X^*\tilde{g}$ is defined by DF and \tilde{g} , after taking a subsequence of $\varepsilon \rightarrow 0$, we know $X^*\tilde{g}$ converges in $\mathcal{C}^{1-\gamma}(\overline{B(0, (1/2))}) \times [-(5\alpha/8), 0]$ topology.

Denote by $\hat{M}_s = \tilde{M}_s \cap \exp_{\bar{x}}\{|x'| < 1\}$, and $\hat{M} = \cup_{s \in [-(\alpha/2), 0]} \hat{M}_s$. By summing up the above discussion, the piece \hat{M} of \tilde{M} containing $(\bar{x}, 0)$ will converge to a solution of the MCF (in the classical sense) which is embedded on the Euclidean space $\mathbb{R}^{\bar{n}}$ with $|\hat{A}_\infty|(O, 0) = 1$ and $|\hat{A}_\infty|(\cdot, s) \leq 4$ on $[-(\alpha/2), 0]$.

On the other hand, let $\tilde{\varphi} = Q^{-n}\varphi = (4\pi(-s))^{-(n/2)} \exp[-(1 + (s/Q^2\varepsilon)) (d_{\tilde{M}}^2(\bar{x}, \cdot)/4(-s) - (n/2\varepsilon)(\bar{t} + Q^{-2}s)]$, note that

$$\begin{aligned} & Q^{-2} \left| H + \left(1 + \frac{1}{\varepsilon}(t - \bar{t}) \right) \frac{d_{\tilde{M}}(\bar{x}, \cdot) \bar{\nabla}^\perp d_{\tilde{M}}(\bar{x}, \cdot)}{2(\bar{t} - t)} \right|_{\tilde{g}}^2 \\ &= \left| \tilde{H} - \left(1 + \frac{s}{Q^2\varepsilon} \right) \frac{d_{\tilde{M}}(\bar{x}, \cdot) \tilde{\nabla}^\perp d_{\tilde{M}}(\bar{x}, \cdot)}{2s} \right|_{\tilde{g}}^2, \\ \psi &= \left(1 - \frac{Q^{-2}d_{\tilde{M}}(\bar{x}, \cdot)^2 + 3n\bar{t} + 3nQ^{-2}s}{\rho^2} \right)_+^3 \rightarrow 1, \\ \tilde{\varphi} &\rightarrow (4\pi(-s))^{-(n/2)} e^{-(|\cdot|^2)/4(-s)} \quad \text{and} \quad \varphi\psi dv = \tilde{\varphi}\psi d\tilde{v}. \end{aligned}$$

Since $\hat{M}_s \subset \tilde{M}_s$, by passing (7.11) to limit, we have

$$\int_{-(1/2)\alpha}^0 \left[\int_{\hat{M}_s^\infty} \left| \hat{H}_\infty - \frac{x^\perp}{2s} \right|^2 (4\pi(-s))^{-(n/2)} e^{-(|x|^2/4(-s))} \right] ds = 0,$$

where we denote the limit of \hat{M}_s by \hat{M}_s^∞ , \hat{H}_∞ the mean curvature on the limit. This implies

$$\hat{H}_\infty = \frac{x^\perp}{2s} \quad \text{for } s \in \left[-\frac{\alpha}{2}, 0\right].$$

The boundedness of the second fundamental form on \hat{M}_0^∞ implies $x^\perp \equiv 0$ on \hat{M}_0^∞ . Since the second fundamental form and its twice covariant derivative of \hat{M}_s^∞ are bounded for $s \in [-(\alpha/2), 0]$, \hat{M}_s^∞ are $C^{4-\gamma}$ submanifolds for any $\gamma > 0$. Moreover by the higher-derivative estimates in Theorem 3.2 (in Euclidean space), \hat{M}_0^∞ is smooth.

Note $0 \in \hat{M}_0^\infty$, after a orthogonal transformation, we may assume $T_0\hat{M}_0^\infty = \{(x_1, x_2, \dots, x_n, 0, \dots, 0)\}$. Clearly we still have the condition $x^\perp \equiv 0$ on \hat{M}_0^∞ . We may write \hat{M}_0^∞ as a graph (at least locally near 0) $(x', f_1(x'), \dots, f_{\bar{n}-n}(x'))$ where $x' = (x_1, \dots, x_n)$. Now $x^\perp = (x', f_1(x'), \dots, f_{\bar{n}-n}(x'))^\perp \equiv 0$ implies $\sum_{p=1}^n (\partial f_i / \partial x_p) x_p = f_i(x')$. So f_i is homogenous of degree 1. Since $Df_i(0) = 0$, we conclude $f_i \equiv 0$. Hence we know \hat{M}_0^∞ is an n -dimensional linear subspace \mathbb{R}^n of $\mathbb{R}^{\bar{n}}$.

This contradicts $|\hat{A}_\infty|(O, 0) = 1$ and we complete the proof of (7.10).

Note that $B_{\bar{M}}(\bar{x}, \rho) \subseteq B_{\bar{M}}(x_0, \rho + (2K + 1)\varepsilon) \subseteq B_{\bar{M}}(x_0, 4\sqrt{\varepsilon})$. Combining (7.10) and monotonicity formula (7.8), we know

$$\begin{aligned} & \int_{M_0 \cap B_{\bar{M}}(x_0, 4\sqrt{\varepsilon})} (4\pi\bar{t})^{-(n/2)} \exp \left[- \left(1 - \frac{\bar{t}}{\varepsilon} \right) \frac{d_{\bar{M}}^2(\bar{x}, x)}{4\bar{t}} \right] dv \\ (7.12) \quad & \geq \int_{M_t} \varphi\psi dv \Big|_{t=\bar{t}-(1/2)\alpha Q^{-2}} \geq (1 + \beta)e^{-(n/2\varepsilon)\bar{t}} \left(1 - \frac{3n\bar{t}}{\rho^2} \right)^3. \end{aligned}$$

By assumption, there is a normal coordinate system $(y^1, \dots, y^{\bar{n}})$ of \bar{M} around x_0 with $T_{x_0}M_0 = \text{span}\{(\partial/\partial y^1), \dots, (\partial/\partial y^n)\}$ and a vector valued function $F : \{y' = (y^1, \dots, y^n) \mid (y^1)^2 + \dots + (y^n)^2 < 1\} \rightarrow \mathbb{R}^{\bar{n}-n}$ with $F(0) = 0$, $|DF|(0) = 0$, $|DF|^2(y') = \sum_{i,\gamma} (\partial F^\gamma / \partial y^i)(\partial F^\gamma / \partial y^i) \leq \delta^2$ such that $M_0 \cap \{|y'| < 1\} = \{(y', F(y')) \mid |y'| < 1\}$. Let $P : \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^n$ be the orthogonal projection into the first n -components. Let $\exp_{x_0}(\bar{y}) = \bar{x}$ and $\bar{y}' = P\bar{y}$. For $x \in B_{\bar{M}}(x_0, 4\sqrt{\varepsilon})$, let $\exp_{x_0}(y) = x$ and $y' = Py$. Since the curvature of \bar{M} is bounded by c_0^2 , by comparison theorem on the ball $B_{T_{x_0}\bar{M}}(o, 4\sqrt{\varepsilon})$, we have

$$(7.13) \quad d_{\bar{M}}(\bar{x}, x) \geq \frac{\sin(4c_0\sqrt{\varepsilon})}{4c_0\sqrt{\varepsilon}} |\bar{y}' - y'| \geq (1 - 3c_0^2\varepsilon) |\bar{y}' - y'| \geq (1 - 3c_0^2\varepsilon) |\bar{y}' - y'|.$$

On the other hand, also by comparison theorem, the Riemannian volume element dv of M_0 satisfies

$$(7.14) \quad \exp_{x_0}^* dv \leq \left[\frac{\sinh(c_0 d_{\bar{M}}(x_0, \cdot))}{c_0 d_{\bar{M}}(x_0, \cdot)} \right]^n dv_{\exp_{x_0}^{-1} M_0} \leq [1 + 16c_0^2 \varepsilon]^n dv_{\exp_{x_0}^{-1} M_0}$$

whenever $x \in M_0 \cap B_{\bar{M}}(x_0, 4\sqrt{\varepsilon})$. By definition, it is clear that

$$(7.15) \quad dv_{\exp_{x_0}^{-1} M_0} \leq (1 + |DF|^2)^{n/2} dy^1 \dots dy^n \leq (1 + \delta^2)^{n/2} dy^1 \dots dy^n.$$

Combining (7.13), (7.14) and (7.15), we have

$$\begin{aligned} & \int_{M_0 \cap B_{\bar{M}}(x_0, 4\sqrt{\varepsilon})} (4\pi\bar{t})^{-(n/2)} \exp \left[- \left(1 - \frac{\bar{t}}{\varepsilon} \right) \frac{d_{\bar{M}}^2(\bar{x}, x)}{4\bar{t}} \right] dv \\ & \leq (1 + \delta^2)^{n/2} (1 + 16c_0^2 \varepsilon)^n (1 - \varepsilon)^{-(n/2)} (1 - 3c_0^2 \varepsilon)^{-n} \\ & \quad \times \int_{(|y^1|^2 + \dots + |y^n|^2)^{1/2} < 4\sqrt{\varepsilon}} \left[\frac{4\pi\bar{t}}{(1 - \varepsilon)(1 - 3c_0^2 \varepsilon)^2} \right]^{-(n/2)} \\ & \quad \times \exp \left[-|y' - y|^2 / \frac{4\bar{t}}{(1 - \varepsilon)(1 - 3c_0^2 \varepsilon)^2} \right] dy^1 \dots dy^n \\ & \leq (1 + \delta^2)^{n/2} (1 + 16c_0^2 \varepsilon)^n (1 - \varepsilon)^{-(n/2)} (1 - 3c_0^2 \varepsilon)^{-n}. \end{aligned}$$

By (7.12) and the fact $\bar{t} \leq \varepsilon^2$, we conclude that

$$(1 + \delta^2)^{(n/2)} (1 + 16c_0^2 \varepsilon)^n (1 - \varepsilon)^{-(n/2)} (1 - 3c_0^2 \varepsilon)^{-n} (1 - 3n\varepsilon)^{-3} e^{n\varepsilon/2} \geq (1 + \beta),$$

which is a contradiction as $\varepsilon, \delta \rightarrow 0$. We complete the proof of the Theorem. □

Theorem 7.5. *Let \bar{M} be an \bar{n} -dimensional manifold satisfying $\sum_{i=0}^3 |\bar{\nabla}^i \bar{\text{Rm}}| \leq c_0^2$ and $\text{inj}(\bar{M}) \geq i_0 > 0$. Then there is $\varepsilon > 0$ with the following property. Suppose we have a smooth solution $M_t \subset \bar{M}$ to the MCF properly embedded in $B_{\bar{M}}(x_0, r_0)$ for $t \in [0, T]$, where $r_0 < (i_0/2)$, $0 < T \leq \varepsilon^2 r_0^2$. We assume that at time zero, $x_0 \in M_0$, and the second fundamental form satisfies $|A|(x) \leq r_0^{-1}$ on $M_0 \cap B_{\bar{M}}(x_0, r_0)$ and assume M_0 is graphic in the ball $B_{\bar{M}}(x_0, r_0)$. Then we have*

$$(7.16) \quad |A|(x, t) \leq (\varepsilon r_0)^{-1}$$

for any $x \in B_{\bar{M}}(x_0, \varepsilon r_0) \cap M_t$, $t \in [0, T]$.

Proof. By scaling we may assume $r_0 = 1$. By Lemma 7.1, for any $\delta > 0$, there is $0 < r_\delta < 1$ such that the connected component of $M_0 \cap B_{\bar{M}}(x_0, (1/96))$ containing x_0 contains a δ -Lipschitz graph of radius $2r_\delta$ at x_0 . By our graphic assumption, we conclude that $M_0 \cap B_{\bar{M}}(x_0, r_\delta)$ is a δ -Lipschitz graph. So Theorem 7.3 is applicable with radius r_δ .

Consequently, for any $\alpha > 0$, there exists an $\varepsilon_\alpha > 0$ such that

$$(7.17) \quad |A|(x, t)^2 \leq \frac{\alpha}{t} + \varepsilon_\alpha^{-2}$$

whenever $x \in M_t \cap B_{\bar{M}}(x_0, \varepsilon_\alpha)$, $t \in [0, \varepsilon_\alpha^2] \cap [0, T]$. Let α be a fixed small constant to be determined later. It turns out that we only need to choose $\alpha = \alpha(c_0, \bar{n}, n)$ finally. Choose $\varepsilon = \min\{\sqrt{\alpha}\varepsilon_\alpha, 10^{-1}\}$. Then by (7.17) we have

$$(7.18) \quad |A|(x, t)^2 \leq \frac{2\alpha}{t}$$

whenever $x \in M_t \cap B_{\bar{M}}(x_0, \varepsilon_\alpha)$, $t \in [0, \varepsilon^2] \cap [0, T]$.

Claim. $|A|(x, t) \leq \varepsilon^{-1}$ holds on $M_t \cap B_{\bar{M}}(x_0, \varepsilon)$, $t \in [0, \varepsilon^2] \cap [0, T]$.

Suppose $|A|(x_1, t_1) > \varepsilon^{-1}$ holds for some point (x_1, t_1) , $x_1 \in M_{t_1} \cap B_{\bar{M}}(x_0, \varepsilon)$, $t_1 \in [0, \varepsilon^2] \cap [0, T]$. We can choose another point (\bar{x}, \bar{t}) , $\bar{x} \in M_{\bar{t}} \cap B_{\bar{M}}(x_0, 4\varepsilon)$, $\bar{t} \in [0, \varepsilon^2] \cap [0, T]$ such that $Q = |A|(\bar{x}, \bar{t}) \geq \varepsilon^{-1}$ and

$$(7.19) \quad |A|(x, t) \leq 4Q$$

whenever $x \in M_t$, $d_{\bar{M}}(\bar{x}, x) \leq Q^{-1}$, $0 \leq t \leq \bar{t}$.

Actually (\bar{x}, \bar{t}) can be constructed as the limit of a finite sequence (x_i, t_i) satisfying $0 \leq t_k \leq t_{k-1}$, $d_{\bar{M}}(x_0, x_k) \leq d_{\bar{M}}(x_0, x_{k-1}) + |A|(x_{k-1}, t_{k-1})^{-1}$, $|A|(x_k, t_k) \geq 4|A|(x_{k-1}, t_{k-1})$. Since

$$|A|(x_k, t_k) \geq 4^{k-1}|A|(x_1, t_1) \geq 4^{k-1}\varepsilon^{-1},$$

$d_{\bar{M}}(x_0, x_k) \leq d_{\bar{M}}(x_0, x_1) + \sum_{i=1}^{\infty} (4^{i-1}|A|(x_1, t_1))^{-1} \leq 3\varepsilon < (1/2)$, and the solution is smooth, the sequence must be finite and the last element fits.

Note that $3n\bar{t}Q^2 \leq 6n\alpha \leq (1/2)$ by choosing $\alpha \leq (1/12n)$. Let $\psi = (1 - (d_{\bar{M}}^2(\bar{x}, \cdot) + 3nt/Q^{-2}))_+^3$, then we have

$$\left(\frac{\partial}{\partial t} - \Delta \right) \psi \leq 0$$

whenever $d_{\bar{M}}(\bar{x}, \cdot)^2 < \min\{(1/c_0^2 e), i_0^2\}$, $t \in [0, \bar{t}]$. On the other hand, by (3.2), the second fundamental form satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right) |A|^2 \leq -|\nabla A|^2 + C(\bar{n})|A|^4 + C(\bar{n})(1 + c_0^2)(|A|^2 + |A|).$$

Hence

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) (\psi|A|^2) &\leq -|\nabla A|^2 \psi + C(\bar{n})|A|^4 \psi + C(\bar{n})(1 + c_0^2)(|A|^2 + |A|)\psi \\ &\quad + 4|\nabla A||A||\nabla \psi| \\ &\leq C(\bar{n})|A|^4 \psi + C(\bar{n})(1 + c_0^2)(|A|^2 + |A|)\psi + 4\frac{|\nabla \psi|^2}{\psi}|A|^2 \\ &\leq C(\bar{n})|A|^4 \psi + C(\bar{n})(1 + c_0^2)(|A|^2 + |A|)\psi \\ (7.20) \quad &\quad + 144Q^2|A|^2\psi^{1/3} \end{aligned}$$

on $[0, \bar{t}]$. By (7.19) and (7.20), we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) (\psi|A|^2) \leq C(\bar{n})Q^4 + C(\bar{n})(1 + c_0^2)(Q + Q^2).$$

From the maximum principle, it follows

$$\begin{aligned} (\psi|A|^2)_{\max}|_{t=\bar{t}} &\leq 1 + C(\bar{n})Q^4\bar{t} + C(\bar{n})(1 + c_0^2)(Q + Q^2)\bar{t} \\ &\leq 1 + 2\alpha C(\bar{n})Q^2 + C(\bar{n})(1 + c_0^2)(\sqrt{2\alpha\bar{t}} + 2\alpha). \end{aligned}$$

Note that

$$(\psi|A|^2)_{\max}|_{t=\bar{t}} \geq \psi|A|^2(\bar{x}, \bar{t}) \geq (1 - 3nQ^2\bar{t})^3 Q^2 \geq (1 - 18n\alpha)Q^2,$$

hence we have

$$(1 - 18n\alpha)Q^2 \leq 1 + 2\alpha C(\bar{n})Q^2 + C(\bar{n})(1 + c_0^2)(\sqrt{2\alpha\bar{t}} + 2\alpha).$$

This implies

$$Q^2 \leq \frac{1 + C(\bar{n})(1 + c_0^2)(\sqrt{2\alpha} + 2\alpha)}{1 - (18n + 2C(\bar{n}))\alpha}.$$

Choosing suitable small $\alpha = \alpha(c_0, \bar{n}, n)$, we have $Q^2 \leq 2$, which is a contradiction with $Q^2 > \varepsilon^{-2}$. So the Claim is proved. \square

We remark that in the above theorem the condition that M_0 is graphic in the ball $B_{\bar{M}}(x_0, r_0)$ can be replaced by any one of the following conditions:

- (i) $d_{\bar{g}}(x, y) \geq (d_{g_0}(x, y)/C)$ for any $x, y \in M_0 \cap B_{\bar{M}}(x_0, r_0)$;
- (ii) there is a $\epsilon_0 > 0$ such that $B_{\bar{M}}(x_0, \epsilon r_0) \cap M_0$ is connected for any $\epsilon \leq \epsilon_0$.

Corollary 7.6. *Let \bar{M} be an \bar{n} -dimensional complete manifold satisfying $\sum_{i=0}^3 |\bar{\nabla}^i \bar{\text{Rm}}| \leq c_0^2$ and $\text{inj}(\bar{M}) \geq i_0 > 0$. Let $X_0 : M \rightarrow \bar{M}$ be an n -dimensional isometrically properly embedded submanifold with bounded second fundamental form $|A| \leq c_0$ in \bar{M} . We assume $M_0 = X_0(M)$ is uniform graphic with some radius $r > 0$. Suppose $X(x, t)$ is a smooth solution to the mean curvature flow (1.1) on $M \times [0, T_0]$ properly embedded in \bar{M} with X_0 as initial data. Then there is $T_1 > 0$ depending upon c_0, i_0, r and the dimension \bar{n} such that*

$$|A|(x, t) \leq 2c_0$$

for all $x \in M, 0 \leq t \leq \min\{T_0, T_1\}$.

Proof. By Theorem 7.5, there is $\epsilon > 0$ such that for any $x_0 \in M$, we have

$$|A|(x, t) \leq \epsilon^{-1}$$

on $B_{\bar{M}}(x_0, \epsilon), t \in [0, \epsilon^2] \cap [0, T]$. Let $[0, \gamma] \subset [0, \epsilon^2] \cap [0, T]$ be the maximal time interval so that the orbit of $x_0, x_0^t \in B_{\bar{M}}(x_0, \epsilon)$ for $t \in [0, \gamma]$. Then by the MCF equation, we know

$$\frac{d}{dt} d_{\bar{M}}(x_0, x_0^t) \leq C\epsilon^{-1},$$

for any $t \in [0, \gamma]$. This implies $\gamma \geq (\epsilon^2/C)$ for some $C = C(n, \bar{n})$. Choosing $\epsilon = (\epsilon/\sqrt{C}), T = \min\{T_0, \epsilon^2\}$, we conclude that the second fundamental forms are uniformly bounded by the constant ϵ^{-1} on $M \times [0, T]$. Once the second fundamental form is bounded, since we assumed $\sum_{i=0}^3 |\bar{\nabla}^i \bar{\text{Rm}}| \leq c_0^2$, we have gradient estimate $|\nabla A| \leq (C/\sqrt{t})$, and hence suitable linear growth function with bounded first and second derivatives can be constructed. Therefore we can apply the maximum principle to the equation of $|A|$ to conclude a uniform estimate $|A| \leq 2c_0$, for any $t \in [0, (1/C(\bar{n})c_0^2)]$. Set $T_1 = \min\{T, (1/C(\bar{n})c_0^2)\}$. The proof is completed. \square

Theorem 1.3 follows as a corollary of Theorem 1.1 and Corollary 7.6.

In [16], Huisken established his monotonicity formula, which was later generalized by Ecker–Huisken [11], Huisken [17], and to parabolic flows on Riemannian manifolds by Hamilton [15]. The remarkable localized monotonicity formula was discovered by Ecker [9].

Finally, we are acknowledged kindly by Prof. Klaus Ecker that part of our arguments of using monotonicity was also carried out before by B. White [25] (see [10]) in different situations. The choice of ψ in inequality (7.6) appeared first in [1] for the case of Euclidean space, see Remark 4.8 of the book [10].

Acknowledgment

We are grateful to Professor Xi-Ping Zhu for useful conversations and encouragement. We are also thankful for beneficial comments from Professor Klaus Ecker who brought our attention to some references. The second author would like to thank Professor Nai-Chung Leung and Professor Luen-Fai Tam for their constant teaching and encouragement, and Professor Mu-Tao Wang for very helpful discussions. The first author is partially supported by Grants 2005-34000-3171404 and 2006-34000-1131040.

References

- [1] K. A. Brakke, *The motion of a surface by its mean curvature*, Mathematical Notes, vol. **20**, Princeton University Press, Princeton, NJ, 1978.
- [2] H. D. Cao and X. P. Zhu, *A complete proof of the Poincaré and geometrization conjectures — application of the Hamilton–Perelman theory of the Ricci flow*, Asian J. Math. **10** (2006), no. 2, 165–492.
- [3] J. Cheeger and D. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland, Amsterdam, 1975.
- [4] B. L. Chen and X. P. Zhu, *Ricci flow with surgery on four manifolds with positive isotropic curvature*, J. Diff. Geom. **74** (2006), 177–264.
- [5] B. L. Chen and X. P. Zhu, *Uniqueness of the Ricci flow on complete noncompact manifolds*, J. Diff. Geom. **74** (2006), 119–154.
- [6] K. S. Chou and X. P. Zhu, *Shortening complete plane curves*, J. Diff. Geom. **50** (1998), 471–504.

- [7] D. De Turck, *Deforming metrics in the direction of their Ricci tensors*, J. Diff. Geom. **18** (1983), 157–162.
- [8] W. Y. Ding and F. H. Lin, *A generalization of Eells-Sampson’s theorem*, J. Partial Diff. Equations **5** (1992), no. 4, 13–22.
- [9] K. Ecker, *A local monotonicity formula for mean curvature flow*, Ann. Math. **154** (2001), 503–525.
- [10] K. Ecker, *Regularity theory for mean curvature flow*, Progress in Non-linear Differential Equations and their Applications, vol. 57, Birkäuser Boston, Inc., Boston, MA, 2004.
- [11] K. Ecker and G. Huisken, *Mean curvature evolution of entire graphs*, Ann. Math. **130** (1989), 453–471.
- [12] K. Ecker and G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), 547–569.
- [13] J. Eells and J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.
- [14] R. S. Hamilton, *A compactness property for solution of the Ricci flow*, Amer. J. Math. **117** (1995), 545–572.
- [15] R. S. Hamilton, *Monotonicity formulas for parabolic flows on manifolds*, Comm. Anal. Geom. **1** (1993), 127–137.
- [16] G. Huisken, *Asymptotic behaviour for singularities of the mean curvature flow*, J. Diff. Geom. **31** (1990), 285–299.
- [17] G. Huisken, *Local and global behaviour of hypersurfaces moving by mean curvature flow*, Proceedings of the Centre for Mathematics and its Applications, vol. 26, Australian National University, 1991.
- [18] H. B. Lawson, Jr. and R. Osserman, *Non existence, non uniqueness and irregularity of solutions to minimal surface system*, Acta Math. **139** (1977), nos 1–2 1–17.
- [19] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Uralceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, vol. 23, AMS, Providence, R.I., 1967, xi + 648 pp.
- [20] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Preprint, arXiv:math. DG/0211159 v1.

- [21] G. Perelman, *Ricci flow with surgery on three manifolds*, Preprint, 2003 arXiv:math. DG/0303109 v1.
- [22] R. Schoen and S. T. Yau, *Lectures on differential geometry*, Conference proceedings and Lecture Notes in Geometry and Topology, vol. **1**, International Press Publications, 1994.
- [23] R. Schoen and S. T. Yau, *Module space of harmonic maps, compact group actions and the topology of manifolds with non-positive curvature*, in 'Lectures on harmonic maps', International Press, 1997.
- [24] M .T. Wang, *The mean curvature flow smoothes Lipschitz submanifolds*, Comm. Anal. Geom. **12** (2004), no 3, 581–599.
- [25] B. White, *A local regularity theorem for classical mean curvature flow*, Preprint 2000.

DEPARTMENT OF MATHEMATICS
SUN YAT-SEN UNIVERSITY
GUANGZHOU 510275
P.R. CHINA
E-mail address: mcsobl@mail.sysu.edu.cn

INSTITUTE OF MATHEMATICAL SCIENCES AND DEPARTMENT OF MATHEMATICS
THE CHINESE UNIVERSITY OF HONG KONG
HONG KONG
P.R. CHINA
E-mail address: lyin@math.cuhk.edu.hk

RECEIVED FEBRUARY 20, 2006