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A theorem of Hopf and the Cauchy–Riemann inequality

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Recently, Abresch and Rosenberg [1] have extended Hopf's Theorem on constant mean curvature to 3-dimensional spaces other than the space forms. Here we show that, rather than assuming constant mean curvature, it suffices to assume an inequality on the differential of the mean curvature.

1. Introduction

In 1951, Hopf [9] published a theorem in a seminal paper on surfaces of constant mean curvature which can be stated as follows. *Let a genus zero compact surface M be immersed in $\mathbb{R}^3$ with constant mean curvature H. Then M is isometric to the standard sphere.* Hopf gave two proofs of this result (see [9] for details). Both proofs depend on the fact that any surface can be given isothermal parameters (u, v), i.e., $ds^2 = \lambda^2(du^2 + dv^2)$, where λ^2 is a function on M, so that M can be viewed as a Riemann surface with local parameter $z = u + iv$.

To fix the notation and for future reference, we will summarize both proofs.

In the first proof, one considers the second fundamental form α in isothermal parameters and takes the $(2, 0)$-component of α, $\alpha^{(2,0)} = (1/2)\psi\, dz\, dz$. It can be shown that the complex function ψ is holomorphic iff $H =$ const. and that the zeroes of ψ are the umbilic points in M. It is also seen that the quadratic form $\alpha^{(2,0)}$ does not depend on the parameter z; hence, it is globally defined on M. It is a known theorem on Riemann surfaces that if the genus g of M is zero, any holomorphic quadratic form vanishes identically. Then $\psi \equiv 0$, i.e., all points of M are umbilics, and hence M is a standard sphere.

From our point of view, the second proof is even more interesting. The quadratic equation $\mathrm{Im}(\psi dz^2) = 0$ determines two fields of directions (the principal directions) the singularities of which are the zeroes of ψ. Since ψ

is holomorphic, if z_0 is a zero of ψ, either $\psi \equiv 0$ in a neighborhood V of z_0 or

$$\psi(z) = (z - z_0)^k f_k(z), \;\; z \in V, \;\; k \geq 1,$$

where f_k is a function of z with $f_k(z_0) \neq 0$. It follows that z_0 is an isolated singularity of the field of directions and its index is $(-k/2)$. Thus, either $\alpha^{(2,0)} \equiv 0$ on M, and we have a standard sphere, or all singularities are isolated and have negative index. Since $g = 0$, the sum of the indices of all singularities for any field of directions is two (hence positive). This is a contradiction, so $\alpha^{(2,0)} \equiv 0$ on M.

Notice that in the second proof the fact that ψ is holomorphic is only used to show that the index of an isolated singularity of the field of directions is negative and that either $\psi \equiv 0$ or the zeroes of ψ are isolated.

Recently, Abresch and Rosenberg [2] considered a surface M immersed in $M^2(c) \times \mathbb{R}$, where $M^2(c)$ is a (complete simply connected) 2-dimensional Riemannian manifold with constant curvature c and introduced on M the quadratic form

$$Q(X, Y) = 2H\alpha(X, Y) - c\langle \xi X, \xi Y \rangle; \tag{1.1}$$

here, X and Y are tangents vectors to M and $\xi \colon M^2(c) \times \mathbb{R} \to \mathbb{R}$ is the natural projection onto $\mathbb{R}$, i.e., $\xi(p, t) = t$, $p \in M^2(c)$, $t \in \mathbb{R}$; we have, for notational simplicity, identified ξ with its differential $d\xi$. Let $Q^{(2,0)}$ be the $(2, 0)$-component of Q.

Abresch and Rosenberg proved that $Q^{(2,0)}$ is holomorphic if $H =$ const. on M, and if M is a genus zero compact surface, then M is an embedded surface invariant by rotations in $M^2(c) \times \mathbb{R}$ (see [1] and the references there for details).

The goal of the present paper is to show that the above result still holds if H is not necessarily constant but its differential satisfies a certain inequality. More precisely, we prove

Theorem 1.1. *Let M be a compact immersed surface of genus zero in $M^2(c) \times \mathbb{R}$. Assume that*

$$|dH| \leq g|Q^{(2,0)}|,$$

where $|dH|$ is the norm of the differential dH of the mean curvature H of M, and g is a continuous, non-negative real function. Then $Q^{(2,0)}$ is identically zero, and M is an embedded surface invariant by rotations in $M^2(c) \times \mathbb{R}$.

A crucial point in our proof is to observe, as already noticed by Eschenburg and Tribuzy [6], that the second proof of Hopf uses only a weak notion of holomorphy and try to enclose that in a form that we can use. The outcome is the following lemma which is an adaptation of results of Chern [5] and Eschenburg and Tribuzy [6].

Main Lemma. *Let $f\colon U \subset \mathbb{C} \to \mathbb{C}$ be a complex function defined in an open set U of the complex plane. Assume that*

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq h(z)\,|f(z)|, \tag{1.2}$$

where h is a continuous, non-negative real function. Assume further that $z = z_0 \in U$ is a zero of f. Then either $f \equiv 0$ in a neighborhood $V \subset U$ of z_0, or

$$f(z) = (z - z_0)^k\, f_k(z), \;\; z \in V, \;\; k \geq 1,$$

where $f_k(z)$ is a continuous function with $f_k(z_0) \neq 0$.

Thus condition (1.2) implies the weak condition of holomorphy that is used in Hopf's second proof. Following Eschenburg and Tribuzy [6], we call (1.2) a *Cauchy–Riemann inequality.*

We will prove the Main Lemma in Section 3 of this paper.

It is interesting to observe the context in which the work [5] of Chern was written. In Hopf's paper of 1951, it was also proved that the theorem would hold for special Weingarten surfaces M provided they were analytic. A *Weingarten surface* is a surface for which there exists a functional relation between the principal curvatures: $W(k_1, k_2) = 0$. If this relation can be solved in, say, k_2 and $dk_2/dk_1 = -1$, when $k_1 = k_2$, we say that the Weingarten surface is *special*, a notion introduced by Chern [4] in 1945.

The requirement that the special Weingarten surfaces were analytic for the validity of Hopf's theorem was removed by Hartman and Wintner [8] in 1954. Their paper is somewhat long. In the following year, Chern gave in [5] a short and elegant proof of the result of Hartman and Wintner. Chern's proof depends on a lemma from which our Main Lemma is an adaptation.

2. Proof of Theorem 1.1 assuming the Main Lemma

Let (u, v) be isothermal parameters in an open set $U \subset M$ and set $z = u + iv$, $dz = 1/\sqrt{2}(du + i\,dv)$, $d\bar{z} = 1/\sqrt{2}(du - i\,dv)$. Set $Q^{(2,0)} = (1/2)\psi(z)dz\,dz$

and assume that there exists a point $z_0 \in U$ such that $\psi(z_0) = 0$. Set

$$Z = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \overline{Z} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Since (u,v) are isothermal parameters, $\langle Z, \overline{Z}\rangle = \lambda^2$, where $\lambda^2 = \langle \partial/\partial u, \partial/\partial u\rangle = \langle \partial/\partial v, \partial/\partial v\rangle$. Notice that $Q(Z,Z) = \psi(z)$ and set $|Q(Z,Z)| = |\psi(z)|$.

For future purposes, it will be convenient to consider the more general ambient space $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is a Riemannian manifold with constant sectional curvature c. Let $M \hookrightarrow M^n(c) \times \mathbb{R}$ be an immersed surface and set, in this new situation,

$$Q(X,Y) = 2\langle \alpha(X,Y), \vec{H}\rangle - c\langle \xi X, \xi Y\rangle, \tag{2.1}$$

where $\vec{H} = HN$ is the mean curvature vector of the immersion, α is the normal-valued second fundamental form and ξ: $M^n(c) \times \mathbb{R} \to \mathbb{R}$ is again the projection $\xi(p,t) = t$.

We first want to compute

$$\frac{\partial \psi}{\partial \bar{z}} = \overline{Z}\, Q(Z,Z) = 2\overline{Z}\langle \alpha(Z,Z), \vec{H}\rangle - c\,\overline{Z}\langle \xi Z, \xi Z\rangle.$$

The first term yields

$$2\overline{Z}\langle \alpha(Z,Z), \vec{H}\rangle = 2\langle \nabla^{\perp}_{\overline{Z}}\, \alpha(Z,Z), \vec{H}\rangle + 2\langle \alpha(Z,Z), \nabla^{\perp}_{\overline{Z}}\vec{H}\rangle.$$

By definition,

$$\big(\nabla^{\perp}_{\overline{Z}}\alpha\big)(Z,Z) = \nabla^{\perp}_{\overline{Z}}\alpha(Z,Z) - 2\alpha\langle \nabla_{\overline{Z}}Z, Z\rangle = \nabla^{\perp}_{\overline{Z}}\alpha(Z,Z),$$

since, as it is easily checked, $\nabla_{\overline{Z}}Z = 0$. Thus

$$2\overline{Z}\langle \alpha(Z,Z), \vec{H}\rangle = 2\langle (\nabla^{\perp}_{\overline{Z}}\alpha)(Z,Z), \vec{H}\rangle + 2\langle \alpha(Z,Z), \nabla^{\perp}_{\overline{Z}}\vec{H}\rangle,$$

hence, by using Codazzi equation, and denoting by $\widetilde{R}$ the curvature of the ambient space,

$$2\overline{Z}\langle \alpha(Z,Z), \vec{H}\rangle = 2\langle \nabla^{\perp}_{Z}\, \alpha(\overline{Z},Z), \vec{H}\rangle + 2\langle \widetilde{R}(\overline{Z},Z)Z, \vec{H}\rangle + 2\langle \alpha(Z,Z), \nabla^{\perp}_{\overline{Z}}\vec{H}\rangle.$$

We now need the following.

Lemma 2.1. *With the above notation,*

$$\langle \widetilde{R}(\overline{Z}, Z)Z, \vec{H}\rangle = c\lambda^2\langle \xi Z, \xi\vec{H}\rangle.$$

Proof of Lemma 2.1. Let $\pi\colon M^n(c)\times\mathbb{R}\to M^n(c)$ be defined by $\pi(p,t)=p$ and identifying, for convenience of notation, π and ξ with their differentials, we obtain that $X=\pi X+\xi X$. Since the ambient space is a product space, we have for its curvature $\widetilde{R}$

$$\langle \widetilde{R}(Z,\overline{Z})Z,\vec{H}\rangle = c\big\{\langle \pi\overline{Z},\pi Z\rangle\langle \pi Z,\pi\vec{H}\rangle - \langle \pi Z,\pi Z\rangle\langle \pi\overline{Z},\pi\vec{H}\rangle\big\}.$$

Let us compute the various terms of $\widetilde{R}$,

$$\begin{aligned}\langle \pi\overline{Z},\pi Z\rangle &= \langle \overline{Z}-\xi\overline{Z}, Z-\xi Z\rangle\\ &= \langle \overline{Z}, Z\rangle - \langle \overline{Z},\xi Z\rangle - \langle \xi\overline{Z}, Z\rangle + \langle \xi\overline{Z},\xi Z\rangle\\ &= \lambda^2 - \langle \pi\overline{Z}+\xi\overline{Z},\xi Z\rangle - \langle \xi\overline{Z},\pi Z+\xi Z\rangle + \langle \xi\overline{Z},\xi Z\rangle\\ &= \lambda^2 - 2\langle \xi\overline{Z},\xi Z\rangle + \langle \xi\overline{Z},\xi Z\rangle = \lambda^2 - \langle \xi\overline{Z},\xi Z\rangle.\end{aligned}$$

On the other hand, since $\langle Z,N\rangle=0$, we have

$$\begin{aligned}\langle \pi Z,\pi\vec{H}\rangle &= \langle Z-\xi Z,\vec{H}-\xi\vec{H}\rangle\\ &= -\langle \xi Z,\vec{H}\rangle - \langle Z,\xi\vec{H}\rangle + \langle \xi Z,\xi\vec{H}\rangle\\ &= -\langle \xi Z,\xi\vec{H}\rangle.\end{aligned}$$

Thus, the first term in the right-hand side becomes

$$\begin{aligned}\langle \pi\overline{Z},\pi Z\rangle\langle \pi Z,\pi\vec{H}\rangle &= -(\lambda_2 - \langle \xi\overline{Z},\xi Z\rangle)(\langle \xi Z,\xi\vec{H}\rangle)\\ &= -\lambda^2\langle \xi Z,\xi\vec{H}\rangle + \langle \xi\overline{Z},\xi Z\rangle\langle \xi Z,\xi\vec{H}\rangle.\end{aligned}$$

For the second term, we have, since $\langle Z,Z\rangle=0$,

$$\begin{aligned}\langle \pi Z,\pi Z\rangle &= \langle Z-\xi Z, Z-\xi Z\rangle\\ &= -\langle Z,\xi Z\rangle - \langle \xi Z, Z\rangle + \langle \xi Z,\xi Z\rangle = -\langle \xi Z,\xi Z\rangle\end{aligned}$$

and

$$\langle \pi\overline{Z},\pi\vec{H}\rangle = \langle \overline{Z}-\xi\overline{Z},\vec{H}-\xi\vec{H}\rangle = -\langle \xi\overline{Z},\xi\vec{H}\rangle,$$

hence

$$\langle \pi Z,\pi Z\rangle\langle \pi\overline{Z},\pi\vec{H}\rangle = \langle \xi Z,\xi Z\rangle\langle \xi\overline{Z},\xi\vec{H}\rangle.$$

It follows that

$$\begin{aligned}\langle \widetilde{R}(Z,\overline{Z})Z,\vec{H}\rangle &= c\big\{(-\lambda^2\langle \xi Z,\xi\vec{H}\rangle + \langle \xi\overline{Z},\xi Z\rangle\langle \xi Z,\xi\vec{H}\rangle \\ &\qquad - \langle \xi Z,\xi Z\rangle\langle \xi\overline{Z},\xi\vec{H}\rangle\big\} = -c\lambda^2\langle \xi Z,\xi\vec{H}\rangle,\end{aligned}$$

where we have taken into account that $\langle \xi X,\xi Y\rangle$ is $\pm|\xi X|\,|\xi Y|$. Finally, by the usual symmetries of the curvature tensor, we conclude that

$$\langle \widetilde{R}(\overline{Z},Z)Z,\vec{H}\rangle = c\lambda^2\langle \xi Z,\xi\vec{H}\rangle.$$

□

Back to the computation of $\overline{Z}Q(Z,Z)$, we have

$$\begin{aligned}\overline{Z}Q(Z,Z) &= 2\langle(\nabla^{\perp}_{\overline{Z}}\,\alpha)(\overline{Z},Z),\vec{H}\rangle + 2c\lambda^2\langle \xi Z,\xi\vec{H}\rangle \\ &\quad + 2\langle\alpha(Z,Z),\nabla^{\perp}_{\overline{Z}}\,\vec{H}\rangle - c\overline{Z}\langle \xi Z,\xi Z\rangle.\end{aligned}$$

To simplify the above expression, we need another lemma.

Lemma 2.2. $c\overline{Z}\langle \xi Z,\xi Z\rangle = 2c\lambda^2\langle \xi Z,\xi\vec{H}\rangle$.

Proof of Lemma 2.2. We first observe that

$$\alpha(Z,\overline{Z}) = \overline{\nabla}_{\overline{Z}}Z - (\nabla_{\overline{Z}}\,Z)^{\mathrm{T}} = \overline{\nabla}_{\overline{Z}}Z,$$

since that, by a simple computation, it is seen that $\nabla_{\overline{Z}}\,Z = 0$. Since the ambient space is a product $M^n(c)\times\mathbb{R}$ with natural projections π and ξ, we can write

$$\overline{\nabla}_{\overline{Z}}Z = \overline{\nabla}_{\overline{Z}}(\xi Z + \pi Z) = \nabla^1_{\overline{Z}}(\xi Z) + \nabla^2_{\overline{Z}}(\pi Z)$$

where ∇^1 and ∇^2 are the connections of $\mathbb{R}$ and $M^n(c)$, respectively. Thus,

$$\xi\,\alpha(Z,\overline{Z}) = \xi\,\overline{\nabla}_{\overline{Z}}\,Z = \xi\,\nabla^1_{\overline{Z}}(\xi Z) + \xi\,\nabla^2_{\overline{Z}}(\pi Z) = \nabla^1_{\overline{Z}}(\xi Z),$$

hence

$$\langle \xi\,\alpha(Z,\overline{Z}),\xi Z\rangle = \langle \nabla^1_{\overline{Z}}(\xi Z),\xi Z\rangle = \langle \nabla_{\overline{Z}}(\xi Z),\xi Z\rangle.$$

Now let us compute $\overline{Z}\langle \xi Z,\xi Z\rangle$.

$$\overline{Z}\langle \xi Z,\xi Z\rangle = 2\langle \overline{\nabla}_{\overline{Z}}(\xi Z),\xi Z\rangle = 2\langle \xi\,\alpha(Z,\overline{Z}),\xi Z\rangle.$$

Set $E = 1/\sqrt{2}\,(e_1 - ie_2)$, where e_1 and e_2 are the unit vectors of $\partial/\partial u$, $\partial/\partial v$, respectively. Thus, $Z = \lambda E$ and

$$\begin{aligned}\alpha(Z,\overline{Z}) = \lambda^2\alpha(E,\overline{E}) &= \lambda^2\alpha\left(\frac{e_1 - ie_2}{\sqrt{2}}, \frac{e_1 + ie_2}{\sqrt{2}}\right)\\ &= \frac{\lambda^2}{2}\{\alpha(e_1,e_1) + \alpha(e_2,e_2)\} = \lambda^2\vec{H},\end{aligned}$$

hence

$$\overline{Z}\langle\xi Z, \xi\overline{Z}\rangle = 2\langle\nabla_{\overline{Z}}(\xi Z), \xi Z\rangle = 2\langle\xi\alpha(Z,\overline{Z}), \xi Z\rangle = 2\lambda^2\langle\xi\vec{H}, \xi Z\rangle$$

and this proves Lemma 2.2. □

It follows from Lemma 2.2 that

$$\overline{Z}\,Q(Z,Z) = 2\langle(\nabla^{\perp}_{\overline{Z}}\,\alpha)(\overline{Z},Z), \vec{H}\rangle + 2\langle\alpha(Z,Z), \nabla^{\perp}_{\overline{Z}}\,\vec{H}\rangle.$$

The first term in the right-hand side can be computed as follows. By definition,

$$\begin{aligned}(\nabla^{\perp}_{\overline{Z}}\,\alpha)(\overline{Z},Z) &= \nabla^{\perp}_{\overline{Z}}(\alpha(\overline{Z},Z)) - \alpha(\nabla_{Z}\overline{Z},Z)\\ -\alpha(\overline{Z},\nabla_{Z}Z) &= Z(\langle\overline{Z},Z\rangle\vec{H}) - \alpha(\overline{Z},\nabla_{Z}Z),\end{aligned}$$

where we have used that $\alpha(Z,\overline{Z}) = \lambda^2\vec{H}$ (see Lemma 2.2) and that $\nabla_{Z}\overline{Z} = 0$. Thus

$$(\nabla^{\perp}_{\overline{Z}}\,\alpha)(\overline{Z},Z) = \langle\nabla_{Z}\overline{Z},Z\rangle\vec{H} + \langle\overline{Z},\nabla_{Z}Z\rangle\vec{H} + \langle\overline{Z},Z\rangle\nabla^{\perp}_{\overline{Z}}\,\vec{H} - \alpha(\overline{Z},\nabla_{Z}Z).$$

Now let E as defined in Lemma 2.2. Then any complex vector X on M is given by $X = \xi E$, where ξ is a complex number. Thus if $Y = \eta\,E$, we obtain

$$\alpha(X,\overline{Y}) = \xi\overline{\eta}\,\alpha(E,\overline{E}) = \langle X,\overline{Y}\rangle\vec{H}.$$

Setting in the above $X = \nabla_{Z}Z$ and $Y = Z$, we have

$$(\nabla^{\perp}_{\overline{Z}}\,\alpha)(\overline{Z},Z) = \langle\overline{Z},Z\rangle\nabla^{\perp}_{\overline{Z}}\,\vec{H} + \langle\overline{Z},\nabla_{Z}Z\rangle\vec{H} - \langle\nabla_{Z}Z,\overline{Z}\rangle\vec{H} = \langle Z,\overline{Z}\rangle\nabla^{\perp}_{\overline{Z}}\,\vec{H}.$$

Coming back to $\overline{Z}\,Q(Z,Z)$, we obtain finally

$$\overline{Z}\,Q(Z,Z) = 2\langle\langle Z,\overline{Z}\rangle\nabla^{\perp}_{\overline{Z}}\,\vec{H}, \vec{H}\rangle + 2\langle\alpha(Z,Z), \nabla^{\perp}_{\overline{Z}}\,\vec{H}\rangle, \tag{2.2}$$

where the right-hand side of the equality is expressed in terms of the covariant derivatives of the mean curvature vector.

Remark 2.3. At this point, we have obtained the following generalization of Theorem 1 of Abresch and Rosenberg [2]. *Let M be an immersed surface in $M^n(c) \times \mathbb{R}$ such that its mean curvature vector $\vec{H}$ is parallel in the normal bundle. Introduce a complex structure in M compatible with the induced metric. Then the $(2,0)$-part of the quadratic form on M*

$$Q(X,Y) = 2\langle \alpha(X,Y), \vec{H}\rangle - c\langle \xi X, \xi Y\rangle$$

is holomorphic. Here α is the second quadratic form of the immersion and ξ is the projection of the ambient space on the factor $\mathbb{R}$. It is a natural question to ask which surfaces in $M^n(c) \times \mathbb{R}$ satisfy the condition that the above quadratic form is holomorphic.

Back to the proof of our Theorem 1.1, we now specialize for $n = 2$ expression (2.2). Since the codimension is now one, $\nabla^{\perp}_{X} N = 0$, for all $X \in TM$, where N is the unit normal vector to the surface M. Thus

$$\nabla^{\perp}_{\overline{Z}} \vec{H} = \nabla^{\perp}_{\overline{Z}}(HN) = (\overline{Z}N)N,$$

and we obtain

$$\overline{Z}\, Q(Z,Z) = 2\lambda^2 Z(H)H + 2\alpha(Z,Z)(\overline{Z}H).$$

Since

$$|Z(H)| = |dH(Z)| \leq |dH|\,|Z| = |dH|\lambda,$$

and similarly for $|\overline{Z}(H)|$, we have

$$|\overline{Z}\, Q(Z,Z)| \leq \big\{2\lambda^3|H| + 2\lambda|\alpha(Z,Z)|\big\}|dH|.$$

By the hypothesis of the Theorem 1.1, $|dH| \leq g|Q^{(2,0)}|$. Thus, we obtain

$$|\overline{Z}\, Q(Z,Z)| \leq h(z)\,|Q(Z,Z)|,$$

where h is continuous and non-negative. Then we can apply the Main Lemma and we obtain the following.

Let $U \subset M$ be an open set covered by isothermal coordinates. Assume that the set of zeros of $Q(Z,Z)$ in U is not empty and let $z_0 \in U$ be a zero of $Q(Z,Z)$. By the Main Lemma, either $Q(Z,Z)$ is identically zero in a neighborhood V of z_0 or this zero is isolated and the index of a direction field determined by $\mathrm{Im}[Q(Z,Z)dz^2] = 0$ is $(-k/2)$ (hence negative). If, for some coordinate neighborhood V of zero, $Q(Z,Z) \equiv 0$, this will be so for the whole M; otherwise, the zeroes on the boundary of V will contradict the Main Lemma. So if $Q(Z,Z)$ is not identically zero, all zeroes are isolated and have negative indices. Since M has genus zero, the sum of the indices of the singularities of any field of directions is 2 (hence positive). This contradiction shows that $Q(Z,Z)$ is identically zero. Using the classification result of Abresch–Rosenberg, Theorem 1.1 follows.

Remark 2.4. The result of Abresch and Rosenberg applies only to genus zero compact surfaces with constant mean curvature. On the other hand, our result states that either the set of zeroes of $Q^{(2,0)}$ is empty, or in a neighborhood V of a zero of $Q^{(2,0)}$ where the condition $|dH| \leq g|Q^{(2,0)}|$ holds, we have two possibilities: (1) $Q^{(2,0)} \equiv 0$ in V or (2) such a zero is an isolated critical point of one of the direction fields given by $\mathrm{Im}[Q(Z,Z)dz^2] = 0$ whose index is negative (these two direction fields are orthogonal since their unit vectors diagonalize the real quadratic form $Q(X,Y)$ in Equation (1)). In this form, the result can be applied to an immersion of a genus one surface M satisfying $|dH| \leq |Q^{(2,0)}|$. Since no torus appears in the classification of surfaces with $|Q^{(2,0)}| \equiv 0$ [1, Theorem 3], and by Poincaré theorem, no isolated zeroes can occur, we conclude that there are no singularities in the field of directions given by $[\mathrm{Im}\, Q(Z,Z)dz^2] = 0$. It follows that there exists a global adapted frame in M that diagonalizes Q (cf. [2, Theorem 3], where a similar result is obtained in $\mathbb{R}^3$ from a different hypothesis).

Remark 2.5. Our theorem also applies to the non-compact case. Consider a disk-type surface Σ with smooth boundary $\partial\Sigma$ satisfying the following conditions: (1) Σ *is regular up to the boundary, i.e., there exists a smooth surface* $\hat{\Sigma} \supset \Sigma \cup \partial\Sigma$; (2) *the boundary* $\partial\Sigma$ *satisfies the equation* $[\mathrm{Im}\, Q(Z,Z)dz^2] = 0$; (3) *in* $\hat{\Sigma}$, $dH \leq g|Q^{(2,0)}|$. Then, by using the Main Lemma and the computations of the present paper, it can be shown that Σ is one of the surfaces that have $|Q^{(2,0)}| \equiv 0$ and are described in Theorem 3 of [1]. If we include the possibility that $\partial\Sigma$ has nonregular points (vertices), we must add the condition; (4) *the number of vertices that have angles* $< \pi$ *is at most* 3, and the result is the same. This is related to a result of Choe in $\mathbb{R}^3$ [10]. Proofs will be given in a forthcoming paper by one of us (M. do Carmo) and I. Fernandez.

3. Proof of the Main Lemma

We can assume that the zero of f is the origin 0 and that U is a disk D of radius R and center 0. We need some auxiliary lemmas. We follow Chern [5] (see also Hartman and Wintner [7] and Carleman [3]).

Lemma 3.1. *Assume the hypothesis of the Main Lemma and the fact that* $\lim_{z\to 0} f(z)/z^{k-1} = 0$*;* $k \geq 1$*. Then* $\lim_{z\to 0} f(z)/z^k$ *exists.*

Lemma 3.2. *Under the hypothesis of the Main Lemma assume that* $\lim_{z\to 0} f(z)/z^{k-1} = 0$; *for all* $k \geq 1$*. Then* $f \equiv 0$ *in some neighborhood of* 0.

From these two lemmas, the Main Lemma follows. Indeed, from Lemma (3.2) we obtain that if f is not identically zero in a neighborhood of 0, there exists a k such that $\lim_{z\to 0} f(z)/z^{k-1} = 0$ but $\lim_{z\to 0} f(z)/z^k$ may not exist. By Lemma (3.1), we know that $\lim_{z\to 0} f(z)/z^k$ exists, hence is non-zero, say c. Thus, we can write

$$f(z) = c\, z^k + R, \quad \lim_{z\to 0} \frac{R}{z^k} = 0$$

or

$$f(z) = z^k f_k(z), \quad f_k(z) = c + \frac{R}{z^k},$$

so that $f_k(0) = c \neq 0$, and this proves our claim.

It remains to prove Lemmas (3.1) and (3.2).

Proof of Lemma 3.1. From now on, we denote by $D_c(\zeta)$ a disk in the plane $\mathbb{C}$ with center ζ and radius c. Let $w \in D_R(0)$, $w \neq 0$, and in $W = D_R(0) - \{D_a(0) \cup D_a(w)\}$ define a differential form

$$\varphi = \frac{f(z)}{z^r(z-w)}\, dz.$$

Since $1/z^r(z-w)$ is holomorphic in W, we obtain

$$d\varphi = \frac{\partial \varphi}{\partial \bar{z}}\, d\bar{z} \wedge dz = -\frac{1}{z^r(z-w)} \frac{\partial f}{\partial \bar{z}}\, dz \wedge d\bar{z}.$$

Now take disks $D_a(0)$ and $D_a(w)$ in $D_R(0)$ and apply Stokes theorem,

$$\text{(3.1)} \qquad \iint_W d\varphi + \int_{\partial D_R(0)} \varphi - \int_{\partial D_a(0)} \varphi - \int_{\partial D_a(w)} \varphi = 0.$$

Let us compute explicitly the integrals in (3.1). Set $g(z) = f(z)/z^r$ and $z = w + ae^{i\theta}$, $0 \leq \theta \leq 2\pi$. Then

$$\int_{\partial D_a(w)} \varphi = \int_{\partial D_a(w)} \frac{g(z)}{(z-w)}\, dz = \int_0^{2\pi} \frac{g(w+ae^{i\theta})}{ae^{i\theta}}\, aie^{i\theta}\, d\theta,$$

and

$$\lim_{a\to 0} \int_{\partial D_a(w)} \varphi = ig(w) \int_0^{2\pi} d\theta = 2\pi i\, f(w) w^{-r}.$$

Next, set $z = ae^{i\theta}$ and, since by hypothesis, $\lim_{z\to 0} f(z)/z^{r-1} = 0$, we obtain

$$\lim_{a\to 0} \int_{\partial D_a(0)} \varphi = \lim_{a\to 0} \int_0^{2\pi} \frac{f(ae^{i\theta})\, i\, d\theta}{a^{r-1} e^{(r-1)i\theta}(ae^{i\theta} - w)} = 0.$$

It follows, by taking the limits in (1) when $a \to 0$, that

$$(3.2)\quad -2\pi i\, f(w) w^{-r} + \int_{\partial D_R(0)} \frac{f(z)dz}{z^r(z-w)} = \iint_{D_R(0)} \frac{1}{z^r(z-w)} \frac{\partial f}{\partial \bar{z}}\, dz \wedge d\bar{z},$$

where the limit in the double integral exists, because the left-hand side is well defined.

Since the function h in the statement of the Main Lemma is continuous, there exists $A > 0$ such that

$$\max_{z \in D_R(0)} h(z) \leq A.$$

Then, it follows from (3.2),

$$(3.3)\quad 2\pi\, |f(w)w^{-r}| \leq \int_{\partial D_R(0)} \frac{|f(z)|\, |dz|}{|z^r|\, |z-w|} + \iint_{D_R(0)} \frac{2A|f(z)|}{|z|^r\, |z-w|}\, du \wedge dv,$$

since $dz \wedge d\bar{z} = -2i\, du \wedge dv$, $z = u + iv$.

Now take $z_0 \in D$ with $z_0 \neq 0$, multiply the above inequality by $1/|w - z_0|$, and integrate it relative to $dx \wedge dy$, where $w = x + iy$. Then, by setting $D_\varepsilon = D_R(0) - D_\varepsilon(z_0)$, we have,

$$\int_{D_\varepsilon} \frac{2\pi |f(w)w^{-r}|}{|w - z_0|}\, dx \wedge dy \leq \int_{D_\varepsilon} \int_{\partial D_R(0)} \frac{|f(z)||dz|}{|z|^r |z-w||w-z_0|}\, dx \wedge dy$$

$$(3.4)\qquad + \int_{D_\varepsilon} \iint_{D_R(0)} \frac{2A|f(z)| du \wedge dv}{|z|^r|z-w||w-z_0|}\, dx \wedge dy$$

We want to estimate the integrals in (3.4). For this, we first observe that

$$\frac{1}{|z-w|\,|w-z_0|} = \frac{1}{|z-z_0|}\left|\frac{1}{(z-w)} + \frac{1}{(w-z_0)}\right| \tag{3.5}$$

and that

$$\int_{D_R(0)} \frac{dx\wedge dy}{|z-w|} \le \int_{D_{2R}(z)} \frac{dx\wedge dy}{|z-w|} = \int_0^{2R}\int_0^{2\pi} \frac{\rho\, d\theta\, d\rho}{\rho} = 4\pi R. \tag{3.6}$$

It follows that, for the first term on the right-hand side of inequality (3.4), we obtain

$$\begin{aligned}\int_{D_\varepsilon}\int_{\partial D_R(0)} \frac{|f(z)|\,|dz|}{|z|^r\,|z-w|\,|w-z_0|}\, dx\wedge dy &\le \int_{D_\varepsilon}\int_{\partial D_R(0)} \frac{|f(z)|\,|dz|}{|z|^r\,|z-z_0|}\,\frac{dx\wedge dy}{|w-z_0|}\\ &+ \int_{D_\varepsilon}\int_{\partial D_R(0)} \frac{|f(z)|\,|dz|}{|z|^r\,|z-z_0|}\,\frac{dx\wedge dy}{|z-w|} \le 8\pi R\int_{\partial D_R(0)} \frac{|f(z)|\,|dz|}{|z|^r\,|z-z_0|},\end{aligned}$$

where we have used (3.5) and (3.6). Similarly, for the second term on the right-hand side of (3.4), we obtain

$$\begin{aligned}&2A\int_{D_\varepsilon}\iint_{D_R(0)} \frac{|f(z)|}{|z|^r}\,\frac{du\wedge dv}{|z-w|\,|w-z_0|}\, dx\wedge dy\\ &\le 16A\pi R\iint_{D_R(0)} \frac{|f(z)|}{|z|^r\,|z-z_0|}\, du\wedge dv.\end{aligned}$$

Thus, we can write the inequality (3.4) as

$$\begin{aligned}2\pi\int_{D_\varepsilon} \frac{|f(w)|\,|w|^{-r}}{|w-z_0|}\, dx\wedge dy &\le 8\pi R\int_{\partial D_R(0)} \frac{f(z)\,|dz|}{|z|^r\,|z-z_0|}\\ &+ 16A\pi R\iint_{D_R(0)} \frac{|f(z)|du\wedge dv}{|z|^r\,|z-z_0|}\end{aligned}$$

or

$$(1-8AR)\iint_{D_R(0)} \frac{|f(z)|}{|z|^r\,|z-z_0|}\, du\wedge dv \le 4R\int_{\partial D_R(0)} \frac{|f(z)|\,|dz|}{|z|^r\,|z-z_0|}. \tag{3.7}$$

Since A does not change if R decreases, we can choose R small enough so that $1-8AR>0$.

Now, the integral in the right-hand side of (3.7) is bounded as $z_0 \to 0$; hence, the same holds for the integral in the left-hand side. Since its integrand increases monotonically as $z_0 \to 0$, we have that

$$\lim_{z_0 \to 0} \iint_{D_R(0)} \frac{f(z)}{|z|^r |z - z_0|} \, du \wedge dv$$

exists. It follows from (3.3) and that $f(w)w^{-r}$ is bounded when $w \to 0$. Thus, from (3.2), we conclude that $f(w)w^{-r}$ exists, as we wished. □

Proof of Lemma 3.2. Assume that f is not identically zero in a neighborhood of 0 and let z_0 be such that $f(z_0) \neq 0$, $|z_0| < R$.

Now by multiplying the inequality (3.3) by $dx \wedge dy$ and integrating, we obtain

$$\begin{aligned} &2\pi(1 - 8AR) \iint_{D_R(0)} |f(w)| \, |w|^{-r} \, dx \, dy \\ &\leq 8\pi R \int_{\partial D_R(0)} \frac{|f(z)| \, |dz|}{|z|^r}, \quad \text{for all } r \geq 1. \end{aligned} \tag{3.8}$$

Notice that, by setting

$$D^* = \left\{ z \in D_{R(0)}; |z| \leq |z_0| \text{ and } |f(z)| \geq \frac{|f(z_0)|}{2} \right\}$$

we obtain

$$\begin{aligned} (1 - 8AR) &\iint_{D_R(0)} |f(z)| \, |z|^{-r} \, du \wedge dv \\ &\geq (1 - 8AR) \iint_{D^*} |f(z)| \, |z|^{-r} \, du \wedge dv \\ &\geq \frac{1 - 8AR}{2} |f(z_0)| \, |z_0|^{-r} \operatorname{vol} D^* = a|z_0|^{-r}, \end{aligned}$$

where $a = 1 - 8AR/2|f(z_0)| \operatorname{vol} D^*$.

On the other hand,

$$4R \int_{\partial D_R(0)} |f(z)| \, |z|^{-r} \, |dz| \leq b \, R^{-r},$$

where

$$b = 4R \max_{\partial D_R(0)} |f(z)| \int_{\partial D_R(0)} |dz|.$$

It follows from those estimates and (3.8) that $a|z_0|^{-r} \leq b\,R^{-r}$, for all r, where a and b do not depend on r. Thus, since $|z_0| < R$,

$$0 \leq \lim_{r\to\infty} \frac{a}{b} \leq \lim_{r\to\infty} \left(\frac{|z_0|}{R}\right)^r = 0.$$

Because $a = 1 - 8AR/2|f(z_0)|$ vol D^*, this implies that $|f(z_0)| = 0$, a contradiction to the definition of z_0. This completes the proof of Lemma (3.2) and of the Main Lemma. □

4. Further results and questions

Bryant proved the following result in [7]. *Let M be a compact surface of genus zero immersed in $\mathbb{R}^3$ and let f be any smooth function defined in an open interval containing $[0,\infty)$. Then if M satisfies a Weingarten relation of the form*

$$H = f(H^2 - K) = f\big(|\alpha^{(2,0)}|^2\big),$$

where $\alpha^{(2,0)}$ is the $(2,0)$-part of the second quadratic form in M, then M is isometric to a sphere.

We can generalize this result as follows.

Proposition 4.1. *Let M be a compact surface of genus zero immersed in $M^2(c) \times \mathbb{R}$ and let f be a smooth function. Assume that*

$$H = f(|Q^{(2,0)}|^2). \tag{4.1}$$

Then $Q^{(2,0)} \equiv 0$ and the conclusion of Theorem 1.1 applies.

Proof. $|dH| = |df|\,|d(|Q^{(2,0)}|^2)|$.
But

$$|Q^{(2,0)}|^2 = Q^{(2,0)}\,\overline{Q}^{(2,0)}.$$

Thus

$$d|Q^{(2,0)}|^2 = dQ^{(2,0)}\,\overline{Q}^{(2,0)} + Q^{(2,0)}\,d\overline{Q}^{(2,0)}$$

and

$$\begin{aligned} |d\,|Q^{(2,0)}|^2| &\leq |dQ^{(2,0)}|\,|\overline{Q}^{(2,0)}| + |Q^{(2,0)}|\,|d\overline{Q}^{(2,0)}| \\ &= |Q^{(2,0)}|\big\{\big(|dQ^{(2,0)}| + |d\overline{Q}^{(2,0)}|\big)\big\}. \end{aligned}$$

It follows that

$$|dH| \leq |df|\big(|dQ^{(2,0)}| + |d\overline{Q}^{(2,0)}|\big)|Q^{2,0)}| = g|Q^{(2,0)}|,$$

where we have set $g = |df|\big(|dQ^{(2,0)}| + |d\overline{Q}^{(2,0)}|\big)$. Therefore, we are in the conditions of Theorem 1.1, and the result follows. □

A natural question is the following. In the case of Bryant's theorem, it is known that $|\alpha^{(2,0)}|^2 = H^2 - K$. It would be interesting to know an expression of $|Q^{(2,0)}|^2$ in terms of simple geometric invariants of M.

Also, from the function f defined in the above Bryant's statement, he constructs a quadratic form globally defined on the Weingarten surface M, not necessarily homeomorphic to a sphere, which is shown to be holomorphic on M. As an application he shows, for instance, that a smooth immersion of a torus T^2 satisfying a Weingarten relation as above is free of umbilics and there exists a global principal adapted frame on T^2.

Is it possible to construct such a holomorphic quadratic form on Weingarten surfaces immersed in $M^2(c) \times \mathbb{R}$? This may turn out to be useful even if one has to consider some restricted Weingarten relation. A relevant geometric problem related to that is *which are the closed immersed surfaces in $M^2(c) \times \mathbb{R}$ with constant Gaussian curvature*?

Remark 4.2. As in Remark 2.4, we only need condition (4.1) in a neighborhood of a point where $Q^{(2,0)} = 0$, and again the result of the above proposition can be stated in a way (cf. Remark 2.4) that it applies to compact surfaces M not necessarily homeomorphic to spheres.

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