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Non-negatively curved Kähler manifolds with average quadratic curvature decay

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Let (M, g) be a complete noncompact Kähler manifold with non-negative and bounded holomorphic bisectional curvature. Extending our techniques developed in [A. Chau and L.-F. Tam. *On the complex structure of Kähler manifolds with non-negative curvature*, J. Differs. Geom. **73** (2006), 491–530.], we prove that the universal cover $\widetilde{M}$ of M is biholomorphic to $\mathbb{C}^n$ provided either that (M, g) has average quadratic curvature decay, or M supports an eternal solution to the Kähler–Ricci flow with non-negative and uniformly bounded holomorphic bisectional curvature. We also classify certain local limits arising from the Kähler–Ricci flow in the absence of uniform estimates on the injectivity radius.

1. Introduction

Generalizing the classical uniformization theorems to higher dimensions is a central problem in the study of complex manifolds. It is a particularly interesting problem on complete Kähler manifolds. In this paper, we are interested in complete noncompact Kähler manifolds with positive curvature. For such manifolds, there is a well-known conjecture by Yau [36] which states that a complete noncompact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{C}^n$. Yau's conjecture in its full generality remains unsolved. The first major results supporting the conjecture were obtained by Mok et al. [22], and the conjecture has since been studied extensively, see [7, 8, 9, 11, 12, 21, 23, 26, 28, 33, 34]. Recently in [9] the authors proved the following.

Theorem 1.1. *Let $(M^n, \widetilde{g})$ be a complete noncompact Kähler manifold with non-negative and bounded holomorphic bisectional curvature and maximal volume growth. Then M is biholomorphic to $\mathbb{C}^n$.*

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Here, maximum volume growth means that

$$Vol(B(p,r)) \geq C_1 r^{2n}; \quad \forall r \in [0,\infty) \tag{1.1}$$

for some $C_1 > 0$ and $p \in M$. This assumption of maximum volume growth is rather strong. Consider the following average quadratic curvature decay condition

$$\frac{1}{V_x(r)} \int_{B_x(r)} R \leq \frac{C_2}{1+r^2} \tag{1.2}$$

for some $C_2 > 0$, all $x \in M$ and all $r > 0$. Here, $B_x(r)$ is the geodesic ball around x with radius r and volume $V_x(r)$ and R is the scalar curvature of M. It was conjectured by Yau that if (1.1) is true, then (1.2) will be satisfied automatically for a complete noncompact Kähler manifold with non-negative holomorphic bisectional curvature. Provided that the curvature is bounded, this was recently confirmed by Ni [25] (this was earlier confirmed by Chen-Tang-Zhu [11] for the case of dimension 2 and Chen-Zhu [13] in all dimensions under the additional condition that the curvature operator is non-negative). In general, (1.2) does not imply (1.1). For example, let M satisfy (1.2) and let M_1 be the product of M and a flat torus. Then M_1 also satisfies (1.2). But M_1 does not have maximal volume growth. However, it is an open question whether (1.2) will imply (1.1) under the additional assumption that M has positive bisectional curvature.

In [10, 22, 23, 26, 28] without assuming the maximum volume growth condition, it was proved that if R decays faster than quadratic, then the manifold M with non-negative holomorphic bisectional curvature must be flat. Hence one would expect that Theorem 1.1 is still true if the maximum volume growth condition is removed and is replaced by the weaker condition (1.2). In this work, we confirm this expectation in the following:

Theorem 1.2. *Suppose (M^n, g) has holomorphic bisectional curvature which is bounded, non-negative and has average quadratic curvature decay. Then M is holomorphically covered by $\mathbb{C}^n$.*

As in [9], we will use the Kähler–Ricci flow:

$$\begin{cases} \frac{\partial}{\partial t}\tilde{g}_{i\bar{j}}(x,t) = -\tilde{R}_{i\bar{j}}(x,t); \\ \tilde{g}_{i\bar{j}}(x,0) = g_{i\bar{j}}(x). \end{cases} \tag{1.3}$$

The main difficulty in proving Theorem 1.2 with the methods in [9] is the lack of good lower bound for the injectivity radius of $\tilde{g}(t)$. Indeed, Theorem 1.2 was proved by the authors in [9] under the additional assumption that the curvature operator is non-negative, in which case a good lower bound on the injectivity radius can be obtained. Such lower bounds can also be obtained if we assume M has maximum volume growth as in Theorem 1.1.

For (M, g) as in Theorem 1.2, it is now well known by [32, 34] (see also [27]) that (1.3) has a long-time solution $\tilde{g}(t)$, $0 \leq t < \infty$. If we let $g(x,t) = \mathrm{e}^{-t}\tilde{g}(x, \mathrm{e}^t)$, then we obtain a solution to the normalized Kähler–Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}}(x,t) = -R_{i\bar{j}}(x,t) - g_{i\bar{j}}(x,t) \tag{1.4}$$

for $-\infty < t < \infty$. Hence Theorem 1.2 can be viewed as a uniformization theorem on *eternal* solutions of (1.4). Motivated by this, we will also prove a uniformization theorem for eternal solutions to the Kähler–Ricci flow (1.3), i.e., a smooth family of complete Kähler metrics $g(t)$ on M satisfying

$$\frac{\partial}{\partial t} g_{i\bar{j}}(x,t) = -R_{i\bar{j}}(x,t) \tag{1.5}$$

for all $t \in (-\infty, \infty)$. We have the following:

Theorem 1.3. *Let $(M, g(t))$ be a complete eternal solution to (1.5) such that for all t, $g(t)$ has non-negative holomorphic bisectional curvature which is uniformly bounded on M independent of t. Then M is holomorphically covered by $\mathbb{C}^n$.*

Remark 1.4. Recall that by [10, 22, 23, 26, 28], if M is complete noncompact with bounded non-negative bisectional curvature and if the curvature decays faster than quadratic in the average sense, then M is flat. Hence Theorem 1.2 addresses the maximal (quadratic) curvature decay case for nonflat M. On the other hand, by the Harnack inequality [4] and the decay estimates in [33, section 6] (see also [27, Corollary 2.1]), it is seen that the average curvature of nonflat (M, g) in Theorem 1.3 cannot decay faster than linearly uniformly at all points and so the decay rate is minimal in some sense by [28]. So Theorem 1.3 addresses the case of minimal (linear) curvature decay.

By comparing (1.4) and (1.5), we may combine Theorems 1.2 and 1.3 as follows:

Theorem 1.5. *Let M^n be a noncompact complex manifold. Suppose there is a smooth family of complete Kähler metrics $g(t)$ on M such that for*

$\kappa = 0$ or 1, $g(t)$ satisfies

$$\frac{\partial}{\partial t} g_{i\bar{j}}(x,t) = -R_{i\bar{j}}(x,t) - \kappa g_{i\bar{j}}(x,t) \tag{1.6}$$

for all $t \in (-\infty, \infty)$ such that for every t, $g(t)$ has uniformly bounded non-negative holomorphic bisectional curvature on M independent of t. Then M is holomorphically covered by $\mathbb{C}^n$.

By [5], if in Theorem 1.5 we also assume that the Ricci curvature is positive and the scalar curvature attains its maximum in spacetime, then $(M, g(t))$ is a gradient Kähler–Ricci soliton of steady type if $\kappa = 0$, and of expanding type if $\kappa = 1$. If this is the case, then one may use the results on gradient Kähler-Ricci solitons in [3, 8] to conclude that M is biholomorphic to $\mathbb{C}^n$. Hence Theorem 1.5 can also be considered as a generalization of the results in [3, 8].

2. Local limit solution

Before we prove the main result, observe that if we take $\pi : \widehat{M} \to M$ to be the universal holomorphic covering of M in Theorem 1.5 and let $\widehat{g}(t) = \pi^*(g(t))$, then $(\widehat{M}, \widehat{g})$ still satisfies the conditions of the theorem. To prove the theorem, it is sufficient to prove that $\widehat{M}$ is biholomorphic to $\mathbb{C}^n$. By [6], we may further assume that the Ricci curvature of $\widehat{g}(x,t)$ is positive for all x and t. Hence from now on we assume that M in Theorem 1.5 is simply connected and $g(t)$ has positive Ricci curvature for all t.

Let $(M, g(t))$ be as above satisfying the conditions of Theorem 1.5. Fix some point $p \in M$, some time sequence $t_k \to \infty$ and consider the sequence $(M, g(t_k + t), p)$ of long-time solutions to the Kähler–Ricci flow centered at p. Suppose the injectivity radius of $g(t)$ at p has a uniform lower bound. Then by Hamilton's compactness [17], this sequence has a convergent subsequence converging to a solution $h(t)$ to the Kähler–Ricci flow on a limit complex manifold N. Furthermore, by Cao's classification of limits for Kähler–Ricci flow [5], this limit must either be a steady or expanding gradient Kähler–Ricci soliton, depending on whether $\lambda = 0$ or $\lambda = 1$. In this section, we show that in the absence of an injectivity radius estimate, we may still have such a soliton limit, but in a local sense. We will consider a certain locally lifted subsequence limit of $(M, g(t_k + t), p)$ around p. Our first goal will be to show that this local limit is also either an expanding or steady gradient Kähler–Ricci soliton in a certain sense (Theorem 2.1). We will then relate this to the local asymptotic behavior of

$g(t)$ at p in our main Theorem 2.2. In the absence of injectivity radius estimates, Glickenstein [16] constructed a global limit solution from a solution to the Ricci flow as above, allowing for the possibility of Gromov Hausdorff convergence to a limiting metric space of dimension lower than that of M. We refer the reader to [16] for details on the construction of this limit and its application, and in particular to [14, 15] for applications in three dimensions. Our local limit is just the first step of Glickenstein's construction and in fact depends only on Proposition 2.1 and the simple fact that a lifting of a solution to the Ricci flow is still a solution to the flow. For recent work relating this and, in general, on the existence and classification of limits to the Ricci flow, we refer to the works of Ye [37] and Lott [20].

By [18, 31, 34], from the time independent bounds on the curvature of $g(t)$, we have corresponding uniform bounds on all covariant derivatives of the curvature by the Kähler–Ricci flow. Hence for $t \geq a$ with $a > -\infty$, we may assume that these bounds on all covariant derivatives of the curvature of $g(t)$ are also time independent. The proof of Proposition 1.2 in [35] then gives (see also [9, Proposition 2.1]):

Proposition 2.1. *There exist positive constants r and C such that for each $t \geq -1$ there is a holomorphic map Φ_t from the Euclidean ball $D(r)$ (centered at the origin of $\mathbb{C}^n$ with radius r) to M satisfying the following:*

(i) *Φ_t is a local biholomorphism from $D(r)$ to M;*

(ii) $\Phi_t(0) = p$;

(iii) $\Phi_t^*(g(t))(0) = g_e$;

(iv) *$\frac{1}{C} g_e \leq \Phi_t^*(g(t)) \leq C g_e$ in $D(r)$.*

(v) *for any $0 < \alpha < 1$, and $k \geq 0$, the standard $C^{k+\alpha}$ norm of $\Phi_t^*(g(t))$ in $D(r)$ is bounded by a constant C' which is independent of $t \geq -1$.*

where g_e is the standard metric on $\mathbb{C}^n$.

Remark 2.2. Proposition 1.2 in [35] only requires the first covariant derivative of the scalar curvature of $g(t)$ to be bounded independent of t. In our case, however, we have bounds on all covariant derivatives of the Riemannian curvature tensor independent of k. Condition (v) is derived by continuing the argument in [35] or [9].

As in [9], the following proposition is crucial:

Proposition 2.3. *Let $\lambda_1(t) \geq \cdots > \lambda_n(t) > 0$ be the eigenvalues of $R_{i\bar{j}}(p,t)$ relative to $g_{i\bar{j}}(p,t)$.*

(i) *For any $\tau > 0$,*

$$\phi = \frac{\det(R_{i\bar{j}}(p,t) + \tau g_{ij})}{\det(g_{i\bar{j}}(p,t))}$$

is nondecreasing in t.

(ii) *There is a constant $C > 0$ such that $\lambda_n(t) \geq C$ for all $t \geq 0$.*

(iii) *For $1 \leq i \leq n$, the limit $\lim_{t\to\infty} \lambda_i(t)$ exists.*

(iv) *Let $\mu_1 > \cdots > \mu_l > 0$ be the distinct limits in (iii) and let $\rho > 0$ be such that the intervals $[\mu_k - \rho, \mu_k + \rho]$ for $1 \leq k \leq l$ are disjoint. For any t, let $E_k(t)$ be the sum of the eigenspaces corresponding to the eigenvalues $\lambda_i(t)$ such that $\lambda_i(t) \in (\mu_k - \rho, \mu_k + \rho)$. Let $P_k(t)$ be the orthogonal projection (with respect to $g(t)$) onto $E_k(t)$. Then there exists $T > 0$ such that if $t > T$ and if $w \in T_p^{(1,0)}(M)$, $|P_k(t)(w)|_t$ is continuous in t, where $|\cdot|_t$ is the length measured with respect to the metric $g(p,t)$.*

Proof. The proof is identical to the proof of Proposition 3.1 in [9] for $\kappa = 1$. Suppose $\kappa = 0$. By Theorem 2.3 in [5], if

$$Z_{i\bar{j}} = \frac{\partial R_{i\bar{j}}}{\partial t} + g^{k\bar{l}} R_{i\bar{l}} R_{k\bar{j}} \tag{2.1}$$

then

$$Z_{i\bar{j}} w^i w^{\bar{j}} \geq 0 \tag{2.2}$$

for any $w \in T^{(1,0)}(M)$. Let $p_{i\bar{j}} = R_{i\bar{j}} + \tau g_{i\bar{j}}$ and denote its inverse by $(p^{i\bar{j}})$. We have

$$\begin{aligned}
\frac{\partial}{\partial t} \log\phi &= p^{i\bar{j}} \frac{\partial}{\partial t} p_{i\bar{j}} - g^{i\bar{j}} \frac{\partial}{\partial t} g_{i\bar{j}} \\
&= p^{i\bar{j}} \left(\frac{\partial}{\partial t} R_{i\bar{j}} - \tau R_{i\bar{j}} \right) + g^{i\bar{j}} R_{i\bar{j}} \\
&\geq p^{i\bar{j}} \left(-g^{k\bar{l}} R_{i\bar{l}} R_{k\bar{j}} - \tau R_{i\bar{j}} \right) + g^{i\bar{j}} R_{i\bar{j}} \\
&= p^{i\bar{j}} \left(-g^{k\bar{l}} R_{i\bar{l}} R_{k\bar{j}} - \tau p_{i\bar{j}} \right) + \tau^2 p^{i\bar{j}} g_{i\bar{j}} + g^{i\bar{j}} R_{i\bar{j}}
\end{aligned} \tag{2.3}$$

Now at the point (p, t), we choose a unitary basis such that $g_{i\bar{j}} = \delta_{ij}$ and $R_{i\bar{j}} = \lambda_i \delta_{ij}$. Then $p_{i\bar{j}} = (\lambda_i + \tau)\delta_{ij}$ and $p^{i\bar{j}} = (\lambda_i + \tau)^{-1}\delta_{ij}$. Hence, we have

$$\begin{aligned}\frac{\partial}{\partial t}\log\phi &\geq -\sum_{i=1}^{n}\frac{\lambda_i^2}{\lambda_i+\tau} - \tau n + \sum_{i=1}^{n}\frac{\tau^2}{\lambda_i+\tau} + \sum_{i=1}^{n}\lambda_i \\ &= \sum_{i=1}^{n}\left(\frac{-\lambda_i^2}{\lambda_i+\tau} - \tau + \frac{\tau^2}{\lambda_i+\tau} + \lambda_i\right) \\ &= 0.\end{aligned} \tag{2.4}$$

From this (i) follows. The proof of (ii)–(iv) is similar to the proof of (ii)–(iv) in Proposition 3.1 of [9]. □

For each k, consider the lifted family of metrics $g_k(t) := \Phi_{t_k}^* g(t_k + t)$ on $D(r)$ for $t \in [-1, \infty)$, say. Then it is easy to see that $g_k(t)$ solves the Kähler–Ricci flow (1.6) on $D(r)$. Then by Proposition 2.1 and the Kähler–Ricci flow it follows that some subsequence of $g_k(t)$ converges to a smooth limit family $h(t)$, uniformly on compact subsets of $D(r) \times (-1, \infty)$. It is easy to see that these are Kähler metrics on D for all t and that $h(t)$ solves (1.6). Moreover, by Proposition 2.3, the eigenvalues of the Ricci tensor $R^h_{i\bar{j}}(t)$ of $h(t)$ at the origin are equal to $\lim_{s\to\infty}\lambda_i(s)$ for any $t \in [0, \infty)$. Therefore, $\mu_1 > \mu_2 > \cdots > \mu_l > 0$ are distinct eigenvalues of $R^h_{i\bar{j}}(t)$ at the origin. By the uniform bounds on the covariant derivatives of the curvature tensor of $h(t)$ in $D(r) \times (-1, \infty)$, and by Proposition 2.3, we may have the following inequality on $D(r)$, for $t \geq -1/2$, and by choosing a smaller r if necessary:

$$R^h_{i\bar{j}} \geq C h_{i\bar{j}}. \tag{2.5}$$

Theorem 2.4. *Let $Rc^h_{i\bar{j}}(t)$ be the Ricci tensor of the metric $h_{i\bar{j}}(t)$ on $D(r)$.*

(i) *For each $t \in [0, \infty)$ we have*

$$Rc^h_{i\bar{j}}(t) + \kappa h_{i\bar{j}}(t) = f_{i\bar{j}}(t)$$

for some smooth real-valued function $f(t)$ on $D(r)$ such that $f_{ij}(t) = 0$ and the gradient of $f(t)$ in $h(t)$ is zero at the origin.

(ii) *Let $\mu_1 > \mu_2 > \cdots > \mu_l > 0$ be as above. For $1 \leq i \leq l$, let E_i be the eigenspace corresponding to μ_i of $Ric^h(0,0)$ at $t = 0$ at the origin with respect to $h(0)$. Then E_i is also the eigenspace corresponding to μ_i of $Ric^h(0,t)$ for all $t \geq 0$ at the origin with respect to $h(t)$, $1 \leq i \leq l$.*

To prove the theorem, we first prove a lemma which is a direct modification of the results in [5]. In the case of $\kappa = 1$, it will be more convenient to consider the transformed metric $\tilde{h}(t) = th(\log t)$ which solves (1.6) on $D(r) \times [e^{-1}, \infty)$ with $\kappa = 0$. It is clear that $\tilde{h}(t)$ is the limit of the transformed sequence $\tilde{g}_k(t) := tg_k(\log t)$ uniform on compact sets of $D(r) \times [e^{-1}, \infty)$ which also satisfy (1.6) with $\kappa = 0$.

Let $Z_{i\bar{j}}$ and $Z^k_{i\bar{j}}$ be the Harnack quadratic tensors corresponding to $\tilde{h}(t)$ and $\tilde{g}_k(t)$, respectively, as defined in Theorem 2.1 in [5]. Namely for any holomorphic vector (V^i) at a point $q \in D(r)$,

$$Z_{i\bar{j}} = \frac{\partial}{\partial t} R^{\tilde{h}}_{i\bar{j}} + \tilde{h}^{l\bar{k}} R^{\tilde{h}}_{i\bar{k}} R^{\tilde{h}}_{l\bar{j}} + R^{\tilde{h}}_{i\bar{j},k} V_{\bar{k}} + R^{\tilde{h}}_{i\bar{j},\bar{k}} V_k + R^{\tilde{h}}_{i\bar{j}k\bar{l}} V_{\bar{k}} V_l + \frac{1}{t} R^{\tilde{h}}_{i\bar{j}} \tag{2.6}$$

and $Z^k_{i\bar{j}}$ is defined similarly. Denote the trace $\tilde{h}^{i\bar{j}} Z_{i\bar{j}}$ of $Z_{i\bar{j}}$ by Z. Note that Z is a smooth function defined on the holomorphic tangent bundle $T^{(1,0)}(D(r))$.

In case $\kappa = 0$, then let $Q_{i\bar{j}}$ and $Q^k_{i\bar{j}}$ be the Harnack quadratic tensors corresponding to $h(t)$ and $g_k(t)$, respectively, as defined in Theorem 2.3 in [5]. Namely for any holomorphic vector (V^i) at a point $x \in D(r)$,

$$Q_{i\bar{j}} = \frac{\partial}{\partial t} R^{h}_{i\bar{j}} + h^{l\bar{k}} R^{h}_{i\bar{k}} R^{h}_{l\bar{j}} + R^{h}_{i\bar{j},k} V_{\bar{k}} + R^{h}_{i\bar{j},\bar{k}} V_k + R^{h}_{i\bar{j}k\bar{l}} V_{\bar{k}} V_l \tag{2.7}$$

and $Q^k_{i\bar{j}}$ is defined similarly. Denote the trace $h^{i\bar{j}} Q_{i\bar{j}}$ of $Q_{i\bar{j}}$ by Q.

Lemma 2.5.

(i) *For any holomorphic vector $V \in T^{(1,0)}(D(r))$, $Z_{i\bar{j}}$ is a non-negative quadratic form. Moreover, if Z is positive at some point $x_0 \in D(r)$ for all $V \in T^{(1,0)}_{x_0}(D(r))$ at $t = t_0$, then Z is positive for all $t > t_0$ and for $V \in T^{(1,0)}(D(r))$.*

(ii) *For any holomorphic vector $V \in T^{(1,0)}(D(r))$, $Q^h_{i\bar{j}}$ is a non-negative quadratic form. Moreover, if Q^h is positive at some point $x_0 \in D(r)$ for all $V \in T^{(1,0)}_{x_0}(D(r))$ at $t = t_0$, then Q^h is positive for all $t > t_0$ and for $V \in T^{(1,0)}(D(r))$.*

Proof. For any holomorphic vector W, $Z^k_{i\bar{j}} W^i W^{\bar{j}} \geq 0$ for all k by Theorem 2.1 in [5]. Since $Z_{i\bar{j}}$ is the limit of the $Z^k_{i\bar{j}}$'s on $D(r)$ for all t, $Z_{i\bar{j}} W^i W^{\bar{j}} \geq 0$. This proves the first statement of (i). The first statement of (ii) can be proved similarly.

To prove the second statement in (i), assume there is some $x_0 \in D(r)$ and $t_0 \geq e^{-1}$ so that $Z > 0$ for all $V \in T^{(1,0)}_{x_0}(D(r))$. Given any $T > t_0$, we note that for $C > 0$ there exists some $K > 0$ such that given any point $(x,t) \in D(r) \times [t_0, T]$ and $V \in T_x^{(1,0)}(D(r))$, with Euclidean length $||V|| > K$, we must have

$$Z > C \tag{2.8}$$

at (x,t) and V. This follows from (2.6), (2.5) and the fact that the curvature tensor of $\tilde{h}(t)$ and its covariant derivatives in time and space are uniformly bounded on $D \times [t_0, T]$ by constants independent of space and time by our estimates on the $\tilde{g}_k' s$. Hence there exist a neighborhood U of x_0 and $\epsilon > 0$ such that $Z \geq \epsilon$ for all holomorphic vector V at $x \in U$ at $t = t_0$.

Choose a smooth function F on $D(r)$ such that $F(x_0) > 0$, F is zero outside a small neighborhood of x_0 and $Z - \frac{F}{t_0^2} \geq 0$ for all V in $T^{(1,0)}(D(r))$ at $t = t_0$.

Let F evolve by the heat equation on $D \times [t_0, T]$ with the following initial and boundary conditions:

$$\begin{aligned} \frac{\partial}{\partial t} F = \Delta_t F \quad &\text{in} \quad D \times [t_0, T] \\ F = 0 \quad &\text{on} \quad \partial D \times [t_0, T] \\ F(x, t_0) = F(x) & \end{aligned} \tag{2.9}$$

where Δ_t is the Laplacian relative to $\tilde{h}(t)$. F is then strictly positive in $D \times (t_0, T]$ by the strong maximum principle [29, Theorem 5, Chapter 3]. We will show that $\tilde{Z} := Z - \frac{F}{t^2}$ is also non-negative for all V and $(x,t) \in D \times (t_0, T]$. Without loss of generality we may assume $\tilde{h}$ is smooth up to the boundary of $D(r)$. Let $\tilde{Z}$ assume its minimum over all $(x,t) \in \overline{D}(r) \times [t_0, T]$ and V, at some point (x, t_1) and some vector V_0. This minimum exists by compactness and by (2.8). Now assume this minimum is negative. Then by the initial condition of F, we must have $t_1 > t_0$. Also, by the non-negativity of Z and the zero Dirichlet boundary condition on F, we must have x strictly inside $D(r)$. As in [5], we may then extend V_0 locally around (x, t_1) in space-time such that (3.2) in [5] holds. At (x, t_1) we then have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_t\right) \tilde{Z} &= \left(\frac{\partial}{\partial t} - \Delta_t\right) Z + 2\frac{F}{t^3} \\ &= Z_{i\bar{j}} R^{\tilde{h}}_{j\bar{i}} - 2\frac{Z}{t} + 2\frac{F}{t^3} \end{aligned} \tag{2.10}$$

$$
\begin{aligned}
&= Z_{i\bar{j}}R^{\tilde{h}}_{j\bar{i}} - 2\frac{\tilde{Z}}{t} \\
&> 0.
\end{aligned}
$$

But this contradicts the fact that $\tilde{Z}$ is minimal at (x, t_1) and V_0. Thus $\tilde{Z}$ is non-negative as claimed, which completes the proof of (i) of the lemma.

We now consider the second statement of (ii). Assume there is some $x_0 \in D(r)$ and $t_0 \geq 0$ such that Q is positive at (x_0, t_0) for all V. As before, for any $T > t_0$ we observe that given any $C > 0$, there exists some $K > 0$ such that given any point $(x, t) \in D(r) \times [t_0, T]$ and $V \in T^{(1,0)}_x(D(r))$ with Euclidean length $||V|| > K$, we must have $Z > C$ at (x, t) and V.

Hence we can choose a smooth function F on D such that $F(x_0) > 0$, F is zero outside a small neighborhood of x_0 and $Q - F \geq 0$ for all V everywhere on $D(r)$ at $t = t_0$. Let F evolve by heat equation on $D \times [t_0, T]$ with the following initial and boundary conditions:

$$
\begin{aligned}
\frac{\partial}{\partial t}F &= \Delta_t F \quad \text{in} \quad D(r) \times [t_0, t] \\
F &= 0 \quad \text{on} \quad \partial D(r) \times [t_0, T] \\
F(x, t_0) &= F(x)
\end{aligned}
\tag{2.11}
$$

where Δ_t is the Laplacian relative to $h(t)$. F is then strictly positive in $D \times (t_0, T]$ by the strong maximum principle. Now given any $\epsilon > 0$, we will show that $\tilde{Q} := Q - F + \epsilon e^t$ is non-negative for all V and $(x, t) \in D(r) \times (t_0, T]$. Letting ϵ approach zero, this will prove that $Q - F$ is non-negative for all V and $(x, t) \in D(r) \times (t_0, T]$, thus proving the lemma.

Let $\tilde{Q}$ assume its minimum over all $(x, t) \in D(r) \times [t_0, T]$ and V, at some point (x, t) and some vector V_0. Now assume this minimum is negative. Then by our initial condition of F, we must have $t > t_0$. Also, by the non-negativity of Q and the zero Dirichlet boundary condition on F, we must have x strictly inside $D(r)$. We may then extend V_0 locally around (x, t) in space-time such that (3.1) in [5] holds. At (x, t) we then have

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_t\right)\tilde{Q} &= \left(\frac{\partial}{\partial t} - \Delta_t\right)Q + \epsilon e^t \\
&= Q_{i\bar{j}}R^{\tilde{h}}_{j\bar{i}} + \epsilon e^t \\
&> 0.
\end{aligned}
\tag{2.12}
$$

But this contradicts the fact that $\tilde{Q}$ is minimal at (x,t) and V_0. Thus $\tilde{Q}$ is non-negative as claimed, which completes the proof of (ii). □

Proof of Theorem 2.4. (i): We begin with the case where $\kappa = 1$ in Theorem 2.4. With the same notations as in the Lemma 2.5, by Proposition 2.3 and the arguments following (3.7) in [9], we see that

$$tR^{\tilde{h}}(0,t)$$

is constant for all $t \in [e^{-1}, \infty)$ where $R^{\tilde{h}}$ is the scalar curvature of $\tilde{h}$. Thus at the space time point $(0,t)$ we have $R^{\tilde{h}} + t\frac{\partial}{\partial t}R^{\tilde{h}} = 0$. Applying Lemma 2.1 and following the exact argument in the proof of Theorem 4.2 in [5], we conclude that for each $t \in [e^{-1}, \infty)$ there is a smooth real-valued function $\tilde{f}(t)$ on $D(r)$ such that the gradient of $\tilde{f}(t)$ is holomorphic and is zero at the origin. Moreover, we have

$$Rc^{\tilde{h}}_{i\bar{j}} = \tilde{f}_{i\bar{j}} + \frac{1}{t}\tilde{h}_{i\bar{j}}. \tag{2.13}$$

Transforming $\tilde{h}$ back to h, it is easy to see that (i) in Theorem 2.4 is true for $\kappa = 1$.

We now consider the case of $\kappa = 0$. By Proposition 2.3 we have $R^h(0,t)$ is constant for $t \in [0, \infty)$, and in particular, $\frac{\partial}{\partial t}R^h = 0$ at the space time point $(0,t)$. Applying Lemma 2.5 and following the exact argument in the proof of Theorem 4.1 in [5], we conclude that for each $t \in [0, \infty)$ there is a smooth real-valued function $f(t)$ on $D(r)$ such that the gradient of $f(t)$ is holomorphic and is zero at the origin. Moreover, we have

$$Rc^h_{i\bar{j}} = f_{i\bar{j}}. \tag{2.14}$$

This completes the proof of (i) in Theorem 2.4.

(ii): Let $\lambda_1 \geq \cdots \geq \lambda_n > 0$ be the eigenvalues of Rc^h at 0. Note that they are independent of t. Suppose $v \in T_0^{1,0}(D(r))$ is not an eigenvector for $Rc^h(0,0)$ for λ_k. We will show that v cannot be an eigenvector for $Rc^h(0,t)$ for all $t \in (0, \infty)$ for λ_k. It is sufficient to prove that the quantity

$$\begin{aligned} F(t) : &= |Rc^h(0,t)(v,\cdot) - \lambda_k h(0,t)(v,\cdot)|^2_{h(t)} \\ &= h^{j\bar{k}}(0,t)\overline{(R^h_{i\bar{j}}(0,t) - \lambda_k h_{i\bar{j}}(0,t))v^i} \\ &\quad \cdot (R^h_{l\bar{k}}(0,t) - \lambda_k h_{l\bar{k}}(0,t))v^l \end{aligned} \tag{2.15}$$

can never be zero since this is zero at t if and only if v is an eigenvector for $Rc^h(0,t)$ with eigenvalue λ_k. Let $t_0 \in [0, \infty)$ be arbitrary. Choose a holomorphic coordinate in $D(r)$ such that at $0 \in D(r)$ we have $h_{i\bar{j}}(t_0) = \delta_{i\bar{j}}$ and $R^h_{i\bar{j}}(t_0) = \lambda_i \delta_{i\bar{j}}$. Let $f(t)$ be as in Lemma 2.5. It is not hard to show that we may choose some $1 > \delta > 0$ such that starting at any point $p \in D(\delta r)$ we may flow along $-\frac{1}{2}\nabla f(t_0)$ for $t \in [0,1]$ while staying inside $D(r)$ and let φ_t be the local biholomorphism determined by the flow. Note that the origin is a fixed point of the flow because $\nabla f(t) = 0$ at the origin. Let $g(t) = \varphi_t^*(h(t_0))$ be the soliton metric on $D(\delta r) \times [0,1)$ with initial condition $g(0) = h(t_0)$. Then at $0 \in D(\delta r)$, in the above coordinates, we have $g_{i\bar{j}}(t) = e^{-(\lambda_i+\kappa)t}\delta_{i\bar{j}}$ and $R^g_{i\bar{j}}(t) = \lambda_i e^{-(\lambda_i+\kappa)t}\delta_{i\bar{j}}$. For any k we then have

$$
\begin{aligned}
G(t) &:= |Rc^g(0,t)(v,\cdot) - \lambda_k g(0,t)(v,\cdot)|^2_{g(t)} \\
&= \sum_{i \neq k} (\lambda_i - \lambda_k)^2 |v^i|^2 e^{-(\lambda_i+\kappa)t}
\end{aligned} \tag{2.16}
$$

and thus

$$
\begin{aligned}
G'(0) &= \sum_{i \neq k} -(\lambda_i + \kappa)(\lambda_i - \lambda_k)^2 |v^i|^2 \\
&\geq -(\lambda_1 + \kappa) G(0).
\end{aligned} \tag{2.17}
$$

From the above equation, the fact that $g(0) = h(t_0)$ on $D(\delta r)$, and the fact that both $g(t)$ and $h(t_0 + t))$ solve (1.6) on $D(\delta r) \times [0,1)$, it follows that $G(0) = F(t_0)$ and $G'(0) = F'(t_0)$. Hence for any choice of k we have $F'(t_0) \geq -(\lambda_1 + \kappa)F(t_0)$. But $t_0 \in [0,\infty)$ is arbitrary. Thus for any k we have $F'(t) \geq -(\lambda_1 + \kappa)F(t)$ for all $t \in [0,\infty)$. It is now easy to see that if $F(0) \neq 0$, then $F(t)$ cannot be zero for any t. This completes the proof of our claim.

From the claim, we conclude that if v is an eigenvector of $h(t)$ for $t > 0$ with eigenvalue λ_k, then v must be in E_k. Since the multiplicity of each eigenvalue μ_k is constant in t, from this it is easy to see the theorem is true. □

Now given $(M, g(t))$ as in the beginning of the section, we denote the eigenvalues of $Rc(p,t)$ by $\lambda_i(t)$ for $i = 1, ..., n$ as before, and we let μ_k, $E_k(t)$ and $P_k(t)$ for $k = 1, ..., l$ be as in Proposition 2.3. We let n_m for $m = 1$, $1, \ldots, l+1$ with $n_1 = 0$ and $n_{l+1} = n$ be such that $\lambda_i \in (\mu_m - \rho, \mu_m + \rho)$ for all $n_m < i \leq n_{m+1}$ and $m = 1, \ldots, l$ and t sufficiently large such that the intervals $[\mu_m - \rho, \mu_m + \rho]$ are disjoint as in Proposition 2.3 part (iv). For any nonzero vector $v \in T^{1,0}_p(M)$, let $v(t) = v/|v|_t$, where $|v|_t$ is the length of v with respect to $g(t)$ and $v_i(t) = P_i(t)v(t)$.

We now show that Theorem 4.1 in [9] is also true for $(M, g(t))$ in our case; that $Rc(p,t)$ can be "diagonalized" simultaneously near infinity and that $g(t)$ is "Lyapunov regular", to borrow a notion from dynamical systems (see [1]).

Theorem 2.6. *Let $(M, g(t))$ be as described in the beginning of the section. Then $V = T_p^{(1,0)}(M)$ can be decomposed orthogonally with respect to $g(0)$ as $V_1 \oplus \cdots \oplus V_l$ so that the following are true:*

(i) *If v is a nonzero vector in V_i for some $1 \le i \le l$, then $\lim_{t\to\infty} |v_i(t)| = 1$ and thus $\lim_{t\to\infty} Rc(v(t), \bar{v}(t)) = \mu_i$ and*

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{|v|_t^2}{|v|_0^2} = -\mu_i - \kappa.$$

Moreover, the convergence are uniform over all $v \in V_i \setminus \{0\}$.

(ii) *For $1 \le i, j \le l$ and for nonzero vectors $v \in V_i$ and $w \in V_j$ where $i \ne j$, $\lim_{t\to\infty} \langle v(t), w(t)\rangle_t = 0$ and the convergence is uniform over all such nonzero vectors v, w.*

(iii) $\dim_{\mathbb{C}}(V_i) = n_{i+1} - n_i$ *for each i.*

(iv)
$$\sum_{i=1}^{l} (-\mu_i - \kappa) \dim_{\mathbb{C}} V_i = \lim_{t\to\infty} \frac{1}{t} \log \frac{\det(g_{i\bar{j}}(t))}{\det(g_{i\bar{j}}(0))}.$$

Proof. Let $t_k \to \infty$ and construct g_k with limit $h(t)$ as in Theorem 2.4. Observe that since $h(t)$ is a smooth limit of the $g_k(t)'s$ on $D(r) \times [0, \infty)$, the analog of Lemma 3.2 in [9] is true in our case. Using this and (ii) in Theorem 2.4, we may prove the theorem by contradiction exactly as in the proof of Theorem 4.1 in [9]. □

3. Transition maps

Let $(M, g(t))$ be as in Theorem 1.5 and let $p \in M$ be fixed. In addition, we will assume that $R_{i\bar{j}} \ge a' g_{i\bar{j}}$ for some $a' > 0$ at p and $t = 0$. Then by Proposition 2.3, there exists $a > 0$ such that

$$R_{i\bar{j}} \ge a g_{i\bar{j}} \tag{3.1}$$

at p for all $t \ge 0$. Since the covariant derivatives Riemannian curvature tensor Rm_t of $g(t)$ is uniformly bounded, we can conclude that by choosing a possibly smaller $a > 0$, that (3.1) is still true in $B_t(p, R)$ for some $R > 0$ independent of t, where $B_t(p, R)$ is the geodesic ball of radius R with center at p with respect to the metric $g(t)$. Since $B_{t_1}(p, R) \subset B_{t_2}(p, R)$ for $t_2 \ge t_1$

as $g(t)$ is shrinking, we have

$$L_{t_2}(\gamma) \le e^{-\frac{a}{2}(t_2-t_1)} L_{t_1}(\gamma) \tag{3.2}$$

for any curve in $B_{t_1}(p,R)$. Here L_t denotes the length function with respect to $g(t)$.

Recall that by Proposition 2.1, there exist $r>0$ and $C>0$ independent of $t\ge 0$ and a holomorphic maps $\Phi_t : D(r) \to M$ with the following properties:

(i) Φ_t is a local biholomorphism from $D(r)$ onto its image;

(ii) $\Phi_t(0) = p$;

(iii) $\Phi_t^*(g(t))(0) = g_e$;

(iv) $\frac{1}{C} g_e \le \Phi_t^*(g(t)) \le C g_e$ in $D(r)$.

where g_e is the standard metric on $\mathbb{C}^n$. Let $T>0$ and denote Φ_{iT} simply by Φ_i. In this section, we want to construct injective holomorphic maps F_i from $D(\rho)$ to $D(\rho)$ for some ρ such that $\Phi_i = \Phi_{i+1} \circ F_{i+1}$. We should emphasize that Φ_i may not be a covering map.

In this section, we always assume that $t \ge 0$. By property (iv) and reducing r if necessary, we may assume that $\Phi_t(D(r)) \subset B_t(p,R)$, where $R>0$ is such that (3.2) is true. In fact, we have the following:

Lemma 3.1. *For any $0<\rho<r$, there exists $R_1>0$ independent of t such that*

$$B_t\left(p, \frac{1}{R_1}\right) \subset \Phi_t(D(\rho)) \subset B_t(p,R_1) \tag{3.3}$$

Proof. By (iv) above, it is easy to see that

$$\Phi_t(D(\rho)) \subset B_t(p, C_1)$$

for some $C_1>0$ independent of t, where $B_t(p,r)$ is the geodesic ball with radius r centered at p with respect to $g(t)$. On the other hand, $\widehat{B}_t(0,C_2) \subset D(\rho)$ for some $C_2>0$ independent of t, where $\widehat{B}_t(0,C_2)$ is the geodesic ball with radius C_2 centered at 0 with respect to $\Phi_t^*(g(t))$. Hence

$$\Phi_t(D(\rho)) \supset \Phi_t(\widehat{B}_t(0,C_2)) \supset B_t(p,C_2).$$

From this it is easy to see the lemma follows. ∎

Lemma 3.2. *For any $0 < \rho \le r$, where r is as in (i)–(iv), there exists $\rho_1 > 0$ independent of t, satisfying the following for any t: Let γ be a smooth curve in M with $\gamma(0) = q$ such that $q \in B_t(p, \rho_1)$ and $L_t(\gamma) < \rho_1$. Then $\Phi_t(z) = q$ for some $z \in D(\frac{\rho}{8})$, and for all such z there is a unique lift $\widetilde{\gamma}$ of γ by Φ_t so that $\widetilde{\gamma}(0) = z$ and $\widetilde{\gamma} \subset D(\frac{\rho}{2})$.*

Proof. Let $\rho_1 > 0$ be determined later. Let $q \in B_t(p, \rho_1)$ and let $\gamma(s)$, $0 \le s \le \rho_1$ be a curve from q parametrized by arc-length with respect to $g(t)$. Suppose $z \in D(\frac{\rho}{8})$ with $\Phi_t(z) = q$

Since Φ_t is a local biholomorphism, there exists $s_0 > 0$ and a curve $\widetilde{\gamma}$ from z with $\widetilde{\gamma} \subset D(\frac{1}{8}\rho)$ such that $\Phi_t \circ \widetilde{\gamma} = \gamma$ on $[0, s_0]$. Let A be the set of s, such that γ has a lift $\widetilde{\gamma}$ in $D(\frac{1}{2}\rho)$ on $[0, s]$ with $\widetilde{\gamma}(0) = z$. Since Φ_t is a local biholomorphism, A is open in $[0, \rho_1]$. Suppose $s_k \to s$ and $s_k \in A$. By (iv), there is a constant $C > 0$ which is independent of t such that

$$C^{-1} L_e(\widetilde{\gamma}|_{[0,s_k]}) \le L_t(\gamma|_{[0,s_k]}) \le \rho_1,$$

where L_e is the length with respect to g_e. Hence

$$L_e(\widetilde{\gamma}|_{[0,s_k]}) \le C\rho_1 \le \frac{1}{4}\rho$$

if $\rho_1 < \frac{1}{4C}\rho$. Note that since $\widetilde{\gamma}(0) = z \in D(\frac{1}{8}\rho)$, we may assume that $\widetilde{\gamma}(s_k) \to z_1$ for some $z_1 \in \overline{D(\frac{3}{8}\rho)}$. From this it is easy to see that γ can be lifted up to s while staying in $D(\frac{1}{2}\rho)$. Hence A is also closed. Since Φ_t is a local biholomorphism, the lifting must be unique. In particular, by choosing a smaller ρ_1 which is independent of t, we conclude that every minimal geodesic from p with length less than ρ_1 can be lifted to a curve in $D(\frac{1}{8}\rho)$. Hence for all $q \in B_t(p, \rho_1)$, there is a point $z \in D(\frac{1}{8}\rho)$ such that $\Phi_t(z) = q$. This completes the proof of the lemma. □

Lemma 3.3. *Fix $t \ge 0$. Let $0 < \rho \le r$ be given and let ρ_1 be as in Lemma 3.2. Given any $\epsilon > 0$, there exists $\delta > 0$ satisfying the following properties:*

Let $\gamma(\tau)$, $\beta(\tau)$, $\tau \in [0, 1]$ be smooth curves from $q \in B_t(p, \rho_1)$ with length less than ρ_1 and let $z_0 \in D(\frac{1}{8}\rho)$ with $\Phi_t(z_0) = q$. Let $\widetilde{\gamma}$, $\widetilde{\beta}$ be the liftings from z_0 of γ and β as described in Lemma 3.2. Suppose $d_t(\gamma(\tau), \beta(\tau)) < \delta$ for all $\tau \in [0, 1]$, then $d_e(\widetilde{\gamma}(1), \widetilde{\beta}(1)) < \epsilon$. Here d_t is the distance in $g(t)$ and d_e is the Euclidean distance.

Proof. Since Φ_t is a local biholomorphism, there is $\sigma > 0$ such that Φ_t is a biholomorphism onto its image when restricted on $D(z, \sigma)$ for all $z \in \overline{D(\frac{1}{2}\rho)}$, where $D(z, \sigma)$ is the Euclidean ball with center at z and radius σ.

Let q, z_0, γ, β, $\widetilde{\gamma}$ and $\widetilde{\beta}$ as in the lemma. Since $\widetilde{\gamma} \subset D(\frac{1}{2}\rho)$, by property (iv) of Φ_t, there exists $C_1 > 0$ such that

$$\Phi_t(D(\widetilde{\gamma}(\tau)), \sigma) \supset B_t(\gamma(\tau), C_1^{-1}\sigma) \tag{3.4}$$

and

$$d_e(\widetilde{\gamma}(\tau), z) \leq C_1 d_t(\gamma(\tau), \Phi_t(z)) \tag{3.5}$$

for all $z \in D(\widetilde{\gamma}(\tau), \sigma)$ with $\Phi_t(z) \in B_t(\gamma(\tau), C_1^{-1}\sigma)$. Note that C_1 is independent of the curves γ and β.

Given $0 < \epsilon < \sigma$, let $0 < \delta < C_1^{-1}\epsilon < C_1^{-1}\sigma$. Suppose β and $\widetilde{\beta}$ are as in the lemma such that $d_t(\gamma(\tau), \beta(\tau)) < \delta$ for all τ. Since $\widetilde{\gamma}(0) = \widetilde{\beta}(0) = z_0$, we have $d_e(\widetilde{\gamma}(\tau), \widetilde{\beta}(\tau)) < \epsilon$ in $[0, \tau_0]$ for some $\tau_0 > 0$. Let A be the set τ in $[0, 1]$ such that $d_e(\widetilde{\gamma}(\tau'), \widetilde{\beta}(\tau')) < \epsilon$ for all $\tau' \in [0, \tau]$. Then A is nonempty and is open. Suppose $\tau_k \in A$ and $\tau_k \to \tau$. Then

$$d_e(\widetilde{\gamma}(\tau), \widetilde{\beta}(\tau)) \leq \epsilon.$$

Since $\epsilon < \sigma$, $\delta < C_1^{-1}\sigma$, by (3.5) and the fact that $\Phi_t(\widetilde{\gamma}(\tau)) = \gamma(\tau)$, $\Phi_t(\widetilde{\beta}(\tau)) = \beta(\tau)$, we have

$$d_e(\widetilde{\gamma}(\tau), \widetilde{\beta}(\tau)) \leq C_1 d_t(\gamma(\tau), \beta(\tau)) \leq C_1\delta < \epsilon.$$

Hence $\tau \in A$ and $A = [0, 1]$. This completes the proof of the lemma. □

Apply Lemma 3.2 to $\rho = r$ and choose ρ_1 as in the lemma. Note that ρ_1 is independent of i and T. For any $z \in D(r)$, let $\gamma^*(\tau)$, $0 \leq \tau \leq 1$, be the line segment from 0 to z, and let $\gamma = \Phi_i \circ \gamma^*$. By (3.2) and property (iv) for Φ_t, there is a constant $C_1 > 0$ independent of i and T such that

$$L_{i+1}(\gamma) \leq e^{-\frac{a}{2}T} L_i(\gamma) \leq C_1 e^{-\frac{a}{2}T} r. \tag{3.6}$$

Now we choose $T > 0$ large enough so that $C_1 e^{-\frac{a}{2}T} r < \rho_1$. Then by Lemma 3.2, there is a unique lift $\widetilde{\gamma}$ of γ by Φ_{i+1} so that $\widetilde{\gamma}(0) = 0$ and $\widetilde{\gamma} \subset D(\frac{r}{2})$. We then define $F_{i+1}(z) = \widetilde{\gamma}(1)$. F_{i+1} is then well-defined by the uniqueness of the lifting. We have:

Lemma 3.4. *The maps F_{i+1} satisfy the following:*

(a) $F_{i+1} : D(r) \to D(\frac{r}{2})$, $F_{i+1}(0) = 0$ *and* $\Phi_i = \Phi_{i+1} \circ F_{i+1}$.

(b) *For each i, F_{i+1} is a local biholomorphism.*

(c)

$$b_1|v| \le |F'_{i+1}(0)v| \le b_2|v|$$

for some $0 < b_1 \le b_2 < 1$ *for all* i *and for all vector* $v \in \mathbb{C}^n$*, where* F' *is the Jacobian of* F*.*

(d) *There exist* $r > r_1 > 0$ *and* $0 < \theta < 1$ *independent of* i *such that*

$$|F_{i+1}(z)| \le \theta|z|$$

for all i *and for all* $z \in D(r_1)$ *and* F_{i+1} *is injective on* $D(r_1)$*.*

Proof. (a) follows immediately from the definition of F_{i+1}.

To prove (b), let us first prove that F_{i+1} is continuous. Let $z_0 \in D(r)$, $\gamma^*(\tau)$, $0 \le \tau \le 1$, be the line segment from 0 to z_0, $\gamma = \Phi_i \circ \gamma^*$ and $\widetilde{\gamma}$ is the lift of γ by Φ_{i+1} with $\widetilde{\gamma}(0) = 0$. Let $w = \widetilde{\gamma}(1) = F_{i+1}(z_0)$. Given $\epsilon > 0$, let $\delta > 0$ be as in Lemma 3.3 for Φ_{i+1}.

We may assume that $|z_0| \le 1 - \eta$ for some $\eta > 0$. Since Φ_i is uniformly continuous on $D(1 - \eta/2)$, there exists $\sigma > 0$ such that if $|z_1 - z_2| < \sigma$, $z_1, z_2 \in D(1 - \eta/2)$, then $d_i(\Phi_i(z_1), \Phi_i(z_2)) < \delta$.

Moreover, it is easy to see that we can find $\delta' > 0$ such that if $|z_0 - \zeta| < \delta'$, then the ray β^* defined on $[0, 1]$ from 0 to ζ satisfies $|\gamma^*(\tau) - \beta^*(\tau)| < \sigma$ and $\beta^* \subset D(1 - \eta/2)$. Let ζ be such a point in $D(r)$ with β^* as above and let $\beta = \Phi_i \circ \beta^*$. Hence we have

$$d_{i+1}(\gamma(\tau), \beta(\tau)) \le d_i(\gamma(\tau), \beta(\tau)) < \delta.$$

By Lemma 3.3, if $\widetilde{\beta}$ is the lift of β by Φ_{i+1} with $\widetilde{\beta}(0) = 0$, then $|\widetilde{\gamma}(1) - \widetilde{\beta}(1)| < \epsilon$. That is to say, $|F_{i+1}(z_0) - F_{i+1}(\zeta)| < \epsilon$ and F_{i+1} is continuous.

Now it is easy to see that F_{i+1} is a local biholomorphism. In fact, suppose $F_{i+1}(z) = w$ and suppose $\Phi_i(z) = x$ and $\Phi_{i+1}(w) = y$. Let $\epsilon_1 > 0$ be such that Φ_i and Φ_{i+1} are biholomorphisms when restricted on $D(z, \epsilon_1)$ and $D(w, \epsilon_1)$, respectively. Since F_{i+1} is continuous, we can find $0 < \delta_1 < \epsilon_1$ such that $F_{i+1}(D(z, \delta_1)) \subset D(w, \epsilon_1)$. Since $\Phi_i = \Phi_{i+1} \circ F_{i+1}$, we have

$$F_{i+1} = \Phi_{i+1}^{-1} \circ \Phi_i$$

on $B_e(z, \delta_1)$. Hence F_{i+1} is a local biholomorphism.

To prove (c), by properties (ii), (iii) of Φ_t, the fact that $F_{i+1} = \Phi_{i+1}^{-1} \circ \Phi_i$ near the origin, the fact that $R_{i\bar{j}}(t)$ is uniformly bounded and (3.1), it is easy to see that (c) is true.

To prove (d), by gradient estimates, we have

$$F_{i+1}(z) = F_{i+1}(0) + F'_{i+1}(0)z + H_i(z) = F'_{i+1}(0)z + H_i(z),$$

where $|H'_i(z)| \leq C|z|$ for some constant C independent of i on $D(\frac{1}{2}r)$, say. Hence by (c), there exist $r > r_1 > 0$ and $1 > \eta > 0$, independent of i such that $F_{i+1} : D(r_1) \to D(r_1)$ so that

$$|F_{i+1}(z)| \leq \theta |z|$$

for all i and for all $z \in D(r_1)$ and F_{i+1} is injective on $D(r_1)$. □

By this lemma and Theorem 2.6, using the method in [9, §5], see also [19, 30], we can prove the following:

Lemma 3.5. *Let F_i be as in Lemma 3.4. There exist biholomorphisms G_i on $\mathbb{C}^n$ and polynomial maps T_i with the following properties:*

(a) $T_i(0) = 0$, $T'_i(0) = Id$, *and* $\sup_{z \in D(1)} |T_i(z)| \leq C_1$ *for some constant* C_1 *independent of* i.

(b) $G_i(0) = 0$ *and for all open sets,* U *containing the origin*

$$\bigcup_{k=1}^{\infty} (G_k \circ \cdots \circ G_1)^{-1}(U) = \mathbb{C}^n.$$

(c) *There exist* $0 < r_2 < r_1$ *and* $C_2 > 0$ *independent of* $k \geq 1$ *such that*

$$G_{k+1}^{-1} \circ G_{k+2}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+2} \circ F_{k+1}$$

converges uniformly on $D(r_2)$ *as* $l \to \infty$ *to an injective holomorphic map* Ψ_k *such that*

$$D(C_2^{-1} r_2) \subset \Psi_k(D(r_2)) \subset D(C_2 r_2).$$

(d) *By choosing* r_2 *smaller if necessary, we may have that* T_i *is injective on* $D(r_2)$ *such that* T_i^{-1} *is defined on* $D(r_2)$ *and* $T_i^{-1}(D(r_2)) \subset D(r)$ *for all* i.

4. Proof of the main theorem

We are now ready to prove Theorem 1.5. By the remark at the beginning of §2, we may assume that M is simply connected and $g(t)$ has positive Ricci curvature for all t.

Let Φ_i be as in the previous section so that one can define F_i as in Lemma 3.4. Let G_i, T_i, r_2, C_1 and C_2 be as in Lemma 3.5. We want to construct a biholomorphism from $\mathbb{C}^n$ onto M as follows: Let $\Omega_i = (G_i \circ \cdots \circ G_1)^{-1}(D(r_2))$ and define

$$S_i = \Phi_i \circ T_i^{-1} \circ G_i \circ \cdots \circ G_1,$$

which is defined on Ω_i by Lemma 3.5.

Theorem 1.5 will be proved if we can prove that S_i converges to a bihololomorphism from $\mathbb{C}^n$ onto M. We will prove this in several steps as described in the following lemmas.

Lemma 4.1. *For all $z \in \mathbb{C}^n$, $\lim_{i\to\infty} S_i(z) = S(z)$ exists.*

Proof. Let k be fixed and consider $U_k = (G_k \circ \cdots \circ G_1)^{-1}(\frac{1}{2C_2}D(r_2))$, where C_2, r_2 are as in Lemma 3.5(c). Let Ψ_k as in Lemma 3.5(c). Since the convergence in the lemma is uniform in $D(r_2)$, and $G_{k+1}^{-1} \circ G_{k+2}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+2} \circ F_{k+1}$ and Ψ_k are injective, we can find $0 < \rho < r_2$ and l_0 such that

$$\begin{aligned} &G_{k+1}^{-1} \circ G_{k+2}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+2} \circ F_{k+1}(D(\rho)) \\ &\quad \supset D\left(\frac{1}{2C_2} r_2\right) \end{aligned} \tag{4.1}$$

if $l \geq l_0$. Hence for every $l \geq l_0$ we have: for every $z \in U_k$, there exists $\zeta_l \in D(\rho)$ such that

$$G_1^{-1} \circ \cdots \circ G_k^{-1} \circ G_{k+1}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+2} \circ F_{k+1}(\zeta_l) = z. \tag{4.2}$$

Hence

$$\begin{aligned} S_{k+l}(z) &= \Phi_{k+l} \circ T_{k+l}^{-1} \circ G_{k+l} \circ \cdots \circ G_1(z) \\ &= \Phi_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+2} \circ F_{k+1}(\zeta_l) \\ &= \Phi_k(\zeta_l). \end{aligned} \tag{4.3}$$

Take two such subsequences ζ_{l_j} and $\zeta_{l'_j}$ such that $\zeta_{l_j} \to w$ and $\zeta_{l'_j} = w'$ as $j \to \infty$ with $w, w' \in \overline{D(\rho)}$. Since the convergence in Lemma 3.5 is uniform, by (4.2) we have

$$G_1^{-1} \circ \cdots \circ G_k^{-1} \circ \Psi_k(w) = z = G_1^{-1} \circ \cdots \circ G_k^{-1} \circ \Psi_k(w').$$

Hence we must have $w = w'$ and so $\zeta_l \to w$ as $l \to \infty$. By (4.3), we have

$$\begin{aligned} \lim_{l\to\infty} S_{k+l}(z) &= \Phi_k(w) \\ &= \Phi_k \circ \Psi_k^{-1} \circ G_k \circ \cdots \circ G_1(z). \end{aligned} \tag{4.4}$$

Hence $S = \lim_{i\to\infty} S_i$ exists on U_k. By Lemma 3.5, $\bigcup_k U_k = \mathbb{C}^n$, from this the lemma follows. □

Lemma 4.2. *S is a local biholomorphic map from $\mathbb{C}^n$ into M.*

Proof. This follows immediately from (4.4). □

Lemma 4.3. *For any $\epsilon > 0$, $\bigcup_k \Phi_k(D(\epsilon)) = M$.*

Proof. Since the Ricci curvature of $g(0)$ is positive, for any $R > 0$, we have $R_{i\bar{j}}(x,0) \geq a g_{i\bar{j}}(x,0)$ for some $a > 0$ for all $x \in B_0(p,R)$, which is the geodesic ball with respect to $g(0)$. Let $\lambda_i(x,t)$ be the eigenvalues of $R_{i\bar{j}}(x,t)$ with respect to $g(t)$. Then $\lambda_i(x,t) \leq C$ for some constant C independent of x and t. On the other hand, by Proposition 2.6, we have $\prod_{i=1}^n \lambda_i(x,t) \geq a^n$ for $x \in B_0(p,R)$. Hence there exists $b > 0$ independent of t such that

$$R_{i\bar{j}}(x,t) \geq b g_{i\bar{j}}(x,t)$$

for all $t \geq 0$ and $x \in B_0(p,R)$. By the Kähler–Ricci flow equation, we conclude that

$$B_0(p,R) \subset B_t(p, e^{-\frac{b+\kappa}{2}t} R).$$

From this and Lemma 3.1 the lemma follows. □

Lemma 4.4. *S is surjective.*

Proof. From the proof of Lemma 4.1, we conclude that

$$S(\mathbb{C}^n) \supset \Phi_k \circ \Psi_k^{-1}\left(D\left(\frac{1}{2C_2} r_2\right)\right)$$

for all k. From this, Lemma 4.3 and the proof of Lemma 3.5 (c) (see [9, §5]), it is easy to see that $S(\mathbb{C}^n) = M$. □

Lemma 4.5. *S is injective.*

Proof. Suppose there exists distinct $z_1, z_2 \in \mathbb{C}^n$ such that $S(z_1) = S(z_2) = q$. Let $\sigma(\tau)$, $0 \le \tau \le 1$ be the line segment from z_1 to z_2. Let $\gamma = S \circ \sigma$. Then γ is a smooth closed curve in M starting from q. Since M is simply connected, we can find a smooth homotopy $\alpha(s, \tau)$, $0 \le s, \tau \le 1$, with $\alpha(0, \tau) = \gamma(\tau)$, $\alpha(1, \tau) = q$ (the constant map), and $\alpha(s, 0) = \alpha(s, 1) = q$ for all s.

By Lemma 3.5, there exists $0 < \rho < r_2$ and $\eta > 0$ which are independent of i such that Ψ_i^{-1} is defined on $D(\eta)$ and

$$\Psi_i^{-1}(D(\eta)) \subset D\left(\frac{1}{8}\rho\right) \tag{4.5}$$

For $\rho > 0$ let $\rho_1 > 0$ be such that Lemma 3.2 is true. By the proof of Lemma 4.3, there exists k_0 such that if $k \ge k_0$, then $q \in B_k(p, \rho_1)$ and $L_k(\alpha(s, \cdot)) < \rho_1$ for all s, where $B_k(p, \rho_1)$ is the geodesic ball of radius ρ_1 at p relative to $g(kT)$, and L_k is the length with respect to $g(kT)$. We may also assume that $\sigma \subset G_1^{-1} \circ \cdots \circ G_k^{-1}(D(\eta))$, provided k_0 is large enough.

Now fix $k \ge k_0$. Let $\widetilde{\gamma} = \Psi_k^{-1} \circ G_k \circ \cdots \circ G_1 \circ \sigma$. Then $\widetilde{\gamma} \subset D(\frac{\rho}{8})$ by (4.5) and it is a lift of γ by Φ_k by (4.4). Moreover, if $\widetilde{\gamma}(0) = w_1$, and $\widetilde{\gamma}(1) = w_2$, then $w_1 \ne w_2$ because G_i and Ψ_k are injective. Since $\Phi_k(w_1) = \Phi_k(w_2) = q \in B_k(p, \rho_1)$, by Lemma 3.2, for any s, there is a lift $\widetilde{\beta}_s(\tau)$ of $\alpha(s, \tau)$ by Φ_k such that $\widetilde{\beta}_s(0) = w_1$ and $\widetilde{\beta}_s \subset D(\frac{\rho}{2})$.

We claim that $\widetilde{\beta}_s(1) = w_2$. Let $\epsilon > 0$ be such that Φ_k is a biholomorphism onto its image when restricted on $B_e(w_2, \epsilon)$. For such $\epsilon > 0$, let $\delta > 0$ be as in Lemma 3.3. On the other hand, let $\xi > 0$ be such that

$$d_k(\alpha(s_1, \tau), \alpha(s_2, \tau)) < \delta \tag{4.6}$$

for all τ, if $|s_1 - s_2| \le \xi$.

Hence by Lemma 3.3, if $0 \le s \le \xi$ then $\widetilde{\beta}_s(1) \in B_e(w_2, \epsilon)$. But $\Phi_k(\widetilde{\beta}_s(1)) = \alpha(s, 1) = q$, and Φ_k is injective on $B_e(w_2, \epsilon)$. Thus we have $\widetilde{\beta}_s(1) = w_2$ for $0 \le s \le \xi$. In particular, $\widetilde{\beta}_\xi(1) = w_2$. By (4.6) and Lemma 3.3, we can argue as before and conclude that $\widetilde{\beta}_{2\xi}(1) = w_2$. Continue in this way, we have that $\widetilde{\beta}_1(1) = w_2$. On the other hand, $\widetilde{\beta}_1$ is a lift of $\alpha(1, \cdot)$. Hence $\Phi_k(\widetilde{\beta}_1(\tau)) = q$ for all τ. Since $w_1 \ne w_2$, there is τ with $\tilde{\beta}_1(\tau) \ne w_2$ and $\tilde{\beta}_1(\tau) - w_2| \le \frac{\epsilon}{2}$. This is impossible because Φ_k is a injective on $B_e(w_2, \epsilon)$. □

Theorem 1.5 now follows from Lemmas 4.2, 4.4 and 4.5.

Corollary 4.6. *Let $(M, g(t))$ be as in Theorem 1.5 and assume that $g(0)$ has positive Ricci curvature at some point $p \in M$. Then M is biholomorphic to $\mathbb{C}^n$. In particular, if (M, g) is as in Theorem 1.2 and the Ricci curvature of g is positive at some point, then M is biholomorphic to $\mathbb{C}^n$.*

Proof. Let $(M, g(t))$ as in Theorem 1.5. We begin by proving that M has finite fundamental group. Suppose M has infinite fundamental group and let $\widetilde{M}$ be the universal cover of M. Then $\widetilde{M}$ contains infinitely many preimages $p_0, p_1, \dots$ of $p \in M$, and for simplicity we will denote p_0 simply by p. We may also pull $g(t)$ back to obtain a solution to (1.6) on $\widetilde{M}$, and we denote this solution again by $g(t)$. Note that by [6], $(\widetilde{M}, g(t))$ must have positive Ricci curvature on all of $\widetilde{M}$ and for all $-\infty < t < \infty$ since $\widetilde{M}$ is simply connected and $Ric(p, 0) > 0$ by assumption. Now for $(\widetilde{M}, g(t), p)$ consider $\Phi_t : D(r) \to \widetilde{M}$ be as in Proposition 2.1. Then by choosing r smaller, if necessary, we may have the following for all $t \geq 0$:

(a1) $Ric_t \geq c > 0$ in $D(r)$ with respect to $\Phi_t^*(g(t))$ for some constant c.

(a2) There exists $0 < \delta < 1$ such that any point in $D(\delta r)$ can be joined by a geodesic from p in $D(\frac{2}{3}r)$ with respect to $\Phi_t^*(g(t))$.

(a3) There exists $C > 1$ such that $B_t(p, \frac{1}{C}\rho) \subset \Phi_t(D(\rho)) \subset B_t(p, C\rho)$ for all $\rho < r$.

(a1) follows from Propositions 2.1 and 2.3 and the fact that the covariant derivatives of the curvature tensor is bounded. (a2) follows from Proposition 2.1 and the construction of Φ_t through the exponential map so that the injectivity radius of $\Phi_t^*(g(t))$ at the origin is uniformly bounded below by a positive constant. (a3) follows from the proof of Lemma 3.1.

Note that we must have $\lim_{k\to\infty} d_{g(0)}(p_k, p) = \infty$. Also, by the proof of Lemma 4.3, we know that the sets $B_t(p, \epsilon r)$, where $\epsilon = \frac{\delta}{C}$, exhaust M as $t \to \infty$. Thus, we can find a subsequence of p_k, also denoted by p_k, and a sequence $t_k \to \infty$ such that $p_k \in \partial B_{t_k}(p, \epsilon r)$.

Now consider the sequence of pull-backs $\Phi_{t_k}^*(g(t_k + t)$ on $D(r)$. Then by Theorem 2.4 we may assume the sequence converges on $D(r)$ to some $h(t)$ satisfying $R_{i\bar{\jmath}} + \kappa h_{i\bar{\jmath}} = f_{i\bar{\jmath}}$ at $t = 0$, where f is smooth, $f_{ij} = 0$ and $\nabla f(0) = 0$. Now for each k, let z_k be an inverse image of p_k under Φ_{t_k} (which exists by our construction of p_k). Then $C^{-2}\delta \leq |z_k| \leq \delta r$ for all k by (a3), and hence we may also assume that $z_k \to w$, where $C^{-2}\delta r \leq |w| \leq \delta r$.

Moreover, if σ is the covering map that maps p_k to p, then $g(t) = \sigma^*(g(t))$.

For any p_k, let $F : \widetilde{M} \to \widetilde{M}$ be a deck transformation taking p_k to p. Then for every t, F is an isometry with respect to $g(t_k + t)$. Hence, for every k, $\Phi_{t_k}^*(g(t_k + t))$ is the 'same' at the origin o and z_k, and in particular $R(0, t) = R(z_k, t)$ for all t. Thus, by letting $k \to \infty$, we have that $R(w, t)$ is constant in time since $R(o, t)$ is constant in time. Furthermore, by the proof of Theorem 2.4, we may have that $\nabla f(w) = 0$.

Now by (a2), we can join o to w by a geodesic w.r.t. $h(0)$ parametrized by arc-length. As in [18: Proof of Theorem 20.1], we may compute in real coordinates x_α as follows:

$$\frac{d^2 f}{ds^2} = D_\alpha D_\beta f \cdot \frac{dx_\alpha}{ds} dx_\beta > 0$$

since $R_{i\bar{j}} > 0$, $R_{i\bar{j}} + \kappa h_{i\bar{j}} = f_{i\bar{j}}$ and $f_{ij} = 0$. But this is impossible as

$$\frac{df}{ds} = 0$$

at the end points o and w. We have thus proved, by contradiction, that the fundamental group of M must be finite.

It now follows by a well-known result (see [2, p. 35 and p. 40]) that M must be simply connected, and thus biholomorphic to $\mathbb{C}^n$ by the conclusion of Theorem 1.5. This completes the proof of the corollary. □

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