COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 15, Number 1, 77–119, 2007

Newton polygon and string diagram

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In this paper we study the moment map images of curves in toric surfaces. We are particularly interested in the situations when we can perturb the moment map so as to make the image of algebraic curves to be a graph.

1. Introduction

In this paper, we study the moment map image of algebraic curves in toric surfaces. We are particularly interested in the situations that we are able to perturb the moment map so that the moment map image of the algebraic curve is a graph. To put our problem into proper context, let us start with $\mathbb{CP}^2$.

Consider the natural real n-torus (T^n) action on $\mathbb{CP}^n$ given by

$$e^{i\theta}(x) = (e^{i\theta_1}x_1, e^{i\theta_2}x_2, \ldots, e^{i\theta_n}x_n).$$

The T^n acts as symplectomorphisms with respect to the Fubini-Study Kählerform

$$\omega_{\mathrm{FS}} = \partial\bar{\partial}\log(1+|x|^2).$$

The corresponding moment map is

$$F(x) = \left(\frac{|x_1|^2}{1+|x|^2}, \frac{|x_2|^2}{1+|x|^2}, \ldots, \frac{|x_n|^2}{1+|x|^2}\right),$$

which is easy to see if we write ω_{FS} in polar coordinates.

$$\omega_{\mathrm{FS}} = \partial\bar{\partial}\log(1+|x|^2) = i\sum_{k=1}^{n} d\theta_k \wedge d\left(\frac{|x_k|^2}{1+|x|^2}\right).$$

Notice that the moment map F is a Lagrangian torus fibration and the image of the moment map $\Delta = \mathrm{Image}(F)$ is an n-simplex.

In the case of $\mathbb{CP}^2$, $\Delta = \mathrm{Image}(F)$ is a two-simplex, i.e., a triangle. Let $p(z)$ be a homogeneous polynomial. p defines an algebraic curve C_p in $\mathbb{CP}^2$. We want to understand the image of C_p in Δ under the moment map F.

In quantum mechanics, particle interactions are characterized by Feynman diagrams (one-dimensional graphs with some external legs). In string theory, point particles are replaced by circles (string!) and Feynman diagrams are replaced by string diagrams (Riemann surfaces with some marked points). Feynman diagrams in string theory are considered as some low energy limit of string diagrams. *Fattening* the Feynman diagrams by replacing points with small circles, we get the corresponding string diagrams. On the other hand, string diagrams can get "thin" in many ways to degenerate to different Feynman diagrams.

Our situation is a very good analog of this picture. The complex curve C_p in $\mathbb{CP}^2$ can be seen as a string diagram with the intersection points with the three distinguished coordinate $\mathbb{CP}^1$'s (that are mapped to $\partial\Delta$) as marked points. The image of C_p under F can be thought of as some "fattening" of a Feynman diagram Γ in Δ with external points in $\partial\Delta$.

When p is of degree d, the genus of C_p is

$$g = \frac{(d-1)(d-2)}{2}.$$

Generically, C_p will intersect with $\mathbb{CP}^1$ at d points. Ideally, the image of C_p under the moment map will have g holes in Δ and d external points in each edge of Δ. In general $F(C_p)$ can have smaller number of holes. In fact, $F(C_p)$ has at most g holes. (For more detail, please see the "Note on the literature" in the end of the introduction.)

In this paper, we will be interested in constructing examples of C_p such that $F(C_p)$ will have exactly g holes in Δ and d external points in each edge of Δ. Namely, the case when $F(C_p)$ resembles classical Feynman diagrams the most. (Sort of the most classical string diagram.) These examples will be constructed for any degree in Section 2.

Our interest on this problem comes from our work on Lagrangian torus fibration of Calabi–Yau manifolds and symplectic version of SYZ conjecture ([10]) on mirror symmetry. In [5–7], we mainly concern the case of quintic curves in $\mathbb{CP}^2$. The generalization to curves in toric surfaces will be useful in [8,9]. The algebraic curves and their images under the moment map arise as the singular set and singular locus of our Lagrangian torus fibrations.

As we mentioned, $F(C_p)$ can be rather chaotic for general curve C_p. The condition for $F(C_p)$ to resemble a classical Feynman diagram is related to the concept of "near the large complex limit," which is explained in Section 3. (Through discussion with Qin Jing, it is apparent that near the large complex limit is equivalent to near the Zero-dimensional strata in $\overline{\mathcal{M}_g}$, the moduli space of stable curves of genus g. These points in $\overline{\mathcal{M}_g}$ are represented by stable curves, whose irreducible components are all $\mathbb{CP}^1$ with three marked points.) It turns out that our construction of "graph-like" string diagrams for curves in $\mathbb{CP}^2$ can be generalized to curves in general two-dimensional toric varieties using localization technique. More precisely, in the moduli space of curves in a general two-dimensional toric variety, when the curve C_p is close enough to the so-called "large complex limit" (analogous to classical limit in physics) in suitable sense, $F(C_p)$ will resemble a fattening of a classical Feynman diagram. This result will be made precise and proved in theorems 3.10 and 3.14 of Section 3. Examples constructed in Section 2 are special cases of this general construction.

One advantage of string theory over classical quantum mechanic is that the string diagrams (marked Riemann surfaces) are more natural than Feynman diagrams (graphs). For instance, one particular topological type of string diagram under different classical limit can degenerate into very different Feynman diagrams, therefore unifying them. In our construction, there is a natural partition of the moduli space of curves such that in different part the limiting Feynman diagrams are different. We will discuss this natural partition of the moduli space and different limiting Feynman diagrams also in Section 3.

Of course, ideally, it will be interesting if $F(C_p)$ is actually a one-dimensional Feynman diagram Γ in Δ. This will not be true for the moment map F. A natural question is: "Can one perturb the moment map F to $\hat{F}$ so that $\hat{F}(C_p) = \Gamma$?" (Notice that the moment map of a torus action is equivalent to a Lagrangian torus fibration. We will use the two concepts interchangeably in this paper.) Such perturbation is not possible in the smooth category. But when $F(C_p)$ resembles a classical Feynman diagram Γ close enough, we can perturb F suitably as a moment map, so that the perturbed moment map $\hat{F}$ is piecewise smooth and satisfies $\hat{F}(C_p) = \Gamma$. This perturbation construction is explicitly done for the case of line in $\mathbb{CP}^2$ in Section 4 (Theorems 4.3 and 4.8). The general case is dealt with in Section 5 (Theorems 5.11 and 5.14) combining the localization technique in Section 3 and the perturbation technique in Section 4. (In particular, optimal smoothness for $\hat{F}$ is achieved in Theorems 4.8 and 5.14.)

Note on the literature: Our work on Newton polygon and string diagram was motivated by and was an important ingredient of our construction of Lagrangian torus fibrations of Calabi-Yau manifolds [5–9]. After reading my preprint, Prof. Y.-G. Oh pointed out to me the work of Mikhalkin [4], through which I was able to find the literature of our problem. The image of curves under the moment map was first investigated in [3], where it was called "amoeba". Legs of amoeba are already understood in [3]. The problem of determining holes in amoeba was posed in [3, Remark 1.10, p. 198] as a difficult and interesting problem. Work of Mikhalkin [4] that was published in 2000 and works [2, 11] mentioned in its reference point out some previous progress on this problem of determining holes in amoeba aimed at very different applications, which nevertheless is very closely related to our work. Most of the ideas in Sections 2 and 3 are not new and appeared in one form or the other in these previous works mentioned. For example, our localization technique used in Section 3 closely resemble the curve patching idea of Viro (which apparently appeared much earlier) in different context as described in [4]. Due to different purposes, our approach and results are of somewhat distinctive flavor. To our knowledge, our discussion in Sections 4 and 5 on symplectic deformation to Lagrangian fibrations with the image of curve being graph, which is important for our applications, was not discussed before and is essentially new. I also want to mention that according to the description in [4] of a result of Forsberg et al. [2], one can derive that there are at most $g = \frac{(d-1)(d-2)}{2}$ holes in $F(C_p)$ for degree d curve C_p, which I initially conjectured to be true.

Note on the figures: The figures of moment map images of curves as fattening of graphs in this paper are somewhat idealized topological illustration. Some part of the edges of the image that are straight or convex could be curved or concave in more accurate picture. Of course, such inaccuracy will not affect our mathematical argument and the fact that moment map images of curves are fattening of graphs.

Notion of convexity: A function $y = f(x)$ will be called convex if the set $\{(x, y) : y \geq f(x)\}$ is convex. We are aware such functions have been called concave by some authors.

2. The construction for curves in $\mathbb{CP}^2$

To understand our problem better, let us look at the example of Fermat type polynomial

$$p = z_1^d + z_2^d + z_3^d.$$

It is not hard to see that for any d, $F(C_p)$ will look like a curved triangle with only one external point in each edge of Δ and no hole at all (figure 1). (This example is in a sense a string diagram with the most quantum effect.)

From this example, it is not hard to imagine that for most polynomials, chances are the number of holes will be much less than g. Any attempt to construct examples with the maximal number of holes will need special care, especially if one wants the construction for general degree d.

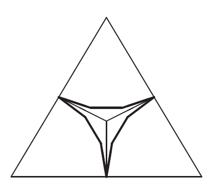

Figure 1: $F(C_p)$ of $p(z) = z_1^d + z_2^d + z_3^d$.

Let $[z] = [z_1, z_2, z_3]$ be the homogeneous coordinate of $\mathbb{CP}^2$. Then a general homogeneous polynomial of degree d in z can be expressed as

$$p(z) = \sum_{I \in N^d} a_I z^I,$$

where

$$N^d = \{I = (i_1, i_2, i_3) \in \mathbb{Z}^3 | |I| = i_1 + i_2 + i_3 = d, I \geq 0\}$$

is the Newton polygon of degree d homogeneous polynomials. In our case, N^d is a triangle with $d+1$ lattice points on each side. Denote $E = (1, 1, 1)$.

To describe our construction, let us first notice that N^d can be naturally decomposed as a union of "hollow" triangles as follows:

$$N^d = \bigcup_{k=0}^{[d/3]} N_k^d,$$

where

$$N_k^d = \{I \in N^d | I \geq kE, I \not\geq (k+1)E\}.$$

On the other hand, the map $I \to I + E$ naturally defines an embedding $i : N^d \to N^{d+3}$. From this point of view, $N_0^d = N^d \backslash N^{d-3}$ and $N_k^d = i(N_{k-1}^{d-3})$ for $k \geq 1$.

When $d = 1$, $g = 0$ and a generic degree 1 polynomial can be reduced to

$$p = z_1 + z_2 + z_3.$$

$F(C_p)$ is a triangle with vertices as middle points of edges of Δ. This clearly satisfies our requirement, namely, with $g = 0$ holes.

For $d \geq 2$, the first problem is to make sure that the external points are distinct and as far apart as possible. For this purpose, we want to consider homogeneous polynomials with two variables. A nice design is to consider

$$q_d(z_1, z_2) = \prod_{i=1}^{d} (z_1 + t_i z_2) = \sum_{i=0}^{d} b_i z_1^{d-i} z_2^i$$

such that $t_{d-i+1} = \frac{1}{t_i} \geq 1$. Then $b_0 = b_d = 1$ and $b_{d-i} = b_i \geq 1$ for $i \geq 1$. We can adjust t_i for $1 \leq i \leq [d/2]$ suitably to make them far apart. (For example, one may assume $F(t_i) = \frac{2i-1}{2d}$, where $F(t) = \frac{t^2}{1+t^2}$ is the moment map for $n = 1$.) Now we can define a degree d homogeneous polynomial in three variables such that coefficients along each edge of N_d is assigned according to q and coefficients in the interior of N_d vanish. We will still denote this polynomial by q_d. Then we have

Theorem 2.1. *For*

$$p_d(z) = \sum_{k=0}^{[d/3]} c_k q_{d-3k}(z) z^{kE},$$

where $c_0 = 1$ and $c_k > 0$, if c_k is big enough compared to c_{k-1}, then $F(C_{p_d})$ has exactly g holes and d external points in each edge of Δ.

Before proving the theorem, let us analyse some examples that give us better understanding of the theorem. Figures 2–5 are some examples of Feynman diagrams corresponding to $1 \leq d \leq 5$. (The case $d = 5$ is the case we are interested in mirror symmetry.) The image of the corresponding string diagrams under the moment map F are some fattened version of these Feynman diagram. For example one can see genus of the corresponding Riemann surfaces from these diagrams. When $d = 1, 2$ there are no holes in the diagram and genus equal to zero. When $d = 5$ there are six holes and the corresponding Riemann surface are genus six curves. These diagrams give a very nice interpretation of genus formula for planar curve. (In my opinion, also a good way to remember it!)

To justify our claim, we first analyse it case by case. When $d = 1$, $g = 0$ and a generic polynomial can be reduced to

$$p_1(z) = z_1 + z_2 + z_3.$$

$F(C_p)$ is a triangle with vertices as middle points of edges of Δ. This clearly is a fattened version of the first diagram in the next picture.

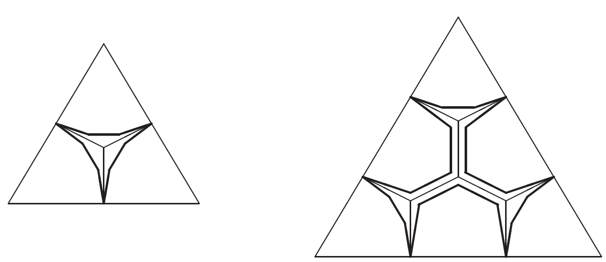

Figure 2: Degree $d = 1, 2$.

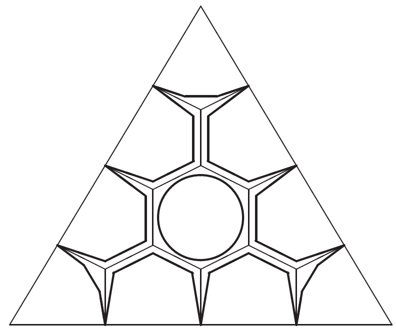

Figure 3: Degree $d = 3$.

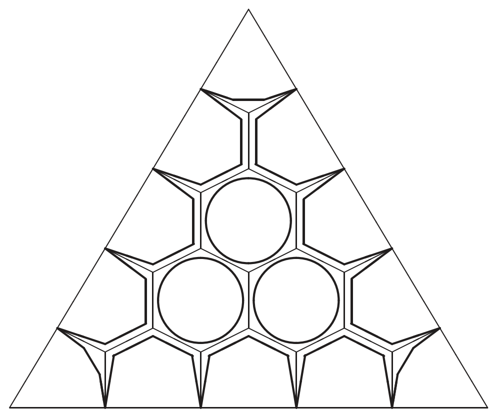

Figure 4: Degree $d = 4$.

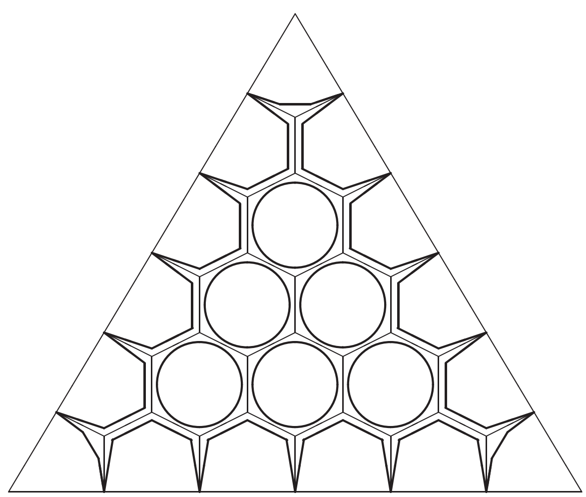

Figure 5: Degree $d = 5$.

When $d = 2$, $g = 0$ and we may take polynomial

$$p_2(z) = (z_1^2 + z_2^2 + z_3^2) + \frac{5}{2}(z_1z_2 + z_2z_3 + z_3z_1).$$

When $z_3 = 0$

$$q_2(z_1, z_2) = z_1^2 + z_2^2 + \frac{5}{2}z_1z_2 = \left(z_1 + \frac{1}{2}z_2\right)(z_1 + 2z_2).$$

Image of $\{q_2(z_1, z_2) = 0\}$ under F are two lines coming out of the edge $r_3 = 0$ starting from the two points $\frac{r_1}{r_2} = 2, \frac{1}{2}$. When z_3 is small, $p_2(z)$ is a small perturbation of $q_2(z_1, z_2)$. By this argument, it is clear that $F(C_2)$ is a fattening of the second diagram figure 2 near the boundary of the triangle. Since in our case $g = 0$. It is not hard to conceive or (if you are more strict) to find a way to prove that $F(C_2)$ is a fattening of the second diagram in figure 2.

When $d = 3$, $g = 1$. We can consider

$$\begin{aligned} p_3(z) = (z_1^3 + z_2^3 + z_3^3) + \frac{7}{2}(z_1^2z_2 + z_2^2z_3 + z_3^2z_1 \\ + z_1z_2^2 + z_2z_3^2 + z_3z_1^2) + bz_1z_2z_3. \end{aligned}$$

We can use similar idea as in the previous case to explain the behavior of $F(C_3)$ near the edges. The main point for this case is to explain how the hole in the center arises. For this purpose, we introduce the following function:

$$\rho_p(r) = \inf_{F(z)=r} |p(z)|.$$

This function takes non-negative value, and

$$F(C_p) = \{[r] | \rho_p(r) = 0\}.$$

ρ also satisfies

$$\begin{aligned} \rho_{p_1p_2} &= \rho_{p_1}\rho_{p_2}, \\ \rho_{p_1+p_2} &\leq \rho_{p_1} + \rho_{p_2}. \end{aligned}$$

An important thing to notice is that $\rho_{z_1z_2z_3}(r) = r_1r_2r_3$ is a function that vanishes at the edges of the triangle and not vanishing anywhere in the interior of the triangle, sort of a bump function. When b is large, ρ_{p_3} will be dominated by $br_1r_2r_3$ away from the edges, which will be positive around center. Therefore $F(C_3)$ will have a hole in the center, which becomes large when b gets large.

When $d = 4$, $g = 3$. We can consider

$$p_4(z) = q_4(z) + bz_1z_2z_3p_1(z).$$

The key point is to understand how the three holes appear. For this purpose, we need to go back to the case when $d = 1$. Notice that $\rho_{z_1z_2z_3p_1} = r_1r_2r_3\rho_{p_1}$ is positive in the three regions as indicated in the diagram for $d = 1$, and it is zero at the boundary of the three regions. When b is large, this term dominates ρ_{p_4} in the interior of the triangle and produces the three holes. Similar discussion as before implies that q_4 will take care of edges.

When $d = 5$, $g = 6$. We need to go back to the case $d = 2$. The discussion is very similar to the previous case, we will omit.

Proof of Theorem 2.1. We prove by induction. For this purpose, notice that we can define $p_d(z)$ alternatively by induction

$$p_d(z) = q_d(z) + b_dz^Ep_{d-3}(z).$$

We need to show that when b_d are large enough for any d, $F(C_{p_d})$ will have $g = \frac{(d-1)(d-2)}{2}$ holes and d external legs in each edge.

Assume above statement is true for $p_{d-3}(z)$, then $F(C_{p_{d-3}})$ will have $g = \frac{(d-4)(d-5)}{2}$ holes and $d-3$ external legs in each edge. It is easy to see that $F(C_{z^Ep_{d-3}})$ will have $g = \frac{(d-4)(d-5)}{2}$ interior holes and $3(d-3)$ side holes that are partly bounded by edges. We are expecting that by adding q_d term, side holes will become interior hole and there will be d external legs on each edge.

Discussion in previous special examples will more or less do this. Here we can do better. We can actually write down explicitly the behavior of $F(C_p)$ near edges. For example, near the edge $z_3 = 0$, $p_d(z) = 0$ can be rewritten as

$$z_3 = -\frac{q_d(z)}{b_dz_1z_2p_{d-3}(z)}.$$

This is a graph over the coordinate line $z_3 = 0$ within say $|z_3| \leq \epsilon \min(|z_1|, |z_2|)$ and away from $z_1 = 0$, $z_2 = 0$ and $d-3$ leg points of $p_{d-3}(z)$. It will be clearer to discuss under local coordinate say $x_1 = \frac{z_1}{z_2}$, $x_3 = \frac{z_3}{z_2}$. We will use the same symbol for homogeneous polynomials and the corresponding inhomogeneous polynomials. Then under this inhomogeneous coordinate

$$x_3 = -\frac{q_d(x_1, x_3)}{b_dx_1p_{d-3}(x_1, x_3)}.$$

Asymptotically, near $x_3 = 0$

$$x_3 = -\frac{q_d(x_1)}{b_d x_1 q_{d-3}(x_1)}.$$

From previous notation $q_d(x_1) = q_d(x_1, 0) = p_d(x_1, 0)$, and

$$q_d(x_1) = \prod_{i=1}^{d} (x_1 - t_{d,i}).$$

Recall that we require $|t_{d,i}|$ to be as far apart as possible for different d, i. From this explicit expression, it is easy to see that near $z_3 = 0$ (say $|x_3| \leq \epsilon$) and away from $z_2 = 0$, C_{p_d} is a graph over the $\mathbb{CP}^1$ $(z_3 = 0)$ away from disks

$$|x_1 - t_{d-3,i}| \leq \frac{q_d(t_{d-3,i})}{t_{d-3,i} q'_{d-3}(t_{d-3,i})} \frac{1}{\epsilon b_d} \quad \text{for } 1 \leq i \leq d-3,$$

and

$$|x_1| \leq \frac{q_d(0)}{q_{d-3}(0)} \frac{1}{\epsilon b_d}.$$

Recall b_d is supposed to be large. Here we further require the choice of ϵ to satisfy ϵ is small and ϵb_d is large. Therefore, all these holes are very small. It is easy to see that the $d-3$ small circles centered around the roots of q_{d-3} will connect with $d-3$ legs of $C_{p_{d-3}}$. In this way, the side holes of $F(C_{z^E p_{d-3}})$ will become interior holes of $F(C_p)$. Together with original interior holes they add up to

$$\frac{(d-4)(d-5)}{2} + 3(d-3) = \frac{(d-1)(d-2)}{2} = g_d$$

interior holes for $F(C_{p_d})$. d zeros of q_d along each edge will produce for us the d exterior legs on each edge. Namely $F(C_d)$ is fattening of the Feynman diagrams as described in previous pictures. $\square$

3. Newton polygon and string diagram

The result in the previous section is actually special cases of a more general result on curves in toric surfaces. When the coefficients of the defining equation of a curve in a general toric surface satisfy certain convexity conditions (in physical term: near the large complex limit), the moment map image (amoeba) of the curve in the toric surface will also resemble the fattening of a graph. The key idea that enables such generalization is the so-called "localization technique" that reduces the amoeba of our curve near the large complex limit locally to the amoeba of a line, which is well understood.

We start with toric terminologies. Let M be a rank 2 lattice and $N = M^\vee$ denotes the dual lattice. For any $\mathbb{Z}$-module $\mathbb{A}$, let $N_{\mathbb{A}} = N \otimes_{\mathbb{Z}} \mathbb{A}$. Given an integral polygon $\Delta \subset M$, we can naturally associate a fan Σ by the construction of normal cones. For a face α of the polygon Δ, define the normal cone of α

$$\sigma_\alpha := \{n \in N | \langle m', n\rangle \leq \langle m, n\rangle \quad \text{for all } m' \in \alpha, m \in \Delta\}.$$

Let Σ denote the fan that consists of all these normal cones. We are interested in the corresponding toric variety P_Σ. Let $\Sigma(1)$ denote the collection of one-dimensional cones in the fan Σ, then any $\sigma \in \Sigma(1)$ determines a $N_{\mathbb{C}^*}$-invariant Weil divisor D_σ.

For $m \in M$, $s_m = \mathrm{e}^{\langle m,n\rangle}$ defines a monomial function on $N_{\mathbb{C}^*}$ that extends to a meromorphic function on P_Σ. Let e_σ denote the unique primitive element in $\sigma \in \Sigma(1)$. The Cartier divisor

$$(s_m) = \sum_{\sigma \in \Sigma(1)} \langle m, e_\sigma\rangle D_\sigma.$$

Consider the divisor

$$D_\Delta = \sum_{\sigma \in \Sigma(1)} l_\sigma D_\sigma, \quad \text{where} l_\sigma = -\inf_{m \in \Delta} \langle m, e_\sigma\rangle.$$

The corresponding line bundle $L_\Delta = \mathcal{O}(D_\Delta)$ can be characterized by the piecewise linear function p_Δ on N that satisfies $p_\Delta(e_\sigma) = l_\sigma$ for any $\sigma \in \Sigma(1)$. It is easy to see that p_Δ is strongly convex with respect to the fan Σ, hence L_Δ is ample on P_Σ. Since $(s_m) + D_\Delta$ is effective if and only if $m \in \Delta$, $\{s_m\}_{m\in\Delta}$ can be identified with the set of $N_{\mathbb{C}^*}$-invariant holomorphic sections of L_Δ. In this sense, the polygon Δ is usually called the *Newton polygon* of the line bundle L_Δ on P_Σ. A general section of L_Δ can be expressed as

$$s = \sum_{m \in \Delta} a_m s_m.$$

$C_s = s^{-1}(0)$ is a curve in P_Σ. We can consider the image of the curve C_s under some moment map of P_Σ. The problem we are interested in is when this image will form a fattening of a graph. The case discussed in the last section is a special case of this problem, corresponding to the situation of $P_\Sigma \cong \mathbb{CP}^2$ and $L_\Delta \cong \mathcal{O}(k)$.

With $w = \{w_m\}_{m\in\Delta} \in \tilde{N}_0 \cong \mathbb{Z}^\Delta$, we can define an action of $\delta \in \mathbb{R}_+$ on sections of L_Δ.

$$s^{\delta^w} = \delta(s) = \sum_{m\in\Delta} (\delta^{w_m} a_m) s_m.$$

$$A = \{\{l + \langle m, n\rangle\}_{m\in\Delta} \in \tilde{N}_0 : (l,n) \in N^+ = \mathbb{Z} \oplus N\} \subset \tilde{N}_0$$

is the sublattice of affine functions on Δ. An element $[w] \in \tilde{N} = \tilde{N}_0/A$ can be viewed as an equivalent class of $\mathbb{Z}$-valued functions $w = (w_m)_{m\in\Delta}$ on Δ modulo the restriction of affine function on M.

When $w = \{w_m\}_{m\in\Delta} \in \tilde{N}_0$ is a strictly convex function on Δ, w determines a simplicial decomposition Z of Δ. Clearly every representative of $[w] \in \tilde{N}$ determines the same simplicial decomposition Z of Δ. Let $\tilde{S}$ (resp. $\tilde{S}^{\rm top}$) be the set of $S \subset \Delta$ that forms a simplex (resp. top dimensional simplex) containing no other integral points. Then Z can be regarded as a subset of $\tilde{S}$. Let $Z^{\rm top} = Z \cap \tilde{S}^{\rm top}$.

From now on, assume $|a_m| = 1$ for all $m \in \Delta$. $|s_m| = |\mathrm{e}^{\langle m,n\rangle}|$ is a function on $N_{\mathbb{C}^*} \subset P_\Sigma$. Let

$$h_{\delta^w} = \log |s^{\delta^w}|^2_\Delta, \quad \text{where} |s^{\delta^w}|^2_\Delta = \sum_{m\in\Delta} |s^{\delta^w}|^2_m, \quad |s^{\delta^w}|_m = |s^{\delta^w}_m| = |\delta^{w_m} s_m|.$$

$\omega_{\delta^w} = \partial\bar\partial h_{\delta^w}$ naturally defines a $N_{\mathbb{S}}$-invariant Kählerform on P_Σ, where $\mathbb{S}$ denotes the unit circle in $\mathbb{C}^*$ as $\mathbb{Z}$-submodule.

Choose a basis n_1, n_2 of N, then $n \in N_{\mathbb{C}}$ can be expressed as

$$n = \sum_{k=1}^{2} (\log x_k) n_k = \sum_{k=1}^{2} (\log r_k + i\theta_k) n_k.$$

Under this local coordinate, the Kählerform ω_{δ^w} can be expressed as

$$\omega_{\delta^w} = \partial\bar\partial h_{\delta^w} = i \sum_{k=1}^{2} d\theta_k \wedge dh_k, \quad \text{where } h_k = |x_k|^2 \frac{\partial h_{\delta^w}}{\partial |x_k|^2}.$$

It is straightforward to compute that

$$h_k = \sum_{m\in\Delta} \langle m, n_k\rangle \rho_m, \quad \text{where } \rho_m = \frac{|s^{\delta^w}|^2_m}{|s^{\delta^w}|^2_\Delta}.$$

Consequently,

$$\omega_{\delta^w} = i\sum_{m\in\Delta} d\langle m,\theta\rangle \wedge d\rho_m, \quad \text{where } \theta = \sum_{k=1}^{2}\theta_k n_k.$$

Lemma 3.1. *The moment map is*

$$F_{\delta^w}(x) = \sum_{m\in\Delta}\rho_m(x)m,$$

which maps P_Σ to Δ.

By this map, $N_{\mathbb{S}}$-invariant functions h, h_k, ρ_m on P_Σ can all be viewed as functions on Δ. We have

Lemma 3.2. *ρ_m as a function on Δ achieves its maximum exactly at $m\in\Delta$.*

Proof. By $x_k\dfrac{\partial|s^{\delta^w}|_m^2}{\partial x_k} = \langle m,n_k\rangle|s^{\delta^w}|_m^2$, ρ_m achieves maximal implies

$$\begin{aligned}\sum_{k=1}^{2}x_k\frac{\partial\rho_m}{\partial x_k}m_k &= \rho_m\sum_{m'\in\Delta}\sum_{k=1}^{2}(\langle m,n_k\rangle - \langle m',n_k\rangle)\rho_{m'}m_k\\ &= \rho_m\sum_{m'\in\Delta}(m-m')\rho_{m'} = \rho_m(m-F_{\delta^w}(x)) = 0.\end{aligned}$$

Therefore $F_{\delta^w}(x) = m$ when ρ_m achieves maximal. □

Lemma 3.3. *For any subset $S\subset\Delta$, $\rho_S = \sum_{m\in S}\rho_m$ as a function on Δ achieves maximum in the convex hull of S. At the maximal point of ρ_S*

$$F_{\delta^w}(x) = \sum_{m\in S}\rho_m^S m = \sum_{m\notin S}\rho_m^{S^c}m, \quad \textit{where } \rho_m^S = \frac{\rho_m}{\rho_S}, \quad S^c = \Delta\setminus S.$$

Proof. By $x_k\dfrac{\partial|s^{\delta^w}|_m^2}{\partial x_k} = \langle m,n_k\rangle|s^{\delta^w}|_m^2$, $\rho_S = \sum_{m\in S}\rho_m$ achieves maximal implies

$$\begin{aligned}\sum_{m\in S}\sum_{k=1}^{2}x_k\frac{\partial\rho_m}{\partial x_k}m_k &= \sum_{m\in S}\rho_m\sum_{m'\in\Delta}\sum_{k=1}^{2}(\langle m,n_k\rangle - \langle m',n_k\rangle)\rho_{m'}m_k\\ &= \sum_{m\in S}\rho_m\sum_{m'\in\Delta}(m-m')\rho_{m'} = \sum_{m\in S}\rho_m(m-F_{\delta^w}(x)) = 0.\end{aligned}$$

Therefore

$$F_{\delta^w}(x) = \sum_{m\in S} \rho_m^S m = \sum_{m\in S} \frac{|s^{\delta^w}|_m^2}{|s^{\delta^w}|_S^2} m, \quad \text{where } |s^{\delta^w}|_S^2 = \sum_{m\in S} |s^{\delta^w}|_m^2,$$

when $\rho_S = \sum_{m\in S} \rho_m$ achieves maximal. It is easy to derive

$$F_{\delta^w}(x) = \sum_{m\in S} \rho_m^S m = \sum_{m\notin S} \rho_m^{S^c} m.$$

□

Lemma 3.4. *There exists a constant $a > 0$ (independent of δ) such that for any $x \in P_\Sigma$ the set*

$$S_x = \{m \in \Delta | \rho_m(x) > \delta^a\}$$

is a simplex in Z.

Proof. Take a maximal subset $\tilde{S}_x \subset S_x$ that forms a simplex, which is allowed to contain no integral points in $S_x \setminus \tilde{S}_x$. Clearly, S_x is in the affine span of $\tilde{S}_x$ in M. (Without loss of generality, we will assume that $\tilde{S}_x$ forms a top dimensional simplex in M. Otherwise, we need to restrict our argument to the affine span of $\tilde{S}_x$ in M.) For any $m \in \Delta$, there exists a unique expression

$$s_m = \delta^{w_m} \prod_{\tilde{m}\in\tilde{S}_x} s_{\tilde{m}}^{l_{\tilde{m}}}.$$

Correspondingly

$$\rho_m = \delta^{2w_m} \prod_{\tilde{m}\in\tilde{S}_x} \rho_{\tilde{m}}^{l_{\tilde{m}}}.$$

For $m \in \Delta$ satisfying $w_m < 0$,

$$\rho_m(x) \geq \delta^{2w_m + a\sum_{\tilde{m}\in\tilde{S}_x} \max(0, l_{\tilde{m}})} > 1$$

for $a > 0$ small. Therefore we may assume $w_m \geq 0$ for all $m \in \Delta$. Since $\{w_m\}_{m\in\Delta}$ is convex and generic, we have $\tilde{S}_x \in Z$. For m not in the simplex spanned by $\tilde{S}_x$, $w_m > 0$, we have

$$\rho_m \leq \delta^{2w_m + a\sum_{\tilde{m}\in\tilde{S}_x} \min(0, l_{\tilde{m}})} \leq \delta^a.$$

for $a > 0$ small. Therefore $S_x = \tilde{S}_x \in Z$. □

The following proposition is a direct corollary of Lemma 3.4.

Proposition 3.5. *For $S \in Z$ and $x \in P_\Sigma$, assume that $\rho_m(x) > \epsilon$ for all $m \in S$. Then $\rho_m(x) = O(\delta^+)$ for all $m \notin S$ such that $S \cup \{m\} \notin Z$.*

Remark 3.6. In this paper, $O(\delta^+)$ denotes a quantity bounded by $A\delta^a$ for some universal positive constants A, a that only depend on w and Δ. In this paper, the relation between ϵ and δ is that we will take ϵ as small as we want and then take δ as small as we want depending on ϵ. Geometrically, the metric ω_{δ^w} develop necks that have scale $\delta^{a'}$ for some $a' > a$. ϵ is the gluing scale in Section 5 that satisfies $\epsilon \geq \delta^a$. For this section, it is sufficient to take $\epsilon = \delta^a$, which we will assume. In particular, $O(\delta^+) = O(\epsilon)$ in this section.

For $S \in Z$, we have two $N_{\mathbb{S}}$-invariant Kählerforms

$$\omega^S_{\delta^w} = \partial\bar{\partial} h^S_{\delta^w}, \quad \omega_S = \partial\bar{\partial} h_S, \quad \text{where } h^S_{\delta^w} = \log |s^{\delta^w}|^2_S \ \ h_S = \log |s|^2_S.$$

The corresponding moment maps are

$$F^S_{\delta^w} = \sum_{m \in S} \rho^S_m m \quad \text{and} \quad F_S = \sum_{m \in S} \frac{|s|^2_m}{|s|^2_S} m.$$

The two $N_{\mathbb{S}}$-invariant Kählerforms and their moment maps coincide if only if $w_m = 0$ for $m \in S$.

Apply Lemma 3.4, we have

Proposition 3.7. *For any $x \in P_\Sigma$, $|\omega^{S_x}_{\delta^w}(x) - \omega_{\delta^w}(x)| = O(\delta^+)$ and $|F^{S_x}_{\delta^w}(x) - F_{\delta^w}(x)| = O(\delta^+)$.*

For each simplex $S \in Z$, let

$$U^S_\epsilon = \{x \in P_\Sigma | \rho_S(x) > 1 - |\Delta|\epsilon, \rho_m(x) > \epsilon, \text{for } m \in S\},$$

where $|\Delta|$ denotes the number of integral points in Δ. The definition clearly implies the following.

Proposition 3.8. *For any $x \in U^S_\epsilon$, $|\omega^S_{\delta^w}(x) - \omega_{\delta^w}(x)| = O(\epsilon)$ and $|F^S_{\delta^w}(x) - F_{\delta^w}(x)| = O(\epsilon)$.*

Proposition 3.9.

$$P_\Sigma = \bigcup_{S \in Z} U^S_\epsilon.$$

Namely, $\{U^S_\epsilon\}_{S \in Z}$ is an open covering of P_Σ.

Proof. For any $x \in P_\Sigma$, let S contain those $m \in \Delta$ such that $\rho_m(x) > \epsilon$, then $\sum_{m \notin S} \rho_m(x) \leq |\Delta|\epsilon$. Lemma 3.4 implies that $S \subset S_x \in Z$ is a simplex. Consequently, $S \in Z$, $x \in U_\epsilon^S$. $\square$

Recall $C_{s^{\delta^w}} = (s^{\delta^w})^{-1}(0)$. We have

Proposition 3.10. *The image $F_{\delta^w}(C_{s^{\delta^w}})$ is independent of the choice of $w = (w_m)_{m\in\Delta}$ as a representative of an element $[w] \in \tilde{N} = \tilde{N}_0/A$.*

Proof. Assume that $\tilde{w} = (\tilde{w}_m)_{m\in\Delta}$ is another representative of $w \in \tilde{N} = \tilde{N}_0/A$. Then there exists $(l, n) \in \mathbb{Z} \oplus N$ such that $\tilde{w}_m = w_m - \langle m, n\rangle + l$. For $x \in N_{\mathbb{C}^*}$, let $\tilde{x} = x + n \log \delta$, then $s_m(\tilde{x}) = \delta^{\langle m,n\rangle} s_m(x)$ and $s_m^{\delta^{\tilde{w}}}(\tilde{x}) = \delta^{\tilde{w}_m} s_m(\tilde{x}) = \delta^l \delta^{w_m} s_m(x) = \delta^l s_m^{\delta^w}(x)$. Hence

$$s^{\delta^{\tilde{w}}}(\tilde{x}) = \sum_{m\in\Delta} a_m s_m^{\delta^{\tilde{w}}}(\tilde{x}) = \delta^l \sum_{m\in\Delta} a_m s_m^{\delta^w}(x) = \delta^l s^{\delta^w}(x),$$

and the transformation $x \to \tilde{x}$ maps $C_{s^{\delta^w}}$ to $C_{s^{\delta^{\tilde{w}}}}$. On the other hand,

$$|s_m^{\delta^{\tilde{w}}}(\tilde{x})|^2 = \delta^{2l} |s_m^{\delta^w}(x)|^2, \quad |s^{\delta^{\tilde{w}}}(\tilde{x})|_\Delta^2 = \delta^{2l} |s^{\delta^w}(x)|_\Delta^2,$$
$$F_{\delta^{\tilde{w}}}(\tilde{x}) = \sum_{m\in\Delta} \frac{|s_m^{\delta^{\tilde{w}}}(\tilde{x})|^2}{|s^{\delta^{\tilde{w}}}(\tilde{x})|_\Delta^2} m = \sum_{m\in\Delta} \frac{|s_m^{\delta^w}(x)|^2}{|s^{\delta^w}(x)|_\Delta^2} m = F_{\delta^w}(x).$$

Therefore $F_{\delta^w}(C_{s^{\delta^w}}) = F_{\delta^{\tilde{w}}}(C_{s^{\delta^{\tilde{w}}}})$. $\square$

For each simplex $S \in Z$, let $C_S = s_S^{-1}(0)$, where $s_S = \sum_{m\in S} a_m s_m$, and let Γ_S denote the union of all the simplices in the baricenter subdivision of S not containing the vertex of S. Then

$$\Gamma_Z = \bigcup_{S\in Z} \Gamma_S \tag{3.1}$$

is a graph in Δ. We have

Theorem 3.11.
$$\lim_{\delta\to 0} F_{\delta^w}(C_{s^{\delta^w}}) = \bigcup_{S\in Z} F_S(C_S)$$
is a fattening of Γ_Z. Consequently, for $\delta \in \mathbb{R}_+$ small, $F_{\delta^w}(C_{s^{\delta^w}})$ is a fattening of Γ_Z.

Proof. For $x \in P_\Sigma$, according to Proposition 3.9, there exists $S \in Z$ such that $x \in U_\epsilon^S$. Since S is a simplex, w can be adjusted by elements in A so that

$w_m = 0$ for $m \in S$ and $w_m < 0$ for $m \notin S$. According to Proposition 3.10, $F_{\delta^w}(C_{s^{\delta^w}})$ is unchanged under such adjustment of w. Such adjustment enables us to isolate the discussion to one simplex at a time. For this adjusted weight w, $\omega^S_{\delta^w} = \omega_S$ and $F^S_{\delta^w} = F_S$. Proposition 3.8 implies that for $x \in U^S_\epsilon$, $F_{\delta^w}(x)$ can be approximated (up to ϵ-terms) by $F_S(x) = F^S_{\delta^w}(x)$.

Since $w_m < 0$ for $m \notin S$, we have $|s^{\delta^w} - s_S| = O(\delta^+)$ on U^S_ϵ. $C_{s^{\delta^w}} \cap U^S_\epsilon$ can be approximated (up to $O(\delta^+)$-terms) by $C_S \cap U^S_\epsilon$. Consequently, $F_{\delta^w}(C_{s^{\delta^w}} \cap U^S_\epsilon)$ is an $O(\epsilon)$-approximation of $F_{\delta^w}(C_S \cap U^S_\epsilon)$. Patch such local results together, we get

$$\lim_{\delta \to 0} F_{\delta^w}(C_{s^{\delta^w}}) = \bigcup_{S \in Z} F_S(C_S).$$

In fact, $\lim_{\delta \to 0} C_{s^{\delta^w}} = \bigcup_{S \in Z^{\rm top}} C_S$, where on the right-hand side, when $S_1 \cap S_2$ is a one-simplex, the marked points of C_{S_1} and C_{S_2} corresponding to $S_1 \cap S_2$ are identified. This limit can be understood in the moduli space $\overline{\mathcal{M}}_g$ of stable curves.

When $S \in Z$ is a one-simplex, $F_S(C_S) = \Gamma_S$ is the baricenter of S. When $S \in Z$ is a two-simplex, let m^0, m^1, m^2 be the vertices of the simplex S. Under the coordinate $x_k = (a_{m^k} s_{m^k})/(a_{m^0} s_{m^0})$ for $k = 1, 2$, $C_S = \{x_1 + x_2 + 1\}$ and $F_S(x) = \sum_{k=0}^2 \frac{|x_k|^2}{|x|^2} m^k$, where $x_0 = 1$ and $|x|^2 = 1 + |x_1|^2 + |x_2|^2$. $F_S(C_S)$ is just the curved triangle in the simplex $S \subset \Delta$ as illustrated in the first picture in figure 1, which is clearly a fattening of the Y shaped graph Γ_S. Consequently, $\bigcup_{S \in Z} F_S(C_S)$ is a fattening of $\Gamma_Z = \bigcup_{S \in Z} \Gamma_S$. $\square$

Remark 3.12. The result in this theorem is essentially known to Viro in a somewhat different but equivalent form as described in [4].

Remark 3.13. To achieve the pictures of images of curves in figures 3, 4 and 5 in the last section, it is necessary to use the moment map introduced in this section. If the moment map of the standard Fubini-Study metric is used, the pictures will look more like hyperbolic metric, more precisely, the holes around center of the polygon will be larger and near the boundary of the polygon will be smaller.

Example 3.14. The Newton polygon corresponding to an ample line bundle L over P_Σ as $\mathbb{P}^2$ with three points blown up. Let E_1, E_2, E_3 be the three

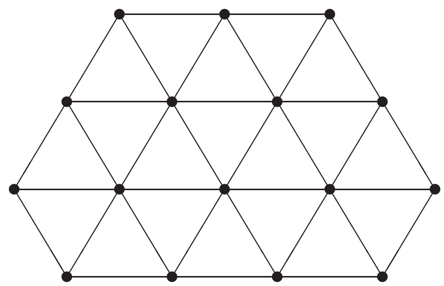

Figure 6: The standard simplicial decomposition.

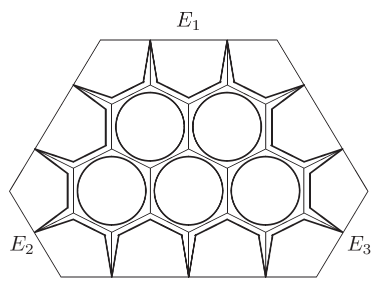

Figure 7: $F_s(C_s)$.

exceptional divisors, then

$$L \cong \pi^*(\mathcal{O}(5)) \otimes \mathcal{O}(-2E_1 - E_2 - E_3),$$

where $\pi : P_\Sigma \to \mathbb{P}^2$ is the natural blow up. Choose a section s of this line bundle near the large complex limit corresponding to the above standard simplicial decomposition of the Newton polygon (figure 6). Then the curve $C_s = s^{-1}(0)$ cut out by the section s will be mapped to figure 7 under corresponding moment map F_s.

3.1. Secondary fan

Theorem 3.11 can be better understood in the context of the secondary fan. To begin with, we consider the space $\mathcal{M}_\Delta$ of curves C_s modulo the equivalent relations of toric actions. With a little abuse of notation, we will call $\mathcal{M}_\Delta$ the toric moduli space of curves C_s with the Newton polygon Δ. Let $\tilde{M}_0 \cong \mathbb{Z}^\Delta$ be the dual lattice of $\tilde{N}_0 = \{w = (w_m)_{m\in\Delta} \in \mathbb{Z}^\Delta\} \cong \mathbb{Z}^\Delta$.

Recall that $\tilde{N} = \tilde{N}_0/A$. The dual lattice $\tilde{M} = A^{\perp}$. We have the natural identification

$$\mathcal{M}_\Delta \cong \tilde{N}_{\mathbb{C}^*} = \mathrm{Spec}(\mathbb{C}[\tilde{M}]) \cong (\mathbb{C}^*)^\Delta / N^+_{\mathbb{C}^*}.$$

To make sense of the large complex limit, we need the compactification $\overline{\mathcal{M}_\Delta}$ of $\mathcal{M}_\Delta$ determined by the so-called secondary fan.

For general $[w] \in \tilde{N}$, $w = (w_m)_{m\in\Delta}$ is not convex on Δ. Let $\hat{w} = (\hat{w}_m)_{m\in\Delta}$ be the convex hull of w. When w is generic, $\hat{w}$ determines a simplicial decomposition Z_w of Δ. (It is easy to observe that Z_w is independent of the choice of representative w in the equivalent class $[w]$.) Let $\hat{S}$ be the set of $S \subset \Delta$ that forms an r-dimensional simplex. Then Z_w can be regarded as a subset of $\hat{S}$. Let $\hat{Z}$ denote the set of all Z_w for $[w] \in \tilde{N}$. For $Z \in \hat{Z}$, let $\tau_Z \subset \tilde{N}$ be the closure of the set of all $[w] \in \tilde{N}$ such that $Z_w = Z$. Each τ_Z is a convex integral top dimension cone in $\tilde{N}$. The union of all τ_Z is exactly $\tilde{N}$. Let $\hat{\Sigma}$ be the fan whose cones are subcones of the top dimensional cones $\{\tau_Z\}_{Z\in\hat{Z}}$. $\hat{\Sigma}$ is a complete fan.

Let $\tilde{Z}$ be the set of simplicial decompositions $Z_w \subset \tilde{S}$ of Δ that is determined by some strictly convex function $w = (w_m)_{m\in\Delta}$ on Δ. For $Z \in \tilde{Z}$, let $\tau_Z \subset \tilde{N}$ be the set of $[w] \in \tilde{N}$, where $w = (w_m)_{m\in\Delta}$ is a piecewise linear convex function on Δ with respect to the simplicial decomposition Z. Each τ_Z is a integral top dimension cone in $\tilde{N}$. The union of all τ_Z

$$\tau = \bigcup_{Z\in\tilde{Z}} \tau_Z$$

is exactly the convex cone of all $[w] \in \tilde{N}$, where $w = (w_m)_{m\in\Delta}$ is a piecewise linear convex function on Δ. Let $\tilde{\Sigma}$ be the fan whose cones are subcones of the top dimensional cones $\{\tau_Z\}_{Z\in\tilde{Z}}$. $\tilde{\Sigma}$ is a subfan of the complete fan $\hat{\Sigma}$.

The fan $\hat{\Sigma}$ is the so-called *secondary fan.* (For more detail about the secondary fan, please refer to the book [3]. [1] contains some application of secondary fan to mirror symmetry.) $\hat{\Sigma}$ naturally determines the compactification $\overline{\mathcal{M}_\Delta} = P_{\hat{\Sigma}}$. We will call $\tilde{\Sigma}$ the *partial secondary fan.* $\tilde{\Sigma}$ determines the partial compactification $\widetilde{\mathcal{M}_\Delta} = P_{\tilde{\Sigma}}$. For each $Z \in \hat{Z}$, the top dimensional cone τ_Z determines a single fixed point $s^Z_\infty \in \overline{\mathcal{M}_\Delta} \backslash \mathcal{M}_\Delta$ of the $\tilde{N}_{\mathbb{C}^*}$ action. We will call such s^Z_∞ a *large complex limit* point. The set of different large complex limit points is parameterized by the set of simplicial decomposition $\hat{Z}$. Each large complex limit point s^Z_∞ possesses a cell neighborhood $\overline{\tau^{\mathbb{C}}_Z} \subset \overline{\mathcal{M}_\Delta}$, where $\tau^{\mathbb{C}}_Z = \tau_Z \otimes_{\mathbb{Z}_{\geq 0}} \mathbb{C}_+ \subset \tilde{N}_{\mathbb{C}^*}$, $\mathbb{Z}_{\geq 0}$ acts trivially on $\mathbb{C}_+ = \{z \in$

$\mathbb{C}^* : |z| \geq 1\}$. We have the following natural cell decomposition of $\overline{\mathcal{M}_\Delta}$

$$\overline{\mathcal{M}_\Delta} = \bigcup_{Z \in \tilde{Z}} \overline{\tau_Z^{\mathbb{C}}}.$$

Given a simplicial decomposition $Z \in \hat{Z}$ of Δ, let τ_Z^0 denote the interior of τ_Z. Any $[w] \in \tau_Z^0$ can be represented by a strongly convex piecewise linear function $w = (w_m)_{m \in \Delta}$ on Δ with respect to Z. It is easy to see that when δ approaches 0, $C_{s^{\delta^w}}$ will approach the large complex limit point s_∞^Z in $\overline{\mathcal{M}_\Delta}$. In such situation, we will say that $C_{s^{\delta^w}}$ or s_δ is *near the large complex limit* point (determined by Z), when δ is small.

Theorem 3.11 applies to each of such large complex limit point s_∞^Z in $\widetilde{\mathcal{M}_\Delta}$ for $Z \in \tilde{Z}$, and can be rephrased as: when the string diagrams $C_{s^{\delta^w}}$ approach the large complex limit point s_∞^Z in $\widetilde{\mathcal{M}_\Delta}$ as $\delta \to 0$, the amoebas $F_{\delta^w}(C_{s^{\delta^w}})$ of the string diagrams $C_{s^{\delta^w}}$ converge to the Feynman diagram Γ_Z.

Theorem 3.11 can be generalized to the full compactification $\overline{\mathcal{M}_\Delta}$ without additional difficulty.

Theorem 3.15. *For $Z \in \hat{Z}$, when the string diagrams $C_{s^{\delta^w}}$ approach the large complex limit point s_∞^Z in $\overline{\mathcal{M}_\Delta}$ as $\delta \to 0$, the amoebas $F_{\delta^w}(C_{s^{\delta^w}})$ of the string diagrams $C_{s^{\delta^w}}$ converge to the Feynman diagram Γ_Z.*

Proof. It is straightforward to generalize Lemma 3.4, Propositions 3.7, 3.8, 3.9, 3.10 and in particular, Theorem 3.11 to the case when $Z \in \hat{Z}$. The arguments are literally the same with the understanding that S considered as a subset in M contains only the integral vertex points of the simplex S, not any other integral points in the simplex S. □

Remark 3.16. Theorem 3.11 is used in [7] to construct Lagrangian torus fibration for quintic Calabi-Yau manifolds near large complex limit in the partial secondary fan compactification. Theorem 3.15 can be used to construct similar Lagrangian torus fibration for quintic Calabi-Yau manifolds near large complex limit that is not necessarily in the partial secondary fan compactification. According to [1], a large complex limit in the partial secondary fan compactification, under the mirror symmetry, corresponds to large radius limit of a Kählercone of the mirror Calabi-Yau manifold, while a large complex limit not in the partial secondary fan compactification, under the mirror symmetry, may correspond to large radius limit of some other physical model like Landau–Ginzberg model, etc.

4. The pair of pants and the three-valent vertex of a graph

In Feynman diagram, a three-valent vertex represents the most basic particle interaction. In string theory, the corresponding string diagram is the pair of pants, which can be represented by a general line in $\mathbb{CP}^2$ with the three punctured points being the intersection points of this line with the three coordinate lines. In this section, we will describe an analogue of this picture in our situation. More precisely, the standard moment map maps a general line to a fattening of the three-valent vertex neighborhood of a graph. In this section, we will explicitly perturb the moment map, so that the perturbed moment map will map the general line to the three-valent vertex neighborhood, i.e., a Y shaped graph.

4.1. The piecewise smooth case

Consider $\mathbb{CP}^2$ with the Fubini-Study metric and the curve C_0: $z_0 + z_1 + z_2 = 0$ in $\mathbb{CP}^2$. We have the torus fibration $F : \mathbb{CP}^2 \to \mathbb{R}^+\mathbb{P}^2$ defined as

$$F([z_1, z_2, z_3]) = [|z_1|, |z_2|, |z_3|].$$

Under the inhomogeneous coordinate $x_i = z_i/z_0$, locally we have

$$F\colon \mathbb{C}^2 \to (\mathbb{R}^+)^2, \quad F(x_1, x_2) = (r_1, r_2),$$

where $x_k = r_k e^{i\theta_k}$. The image of $C_0 : x_1 + x_2 + 1 = 0$ under F is

$$\tilde{\Gamma} = \{(r_1, r_2) | r_1 + r_2 \geq 1, r_1 \leq r_2 + 1, r_2 \leq r_1 + 1\}.$$

C_0 is a symplectic submanifold. We want to deform C_0 symplectically to C_1 whose image under F is expected to be

$$\Gamma = \{(r_1, r_2) | 0 \leq r_2 \leq r_1 = 1 \quad \text{or } 0 \leq r_1 \leq r_2 = 1 \quad \text{or } r_1 = r_2 \geq 1\}.$$

A moment of thought suggests taking $C_t = \mathcal{F}_t(C_0)$, where

$$\mathcal{F}_t(x_1, x_2) = \left(\left(\frac{\max(1, r_2)}{\max(r_1, r_2)} \right)^t x_1, \left(\frac{\max(1, r_1)}{\max(r_1, r_2)} \right)^t x_2 \right).$$

The Kählerform of the Fubini-Study metric can be written as

$$\omega_{\mathrm{FS}} = \frac{dx_1 \wedge d\bar{x}_1 + dx_2 \wedge d\bar{x}_2 + (x_2\, dx_1 - x_1\, dx_2) \wedge (\bar{x}_2\, d\bar{x}_1 - \bar{x}_1\, d\bar{x}_2)}{(1 + |x|^2)^2}.$$

Lemma 4.1. *ω_{FS} restricts to a symplectic form on $C_t \setminus \mathrm{Sing}(C_t)$, where $\mathrm{Sing}(C_t) := \{x \in C_t : (r_1 - 1)(r_2 - 1)(r_1 - r_2) = 0\}$. More precisely, there exists $c > 1$ such that $\frac{1}{c}\omega_{\mathrm{FS}} \leq \mathcal{F}_t^* \omega_{\mathrm{FS}} \leq c\omega_{\mathrm{FS}}$ on C_0 for all $t \in [0, 1]$.*

Proof. Due to the symmetries of permuting $[z_0, z_1, z_2]$, to verify that C_t is symplectic, we only need to verify for one region out of six. Consider $1 \geq |x_2| \geq |x_1|$, where

$$C_t = \left\{ \left(\left(\frac{1}{r_2}\right)^t x_1, \left(\frac{1}{r_2}\right)^t x_2 \right) : x_1 + x_2 + 1 = 0 \right\}.$$

$x_1 + x_2 + 1 = 0$ implies that

$$dx_1 = -dx_2.$$

Recall that

$$\frac{dr_k}{r_k} = \mathrm{Re}\left(\frac{dx_k}{x_k}\right), \quad d\theta_k = \mathrm{Im}\left(\frac{dx_k}{x_k}\right).$$

Consequently

$$\begin{aligned} \frac{dr_1}{r_1} &= \mathrm{Re}\left(\frac{dx_1}{x_1}\right) = -\mathrm{Re}\left(\left(\frac{x_2}{x_1}\right)\frac{dx_2}{x_2}\right), \\ \frac{dr_2}{r_2} &= \mathrm{Re}\left(\frac{dx_2}{x_2}\right) = -\mathrm{Re}\left(\left(\frac{x_1}{x_2}\right)\frac{dx_1}{x_1}\right). \end{aligned}$$

We have

$$\begin{aligned} d\left(\left(\frac{1}{r_2}\right)^t x_1\right) &= \left(\frac{1}{r_2}\right)^t \left(dx_1 - tx_1\frac{dr_2}{r_2}\right), \quad d\left(\left(\frac{1}{r_2}\right)^t x_1\right) \wedge d\left(\left(\frac{1}{r_2}\right)^t \bar{x}_1\right) \\ &= \left(\frac{1}{r_2}\right)^{2t} \left(dx_1 \wedge d\bar{x}_1 + t(x_1\, d\bar{x}_1 - \bar{x}_1\, dx_1) \wedge \frac{dr_2}{r_2}\right) \\ &= \left(\frac{1}{r_2}\right)^{2t} \left(1 + t\,\mathrm{Re}\left(\frac{x_1}{x_2}\right)\right) dx_1 \wedge d\bar{x}_1, \\ d\left(\left(\frac{1}{r_2}\right)^t x_2\right) &= \left(\frac{1}{r_2}\right)^t \left(dx_2 - tx_2\frac{dr_2}{r_2}\right) \end{aligned}$$

$$d\left(\left(\frac{1}{r_2}\right)^t x_2\right) \wedge d\left(\left(\frac{1}{r_2}\right)^t \bar{x}_2\right)$$
$$= \left(\frac{1}{r_2}\right)^{2t}\left(dx_2 \wedge d\bar{x}_2 + t(x_2\, d\bar{x}_2 - \bar{x}_2\, dx_2) \wedge \frac{dr_2}{r_2}\right)$$
$$= \left(\frac{1}{r_2}\right)^{2t}(1-t)\, dx_2 \wedge d\bar{x}_2, \left(\left(\frac{1}{r_2}\right)^t x_2\right) d\left(\left(\frac{1}{r_2}\right)^t x_1\right)$$
$$- \left(\left(\frac{1}{r_2}\right)^t x_1\right) d\left(\left(\frac{1}{r_2}\right)^t x_2\right)$$
$$= \left(\frac{1}{r_2}\right)^{2t}(x_2\, dx_1 - x_1\, dx_2) = \left(\frac{1}{r_2}\right)^{2t} x_2\left(1 + \left(\frac{x_1}{x_2}\right)\right) dx_1$$
$$= -\left(\frac{1}{r_2}\right)^{2t} dx_1.$$

By restriction to C_t and use the fact that $1 + \mathrm{Re}(\frac{x_1}{x_2}) \geq \frac{1}{2}$ on C_0, we get

$$\frac{(\mathcal{F}_t^*\omega_{\mathrm{FS}})|_{C_0}}{dx_1 \wedge d\bar{x}_1} = \frac{(1-t)(1/r_2)^{2t} + (1/r_2)^{2t}(1 + t\,\mathrm{Re}(x_1/x_2)) + (1/r_2)^{4t}}{\left(1 + (1/r_2)^{2t}(r_1^2 + r_2^2)\right)^2} \geq \frac{1}{6}.$$
$$\frac{(\mathcal{F}_t^*\omega_{\mathrm{FS}})|_{C_0}}{\omega_{\mathrm{FS}}|_{C_0}} = \frac{(1-t)(1/r_2)^{2t} + (1/r_2)^{2t}(1 + t\,\mathrm{Re}(x_1/x_2)) + (1/r_2)^{4t}}{3(1/(1 + r_1^2 + r_2^2)) + (1/r_2)^{2t}(r_1^2 + r_2^2)/(1 + r_1^2 + r_2^2))^2} \geq \frac{1}{2}.$$

These computations show that C_t is symplectic in the region $r_1 < r_2 < 1$. By symmetry, we can see that C_t is symplectic in the other five regions. □

Proposition 4.2. *$\mathcal{F}_t^*\omega_{\mathrm{FS}}$ is a piecewise smooth continuous symplectic form on C_0 for any $t \in [0, 1]$.*

Proof. In light of Lemma 4.1, only continuity need comment. This is an easy consequence of the invariance of $\mathcal{F}_t^*\omega_{\mathrm{FS}}$ under the symmetries of mutating the coordinate $[z_0, z_1, z_2]$. □

Theorem 4.3. *There exists a family of piecewise smooth Lipschitz Hamiltonian diffeomorphism $H_t\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ such that H_t is smooth away from $\mathrm{Sing}(C_0)$, $H_t(C_0) = C_t$, $H_t(\mathrm{Sing}(C_0)) = \mathrm{Sing}(C_t)$ and H_t is identity away from an arbitrary small neighborhood of $C_{[0,1]} := \bigcup_{t\in[0,1]} C_t$. In particular H_t leaves $\partial\mathbb{CP}^2$ (the union of the three coordinate $\mathbb{CP}^1$'s) invariant. The perturbed moment map (Lagrangian fibration) $\hat{F} = F \circ H_1$ satisfies $\hat{F}(C_0) = \Gamma$ (the Y shaped graph with a three-valent vertex v_0).*

Proof. Lemma 4.1 implies that C_t's are piecewise smooth symplectic submanifolds in $\mathbb{CP}^2$. Each C_t is a union of six pieces of smooth symplectic submanifolds with boundaries and corners. The six pieces have equal area (equal to one-sixth of the total area of C_t), which is independent of t. C_0 is symplectic isotopic to C_1 via the family $\{C_t\}$. By extension theorem (corollary 6.3) in [6], we may construct a piecewise smooth Lipschitz Hamiltonian diffeomorphism $H_t : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $H_t(C_0) = C_t$. Corollary 6.3 in [6] can further ensure that H_t leaves $\partial\mathbb{CP}^2$ invariant as desired.

More precisely, the proof of Corollary 6.3 in [6] is separated into two steps. In the first step, one modify the symplectic isotopy (see Section 6 of [6] for definition) $\mathcal{F}_t : C_0 \to C_t$ into a symplectic flow while keeping the restriction of $\mathcal{F}_t$ to the boundaries of the six pieces unchanged. (One in fact first modify $\mathcal{F}_t$ in one of the six pieces, then extend the modification symmetrically to the other pieces.) In particular, $C_t \cap \partial\mathbb{CP}^2$ is fixed by the symplectic flow. In the second step, Theorem 6.9 in [6] is applied to extend the symplectic flow to $\mathbb{CP}^2$ while keeping $\partial\mathbb{CP}^2$ fixed. The construction in effect ensures that $H_t|_{\mathrm{Sing}(C_0)} = \mathcal{F}_t|_{\mathrm{Sing}(C_0)}$ and H_t is smooth away from $\mathrm{Sing}(C_0)$. □

Similar construction can be carried out for degree d Fermat type curves. (The case of $d = 5$ is carried out in [6].)

Let $\mu_r := C_0 \cap \hat{F}^{-1}(r)$ for $r \in \Gamma$. When $d = 1$, for r being one of the three boundary points of Γ, μ_r is a point. For r in smooth part of Γ, μ_r is a circle. For r being the unique singular point of Γ, which in quantum mechanics usually indicate the particle interaction point, μ_r is of "Θ" shape. Figure 8 indicates the simplest string interaction.

When $d = 5$, for r being one of the three boundary points of Γ, μ_r is five points. For r in smooth part of Γ, μ_r is five circles. For r being the unique singular point of Γ, which in quantum mechanics usually indicate the particle interaction point, μ_r is a graph in two-torus as indicated in figure 9, which is much more complicated than $d = 1$ case. This picture indicates sort of degenerate multi-particle string interaction with multiplicity.

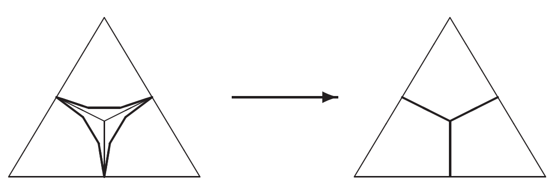

Figure 8: $F(C_p)$ of $p = z_1^d + z_2^d + z_3^d$ perturbed to $\hat{F}(C_p) = \Gamma$.

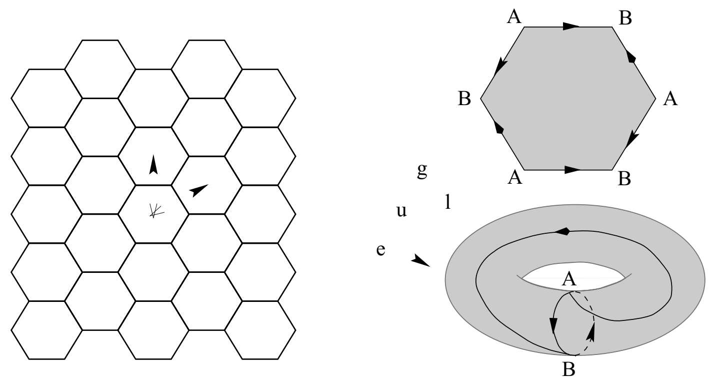

Figure 9: $\hat{F}^{-1}(\mathrm{Sing}(\Gamma))$ for $d = 5$ and $d = 1$.

4.2. The smooth case

Notice that C_t in Section 4.1 is not smooth on $\mathrm{Sing}(C_t)$. In this section, we will make C_t smooth. The trade-off is that $F(C_1) = \Gamma$ except in a small neighborhood of the vertex of the graph Γ, where $F(C_1)$ is a fattening of Γ. To modify the definition of C_t to make it smooth, consider real function $h(a) \geq 0$ such that $h(a) + h(-a) = 1$ for all a and $h(a) = 0$ for $a \leq -\epsilon$. Then consequently, $h(a) = 1$ for $a \geq \epsilon$ and $h(a) \leq 1$.

We may modify the definition of C_t to consider $C_t = \tilde{\mathcal{F}}_t(C_0)$, where

$$\tilde{\mathcal{F}}_t(x_1, x_2) = \left(\left(\frac{\eta_1}{\eta_0}\right)^t x_1, \left(\frac{\eta_2}{\eta_0}\right)^t x_2\right),$$

$$\eta_2 = r_1^{h(\log r_1)}, \quad \eta_1 = r_2^{h(\log r_2)}, \quad \eta_0 = r_1 \left(\frac{r_2}{r_1}\right)^{h(\log(r_2/r_1))}.$$

C_t is now smooth and is only modified in a ϵ-neighborhood of $\mathrm{Sing}(C_t)$.

Assume $\lambda(a) = h(a) + h'(a)a$, $\lambda_0 = \lambda(\log r_2 - \log r_1)$, $\lambda_1 = \lambda(\log r_1)$, $\lambda_2 = \lambda(\log r_2)$. Then

$$\frac{d\eta_2}{\eta_2} = \lambda_1 \frac{dr_1}{r_1}, \quad \frac{d\eta_1}{\eta_1} = \lambda_2 \frac{dr_2}{r_2}, \quad \frac{d\eta_0}{\eta_0} = \frac{dr_1}{r_1} + \lambda_0 \left(\frac{dr_2}{r_2} - \frac{dr_1}{r_1}\right);$$

$$d\left(\left(\frac{\eta_1}{\eta_0}\right)^t x_1\right) = \left(\frac{\eta_1}{\eta_0}\right)^t \left(dx_1 + t x_1 \left(\frac{d\eta_1}{\eta_1} - \frac{d\eta_0}{\eta_0}\right)\right);$$

$$\frac{d\eta_1}{\eta_1} - \frac{d\eta_0}{\eta_0} = -(1 - \lambda_0)\frac{dr_1}{r_1} - (\lambda_0 - \lambda_2)\frac{dr_2}{r_2};$$

$$
\begin{aligned}
d\left(\left(\frac{\eta_1}{\eta_0}\right)^t x_1\right) \wedge d\left(\left(\frac{\eta_1}{\eta_0}\right)^t \bar{x}_1\right)
&= \left(\frac{\eta_1}{\eta_0}\right)^{2t} \Big(dx_1 \wedge d\bar{x}_1 - t(x_1\, d\bar{x}_1 - \bar{x}_1\, dx_1) \\
&\quad \wedge \left(\frac{d\eta_1}{\eta_1} - \frac{d\eta_0}{\eta_0}\right)\Big) \\
&= \left(\frac{\eta_1}{\eta_0}\right)^{2t} \left(1 - (1-\lambda_0)t + (\lambda_0 - \lambda_2)t \operatorname{Re}\left(\frac{x_1}{x_2}\right)\right) dx_1 \wedge d\bar{x}_1; \\
d\left(\left(\frac{\eta_2}{\eta_0}\right)^t x_2\right) &= \left(\frac{\eta_2}{\eta_0}\right)^t \left(dx_2 + t x_2 \left(\frac{d\eta_2}{\eta_2} - \frac{d\eta_0}{\eta_0}\right)\right); \\
\frac{d\eta_2}{\eta_2} - \frac{d\eta_0}{\eta_0} &= -(1 - \lambda_0 - \lambda_1)\frac{dr_1}{r_1} - \lambda_0 \frac{dr_2}{r_2}; \\
d\left(\left(\frac{\eta_2}{\eta_0}\right)^t x_2\right) \wedge d\left(\left(\frac{\eta_2}{\eta_0}\right)^t \bar{x}_2\right)
&= \left(\frac{\eta_2}{\eta_0}\right)^{2t} \Big(dx_2 \wedge d\bar{x}_2 + t(x_2\, d\bar{x}_2 - \bar{x}_2\, dx_2) \\
&\quad \wedge \left(\frac{d\eta_2}{\eta_2} - \frac{d\eta_0}{\eta_0}\right)\Big) \\
&= \left(\frac{\eta_2}{\eta_0}\right)^{2t} \left(1 + (1 - \lambda_0 - \lambda_1)t \operatorname{Re}\left(\frac{x_2}{x_1}\right) - \lambda_0 t\right) dx_2 \wedge d\bar{x}_2; \\
\alpha &= \left(\left(\frac{\eta_2}{\eta_0}\right)^t x_2\right) d\left(\left(\frac{\eta_1}{\eta_0}\right)^t x_1\right) \\
&\quad - \left(\left(\frac{\eta_1}{\eta_0}\right)^t x_1\right) d\left(\left(\frac{\eta_2}{\eta_0}\right)^t x_2\right) \\
&= \left(\frac{\eta_2\eta_1}{\eta_0^2}\right)^t \left(x_2\, dx_1 - x_1\, dx_2 + t x_1 x_2 \left(\frac{d\eta_1}{\eta_1} - \frac{d\eta_2}{\eta_2}\right)\right) \\
&= \left(\frac{\eta_2\eta_1}{\eta_0^2}\right)^t \left(-dx_1 + t x_1 x_2 \left(\lambda_2 \frac{dr_2}{r_2} - \lambda_1 \frac{dr_1}{r_1}\right)\right); \\
\alpha \wedge \bar{\alpha} &= \left(\frac{\eta_2\eta_1}{\eta_0^2}\right)^{2t} \Big(dx_1\, d\bar{x}_1 - t\left(\bar{x}_1\bar{x}_2\, dx_1 - x_1 x_2\, d\bar{x}_1\right) \\
&\quad \left(\lambda_2 \frac{dr_2}{r_2} - \lambda_1 \frac{dr_1}{r_1}\right)\Big) \\
&= \left(\frac{\eta_2\eta_1}{\eta_0^2}\right)^{2t} \left(1 + t\left(\lambda_2 \operatorname{Re}(x_1) + \lambda_1 \operatorname{Re}(x_2)\right)\right) dx_1\, d\bar{x}_1.
\end{aligned}
$$

By restriction to C_t we get

$$
\begin{aligned}
\frac{(\tilde{\mathcal{F}}_t^*\omega_{\mathrm{FS}})|_{C_0}}{dx_1\wedge d\bar{x}_1} &= \Bigg[\left(\frac{\eta_2}{\eta_0}\right)^{2t}\left(1+(1-\lambda_0-\lambda_1)t\,\mathrm{Re}\left(\frac{x_2}{x_1}\right)-\lambda_0 t\right)\\
&\quad+\left(\frac{\eta_1}{\eta_0}\right)^{2t}\left(1-(1-\lambda_0)t+(\lambda_0-\lambda_2)t\,\mathrm{Re}\left(\frac{x_1}{x_2}\right)\right)\\
&\quad+\left(\frac{\eta_2\eta_1}{\eta_0^2}\right)^{2t}(1+t(\lambda_2\,\mathrm{Re}(x_1)+\lambda_1\,\mathrm{Re}(x_2)))\Bigg]\Bigg/\\
&\qquad\left(1+\left(\frac{\eta_2}{\eta_0}\right)^{2t}r_2^2+\left(\frac{\eta_1}{\eta_0}\right)^{2t}r_1^2\right)^2\\
&=\frac{\tilde{\omega}_t}{dx_1\wedge d\bar{x}_1}+tR_t,\quad \text{where } R_t=(1-\lambda_0)\\
&\quad R_{t,0}+\lambda_1 R_{t,1}+\lambda_2 R_{t,2},\\
R_{t,0} &= \frac{(\eta_2/\eta_0)^{2t}\,(1+\mathrm{Re}\,(x_2/x_1))-(\eta_1/\eta_0)^{2t}(1+\mathrm{Re}(x_1/x_2))}{(1+(\eta_2/\eta_0)^{2t}\,r_2^2+(\eta_1/\eta_0)^{2t}\,r_1^2)^2},\\
R_{t,1} &= \frac{(\eta_2/\eta_0)^{2t}(\eta_1^{2t}\,\mathrm{Re}(x_2)-\mathrm{Re}(x_2/x_1))}{\left(1+(\eta_2/\eta_0)^{2t}\,r_2^2+(\eta_1/\eta_0)^{2t}\,r_1^2\right)^2},\\
R_{t,2} &= \frac{(\eta_1/\eta_0)^{2t}(\eta_2^{2t}\,\mathrm{Re}(x_1)-\mathrm{Re}(x_1/x_2))}{(1+(\eta_2/\eta_0)^{2t}\,r_2^2+(\eta_1/\eta_0)^{2t}\,r_1^2)^2},\\
\tilde{\omega}_t &= \frac{\begin{array}{c}(\eta_2/\eta_0)^{2t}(1+t\,\mathrm{Re}(x_1/x_2))\\ +(\eta_1/\eta_0)^{2t}(1-t)+(\eta_2\eta_1/\eta_0^2)^{2t}\end{array}}{(1+(\eta_2/\eta_0)^{2t}r_2^2+(\eta_1/\eta_0)^{2t}r_1^2)^2}dx_1\wedge d\bar{x}_1.
\end{aligned}
$$

Proposition 4.4. *C_t is symplectic for $t\in[0,1]$. Namely, C_0 is symplectic isotropic to C_1 via the family $\{C_t\}_{t\in[0,1]}$ of smooth symplectic curves. More precisely, $(\tilde{\mathcal{F}}_t^*\omega_{\mathrm{FS}})|_{C_0}$ is smooth and is an $O(\epsilon)$-perturbation of $(\mathcal{F}_t^*\omega_{\mathrm{FS}})|_{C_0}$.*

Proof. According to Lemma 4.1 and proposition 4.2, it is sufficient to show that $(\tilde{\mathcal{F}}_t^*\omega_{\mathrm{FS}})|_{C_0}$ is an $O(\epsilon)$-perturbation of $(\mathcal{F}_t^*\omega_{\mathrm{FS}})|_{C_0}$.

Since $(\tilde{\mathcal{F}}_t^*\omega_{\mathrm{FS}})|_{C_0}$ and $(\mathcal{F}_t^*\omega_{\mathrm{FS}})|_{C_0}$ coincide away from an ϵ-neighborhood of $\mathrm{Sing}(C_0)$, with the help of symmetry, the cases that remain to be verified are ϵ-neighborhoods of $\{r_1=r_2\leq 1-\epsilon\}$, $\{r_2=1, 0\leq r_1\leq 1-\epsilon\}$ and $\{r_1=r_2=1\}$. On this neighborhoods, it is easy to observe that $\eta_1=1+O(\epsilon)$, $\eta_2=1+O(\epsilon)$, $\eta_1=r_2+O(\epsilon)$. Compare the expressions of $\tilde{\omega}_t$ and $(\mathcal{F}_t^*\omega_{\mathrm{FS}})|_{C_0}$, we have that $\tilde{\omega}_t$ is an $O(\epsilon)$-perturbation of $(\mathcal{F}_t^*\omega_{\mathrm{FS}})|_{C_0}$. Only thing remains to be shown is $R_t=O(\epsilon)$.

In an ϵ-neighborhood of $\{r_1 = r_2 \leq 1-\epsilon\}$, $\eta_k = 1 + O(\epsilon)$ for $k = 1, 2$, $\lambda_1 = \lambda_2 = 0$ and $\mathrm{Re}(\frac{x_2}{x_1}) - \mathrm{Re}(\frac{x_1}{x_2}) = O(\epsilon)$. Consequently, $R_t = t(1-\lambda_0) R_{t,0} = O(\epsilon)$.

In an ϵ-neighborhood of $\{r_2 = 1, 0 \leq r_1 \leq 1-\epsilon\}$, $\eta_k = 1 + O(\epsilon)$ for $0 \leq k \leq 2$, $\lambda_1 = 0$, $1 - \lambda_0 = 0$ and $\mathrm{Re}(x_1) - \mathrm{Re}(\frac{x_1}{x_2}) = O(\epsilon)$. Consequently, $R_t = t\lambda_2 R_{t,2} = O(\epsilon)$.

In an ϵ-neighborhood of $\{r_1 = r_2 = 1\}$, $\eta_k = 1 + O(\epsilon)$ for $0 \leq k \leq 2$, $\mathrm{Re}(\frac{x_2}{x_1}) - \mathrm{Re}(\frac{x_1}{x_2}) = O(\epsilon)$, $\mathrm{Re}(x_1) - \mathrm{Re}(\frac{x_1}{x_2}) = O(\epsilon)$, $\mathrm{Re}(x_2) - \mathrm{Re}(\frac{x_2}{x_1}) = O(\epsilon)$. Consequently, $R_{t,k} = O(\epsilon)$ for $0 \leq k \leq 2$ and $R_t = O(\epsilon)$. $\square$

Theorem 4.5. *There exists a family of Hamiltonian diffeomorphism $H_t : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $H_t(C_0) = C_t$ and H_t is identity away from an arbitrary small neighborhood of $C_{[0,1]}$. The perturbed moment map (Lagrangian fibration) $\hat{F} = F \circ H_1$ is smooth and satisfies $\hat{F}(C_0) = \Gamma$ (the Y shaped graph with a three-valent vertex v_0) away from a small neighborhood of v_0. (H_t can be made to be identity on $\partial\mathbb{CP}^2$ with the expense of smoothness of $\hat{F}$ at $\partial C_0 := \partial\mathbb{CP}^2 \cap C_0$.)*

Proof. Proposition 4.4 implies that C_0 is smoothly symplectic isotropic to C_1 via the family $\{C_t\}_{t\in[0,1]}$. By the extension theorem (Theorem 6.1) in [6], we can get a family of C^∞ Hamiltonian diffeomorphism $H_t : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $H_t(C_0) = C_t$ and H_t is identity away from an arbitrary small neighborhood of $C_{[0,1]}$. To ensure that H_t leaves $\partial\mathbb{CP}^2$ invariant, we need to use the extension theorem (Theorem 6.6) in [6]. Then H_t can only be made C^∞ away from the three intersection points of C_t and $\partial\mathbb{CP}^2$. $\square$

4.3. The optimal smoothness

$\hat{F}$ constructed in Section 4.2 is smooth. ($\hat{F}$ is not smooth at $\partial C_0 = \partial\mathbb{CP}^2 \cap C_0$ if $\hat{F}$ is required to be equal to F on $\partial\mathbb{CP}^2$. This non-smoothness is due to the fact that C_0 is not symplectically normal crossing to $\partial\mathbb{CP}^2$ under ω_{FS} and can be cured by modifying ω_{FS} near ∂C_0 so that C_0 is symplectically normal crossing to $\partial\mathbb{CP}^2$.) The trade off is that $\hat{F}(C_0) = \Gamma$ (the Y shaped graph with a three-valent vertex v_0) away from a small neighborhood of v_0.

$\hat{F}$ constructed in Section 4.1 satisfies $\hat{F}(C_0) = \Gamma$, but is only piecewise smooth and is not smooth at $\mathrm{Sing}(C_0)$. A natural question is: What is the optimal smoothness that $\hat{F}$ can achieve if we insist $\hat{F}(C_0) = \Gamma$? Clearly, $\hat{F}$ cannot be smooth over v_0. In this section, we will show that $\hat{F}$ can be made smooth over Γ away from v_0. More precisely, let $\mathrm{Sing}_0(C_0) =$

$\hat{F}^{-1}(v_0) \cap \mathrm{Sing}(C_0)$, we will show that $\hat{F}$ can be made smooth away from $\mathrm{Sing}_0(C_0)$. ($\hat{F}$ is not smooth at ∂C_0, if $\partial\mathbb{CP}^2$ is required to be fixed under $\hat{F}$.)

Let $b(a)$ be a smooth non-decreasing function satisfying $b(a) = 0$ for $a \leq 0$, $b(a) > 0$ for $a > 0$, $b(a) = 1$ for $a \geq \sqrt{\epsilon}$ and $b'(a) \leq C/\sqrt{\epsilon}$. We may modify the definition of C_t to consider $C_t = \tilde{\mathcal{F}}_t(C_0)$, where

$$\begin{aligned}
\tilde{\mathcal{F}}_t(x_1, x_2) &= \left(\left(\frac{\eta_1}{\eta_0}\right)^t x_1, \left(\frac{\eta_2}{\eta_0}\right)^t x_2\right),\\
\log \eta_2 &= \log r_1 h\left(\frac{\log r_1}{b_1}\right), \quad b_1 = b\left(\log\left(\frac{r_1}{r_2^2}\right)\right),\\
\log \eta_1 &= \log r_2 h\left(\frac{\log r_2}{b_2}\right), \quad b_2 = b\left(\log\left(\frac{r_2}{r_1^2}\right)\right),\\
\log \eta_0 &= \log r_1 h\left(\frac{\log(r_1/r_2)}{b_0}\right) + \log r_2 h\left(\frac{\log(r_2/r_1)}{b_0}\right),\\
b_0 &= b(\log(r_1 r_2)).
\end{aligned}$$

Notice that $\tilde{\mathcal{F}}_t$ here coincides with $\tilde{\mathcal{F}}_t$ in Section 4.2 away from a $\sqrt{\epsilon}$-neighborhood of $\mathrm{Sing}_0(C_0)$, coincides with $\mathcal{F}_t$ in Section 4.1 near $\mathrm{Sing}_0(C_0)$ away from a $\sqrt{\epsilon}$-neighborhood of v_0. Therefore, the only new construction of $\tilde{\mathcal{F}}_t$ is over a $\sqrt{\epsilon}$-neighborhood of v_0.

Assume $\lambda_0 = \lambda\left(\frac{\log r_2 - \log r_1}{b_0}\right)$, $\lambda_1 = \lambda\left(\frac{\log r_1}{b_1}\right)$, $\lambda_2 = \lambda\left(\frac{\log r_2}{b_2}\right)$. Then

$$\begin{aligned}
&\frac{d\eta_2}{\eta_2} = \lambda_1 \frac{dr_1}{r_1} - \beta_1, \quad \frac{d\eta_1}{\eta_1} = \lambda_2 \frac{dr_2}{r_2} - \beta_2,\\
&\frac{d\eta_0}{\eta_0} = \frac{dr_1}{r_1} + \lambda_0\left(\frac{dr_2}{r_2} - \frac{dr_1}{r_1}\right) - \beta_0.
\end{aligned}$$

Lemma 4.6. *$\beta_i = O(\sqrt{\epsilon})$ for $i = 1, 2, 3$.*

Proof.

$$\beta_1 = \left[\frac{\log r_1}{b_1} h'\left(\frac{\log r_1}{b_1}\right)\right]\left[\frac{\log r_1}{b_1} b'\left(\log \frac{r_1}{r_2^2}\right)\right]\left(\frac{dr_1}{r_1} - 2\frac{dr_2}{r_2}\right).$$

Notice that $h'\left(\frac{\log r_1}{b_1}\right) \neq 0$ only when $\frac{\log r_1}{b_1} \leq \epsilon$. Hence

$$\left[\frac{\log r_1}{b_1} h'\left(\frac{\log r_1}{b_1}\right)\right] = O(1), \quad \left[\frac{\log r_1}{b_1} b'\left(\log \frac{r_1}{r_2^2}\right)\right] = O(\sqrt{\epsilon}).$$

Consequently, $\beta_1 = O(\sqrt{\epsilon})$. The verifications for β_2 and β_3 are similar. $\square$

By similar computation as in Section 4.2, we get

$$\frac{(\tilde{\mathcal{F}}_t^*\omega_{\mathrm{FS}})|_{C_0}}{dx_1\wedge d\bar{x}_1} = \frac{\tilde{\omega}_t}{dx_1\wedge d\bar{x}_1} + tR_t + tB_t \geq \frac{1}{6} + O(\sqrt{\epsilon}),$$

where B_t is linear on $\{\beta_i\}_{i=1}^3$ and $B_t = O(\sqrt{\epsilon})$.

Proposition 4.7. *C_t is symplectic for $t\in[0,1]$. Namely, C_0 is symplectic isotropic to C_1 via the family $\{C_t\}_{t\in[0,1]}$ of smooth symplectic curves. More precisely, $(\tilde{\mathcal{F}}_t^*\omega_{\mathrm{FS}})|_{C_0}$ is smooth away from $\mathrm{Sing}_0(C_0)$ and is an $O(\sqrt{\epsilon})$-perturbation of $(\mathcal{F}_t^*\omega_{\mathrm{FS}})|_{C_0}$.*

Proof. This proposition is a direct consequence of the above computation, Lemma 4.1, Propositions 4.2 and 4.4 together with the additional estimate $B_t = O(\sqrt{\epsilon})$ implied by Lemma 4.6. □

Theorem 4.8. *There exists a family of Hamiltonian diffeomorphism H_t: $\mathbb{CP}^2\to\mathbb{CP}^2$ such that $H_t(C_0) = C_t$ and H_t is identity away from an arbitrary small neighborhood of $C_{[0,1]}$. $\hat{F} = F\circ H_1$ satisfies $\hat{F}(C_0) = \Gamma$ (the Y shaped graph with a three-valent vertex) and is smooth away from $\mathrm{Sing}_0(C_0)$. (H_t can be made to be identity on $\partial\mathbb{CP}^2$ with the expense of smoothness of $\hat{F}$ at $\partial\mathbb{CP}^2\cap C_0$.)*

Proof. The proof is essentially the same as the proofs of Theorems 4.3 except here C_t is decomposed into three (instead of six) smooth symmetric pieces, Lemma 4.1 and Proposition 4.2 is replaced by Proposition 4.7 and $\mathrm{Sing}(C_0)$ is replaced by $\mathrm{Sing}_0(C_0)$. □

5. String diagram and Feynman diagram

In this section, we will naturally combine the localization technique of Section 3, which reduces the curves (string diagram) locally to individual pair of pants, with the explicit perturbation technique of Section 4 to perturb the moment map F_{δ^w}, so that the perturbed moment map will map $C_{s^{\delta^w}}$ to a graph. This is a very interesting analogue of the relation of string diagrams in string theory and Feynman diagrams in quantum mechanics.

In general, given a simplicial decomposition $Z\in\tilde{Z}$ of Δ, take a weight $w\in\tau_Z^0$, according to Proposition 3.9, we have

$$P_\Sigma = \bigcup_{S\in Z} U_\epsilon^S.$$

According to results in [6], the perturbation of the moment map can be reduced to the perturbation of the pair $(C_{s^{\delta w}}, \omega_{\delta^w})$ of symplectic curve and symplectic form. For each $S \in Z^{\rm top}$, locally in U^S_ϵ, $(C_{s^{\delta w}} \cap U^S_\epsilon, \omega_{\delta^w}|_{U^S_\epsilon})$ is a close approximation of the line and the Fubini-Study Kählerform discussed in Section 4. Namely, the construction in Section 4 can be viewed as local model for construction here. In the following, we will start with some modification of the local model in Section 4, then we will apply the modified local model to perturb $C_{s^{\delta w}}$. For such purpose, ω_{δ^w} also need to be perturbed suitably.

5.1. Modified local models

Consider a smooth non-negative non-decreasing function $\gamma_\epsilon(u)$, such that $\gamma_\epsilon(u) = 0$ for $\sqrt{u} \leq A_1\epsilon$ and $\gamma_\epsilon(u) = 1$ for $\sqrt{u} \geq A_2\epsilon$. A_1, A_2 are positive constants satisfying $1 < A_1 < A_2 < |\Delta|$. Let $\gamma_{\epsilon,t}(u) = t\gamma_\epsilon(u) + (1-t)$ and

$$\eta_1 = \max(1, r_2), \quad \eta_2 = \max(1, r_1), \quad \eta_0 = \max(r_1, r_2).$$

Proposition 5.1. *$C_t = p_t^{-1}(0)$ is symplectic curve under the Fubini-Study Kählerform for $t \in [0,1]$, where*

$$p_t(x) = \gamma_{\epsilon,t}\left(\frac{r_1^2}{\eta_1^2}\right)x_1 + \gamma_{\epsilon,t}\left(\frac{r_2^2}{\eta_2^2}\right)x_2 + \gamma_{\epsilon,t}\left(\frac{1}{\eta_0^2}\right) = 0.$$

Namely, the family $\{C_t\}_{t\in[0,1]}$ is a symplectic isotopy from $C_0 = \{(x_1, x_2) : x_1 + x_2 + 1 = 0\}$ to

$$C_1 = \left\{(x_1, x_2) : \gamma_\epsilon\left(\frac{r_1^2}{\eta_1^2}\right)x_1 + \gamma_\epsilon\left(\frac{r_2^2}{\eta_2^2}\right)x_2 + \gamma_\epsilon\left(\frac{1}{\eta_0^2}\right) = 0\right\}.$$

Proof. By symmetry, we only need to verify that C_t is symplectic in the region $1 \geq |x_2| \geq |x_1|$, where

$$p_t(x) = \gamma_{\epsilon,t}(|x_1|^2)x_1 + x_2 + 1 = 0.$$

Since C_t is a complex curve away from the region $\{A_1\epsilon \leq |x_1| \leq A_2\epsilon\}$, we only need to verify that $C_t \cap \{A_1\epsilon \leq |x_1| \leq A_2\epsilon\}$ is symplectic.

Recall the Kählerform of the Fubini-Study metric is

$$\omega_{\mathrm{FS}} = \frac{dx_1 \wedge d\bar{x}_1 + dx_2 \wedge d\bar{x}_2 + (x_2\, dx_1 - x_1\, dx_2) \wedge (\bar{x}_2\, d\bar{x}_1 - \bar{x}_1\, d\bar{x}_2)}{(1+|x|^2)^2}.$$

When restricted to $C_t \cap \{A_1\epsilon \le |x_1| \le A_2\epsilon\}$,

$$\begin{aligned}\omega_{\mathrm{FS}} &= \frac{1}{2} dx_1 \wedge d\bar{x}_1 + \frac{1}{4} dx_2 \wedge d\bar{x}_2 + O(\epsilon)\\ &= \left(\frac{1}{2} + \frac{1}{4}[\gamma_{\epsilon,t}(|x_1|^2)^2 + \gamma_{\epsilon,t}(|x_1|^2) t \gamma_\epsilon^{\#}(|x_1|^2)]\right) dx_1 \wedge d\bar{x}_1 + O(\epsilon)\\ &\ge \frac{2+(1-t)^2}{4}\, dx_1 \wedge d\bar{x}_1 + O(\epsilon),\end{aligned}$$

where $\gamma_\epsilon^{\#}(|x_1|^2) = 2|x_1|^2 \gamma_\epsilon'(|x_1|^2)$. Therefore C_t is symplectic. □

Proposition 5.2. *$C_t = \mathcal{F}_t(C_0)$ is symplectic for $t \in [0,1]$, where*

$$\mathcal{F}_t(x_1,x_2) = \left(\left(\frac{\eta_1}{\eta_0}\right)^t x_1, \left(\frac{\eta_2}{\eta_0}\right)^t x_2\right),$$

$$C_0 = \left\{(x_1,x_2) : \gamma_\epsilon\left(\frac{r_1^2}{\eta_1^2}\right) x_1 + \gamma_\epsilon\left(\frac{r_2^2}{\eta_2^2}\right) x_2 + \gamma_\epsilon\left(\frac{1}{\eta_0^2}\right) = 0\right\}. \tag{5.2}$$

Proof. By symmetry, we only need to verify that C_t is symplectic in the region $1 \ge |x_2| \ge |x_1|$, which is one of the six symmetric regions that together form $\mathbb{CP}^2$. In the region $1 \ge |x_2| \ge |x_1|$,

$$C_t = \left\{\left(\left(\frac{1}{r_2}\right)^t x_1, \left(\frac{1}{r_2}\right)^t x_2\right) : \gamma_\epsilon(|x_1|^2)x_1 + x_2 + 1 = 0\right\}$$

$\gamma_\epsilon(|x_1|^2)x_1 + x_2 + 1 = 0$ implies that

$$dx_2 = -\gamma_\epsilon\, dx_1 - x_1\, d\gamma_\epsilon.$$

Hence

$$\frac{dr_2}{r_2} = \mathrm{Re}\left(\frac{dx_2}{x_2}\right) = -\gamma_\epsilon \mathrm{Re}\left(\left(\frac{x_1}{x_2}\right)\frac{dx_1}{x_1}\right) - \mathrm{Re}\left(\frac{x_1}{x_2}\right) d\gamma_\epsilon.$$

$$\begin{aligned} d\left(\left(\frac{1}{r_2}\right)^t x_1\right) \wedge d\left(\left(\frac{1}{r_2}\right)^t \bar{x}_1\right) &= \left(\frac{1}{r_2}\right)^{2t} \\ &\quad \left(dx_1 \wedge d\bar{x}_1 + t(x_1 d\bar{x}_1 - \bar{x}_1 dx_1) \wedge \frac{dr_2}{r_2}\right) \\ &= \left(\frac{1}{r_2}\right)^{2t}\left(1 + t(\gamma_\epsilon + \gamma_\epsilon^{\#})\mathrm{Re}\left(\frac{x_1}{x_2}\right)\right) dx_1 \wedge d\bar{x}_1, \\ d\left(\left(\frac{1}{r_2}\right)^t x_2\right) \wedge d\left(\left(\frac{1}{r_2}\right)^t \bar{x}_2\right) &= \left(\frac{1}{r_2}\right)^{2t} \\ &\quad \left(dx_2 \wedge d\bar{x}_2 + t(x_2 d\bar{x}_2 - \bar{x}_2 dx_2) \wedge \frac{dr_2}{r_2}\right) \\ &= \left(\frac{1}{r_2}\right)^{2t} (1-t) dx_2 \wedge d\bar{x}_2 \\ &= \left(\frac{1}{r_2}\right)^{2t} (1-t)(\gamma_\epsilon^2 + \gamma_\epsilon \gamma_\epsilon^{\#}) dx_1 \wedge d\bar{x}_1 \geq 0, \\ &\quad \left(\left(\frac{1}{r_2}\right)^t x_2\right) d\left(\left(\frac{1}{r_2}\right)^t x_1\right) \\ &\quad - \left(\left(\frac{1}{r_2}\right)^t x_1\right) d\left(\left(\frac{1}{r_2}\right)^t x_2\right) \\ &= \left(\frac{1}{r_2}\right)^{2t} (x_2 dx_1 - x_1 dx_2) = -\left(\frac{1}{r_2}\right)^{2t} (dx_1 + x_1^2 d\gamma_\epsilon). \end{aligned}$$

By restriction to C_t we get

$$\frac{\omega_{\mathrm{FS}}|_{C_t}}{dx_1 \wedge d\bar{x}_1} \geq \frac{\begin{array}{c}(1/r_2)^{2t}(1 + t(\gamma_\epsilon + \gamma_\epsilon^{\#})\mathrm{Re}(x_1/x_2)) \\ +(1/r_2)^{4t}(1 - \mathrm{Re}(x_1)\gamma_\epsilon^{\#})\end{array}}{(1 + r_2^{2-2t} + (r_1/r_2)^{2t} r_1^{2-2t})^2} \geq \frac{1}{6} + O(\epsilon).$$

The reason is that $0 \leq \gamma_\epsilon \leq 1$, $\gamma_\epsilon^{\#}\,\mathrm{Re}\left(\frac{x_1}{x_2}\right) = O(\epsilon)$ and $\mathrm{Re}(x_1)\gamma_\epsilon^{\#} = O(\epsilon)$. Therefore C_t is symplectic. □

Remark 5.3. Let $U_\epsilon^{\mathbb{CP}^2} = \{r_1 \leq \epsilon\eta_1, r_2 \leq \epsilon\eta_2, 1 \leq \epsilon\eta_0\} \subset \mathbb{CP}^2$. It is easy to observe that outside of $U_\epsilon^{\mathbb{CP}^2}$, C_t in Proposition 5.2 is equal to $\{x_2 + 1 = 0\}$

when $|x_1|$ is small, equal to $\{x_1+1=0\}$ when $|x_2|$ is small, equal to $\{x_1+x_2=0\}$ when $|x_1|,|x_2|$ are large. Namely, C_t outside of $U_\epsilon^{\mathbb{CP}^2}$ is toric, $F(C_t \cap (\mathbb{CP}^2 \setminus U_\epsilon^{\mathbb{CP}^2}))$ is 1-dimensional, independent of t and is the union of the three end segments of the Y shaped graph. Also the image of C_1 under any moment map is a one-dimensional graph of Y shape.

The following is the analogue of Theorem 4.3 for our modified local model.

Theorem 5.4. *There exists a family of piecewise smooth Lipschitz Hamiltonian diffeomorphism H_t: $\mathbb{CP}^2 \to \mathbb{CP}^2$ such that $H_t(C_0) = C_t$ and H_t is identity away from an arbitrary small neighborhood of $C_{[0,1]}$ or away from $U_\epsilon^{\mathbb{CP}^2}$. The perturbed moment map (Lagrangian fibration) $\hat{F} = F \circ H_1$ satisfies $\hat{F}(C_0) = \Gamma$ (the Y shaped graph with a three-valent vertex).*

Proof. The proof is essentially the same as the proof of Theorem 4.3 except for the proof of H_t being the identity map when restricted to $\mathbb{CP}^2 \setminus U_\epsilon^{\mathbb{CP}^2}$, which is based on the fact that $\mathcal{F}_t$ restricts to identity map on $C_0 \setminus U_\epsilon^{\mathbb{CP}^2}$. □

To deal with the cases of smooth and optimal smoothness discussed in Sections 4.2 and 4.3, we may take $C_t = \tilde{\mathcal{F}}_t(C_0)$, where we take C_0 in (5.2) and $\tilde{\mathcal{F}}_t$ in either Section 4.2 or Section 4.3. (Notice that in the region where C_0 is modified, $\tilde{\mathcal{F}}_t$ in Sections 4.2and 4.3 coincide.)

Proposition 5.5. *$C_t = \tilde{\mathcal{F}}_t(C_0)$ is symplectic for $t \in [0,1]$, where C_0 is defined in (5.1).*

Proof. By symmetry, we only need to verify that C_t is symplectic in the region, where $|x_1| \leq |x_2| \leq 1$ and $\gamma_\epsilon(|x_1|^2) < 1$. In this region, we have $|x_1| = O(\epsilon)$ and $x_2 = -1 + O(\epsilon)$. Consequently, $\lambda_0 - 1 = \lambda_1 = 0$, $\eta_2 = 1$, $\eta_1 = 1 + O(\epsilon)$ and $\eta_0 = r_2 = 1 + O(\epsilon)$.

$$\frac{d\eta_2}{\eta_2} = 0, \quad \frac{d\eta_1}{\eta_1} = \lambda_2 \frac{dr_2}{r_2}, \quad \frac{d\eta_0}{\eta_0} = \frac{dr_2}{r_2}.$$

$\gamma_\epsilon(|x_1|^2)x_1 + x_2 + 1 = 0$ implies that

$$dx_2 = -\gamma_\epsilon\, dx_1 - x_1\, d\gamma_\epsilon.$$

Hence

$$
\begin{aligned}
\frac{dr_2}{r_2} &= \mathrm{Re}\left(\frac{dx_2}{x_2}\right) = -\gamma_\epsilon \mathrm{Re}\left(\left(\frac{x_1}{x_2}\right)\frac{dx_1}{x_1}\right) - \mathrm{Re}\left(\frac{x_1}{x_2}\right) d\gamma_\epsilon. \\
dx_2 \wedge d\bar{x}_2 &= \gamma_\epsilon(\gamma_\epsilon + \gamma_\epsilon^{\#}) dx_1 \wedge d\bar{x}_1; \\
d\left(\left(\frac{\eta_1}{\eta_0}\right)^t x_1\right) &= \left(\frac{\eta_1}{\eta_0}\right)^t \left(dx_1 + tx_1\left(\frac{d\eta_1}{\eta_1} - \frac{d\eta_0}{\eta_0}\right)\right); \\
\frac{d\eta_1}{\eta_1} - \frac{d\eta_0}{\eta_0} &= -(1-\lambda_2)\frac{dr_2}{r_2}. \\
d\left(\left(\frac{\eta_1}{\eta_0}\right)^t x_1\right) &\wedge d\left(\left(\frac{\eta_1}{\eta_0}\right)^t \bar{x}_1\right) \\
&= \left(\frac{\eta_1}{\eta_0}\right)^{2t} \left(dx_1 \wedge d\bar{x}_1 - t(x_1 d\bar{x}_1 - \bar{x}_1 dx_1) \wedge \left(\frac{d\eta_1}{\eta_1} - \frac{d\eta_0}{\eta_0}\right)\right) \\
&= \left(\frac{\eta_1}{\eta_0}\right)^{2t} \left(1 + (\gamma_\epsilon + \gamma_\epsilon^{\#})(1-\lambda_2)t\,\mathrm{Re}\left(\frac{x_1}{x_2}\right)\right) dx_1 \wedge d\bar{x}_1. \\
d\left(\left(\frac{\eta_2}{\eta_0}\right)^t x_2\right) &= \left(\frac{\eta_2}{\eta_0}\right)^t \left(dx_2 + tx_2\left(\frac{d\eta_2}{\eta_2} - \frac{d\eta_0}{\eta_0}\right)\right); \\
\frac{d\eta_2}{\eta_2} - \frac{d\eta_0}{\eta_0} &= -\frac{dr_2}{r_2}; \\
d\left(\left(\frac{\eta_2}{\eta_0}\right)^t x_2\right) &\wedge d\left(\left(\frac{\eta_2}{\eta_0}\right)^t \bar{x}_2\right) \\
&= \left(\frac{\eta_2}{\eta_0}\right)^{2t} \left(dx_2 \wedge d\bar{x}_2 + t(x_2\, d\bar{x}_2 - \bar{x}_2\, dx_2) \wedge \left(\frac{d\eta_2}{\eta_2} - \frac{d\eta_0}{\eta_0}\right)\right) \\
&= \left(\frac{\eta_2}{\eta_0}\right)^{2t} \gamma_\epsilon(\gamma_\epsilon + \gamma_\epsilon^{\#})(1-t)\, dx_1 \wedge d\bar{x}_1 \geq 0; \\
\alpha &= \left(\left(\frac{\eta_2}{\eta_0}\right)^t x_2\right) d\left(\left(\frac{\eta_1}{\eta_0}\right)^t x_1\right) \\
&\quad - \left(\left(\frac{\eta_1}{\eta_0}\right)^t x_1\right) d\left(\left(\frac{\eta_2}{\eta_0}\right)^t x_2\right) \\
&= \left(\frac{\eta_2\eta_1}{\eta_0^2}\right)^t \left(x_2\, dx_1 - x_1\, dx_2 + tx_1x_2\left(\frac{d\eta_1}{\eta_1} - \frac{d\eta_2}{\eta_2}\right)\right) \\
&= \left(\frac{\eta_2\eta_1}{\eta_0^2}\right)^t \left(-dx_1 + x_1^2\, d\gamma_\epsilon + tx_1x_2\lambda_2\frac{dr_2}{r_2}\right)
\end{aligned}
$$

$$= -\left(\frac{\eta_2\eta_1}{\eta_0^2}\right)^t dx_1 + O(|x_1|).$$
$$\alpha \wedge \bar{\alpha} = \left(\frac{\eta_2\eta_1}{\eta_0^2}\right)^{2t} dx_1 d\bar{x}_1 + O(|x_1|).$$

By restriction to C_t we get

$$\frac{(\tilde{\mathcal{F}}_t^*\omega_{\mathrm{FS}})|_{C_0}}{dx_1 \wedge d\bar{x}_1} \geq \frac{(\eta_1/\eta_0)^{2t} + (\eta_2\eta_1/\eta_0^2)^{2t} + O(|x_1|)}{(1 + (\eta_2/\eta_0)^{2t}r_2^2 + (\eta_1/\eta_0)^{2t}r_1^2)^2} \geq \frac{1}{2} + O(\epsilon).$$
□

For $C_t = \tilde{\mathcal{F}}(C_0)$, where $\tilde{\mathcal{F}}$ is taken from Section 4.3, we have

Theorem 5.6. *$\hat{F}$ in Theorem 5.4 can be made smooth away from* Sing_0 (C_0).

Proof. The proof is essentially the same as the proof of Theorem 4.3 except that C_0 is decomposed into three pieces with boundaries in $\mathrm{Sing}_0(C_0)$. The proof of H_t being the identity map when restricted to $\mathbb{CP}^2 \setminus U_\epsilon^{\mathbb{CP}^2}$ is based on the fact that $\tilde{\mathcal{F}}_t$ restricts to identity map on $C_0 \setminus U_\epsilon^{\mathbb{CP}^2}$. □

Remark 5.7. There is also a version of Theorem 5.6 as analogue of Theorem 4.6 when $\tilde{\mathcal{F}}$ is taken from Section 4.2.

5.2. Perturbation of symplectic curve and form

For $m \in \Delta$, let

$$\Delta_m = \{m' \in \Delta | \{m, m'\} \in Z\}.$$

Choose $\check{\epsilon}$ such that $\delta^a \leq \check{\epsilon} \leq \epsilon$. Define

$$\hat{s}_m = \gamma_\epsilon(\rho_m)s_m, \quad \check{s}_m = [1 - \gamma_{\check{\epsilon}}(\max_{m' \notin \Delta_m}(\rho_{m'}))]s_m,$$
$$\hat{s}^{\delta^w} = \sum_{m\in\Delta} \delta^{w_m} a_m \hat{s}_m, \quad \check{s}^{\delta^w} = \sum_{m\in\Delta} \delta^{w_m} a_m \check{s}_m.$$
$$\check{\omega}_{\delta^w} = \partial\bar{\partial}\check{h}_{\delta^w}, \quad \text{where } \check{h}_{\delta^w} = \log|\check{s}^{\delta^w}|_\Delta^2, \quad |\check{s}^{\delta^w}|_\Delta^2 = \sum_{m\in\Delta} |\delta^{w_m}\check{s}_m|_\Delta^2.$$

Proposition 5.8. *$\check{\omega}_{\delta^w}$ is a Kählerform on P_Σ near $C_t = s_t^{-1}(0)$ for $t \in [0, 1]$, where $s_t = t\hat{s}_{\delta^w} + (1-t)s_{\delta^w}$.*

Proof. For $x \in P_\Sigma$, let $\rho_{m_i}(x)$ for a $m_i \in \Delta$ be the ith largest among $\{\rho_m(x)\}_{m\in\Delta}$. Since S_x is non-empty, we have $m_1 \in S_x$ and $\rho_{m_1}(x) \geq 1/|\Delta| - \epsilon$. If $x \in C_t$, it is easy to derive from the equation of C_t that $\rho_{m_2}(x) \geq 1/|\Delta|^2 - \epsilon/|\Delta|$ and $m_2 \in S_x$ when ϵ is small.

If $\{m_1, m_2\} \not\subset \Delta_m$, then $\max_{m'\notin\Delta_m}(\rho_{m'}(x)) \geq \rho_{m_2}(x) > |\Delta|\check{\epsilon}$ when $\check{\epsilon}$ is small. Hence $\check{s}_m(x) = 0$.

If $\{m_1, m_2\} \subset \Delta_m$ and $\check{s}_m \neq s_m$, then there exists $m' \notin \Delta_m$ such that $\rho_{m'} > \check{\epsilon}$. Hence $\check{S}_x = \{m_1, m_2, m'\}$, $\check{s}_{m'} = s_{m'}$ and $m_3 = m'$, where $\check{S}_x$ is defined as S_x with ϵ replaced by $\check{\epsilon}$. Consequently, $\rho_m(x) = O(\delta^a)$ and $\check{\omega}_{\delta^w}(x)$ is an $O(\delta^a/\check{\epsilon})$-perturbation of $\check{\omega}^{\check{S}_x}_{\delta^w}(x)$. When $\delta^a/\check{\epsilon}$ is small, $\check{\omega}_{\delta^w}$ is a Kählerform at x.

The remaining case is when $\check{s}_m = s_m$ for $m \in S' = \{m_1, m_2, m', m''\}$ and $\check{s}_m = 0$ for $m \notin S'$, where $\{m', m''\}$ is uniquely determined by the relation $\{m_1, m_2\} \subset \Delta_{m'} \cap \Delta_{m''}$. Then $\check{\omega}_{\delta^w}(x) = \omega^{S'}_{\delta^w}(x)$ is clearly Kähler. Therefore $\check{\omega}_{\delta^w}$ is a Kählerform on P_Σ near C_t. □

Proposition 5.9. *C_t is symplectic curve under the Kählerform ω_t for $t \in [0,1]$, where $\omega_t = t\check{\omega}_{\delta^w} + (1-t)\omega_{\delta^w}$. Namely, the family $\{C_t\}_{t\in[0,1]}$ is a symplectic isotopy from $C_0 = C_{s_{\delta^w}}$ to $C_1 = C_{\hat{s}_{\delta^w}}$. Furthermore, there exists smooth symplectomorphisms H_1: $(P_\Sigma, \omega_{\delta^w}) \to (P_\Sigma, \check{\omega}_{\delta^w})$ such that $H_1(C_{s_{\delta^w}}) = C_{\hat{s}_{\delta^w}}$. ($H_1$ can be made to be identity on ∂P_Σ with the expense of smoothness of H_1 at $C_0 \cap \partial P_\Sigma$.)*

Proof. Proposition 5.8 implies that ω_t are Kählerforms on P_Σ near C_t. It is easy to see that s_t is holomorphic outside of the union of U^S_ϵ for $S \in Z^{\text{top}}$, where C_t is automatically symplectic.

For each $S = \{m_0, m_1, m_2\} \in Z^{\text{top}}$, $\{z_i = \delta^{w_{m_i}} a_{m_i} s_{m_i}\}_{i=0}^2$ defines an open embedding $U^S_\epsilon \hookrightarrow \mathbb{CP}^2$, where $[z_0, z_1, z_2]$ is the homogeneous coordinate of $\mathbb{CP}^2$. Using the inhomogeneous coordinates (x_1, x_2) of $\mathbb{CP}^2$ on U^S_ϵ, $\hat{s}_{\delta^w}$ reduces to p_1 in Proposition 5.1 and s_{δ^w} reduces to $p_0 = x_1 + x_2 + 1$ in Proposition 5.1 up to $O(\delta^+)$ terms (Lemma 3.4). Hence C_t here coincides with C_t in Proposition 5.1 inside $U^S_\epsilon \subset \mathbb{CP}^2$. When δ is small, by Proposition 5.1, C_t is symplectic in U^S_ϵ with respect to ω_{FS}. Since $\check{\omega}_{\delta^w} = \omega_{\text{FS}}$ when restricted to U^S_ϵ, C_t is symplectic in U^S_ϵ with respect to $\check{\omega}_{\delta^w}$.

For the second part of the proposition, apply Theorems 6.1 and 6.2 from [6] (which though are conveniently formulated for our application here, are essentially well known along the line of Moser's theorem) to the symplectic isotopic family $\{(C_t, \omega_t)\}_{t\in[0,1]}$, we can construct a smooth symplectomorphism $H_1 : (P_\Sigma, \omega_{\delta^w}) \to (P_\Sigma, \check{\omega}_{\delta^w})$ such that $H_1(C_{s_{\delta^w}}) = C_{\hat{s}_{\delta^w}}$. To satisfy $H_1|_{\partial P_\Sigma} = \mathrm{Id}_{\partial P_\Sigma}$, it is necessary to apply Theorems 6.3 and 6.4 from [6] and H_1 is piecewise smooth, $C^{0,1}$ and is smooth away from $C_0 \cap \partial P_\Sigma$. $\square$

When $S \in Z$ is a one-simplex, Γ_S is just the baricenter of S. Let $s(\Gamma_Z)$ (resp. $e(\Gamma_Z)$) denote the union of Γ_S for those one-simplex $S \in Z$ that is not in $\partial\Delta$ (resp. is in $\partial\Delta$).

Proposition 5.10. *For each $S \in Z^{\mathrm{top}}$, we may modify $C_{\hat{s}_{\delta^w}}$ in U_ϵ^S according to Proposition 5.2, and keep $C_{\hat{s}_{\delta^w}}$ unchanged outside of the union of such U_ϵ^S. In such way, we can construct a family of symplectic curves $\{C_t\}_{t\in[0,1]}$ under the symplectic form $\check{\omega}_{\delta^w}$, such that $C_0 = C_{\hat{s}_{\delta^w}}$ and $F_{\delta^w}(C_1) = \Gamma$ is a graph that coincides with Γ_Z away from an ϵ-neighborhood of $s(\Gamma_Z)$ and is an $O(\epsilon)$-perturbation of Γ_Z.*

Proof. It is straightforward to verify that the deformation defined in the proposition match on overlapping regions. Through similar discussion as in the remark after Proposition 5.2, it is easy to observe that C_t is toric outside of the union of U_ϵ^S for $S \in Z^{\mathrm{top}}$, hence the moment map image of C_t in this region is one-dimensional, independent of t and is inside a small neighborhood of $s(\Gamma_Z) \cap e(\Gamma_Z)$. For each $S \in Z^{\mathrm{top}}$, in U_ϵ^S, as in the proof of Proposition 5.9, we have coordinates (x_1, x_2), which reduces C_t here to $C_t \subset \mathbb{CP}^2$ in Proposition 5.2. Hence the image of $C_1 \cap U_\epsilon^S$ under the moment map coincides with part of $\Gamma_S \subset \Gamma_Z$ according to Proposition 5.2. $\square$

Theorem 5.11. *There exists a piecewise smooth Lagrangian fibration $\hat{F}$ as perturbation of the moment map F_{δ^w} such that $\hat{F}|_{\partial P_\Sigma} = F_{\delta^w}|_{\partial P_\Sigma}$ and $\hat{F}(C_{s^{\delta^w}}) = \Gamma$ is a graph that coincides with Γ_Z away from an ϵ-neighborhood of $s(\Gamma_Z)$ and is an $O(\epsilon)$-perturbation of Γ_Z.*

Proof. According to Proposition 5.9, we can construct a smooth symplectomorphism $H_1\colon (P_\Sigma, \omega_{\delta^w}) \to (P_\Sigma, \check{\omega}_{\delta^w})$ such that $H_1(C_{s_{\delta^w}}) = C_{\hat{s}_{\delta^w}}$. One can make $H_1|_{\partial P_\Sigma} = \mathrm{Id}_{\partial P_\Sigma}$ with the expense of smoothness of H_t at $C_0 \cap \partial P_\Sigma$.

For the symplectic isotopic family $\{C_t\}_{t\in[0,1]}$ under the symplectic form $\check{\omega}_{\delta^w}$ in Proposition 5.10, we may define H_2 in U_ϵ^S for $S \in Z^{\mathrm{top}}$ to be the H_1 in Theorem 5.4 and extend by identity map outside the union of U_ϵ^S for $S \in Z^{\mathrm{top}}$. Then $H_2 : (P_\Sigma, \check{\omega}_{\delta^w}) \to (P_\Sigma, \check{\omega}_{\delta^w})$ is piecewise smooth and $C^{0,1}$ symplectomorphism satisfying $H_2|_{\partial P_\Sigma} = \mathrm{Id}_{\partial P_\Sigma}$, $H_2(C_{\hat{s}_{\delta^w}}) = C_1$ such that $\check{F}_{\delta^w}(C_1) = \Gamma$ is a graph that is an ϵ-perturbation of the graph Γ_Z.

Let $H = H_2 \circ H_1$. Then $H|_{\partial P_\Sigma} = \mathrm{Id}_{\partial P_\Sigma}$ and $\hat{F} = F_{\delta^w} \circ H$ is the desired perturbation of F_{δ^w}. $\square$

Remark 5.12. Theorems 5.11 and 3.11 of this paper are needed for the proofs in [7].

Proposition 5.13. *For each $S \in Z^{\rm top}$, we may modify $C_{\hat{s}_{\delta^w}}$ in U_ϵ^S according to Proposition 5.5, and keep $C_{\hat{s}_{\delta^w}}$ unchanged outside of the union of such U_ϵ^S. In such way, we can construct a family of symplectic curves $\{C_t\}_{t\in[0,1]}$ under the symplectic form $\check{\omega}_{\delta^w}$, such that $C_0 = C_{\hat{s}_{\delta^w}}$ and $F_{\delta^w}(C_1) = \Gamma$ is a graph that coincides with Γ_Z away from an ϵ-neighborhood of $s(\Gamma_Z)$ and is an $O(\epsilon)$-perturbation of Γ_Z.*

Proof. The proof is the same as the proof of Proposition 5.10 except that Proposition 5.2 is replaced with Proposition 5.5. □

Theorem 5.14. *$\hat{F}$ in Theorem 5.11 can be made smooth away from $C_0 \cap \hat{F}^{-1}(v(\Gamma_Z))$ and $C_0 \cap \partial P_\Sigma$, where $v(\Gamma_Z)$ is the set of 3-valent vertices of Γ_Z.*

Proof. The proof is the same as the proof of Theorem 5.11 except that Proposition 5.10 (resp. Theorem 5.4) is replaced with Proposition 5.13 (resp. Theorem 5.6). □

Remark 5.15. In this theorem, $\hat{F}$ achieved optimal smoothness possible. This result is a significant improvement over Theorem 5.11, and should play an important role in improving the Lagrangian torus fibration of quintic Calabi-Yau constructed in [7] to optimal smoothness. We hope to come back to such improvement of [7] in a future paper.

Theorems 5.11 and 5.14 concern the partial secondary fan, where $Z \in \tilde{Z}$. They have natural generalization to the case of secondary fan, where $Z \in \hat{Z}$. Such generalization turns out to be extremely straightforward. The only difference in the argument when $Z \in \hat{Z}$ is that for each $S = \{m_0, m_1, m_2\} \in Z^{\rm top}$, $\{z_i = \delta^{w_{m_i}} a_{m_i} s_{m_i}\}_{i=0}^2$ defines an open covering (instead of embedding) $U_\epsilon^S \hookrightarrow \mathbb{CP}^2$, where $[z_0, z_1, z_2]$ is the homogeneous coordinate of $\mathbb{CP}^2$. Local models in Section 5.1 can be pull back using the open covering maps in the same way as using the open embeddings in the case of $Z \in \tilde{Z}$. With this understanding, it is easy to check that all arguments in the case of $Z \in \tilde{Z}$ can easily be adopted to the case of $Z \in \hat{Z}$. We have

Theorem 5.16. *Theorems 5.11 and 5.14 are also true when $Z \in \hat{Z}$.*

As we did at the end of Section 4.1, we may classify the fibers $\mu_r := C_0 \cap \hat{F}^{-1}(r)$ of the map $\hat{F} : C_0 \to \Gamma$ for $r \in \Gamma$ in the general case. In general,

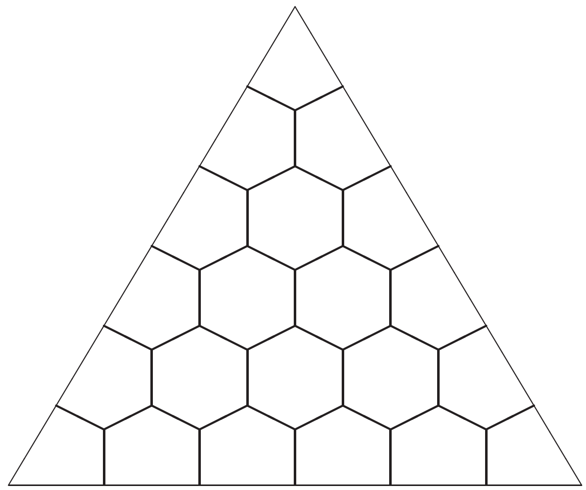

Figure 10: $F(C_p)$ of degree $d = 5$ curve in $\mathbb{CP}^2$ perturbed to $\hat{F}(C_p) = \Gamma$.

when $Z \in \hat{Z}$, μ_r can be several points when r is an end point of Γ. μ_r can be several circles when r is a smooth point of Γ. μ_r can be an abelian multiple cover of the Θ shaped graph in the torus at the right of figure 9 when r is a three-valent vertex of Γ. (The graph illustrated at the left of figure 9 can be viewed as an example of such, which is a $(\mathbb{Z}_5)^2$-cover of the Θ shaped graph.) In the special case when $Z \in \tilde{Z}$, μ_r is a point when r is an end point

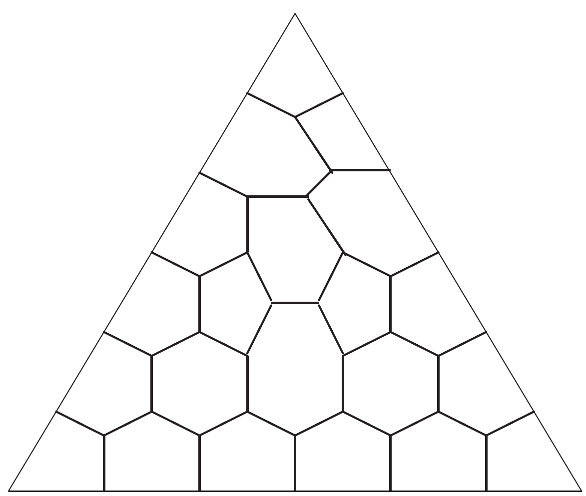

Figure 11: Alternative Γ for degree $d = 5$ curve in $\mathbb{CP}^2$.

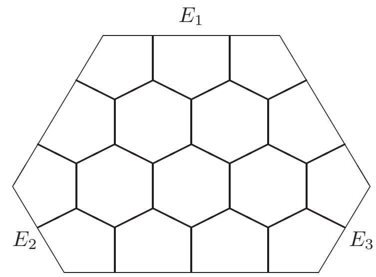

Figure 12: $F_s(C_s)$ in figure 3 perturbed to graph Γ.

of Γ. μ_r is a circle when r is a smooth point of Γ. μ_r is the Θ shaped graph when r is a 3-valent vertex of Γ.

Examples 5.17. Using these theorems, the images of degree $d = 5$ curves in $\mathbb{CP}^2$ under the moment maps as illustrated in figure 2 can be perturbed to figure 10.

This example correspond to the large complex limit with respect to the standard simplicial decomposition of Δ. When approaching different large complex limit in $\overline{\mathcal{M}_g}$ the toric moduli space of stable curves of genus g, the graph Γ will be different and determined by the corresponding simplicial decomposition Z of Δ. Figure 11 is an example for degree $d = 5$ curve in $\mathbb{CP}^2$.

Applying these theorems to the case of curves in the toric surface ($\mathbb{CP}^2$ with three points blown up) as illustrated in figure 3, we will be able to perturb the image of the moment map to figure 12.

Acknowledgment

I would like to thank Prof. S.T. Yau for constant encouragement, Prof. Yong-Geun Oh for pointing out the work of [4] to me. This work was initially done while I was in Columbia University. I am very grateful to Columbia University for excellent research environment. Thanks also go to Qin Jing for stimulating discussions and suggestions. W.-D.R was partially supported by NSF Grant DMS-9703870 and DMS-0104150.

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Received April 12, 2005