A Monge–Ampère-type equation motivated by string theory

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In this paper, we solve a Monge–Ampère-type equation by the continuity method. This equation is motivated by the superstring theory.

1. Introduction

There is a rich class of non-Kähler complex manifolds for dimension greater than two. It is therefore important to construct canonical geometry on such manifolds. Since non-Kähler geometry is not as intuitive as Riemannian geometry, it has been difficult to find a reasonable class of Hermitian metrics that exhibit rich geometry. We believe that metrics motivated by theoretic physics should have good properties. This is especially true for those metrics that are supersymmetric. The work of Strominger did provide such a candidate. In this paper, we provided a smooth solution for Strominger's system. While it is not hard to generalize such system to elliptic fibration over general Calabi–Yau manifolds based on the methods developed here, we shall restrict ourselves to complex three-dimensional examples. We shall discuss the higher dimensional examples in other paper. In the following, we motivate the point of view from string theory.

In the original proposal for compactification of superstring, Candelas et al. [1] constructed the metric product of a maximal symmetric four-dimensional spacetime M with a six-dimensional Calabi–Yau vacuum X as the ten-dimensional space–time; they identified the Yang–Mills connection with the $SU(3)$ connection of the Calabi–Yau metric and set the dilaton to be a constant. Adapting the second author's suggestion of using Uhlenbeck–Yau's theorem on constructing Hermitian-Yang–Mills connections over stable bundles [9], Witten [11] and later Horava–Witten [4] proposed to use higher rank bundles for strong coupled heterotic string theory so that the gauge groups can be $SU(4)$ or $SU(5)$.

At around the same time, Strominger [7] analyzed heterotic superstring background with spacetime supersymmetry and non-zero torsion by allowing

a scalar "warp factor" for the spacetime metric. He considered a ten-dimensional spacetime that is a warped product of a maximal symmetric four-dimensional spacetime M and an internal space X; the metric on $M \times X$ takes the form

$$g^0 = e^{2D(y)} \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{ij}(y) \end{pmatrix}, \quad x \in M, \ y \in X;$$

the connection on an auxiliary bundle is Hermitian-Yang–Mills connection over X:

$$F \wedge \omega^2 = 0, \quad F^{2,0} = F^{0,2} = 0.$$

Here ω is the Hermitian form $\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} \, dz^i \wedge d\bar{z}^j$ defined on the internal space X. In this system, the physical relevant quantities are

$$h = -\sqrt{-1}(\bar{\partial} - \partial)\omega,$$
$$\phi = -\frac{1}{2} \log \|\Omega\| + \phi_0$$

and

$$g_{ij}^0 = e^{2\phi_0} \|\Omega\|^{-1} g_{ij},$$

for a constant ϕ_0.

In order for the ansatz to provide a supersymmetric configuration, one introduces a Majorana–Weyl spinor ϵ so that

$$\delta\psi_M = \nabla_M \epsilon - \frac{1}{8} h_{MNP} \gamma^{NP} \epsilon = 0,$$
$$\delta\lambda = \gamma^M \partial_M \phi \epsilon - \frac{1}{12} h_{MNP} \gamma^{MNP} \epsilon = 0,$$
$$\delta\chi = \gamma^{MN} F_{MN} \epsilon = 0,$$

where ψ_M is the gravitino, λ the dilatino, χ the gluino, ϕ the dilaton and h the Kalb–Ramond field strength obeying

$$dh = \frac{\alpha'}{2}(\operatorname{tr} F \wedge F - \operatorname{tr} R \wedge R).$$

Strominger [7] showed that in order to achieve spacetime supersymmetry, the internal six manifold X must be a complex manifold with a non-vanishing

holomorphic 3-form Ω; and the anomaly cancellation demands that the Hermitian form ω obey[1]

$$\sqrt{-1}\partial\bar{\partial}\omega = \frac{\alpha'}{4}(\operatorname{tr} R \wedge R - \operatorname{tr} F \wedge F)$$

and supersymmetry requires[2]

$$d^*\omega = \sqrt{-1}(\bar{\partial} - \partial) \log \|\Omega\|_\omega.$$

Accordingly, he proposed the system

$$(1.1) \qquad\qquad\qquad\qquad\qquad F_H \wedge \omega^2 = 0;$$

$$(1.2) \qquad\qquad\qquad\qquad\qquad F_H^{2,0} = F_H^{0,2} = 0;$$

$$(1.3) \qquad\qquad \sqrt{-1}\partial\bar{\partial}\omega = \frac{\alpha'}{4}(\operatorname{tr} R \wedge R - \operatorname{tr} F_H \wedge F_H);$$

$$(1.4) \qquad\qquad\qquad d^*\omega = \sqrt{-1}(\bar{\partial} - \partial) \ln \|\Omega\|_\omega.$$

This system gives a solution of a superstring theory with flux that allows non-trivial dilaton field and Yang–Mills field. (It turns out $D(y) = \phi$ and is the dilaton field.) Here ω is the Hermitian form and R is the curvature tensor of the Hermitian metric ω; H is the Hermitian metric and F is its curvature of a vector bundle E; tr is the trace of the endomorphism bundle of either E or TX.

In [5], Li and Yau gave the first irreducible non-singular solution of the supersymmetric system of Strominger for $U(4)$ and $U(5)$ principle bundle. They obtained their solutions by perturbing around the Calabi–Yau vacuum coupled with the sum of tangent bundle and trivial line bundles. In our previous paper [2], we gave, first, examples of solutions to Strominger's system on a non-Kähler manifold, which is a torus bundle over a $K3$ surface. In this paper, we consider the case $\alpha' < 0$. This provides an interesting system of nonlinear partial differential equations.

As in our previous paper, we construct a solution on a torus bundle over the K3 surface constructed by Goldstein and Prokushkin [3]. Let (S, ω_S, Ω_S) be a K3 surface or complex torus with Kähler form ω_S and a non-vanishing

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[1]The curvature F of the vector bundle E in ref. [7] is real, i.e., $c_1(E) = \frac{F}{2\pi}$. But, we are used to taking the curvature F such that $c_1(E) = \frac{\sqrt{-1}}{2\pi}F$. So this equation has modified Equation (2.18) of ref. [7] by a minus sign.

[2]See Equation (56) of ref. [8], which corrects Equation (2.30) of ref. [7] by a minus sign.

holomorphic (2,0)-form Ω_S. Let ω_1 and ω_2 be anti-self-dual (1,1)-forms such that $\frac{\omega_1}{2\pi}$ and $\frac{\omega_2}{2\pi}$ represent integral cohomology classes. Using these two forms, Goldstein and Prokushkin constructed the non-Kähler manifold X such that $\pi : X \to S$ is a holomorphic T^2-fibration over S with hermitian form $\omega_0 = \pi^*\omega_S + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}$ and holomorphic 3-form $\Omega = \Omega_S \wedge \theta$ (for the definition of θ, see Section 2). Our solution to Strominger's system for $\alpha' < 0$ can be constructed as follows.

Let L_1 and L_2 be holomorphic line bundle over S such that their curvature forms are given by $\sqrt{-1}\omega_1$ and $\sqrt{-1}\omega_2$, respectively. Corresponding to these curvature forms, there exist Hermitian metrics h_1 and h_2 on L_1 and L_2. Let $E = L_1 \oplus L_2 \oplus T'S$ and $H_0 = (h_1, h_2, \omega_S)$. Then $F_{H_0} = \mathrm{diag}(\sqrt{-1}\omega_1, \sqrt{-1}\omega_2, R_{\omega_S})$. Let u be any smooth function on S and

$$(1.5) \qquad \omega_u = \pi^*(e^u \omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}.$$

Then $(V = \pi^*E, \pi^*F_{H_0}, X, \omega_u)$ satisfies Strominger's Equations (1.1),(1.2) and (1.4) and we only need to consider Equation (1.3). Because ω_1 and ω_2 are harmonic, we can write ω_1 and ω_2 locally as

$$\omega_1 = \bar{\partial}\xi = \bar{\partial}(\xi_1\, dz_1 + \xi_2\, dz_2)$$

and

$$\omega_2 = \bar{\partial}\zeta = \bar{\partial}(\zeta_1\, dz_1 + \zeta_2\, dz_2),$$

where (z_1, z_2) are local coordinate on S. Let

$$B = \begin{pmatrix} \xi_1 + \sqrt{-1}\zeta_1 \\ \xi_2 + \sqrt{-1}\zeta_2 \end{pmatrix}.$$

Using matrix B, we can compute curvature R_u of the metric ω_u and $R_u \wedge R_u$ as in our previous paper [2]. Let $g = (g_{i\bar{j}})$ if $\omega_S = \frac{\sqrt{-1}}{2}g_{i\bar{j}}\, dz_i \wedge dz_{\bar{j}}$.

Theorem 1.1. $(V = \pi^*E, \pi^*F_{H_0}, X, \omega_u)$ *is a solution of Strominger's system for* $\alpha' = -2$ *if and only if the function u of S satisfies the following Monge–Ampère-type equation:*

$$(1.6) \qquad \triangle e^u \cdot \frac{\omega_S^2}{2!} + \partial\bar{\partial}(e^{-u}\,\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})) + \partial\bar{\partial}u \wedge \partial\bar{\partial}u = 0.$$

For the special case, $\omega_2 = n\omega_1$, $n \in \mathbb{Z}$, $(V, \pi^*F_{H_0}, X, \omega_u)$ is a solution to Strominger's system if and only if the smooth function u on S satisfies the

equation:

$$(1.7) \qquad \triangle \left(e^u + \frac{(1+n^2)}{4} \|\omega_1\|^2_{\omega_S} e^{-u} \right) - 8 \frac{\det(u_{i\bar{j}})}{\det(g_{i\bar{j}})} = 0.$$

In [2], we proved that

$$\mathrm{tr}\,(\bar{\partial} B \wedge \partial B^* \cdot g^{-1})$$

is a globally well-defined (1,1)-form on S and when $\omega_2 = n\omega_1$, $n \in \mathbb{Z}$,

$$\mathrm{tr}\,(\bar{\partial} B \wedge \partial B^* \cdot g^{-1}) = \sqrt{-1} \frac{1+n^2}{4} \|\omega_1\|^2_{\omega_S} \omega_S.$$

Let $f = \frac{(1+n^2)}{4} \|\omega_1\|^2$ and $g'_{i\bar{j}} = (e^u - fe^{-u})g_{i\bar{j}} - 4u_{i\bar{j}}$, then we can rewrite Equation (1.7) as

$$\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} = (e^u - fe^{-u})^2 + 2\{(e^u + fe^{-u})|\nabla u|^2 + e^{-u} \triangle f - 2e^{-u} \nabla u \cdot \nabla f\}.$$

We solve Equation (1.6) by continuity argument [12]. Our main theorem is

Theorem 1.2. *There is a smooth solution of Equation (1.6) such that*

$$e^u \omega_S + \sqrt{-1} e^{-u} \mathrm{tr}(\bar{\partial} B \wedge \partial B^* \cdot g^{-1}) - 2\sqrt{-1}\partial\bar{\partial}u$$

defines a Hermitian metric on S.

Fix a solution u to Equation (1.6). According to Theorem 1.1, $(V, F_{\pi^* H_0}, X, \omega_u)$ defines a reducible solution to Strominger's system. It can be extended to a family of irreducible solution by method of perturbation as was done by Li–Yau [5]:

Theorem 1.3. *Let (E, H_0, S, ω_S) be as above. Fix the holomorphic structure D''_0 of E. Then there is a smooth deformation D''_s of the holomorphic structure D''_0 on E so that there are Hermitian-Yang–Mills metrics H_s on (E, D_s) and smooth functions ϕ_s on S such that*

$$\left(V = \pi^* E, \pi^* D''_s, \pi^* H_s, \pi^*(e^{u+\phi_s}\omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta} \right)$$

are irreducible solutions to Strominger's system for $\alpha' = -2$ on X. Furthermore, $\lim_{s\to 0} \phi_s = 0$ and $\lim_{s\to 0} H_s$ is a regular reducible Hermitian-Yang–Mills connection on $E = L_1 \oplus L_2 \oplus TS$.

2. Geometric model

In this section, we recall the geometric model of Goldstein and Prokushkin [3] for complex non-Kähler manifolds with $SU(3)$ structure. Their results can be stated as follows:

Theorem 2.1 [3]. *Let (S, ω_S, Ω_S) be a Calabi–Yau surface with a non-vanishing holomorphic $(2,0)$−form Ω_S. Let ω_1 and ω_2 be anti-self-dual $(1,1)$-forms on S such that $\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi} \in H^2(S, \mathbb{Z})$. Then there is a three-dimensional complex manifold X such that $\pi : X \to S$ is a holomorphic T^2-fibration over S with the following properties*:

1. *For any real 1-forms α_1 and α_2 defined on an open subset of S satisfying $d\alpha_1 = \omega_1$ and $d\alpha_2 = \omega_2$, there are local coordinates x and y on X such that $dx + i\,dy$ defines a holomorphic 1-form on T^2-fibers and the metric on X has the following form*:

$$(2.1) \qquad g_0 = \pi^* g + (dx + \pi^* \alpha_1)^2 + (dy + \pi^* \alpha_2)^2,$$

   where g is the Ricci-flat metric on S.

2. *X admits a nowhere vanishing holomorphic $(3,0)$-form with unit length*:

$$\Omega = ((dx + \pi^* \alpha_1) + i(dy + \pi^* \alpha_2)) \wedge \pi^* \Omega_S.$$

3. *If either ω_1 or ω_2 represents a non-trivial cohomological class, then X admits no Kähler metric.*

4. *X is a balanced manifold (see [6]), i.e., the Hermitian form*

$$(2.2) \qquad \omega_0 = \pi^* \omega_S + (dx + \pi^* \alpha_1) \wedge (dy + \pi^* \alpha_2)$$

   satisfies the condition that $d\omega_0^2 = 0$.

5. *For any smooth function u on S, the Hermitian metric*

$$\omega_u = \pi^* (e^u \omega_S) + (dx + \pi^* \alpha_1) \wedge (dy + \pi^* \alpha_2)$$

   is conformal balanced and (ω_u, Ω) satisfies Equation (1.4).

Let

$$\theta = (dx + \alpha_1) + \sqrt{-1}(dy + \alpha_2),$$

(2.3)

$$\omega_0 = \omega_S + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}$$

and

(2.4)
$$\omega_u = e^u \omega_S + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}.$$

Let R and R_u be the curvature forms of the Hermitian connections on the holomorphic tangent bundle corresponding to Hermitian metrics ω_0 and ω_u, respectively. Then, we have

Proposition 2.2 [2]**.**

(2.5) $\operatorname{tr} R \wedge R = \pi^* \left(\operatorname{tr} R_S \wedge R_S + 2 \operatorname{tr} \partial\bar{\partial} \left(\bar{\partial}B \wedge \partial B^* \cdot g^{-1} \right) \right).$

Proposition 2.3 [2]**.**

(2.6)
$$\operatorname{tr} R_u \wedge R_u = \pi^* \left(\operatorname{tr} R_S \wedge R_S + 2\partial\bar{\partial}\left(e^{-u} \operatorname{tr}\left(\bar{\partial}B \wedge \partial B^* \cdot g^{-1} \right) \right) \right. $$
$$\left. + 2\partial\bar{\partial}u \wedge \partial\bar{\partial}u \right).$$

Proposition 2.4 [2]**.**   *When* $\omega_2 = n\omega_1$, $n \in \mathbb{Z}$,

(2.7) $\operatorname{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1}) = \dfrac{\sqrt{-1}}{4}(1 + n^2)\|\omega_1\|^2_{\omega_S}\omega_S,$

The Laplace operator $\triangle$ on S is defined by $\triangle = \bar{\partial}^* \circ \bar{\partial}$, where $\bar{\partial}^*$ is defined by the metric ω_S. For any smooth function ψ on S, we have

(2.8) $\sqrt{-1}\partial\bar{\partial}\psi \wedge \omega_S = \triangle\psi \cdot \dfrac{\omega_S^2}{2!}.$

Lemma 2.5 [2]**.**

(2.9) $\sqrt{-1}\partial\bar{\partial}\omega_u = \triangle e^u \cdot \dfrac{\omega_S^2}{2!} + \dfrac{1}{2}(\|\omega_1\|^2_{\omega_S} + \|\omega_2\|^2_{\omega_S})\dfrac{\omega_S^2}{2!}.$

3. Construction of solution to Strominger's system with reducible bundle

In this section, we demonstrate the existence of smooth solutions to Strominger's system.   As $\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi} \in H^{1,1}(S, \mathbb{R}) \cap H^2(S, \mathbb{Z})$, there are holomorphic line bundles L_1 and L_2 over S such that the curvature forms of their

Hermitian connections are $\sqrt{-1}\omega_1$ and $\sqrt{-1}\omega_2$, respectively (see [10]). Associated to these curvature forms, there exist Hermitian metrics h_1 and h_2 on L_1 and L_2. Let

$$(3.1) \qquad\qquad E = L_1 \oplus L_2 \oplus T'S$$

and

$$H_0 = (h_1, h_2, \omega_S).$$

The curvature form F_{H_0} of E is given by

$$(3.2) \qquad\qquad F_{H_0} = \begin{pmatrix} \sqrt{-1}\omega_1 & & \\ & \sqrt{-1}\omega_2 & \\ & & R_S \end{pmatrix},$$

where R_S is the curvature of $T'S$ associate to the Hermitian metric ω_S. Let $V = \pi^* E$ and $F_{\tilde{H}_0} = \pi^* F_{H_0}$.

Lemma 3.1. *For any smooth function u on S, $(V, F_{\tilde{H}_0}, X, \omega_u)$ satisfies Equations (1.1), (1.2) and (1.4).*

Proof. By Theorem 2.1, Equation (1.4) is satisfied. As $F_{\tilde{H}_0}$ is the $(1,1)$-form on X, $F_{\tilde{H}_0}^{2,0} = F_{\tilde{H}_0}^{0,2} = 0$. We also have

$$F_{\tilde{H}_0} \wedge \omega_u^2 = \pi^* (F_{H_0} \wedge e^u \omega_S) \wedge (\pi^*(e^u \omega_S) + \sqrt{-1}\theta \wedge \bar{\theta}) = 0$$

as follows from the equation $\omega_1 \wedge \omega_S = \omega_2 \wedge \omega_S = 0$ and $R_S \wedge \omega_S = 0$. $\qquad\square$

It remains to consider Equation (1.3). We normalize $\alpha' = -2$, then $(V, F_{\tilde{H}_0}, X, \omega_u)$ satisfies the equation

$$(3.3) \qquad\qquad \sqrt{-1}\partial\bar{\partial}\omega_u = \frac{1}{2}(\operatorname{tr} F_{\tilde{H}_0} \wedge F_{\tilde{H}_0} - \operatorname{tr} R_u \wedge R_u)$$

and

$$(3.4) \qquad \begin{aligned} \operatorname{tr} F_{\tilde{H}_0} \wedge F_{\tilde{H}_0} &= -\omega_1 \wedge \omega_1 - \omega_2 \wedge \omega_2 + \operatorname{tr} R_S \wedge R_S \\ &= (\|\omega_1\|^2 + \|\omega_2\|^2)\frac{\omega_S^2}{2!} + \operatorname{tr} R_S \wedge R_S. \end{aligned}$$

Inserting (2.6), (2.9) and (3.4) into (3.3), we can rewrite Equation (3.3) as

$$(3.5) \qquad \triangle e^u \cdot \frac{\omega_S^2}{2!} + \partial\bar{\partial}(e^{-u}\operatorname{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})) + \partial\bar{\partial}u \wedge \partial\bar{\partial}u = 0.$$

In conclusion, we have

Theorem 3.2. $(V, F_{\pi^* H_0}, X, \omega_u)$ *is a solution of Strominger's system for* $\alpha' = -2$ *if and only if the function* u *of* S *satisfies Equation* (*3.5*)*. In particular, when* $\omega_2 = n\omega_1$, $n \in \mathbb{Z}$, $(V, F_{\pi^* H_0}, X, \omega_u)$ *is a solution to Strominger's system if and only if the smooth function* u *on* S *satisfies the equation:*

$$(3.6) \qquad \triangle \left(e^u + \frac{(1+n^2)}{4} \|\omega_1\|_{\omega_S}^2 e^{-u} \right) \frac{\omega_S^2}{2!} + \partial\bar{\partial}u \wedge \partial\bar{\partial}u = 0$$

or

$$(3.7) \qquad \triangle \left(e^u + \frac{(1+n^2)}{4} \|\omega_1\|_{\omega_S}^2 e^{-u} \right) - 8\frac{\det(u_{i\bar{j}})}{\det(g_{i\bar{j}})} = 0.$$

Proof. For the second part of theorem, we note that when $\omega_2 = n\omega_1$, it follows from Proposition 2.4 and (2.8) that

$$\partial\bar{\partial}(e^{-u}\operatorname{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})) = \sqrt{-1}\frac{(1+n^2)}{4}\partial\bar{\partial}(\|\omega_1\|_{\omega_S}^2 e^{-u}) \wedge \omega_S$$
$$= \frac{(1+n^2)}{4} \triangle (\|\omega_1\|_{\omega_S}^2 e^{-u}) \frac{\omega_S^2}{2!}.$$

So, Equation (3.6) follows from (3.5). Equation (3.7) is derived from

$$\partial\bar{\partial}u \wedge \partial\bar{\partial}u = 2\det(u_{i\bar{j}})dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$$
$$= -8\frac{\det(u_{i\bar{j}})}{\det(g_{i\bar{j}})} \frac{\omega_S^2}{2!}.$$

$\square$

Theorem 3.3. *There is a smooth solution of Equation* (*3.5*) *such that*

$$e^u\omega_S + \sqrt{-1}e^{-u}\operatorname{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1}) - 2\sqrt{-1}\partial\bar{\partial}u$$

defines a Hermitian metric on S.

Proof. We solve Equation (3.5) by the continuity method. More precisely, we introduce a parameter $t \in [0, 1]$ and consider the following equation

$$(3.8) \quad \sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S + t\partial\bar{\partial}(e^{-u}\operatorname{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})) + \partial\bar{\partial}u \wedge \partial\bar{\partial}u = 0.$$

We shall impose the following:

$$(3.9) \qquad \text{Elliptic condition}: \omega' = e^u\omega_S + \sqrt{-1}te^{-u}\operatorname{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})$$
$$- 2\sqrt{-1}\partial\bar{\partial}u > 0$$

and

(3.10)   Normalization condition : $\int_S e^{-u} \frac{\omega_S^2}{2!} = A,\quad \int_S 1 \frac{\omega_S^2}{2!} = 1.$

Let $C^{k,\alpha}(S)$ be the space of functions whose kth derivatives are Hölder continuous with exponent $0 \le \alpha \le 1$. We consider the solution in the following space

(3.11) $B_A = \{u \in C^{2,\alpha}(S) | u \text{ satisfies the normalization } (3.10)\}$

and

(3.12) $B_{A,t} = \{u \in B_A | u \text{ also satisfies the elliptic condition } (3.9)\}.$

Let

(3.13) $\mathbf{T} = \{t \in [0,1] | \text{ Equation } (3.5) \text{ admits a solution in } B_{A,t}\}.$

Obviously $0 \in \mathbf{T}$ with a solution $u = -\ln A$. Hence, we only need to show that $\mathbf{T}$ is both closed and open in $[0,1]$. This will imply that $1 \in \mathbf{T}$ and that our original equation has a solution in $C^{2,\alpha}$. By the Schauder theory, this solution is smooth. The proof of openness is the same as the proof of our Theorem 18 in [2].

So, we need only to prove that $\mathbf{T}$ is closed. We need to estimate up to third order. The third-order estimate is the same as in our previous paper [2]. Hence we shall provide estimates up to second order. These will be done in Section 4, 5 and 6. In Section 7, we shall explain why the estimates to Equation (3.7) can be generalized to Equation (3.5).   □

4. Zero-order estimate

Let $f = \frac{1+n^2}{4} \|\omega_1\|^2$, where ω_1 is the anti-self dual (1,1)-form on S. Then Equation (3.7) can be written as

(4.1) $\triangle \left(e^u + f e^{-u} \right) - 8 \frac{\det(u_{i\bar{j}})}{\det(g_{i\bar{j}})} = 0.$

The elliptic condition says

(4.2) $\omega' = (e^u - t f e^{-u})\omega_S - 2\sqrt{-1}\partial\bar{\partial}u > 0$

and the normalization is

$$(4.3) \qquad \int_S e^{-u} \frac{\omega_S^2}{2!} = A, \quad \int_S 1 \frac{\omega_S^2}{2!} = 1.$$

In the following, the volume form will be $\frac{\omega_S^2}{2!}$ unless it is clear from the context. Multiplying the elliptic condition $e^u - fe^{-u} > \triangle u$ by pe^{-pu}, we obtain

$$p(e^{-pu})(e^u - fe^{-u}) \geq p(e^{-pu}) \triangle u = - \triangle (e^{-pu}) + 4|\nabla (e^{-u})^{p/2}|^2$$

and by integrating, we obtain

$$(4.4) \qquad \int_S |\nabla (e^{-u})^{p/2}|^2 \leq \frac{p}{4} \int_S (e^{-u})^{p-1}.$$

Applying the Sobolev inequality, we can find a constant C depending only on S such that

$$\left(\int_S (e^{-u})^{2p} \right)^{1/2} \leq C \int_S (e^{-u})^p + C \int_S |\nabla (e^{-u})^{p/2}|^2$$

$$(4.5)$$

$$\leq C \int_S (e^{-u})^p + \frac{p}{4} C \int_S (e^{-u})^{p-1}.$$

 In the following, we use the constant in the generic sense. So C may mean different constants in different equations. But it shall be independent of u. By (4.5) and the Hölder inequality, we find

$$\int_S (e^{-u})^{2p} \leq C^2 \left(\int_S (e^{-u})^p \right)^2 + C^2 p^2 \left(\int_S (e^{-u})^{p-1} \right)^2$$

$$(4.6)$$

$$\leq C^2 \left(\int_S (e^{-u})^p \right)^2 + C^2 p^2 \left(\int_S (e^{-u})^p \right)^{2(p-1)/p}.$$

We assume that

$$(4.7) \qquad \int_S e^{-u} = A < 1.$$

There are two cases:

Case (1). For any $p \in \mathbb{Z}$, $\int_S e^{-pu} < 1$. Then, $\left(\int_S (e^{-u})^p\right)^2 < \left(\int_S (e^{-u})^p\right)^{2(p-1)/p}$ and from (4.6),

$$\int_S (e^{-u})^{2p} \leq C^2 p^2 \left(\int_S (e^{-u})^p\right)^{2(p-1)/p}.$$

Let $2p = 2^\beta$, then

$$\int_S (e^{-u})^{2^\beta} \leq C^2 (2^{\beta-1})^2 \left(\int_S (e^{-u})^{2^{\beta-1}}\right)^{2(1-2^{-(\beta-1)})}$$

$$\leq \left(\prod_{b=1}^{\beta-1} C^{2^b}\right) \left(\prod_{b=1}^{\beta-1} \left(2^{(\beta-b)}\right)^{2^b}\right) \left(\int_S (e^{-u})^2\right)^{2^{\beta-1} \cdot \prod_{k=1}^{\beta-1}(1-1/2^k)}$$

$$\leq C^{2^\beta - 2} \cdot 2^{2^{\beta+1}} \left(\int_S (e^{-u})^2\right)^{2^{\beta-1} \cdot \prod_{k=1}^{\beta-1}(1-1/2^k)},$$

where the last inequality follows from

$$(4.8) \qquad\qquad \prod_{b=1}^{\beta-1} \left(2^{\beta-b}\right)^{2^b} \leq 2^{2^{\beta+1}},$$

which can be derived from the following calculation:

$$\prod_{b=1}^{\beta-1} \left(2^{\beta-b}\right)^{2^b} = \prod_{b=1}^{\beta-1} 2^{2^b} \prod_{b=1}^{\beta-1} \left(2^{\beta-(b+1)}\right)^{2^b}$$

$$= 2^{2^\beta - 2} \left(\prod_{b=1}^{\beta-1} \left(2^{\beta-(b+1)}\right)^{2^{b+1}}\right)^{1/2}$$

$$= 2^{2^\beta - 1 - \beta} \left(\prod_{b=1}^{\beta-1} \left(2^{\beta-b}\right)^{2^b}\right)^{1/2}$$

$$\leq 2^{2^\beta} \left(\prod_{b=1}^{\beta-1} \left(2^{\beta-b}\right)^{2^b}\right)^{1/2}.$$

Hence, we obtain

$$\left(\int (e^{-u})^{2^\beta}\right)^{1/2^\beta} \leq C^{1-2^{1-\beta}} \cdot 2^2 \left(\int_S (e^{-u})^2\right)^{1/2 \cdot \prod_{k=1}^{\beta-1}(1-1/2^k)}.$$

Let $\beta \to \infty$, we see that

$$(4.9) \qquad \exp(-\inf u) = \|e^{-u}\|_\infty \leq C \left(\int_S (e^{-u})^2 \right)^{\kappa/2},$$

where

$$(4.10) \qquad \kappa = \prod_{\beta=1}^\infty \left(1 - \frac{1}{2^\beta} \right) > 0.$$

To finish our estimate of $\inf u$, it suffices to estimate $\|e^{-u}\|_2$. When $p = 2$, inequality (4.4) yields

$$(4.11) \qquad \int_S |\nabla (e^{-u})|^2 \leq \frac{1}{2} \int_S e^{-u}.$$

Now from the normalization condition (4.3), we have $\int_S(e^{-u} - A) = 0$. Poincare inequality and (4.11) then imply

$$\int_S |(e^{-u} - A)|^2 \leq C \int_S |\nabla(e^{-u} - A)|^2 \leq CA$$

and

$$(4.12) \qquad \int_S (e^{-u})^2 \leq A^2 + CA \leq CA.$$

Combining (4.9) and (4.12), we get

$$(4.13) \qquad \exp(-\inf u) = \|e^{-u}\|_\infty \leq C_1 A^{\kappa/2}$$

and

$$(4.14) \qquad \inf u \geq -\ln C_1 - \frac{\kappa}{2} \ln A.$$

Case (2). There is an integer p such that $\int_S e^{-pu} > 1$. Let p_0 be the first such integer. Then, for any $p > p_0$, the Hölder inequality implies that

$$\int_S e^{-pu} \geq \left(\int_S e^{-p_0 u} \right)^{p/p_0} > 1.$$

From (4.5),

$$\int_S (e^{-u})^{2p} \leq C^2 p^2 \left(\int_S (e^{-u})^p \right)^2 \qquad \text{for } p \geq p_0$$

and

$$\int_S (e^{-u})^{2p} \le C^2 p^2 \left(\int_S (e^{-u})^p \right)^{2(p-1)/p} \qquad \text{for } p < p_0.$$

From these two inequalities, as was discussed in case (1), we obtain the estimate of $\inf u$. Furthermore, we have bound (4.14) with the same κ satisfying (4.10).

Next, we estimate $\sup_S u$. Let $g' = \frac{\sqrt{-1}}{2} g'_{i\bar{j}} \, dz_i \wedge dz_{\bar{j}}$, where

$$g'_{i\bar{j}} = (e^u - f e^{-u}) g_{i\bar{j}} - 4\alpha u_{i\bar{j}}.$$

We note that

$$\frac{\omega'^2}{2!} = \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \frac{\omega_S^2}{2!}.$$

The matrix $(g'^{i\bar{j}})$ satisfies the equation

$$\sum g'^{i\bar{j}} g_{i\bar{j}} = \delta^{\bar{j}}_{\bar{k}}.$$

So,

$$g'^{1\bar{1}} = \frac{g'_{2\bar{2}}}{\det g'_{i\bar{j}}}, \quad g'^{1\bar{2}} = -\frac{g'_{2\bar{1}}}{\det g'_{i\bar{j}}}, \quad g'^{2\bar{1}} = -\frac{g'_{1\bar{2}}}{\det g'_{i\bar{j}}}, \quad g'_{2\bar{2}} = \frac{g'_{1\bar{1}}}{\det g'_{i\bar{j}}}.$$

We define the operator

$$P(\varphi) \frac{\omega'^2}{2!} = \sqrt{-1} \partial \bar{\partial} \varphi \wedge \omega'$$

so that $P(\varphi) = 2 g'^{i\bar{j}} \varphi_{i\bar{j}}$. By computation,

(4.15)
$$\int_S P(e^{ku}) \frac{\det g'}{\det g} \frac{\omega^2}{2!}$$

$$= \int_S 2 g'^{i\bar{j}} \frac{\partial^2 (e^{ku})}{\partial z_i \partial \bar{z}_j} \frac{\det g'}{\det g} \frac{\omega^2}{2!}$$

$$= 2k^2 \int_S g'^{i\bar{j}} e^{ku} \frac{\partial u}{\partial z_i} \frac{\partial u}{\partial \bar{z}_j} \frac{\omega^2}{2!} + 2k \int_S g'^{i\bar{j}} e^{ku} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \frac{\det g'}{\det g} \frac{\omega^2}{2!}$$

$$\ge k \int_S e^{ku} \left(2 g'^{i\bar{j}} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \frac{\det g'}{\det g} \right) \frac{\omega^2}{2!}.$$

Applying the equation, we find

$$2g'^{i\bar{j}}\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\frac{\det g'}{\det g} = \left(2g'_{2\bar{2}}\frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1} + 2g'_{1\bar{1}}\frac{\partial^2 u}{\partial z_2 \partial \bar{z}_2}\right.$$

(4.16)
$$\left. -2g'_{1\bar{2}}\frac{\partial^2 u}{\partial z_2 \partial \bar{z}_1} - 2g'_{2\bar{1}}\frac{\partial^2 u}{\partial z_1 \partial \bar{z}_2}\right)\frac{1}{\det g}$$

$$= 2(e^u - fe^{-u})g^{i\bar{j}}\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} - 16\frac{\det u_{i\bar{j}}}{\det g}$$

$$= (e^u - fe^{-u})\triangle u - 2\triangle(e^u + fe^{-u}).$$

Inserting (4.16) into (4.15), we get

$$\int_S P(e^{ku})\frac{\det g'}{\det g}\frac{\omega^2}{2!} \geq k\int_S e^{ku}(e^u - fe^{-u})\triangle u\frac{\omega^2}{2!}$$

(4.17)
$$- 2k\int_S e^{ku}\triangle(e^u + fe^{-u})\frac{\omega^2}{2!}.$$

On the other hand, from the definition of the operator P, we have

$$\int_S P(e^{ku})\frac{\det g'}{\det g}\frac{\omega^2}{2!} = \int_S P(e^{ku})\frac{\omega'^2}{2!} = \int_S \sqrt{-1}\partial\bar{\partial}(e^{ku})\wedge\omega'$$

$$= \int_S \sqrt{-1}\partial\bar{\partial}(e^{ku})$$

$$\wedge((e^u - fe^{-u})\omega - 2\sqrt{-1}\partial\bar{\partial}u)$$

(4.18)
$$= \int_S (e^u - fe^{-u})\triangle(e^{ku})\frac{\omega^2}{2!}$$

$$+ 2\int_S \partial\bar{\partial}(e^{ku})\wedge\partial\bar{\partial}u$$

$$= k\int_S e^{ku}(e^u - fe^{-u})\triangle u\frac{\omega^2}{2!}$$

$$+ k^2\int_S (e^u - fe^{-u})e^{ku}|\nabla u|^2\frac{\omega^2}{2!},$$

where $|\nabla u|^2 = 2g^{i\bar{j}}\partial_i u \partial_{\bar{j}} u$. Combining (4.17) and (4.18), we see that

$$
\begin{aligned}
(4.19)\quad & k^2 \int_S (e^u - fe^{-u})e^{ku}|\nabla u|^2 \frac{\omega^2}{2!} \\
& \geq -2k \int_S e^{ku} \triangle (e^u + fe^{-u})\frac{\omega^2}{2!} \\
& = -2k \int_S e^{ku}(e^u - fe^{-u}) \triangle u \frac{\omega^2}{2!} - 2k \int_S e^{ku}(e^u + fe^{-u})|\nabla u|^2 \\
& \quad - 2k \int_S e^{(k-1)u} \triangle f + 4k \int_S e^{(k-1)u} \nabla u \cdot \nabla f.
\end{aligned}
$$

Integration by part, we find

$$
\begin{aligned}
(4.20)\quad & -2k \int_S e^{ku}(e^u - fe^{-u})(\triangle u)\frac{\omega^2}{2!} \\
& = 2k(k+1) \int_S e^{(k+1)u}|\nabla u|^2 - 2k(k-1) \int_S fe^{(k-1)u}|\nabla u|^2 \\
& \quad - \frac{2k}{k-1} \int_S e^{(k-1)u} \triangle f - 4k \int_S e^{(k-1)u} \nabla u \cdot \nabla f.
\end{aligned}
$$

Inserting (4.20) into (4.19) and applying Schwarz' inequality, we get

$$
\begin{aligned}
& k^2 \int_S (e^u - fe^{-u})e^{ku}|\nabla u|^2 \frac{\omega^2}{2!} \\
& \geq 2k^2 \int_S e^{(k+1)u}|\nabla u|^2 \\
& \quad - 2k^2 \int_S e^{(k-1)u}f|\nabla u|^2 - 2k\left(1 + \frac{1}{k-1}\right) \int_S e^{(k-1)u} \triangle f
\end{aligned}
$$

and

$$
\begin{aligned}
(4.21)\quad & 2k\left(1 + \frac{1}{k-1}\right) \int_S e^{(k-1)u} \triangle f \\
& \geq k^2 \int_S e^{(k+1)u}|\nabla u|^2 - k^2 \int_S e^{(k-1)u}f|\nabla u|^2.
\end{aligned}
$$

If we take $A > 0$ small enough such that

$$
(4.22) \qquad\qquad\qquad C_1^{-1} A^{-\kappa/2} > 1 + \sup f,
$$

then from (4.13) we see that $\inf e^u > 1 + \sup f$ and

(4.23)
$$k^2 \int_S e^{(k+1)u} |\nabla u|^2 - k^2 \int_S e^{(k-1)u} f |\nabla u|^2$$
$$\geq Ck^2 \int_S e^{(k+1)u} |\nabla u|^2 = \frac{4Ck^2}{(k+1)^2} \int_S |\nabla (e^u)^{(k+1)/2}|^2.$$

Equations (4.21) and (4.23) imply that for all $k \geq 1$,

(4.24)
$$\int_S |\nabla (e^u)^{(k+1)/2}|^2 \leq C(k+1) \int_S e^{(k-1)u}.$$

Now applying the Sobolev inequality and $\inf u > 0$, we get

(4.25)
$$\left(\int (e^u)^{2k} \right)^{1/2} \leq C \int (e^u)^k + C \int |\nabla (e^u)^{k/2}|^2$$
$$\leq C \int (e^u)^k + Ck \int (e^u)^{k-2} \leq Ck \int_S (e^u)^k.$$

Take $2k = 2^\beta$. Equation (4.25) implies

$$\int_S (e^u)^{2^\beta} \leq C(2^{\beta-1})^2 \left(\int_S (e^u)^{2^{\beta-1}} \right)^2$$
$$\leq C^{2^\beta - 2} \prod_{b=1}^{\beta-1} \left(2^{\beta-b} \right)^{2^b} \left(\int_S (e^u)^2 \right)^{2^{\beta-1}}$$
$$\leq C^{2^\beta - 2} 2^{2^{\beta+1}} \left(\int_S (e^u)^2 \right)^{2^{\beta-1}},$$

where the last inequality follows from (4.8). Hence,

$$\left(\int_S (e^u)^{2^\beta} \right)^{1/2^\beta} \leq C \left(\int_S (e^u)^2 \right)^{1/2}.$$

Let $\beta \to \infty$, then we get

(4.26)
$$\sup u = \|u\|_\infty = C \left(\int_S (e^u)^2 \right)^{1/2}.$$

Now we estimate $\|e^u\|_2$. When $k = 1$, inequality (4.24) yields

(4.27)
$$\int_S |\nabla (e^u)|^2 \leq C.$$

Let $M_u = \int_S e^u$, then $\int_S (e^u - M_u) = 0$. Applying the Poincare inequality and (4.27), we get

$$\int_S (e^u)^2 - \left(\int_S (e^u)\right)^2 = \int_S (e^u - M_u)^2$$
$$\leq C \int_S |\nabla (e^u - M_u)|^2$$
$$= C \int_S |\nabla e^u|^2 \leq C.$$

Hence, there is a constant C_2 depending on S, f and A (recall (4.22)) such that

$$(4.28) \qquad \int_S (e^u)^2 \leq \left(\int_S (e^u)\right)^2 + C_2.$$

Let $U_1 = \left\{ x \in S | \exp(-u(x)) \geq \frac{A}{2} \right\}$ and $U_2 = \left\{ x \in S | \exp(-u(x)) < \frac{A}{2} \right\}$. Then, from (4.13), we have

$$A = \int_S e^{-u} = \int_{U_1} e^{-u} + \int_{U_2} e^{-u} < \int_{U_1} e^{-\inf u} + \int_{U_2} \frac{A}{2}$$
$$= e^{-\inf u} \operatorname{Vol}(U_1) + \frac{A}{2}(1 - \operatorname{Vol}(U_2))$$
$$\leq \left(C_1 A^{\kappa/2} - \frac{A}{2} \right) \operatorname{Vol}(U_1) + \frac{A}{2}.$$

As $0 < \kappa < 1$ and $0 < A < 1$, we can choose A small enough such that

$$(4.29) \qquad\qquad A < (2C_1)^{1/(1-\kappa)/2}.$$

Then,

$$\operatorname{Vol}(U_1) \geq \frac{A/2}{C_1 A^{\kappa/2} - A/2} = \frac{1}{2C_1 A^{\kappa/2 - 1} - 1} > 0,$$

and hence

$$(4.30) \qquad \operatorname{Vol}(U_2) = (1 - \operatorname{Vol}(U_1)) < 1 - \frac{1}{2C_1 A^{\kappa/2 - 1} - 1} < 1.$$

Now we want to use (4.28) and (4.30) to estimate $\|e^u\|_2$. Applying Young's inequality and the Hölder inequality, we find

$$
\begin{aligned}
\left(\int_S e^u\right)^2 &= \left(\int_{U_1} e^u + \int_{U_2} e^u\right)^2 \\
&\leq \left(1+\frac{1}{\epsilon}\right)\left(\int_{U_1} e^u\right)^2 + (1+\epsilon)\left(\int_{U_2} e^u\right)^2 \\
&\leq \left(1+\frac{1}{\epsilon}\right)\left(\int_{U_1} e^{2u}\right)\mathrm{Vol}(U_1) \\
&\quad + (1+\epsilon)\mathrm{Vol}(U_2)\left(\int_{U_2} e^{2u}\right) \\
&\leq \left(1+\frac{1}{\epsilon}\right)\left(\frac{2}{A}\right)^2 + (1+\epsilon)\mathrm{Vol}(U_2)\left(\int_{U_2} e^{2u}\right) \\
&\leq \left(1+\frac{1}{\epsilon}\right)\left(\frac{2}{A}\right)^2 + (1+\epsilon)\mathrm{Vol}(U_2)\left(\int_S e^{2u}\right).
\end{aligned}
$$

(4.31)

Inserting (4.28) and (4.30) into (4.31),

$$
\begin{aligned}
\left(\int_S e^u\right)^2 &\leq \left(1+\frac{1}{\epsilon}\right)\left(\frac{2}{A}\right)^2 + C_2(1+\epsilon)\left(1-\frac{1}{2C_1 A^{\frac{\kappa}{2}-1}-1}\right) \\
&\quad + (1+\epsilon)\left(1-\frac{1}{2C_1 A^{\frac{\kappa}{2}-1}-1}\right)\left(\int_S e^u\right)^2.
\end{aligned}
$$

Taking ϵ small enough such that

$$
(1+\epsilon)\left(1-\frac{1}{2C_1 A^{\kappa/2-1}-1}\right) < 1,
$$

then we get

$$
\left(\int_S e^u\right)^2 < \frac{(1+1/\epsilon)\,(2/A)^2 + C_2(1+\epsilon)\left(1-1/(2C_1 A^{\kappa/2-1}-1)\right)}{1-(1+\epsilon)\left(1-1/(2C_1 A^{\kappa/2-1}-1)\right)}.
$$

Now estimate of $\int_S (e^u)^2$ follows from (4.28) and the estimate of $\sup u$ follows from (4.26).

5. Gradient estimate

Suppose the function $\ln|\nabla u|^2 + \ln v(u)$ achieves its maximum at the point $q_1 \in S$, where v is a positive function of u to be determined. Then, at q_1,

we have

$$(5.1) \qquad \bigtriangledown(|\bigtriangledown u|^2) = -\left(\frac{v'(u)}{v(u)}|\bigtriangledown u|^2\right)\bigtriangledown u.$$

We may choose the normal coordinate at q_1 such that $\frac{\partial u}{\partial z_1} \neq 0$ and $\frac{\partial u}{\partial z_2} = 0$. As u is real, we can assume that $\frac{\partial u}{\partial x_1} > 0$ and $\frac{\partial u}{\partial y_1} = 0$. Thus, we can assume that at q_1,

$$(5.2) \qquad \partial_1 u \partial_{\bar{1}} u = \partial_1 u \partial_1 u = \partial_{\bar{1}} u \partial_{\bar{1}} u = \frac{1}{2}|\bigtriangledown u|^2.$$

From (5.1), we get

$$(5.3) \qquad \partial_1 \partial_1 u + \partial_1 \partial_{\bar{1}} u = \partial_{\bar{1}} \partial_{\bar{1}} u + \partial_1 \partial_{\bar{1}} u = -\frac{1}{2}\frac{v'(u)}{v(u)}|\bigtriangledown u|^2$$

and

$$(5.4) \qquad \partial_1 \partial_2 u + \partial_2 \partial_{\bar{1}} u = \partial_{\bar{1}} \partial_{\bar{2}} u + \partial_1 \partial_{\bar{2}} u = 0.$$

By direct calculation, using (5.2), we see that at q_1,

$$
\begin{aligned}
&P(\ln|\bigtriangledown u|^2)|\bigtriangledown u|^2\frac{\det g'}{\det g} \\
&\qquad = 2g'^{i\bar{j}}\left\{\frac{\partial^2|\bigtriangledown u|^2}{\partial z_i \partial \bar{z}_j} - \frac{\partial|\bigtriangledown u|^2}{\partial z_i}\frac{\partial|\bigtriangledown u|^2}{\partial \bar{z}_j}\cdot\frac{1}{|\bigtriangledown u|^2}\right\}\frac{\det g'}{\det g} \\
&\qquad = 4g'^{i\bar{j}}(\partial_i \partial_{\bar{j}} \partial_p u \partial_{\bar{p}} u + \partial_i \partial_{\bar{j}} \partial_{\bar{p}} u \partial_p u)\frac{\det g'}{\det g} \\
(5.5) \\
&\qquad \quad - 4g'^{i\bar{j}}(\partial_i \partial_{\bar{1}} u \partial_{\bar{1}} \partial_{\bar{j}} u + \partial_i \partial_1 u \partial_1 \partial_{\bar{j}} u)\frac{\det g'}{\det g} \\
&\qquad \quad + 4g'^{i\bar{j}}(\partial_i \partial_{\bar{2}} u \partial_2 \partial_{\bar{j}} u)\frac{\det g'}{\det g} + 2g'^{i\bar{j}}R_{i\bar{j}}^{1\bar{1}}|\bigtriangledown u|^2\frac{\det g'}{\det g} \\
&\qquad \quad + 4g'^{i\bar{j}}(\partial_i \partial_2 u \partial_{\bar{2}} \partial_{\bar{j}} u)\frac{\det g'}{\det g}.
\end{aligned}
$$

The last term $4g'^{i\bar{j}}(\partial_i\partial_2 u\partial_{\bar{2}}\partial_{\bar{j}}u)\frac{\det g'}{\det g} \geq 0$. So, we only need to estimate the first four terms. By equation and (5.1), the first term in (5.5) is

$$4g'^{i\bar{j}}(\partial_i\partial_{\bar{j}}\partial_p u\partial_{\bar{p}}u + \partial_i\partial_{\bar{j}}\partial_{\bar{p}}u\partial_p u)\frac{\det g'}{\det g}$$

$$= 2(e^u - fe^{-u})\bigtriangledown\triangle u\cdot\bigtriangledown u - 16\left(\bigtriangledown\left(\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}\right)\cdot\bigtriangledown u\right)$$

$$= 2(e^u - fe^{-u})\{\bigtriangledown\triangle u\cdot\bigtriangledown u\} - 2\{\bigtriangledown\triangle(e^u + fe^{-u})\cdot\bigtriangledown u\}$$

$$= -2(e^u + fe^{-u})\triangle u|\bigtriangledown u|^2 - 2(e^u - fe^{-u})|\bigtriangledown u|^4$$

$$\quad - 4e^{-u}(\bigtriangledown u\cdot\bigtriangledown f)|\bigtriangledown u|^2 + 2e^{-u}\triangle f|\bigtriangledown u|^2$$

(5.6)

$$\quad - 2(e^u + fe^{-u})(\bigtriangledown|\bigtriangledown u|^2\cdot\bigtriangledown u) + 2e^{-u}\triangle u(\bigtriangledown u\cdot\bigtriangledown f)$$

$$\quad - 2e^{-u}(\bigtriangledown u\cdot\bigtriangledown\triangle f) + 4e^{-u}(\bigtriangledown(\bigtriangledown u\cdot\bigtriangledown f)\cdot\bigtriangledown u)$$

$$= -2(e^u + fe^{-u})\triangle u|\bigtriangledown u|^2 - 2(e^u - fe^{-u})|\bigtriangledown u|^4$$

$$\quad - 4e^{-u}(\bigtriangledown u\cdot\bigtriangledown f)|\bigtriangledown u|^2 + 2e^{-u}\triangle f|\bigtriangledown u|^2$$

$$\quad + 2\frac{v'(u)}{v(u)}(e^u + fe^{-u})|\bigtriangledown u|^4 + 2e^{-u}\triangle u(\bigtriangledown u\cdot\bigtriangledown f)$$

$$\quad - 2e^{-u}(\bigtriangledown u\cdot\bigtriangledown\triangle f) + 4e^{-u}(\bigtriangledown(\bigtriangledown u\cdot\bigtriangledown f)\cdot\bigtriangledown u).$$

But from (5.2), (5.3) and (5.4),

(5.7)

$$\bigtriangledown(\bigtriangledown u\cdot\bigtriangledown f)\cdot\bigtriangledown u = \left\{-\frac{\sqrt{2}}{4}\frac{v'(u)}{v(u)}|\bigtriangledown u|(\partial_1 f + \partial_{\bar{1}}f)\right.$$

$$\left. + \frac{1}{2}(\partial_1\partial_1 f + 2\partial_1\partial_{\bar{1}}f + \partial_{\bar{1}}\partial_{\bar{1}}f)\right\}|\bigtriangledown u|^2$$

$$\geq -(C_3|\bigtriangledown u| + C_3)|\bigtriangledown u|^2,$$

where in the last inequality C_3 depends only on $\sup u$, $\inf u$ and f. In the following, C_3 may mean different constants in the different equations that depend only on $\sup u$, $\inf u$, f and S. Inserting (5.7) into (5.6) and applying the Schwarz inequality, the first term in (5.5) can be estimated to be

$$4g'^{i\bar{j}}(\partial_i\partial_{\bar{j}}\partial_p u\partial_{\bar{p}}u + \partial_i\partial_{\bar{j}}\partial_{\bar{p}}u\partial_p u)\frac{\det g'}{\det g}$$

$$\geq -2(e^u - fe^{-u})|\bigtriangledown u|^4 + 2\frac{v'(u)}{v(u)}(e^u + fe^{-u})|\bigtriangledown u|^4$$

(5.8)

$$\quad + 2(e^u + fe^{-u})(e^u - fe^{-u} - \triangle u)|\bigtriangledown u|^2$$

$$- (e^u - fe^{-u} - \triangle u)(C_3|\nabla u| + C_3)$$
$$- C_3|\nabla u|^3 - C_3|\nabla u|^2 - C_3|\nabla u| - C_3.$$

Next we compute the second term in (5.5):

(5.9)
$$- 4g'^{i\bar{j}}(\partial_i\partial_{\bar{1}}u\partial_{\bar{1}}\partial_{\bar{j}}u + \partial_i\partial_1 u\partial_1\partial_{\bar{j}}u)\frac{\det g'}{\det g}$$
$$= -4(e^u - fe^{-u})(\partial_i\partial_{\bar{1}}u\partial_{\bar{1}}\partial_{\bar{i}}u + \partial_i\partial_1 u\partial_1\partial_{\bar{i}}u)$$
$$+ 2 \times 8\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}(\partial_{\bar{1}}\partial_{\bar{1}}u + \partial_1\partial_1 u)$$
$$= -4(e^u - fe^{-u})(\partial_i\partial_{\bar{1}}u\partial_{\bar{1}}\partial_{\bar{i}}u + \partial_i\partial_1 u\partial_1\partial_{\bar{i}}u)$$
$$+ 2(e^u - fe^{-u})(\triangle u)(\partial_{\bar{1}}\partial_{\bar{1}}u + \partial_1\partial_1 u)$$
$$+ 2\{(e^u + fe^{-u})|\nabla u|^2 + e^{-u}\triangle f$$
$$- 2e^{-u}\nabla u \cdot \nabla f\}(\partial_{\bar{1}}\partial_{\bar{1}}u + \partial_1\partial_1 u).$$

The first two terms in (5.9) are equal to

(5.10)
$$- 4(e^u - fe^{-u})(\partial_i\partial_{\bar{1}}u\partial_{\bar{1}}\partial_{\bar{i}}u + \partial_i\partial_1 u\partial_1\partial_{\bar{i}}u)$$
$$+ 2(e^u - fe^{-u})(\triangle u)(\partial_{\bar{1}}\partial_{\bar{1}}u + \partial_1\partial_1 u)$$
$$= 4(e^u - fe^{-u})(\partial_{\bar{1}}\partial_{\bar{1}}u + \partial_1\partial_1 u)\partial_2\partial_{\bar{2}}u$$
$$- 4(e^u - fe^{-u})(\partial_2\partial_{\bar{1}}u\partial_{\bar{1}}\partial_{\bar{2}}u + \partial_2\partial_1 u\partial_1\partial_{\bar{2}}u).$$

From (5.3) we find

(5.11)
$$\partial_1\partial_1 u + \partial_{\bar{1}}\partial_{\bar{1}}u = -\frac{v'(u)}{v(u)}|\nabla u|^2 - 2\partial_1\partial_{\bar{1}}u.$$

Inserting (5.11) and (5.4) into (5.10), and using equation and the Schwarz inequality, we get

(5.12)
$$- 4(e^u - fe^{-u})(\partial_i\partial_{\bar{1}}u\partial_{\bar{1}}\partial_{\bar{i}}u + \partial_i\partial_1 u\partial_1\partial_{\bar{i}}u)$$
$$+ 2(e^u - fe^{-u})(\triangle u)(\partial_{\bar{1}}\partial_{\bar{1}}u + \partial_1\partial_1 u)$$
$$= -4(e^u - fe^{-u})\frac{v'(u)}{v(u)}u_{2\bar{2}}|\nabla u|^2 - (e^u - fe^{-u})^2(\triangle u)$$
$$- (e^u - fe^{-u})\{(e^u + fe^{-u})|\nabla u|^2$$
$$+ e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f\}$$

$$\geq -4(e^u - fe^{-u})\frac{v'(u)}{v(u)}u_{2\bar{2}}|\nabla u|^2$$

$$+ (e^u - fe^{-u})^2(e^u - fe^{-u} - \triangle u)$$

$$- C_3|\nabla u|^2 - C_3|\nabla u| - C_3$$

$$\geq -4(e^u - fe^{-u})\frac{v'(u)}{v(u)}u_{2\bar{2}}|\nabla u|^2 - C_3|\nabla u|^2 - C_3|\nabla u| - C_3.$$

Applying (5.11) and the Schwarz inequality, the third term in (5.9) can be estimated:

$$2\{(e^u + fe^{-u})|\nabla u|^2 + e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f\}(\partial_{\bar{1}}\partial_{\bar{1}}u + \partial_1\partial_1 u)$$

$$(5.13) \quad \geq -2(e^u + fe^{-u})\frac{v'(u)}{v(u)}|\nabla u|^4 - C_3|\nabla u|^3 - C_3|\nabla u|^2$$

$$- 4\{(e^u + fe^{-u})|\nabla u|^2 + e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f\}u_{1\bar{1}}.$$

Inserting (5.12) and (5.13) into (5.9), we get the following estimate of the second term in (5.5)

$$- 4g'^{i\bar{j}}(\partial_i\partial_{\bar{1}}u\partial_{\bar{1}}\partial_{\bar{j}}u + \partial_i\partial_1 u\partial_1\partial_{\bar{j}}u)\frac{\det g'}{\det g}$$

$$(5.14) \quad \geq -2(e^u + fe^{-u})\frac{v'(u)}{v(u)}|\nabla u|^4 - 4(e^u - fe^{-u})\frac{v'(u)}{v(u)}u_{2\bar{2}}|\nabla u|^2$$

$$- 4\{(e^u + fe^{-u})|\nabla u|^2 + e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f\}u_{1\bar{1}}$$

$$- C_3|\nabla u|^3 - C_3|\nabla u|^2 - C_3|\nabla u| - C_3.$$

The third term in (5.5) is

$$4g'^{i\bar{j}}(\partial_i\partial_{\bar{2}}u\partial_2\partial_{\bar{j}}u) \cdot \frac{\det g'}{\det g}$$

$$= -4(e^u - fe^{-u})\det(u_{i\bar{j}}) + 2(e^u - fe^{-u})(\triangle u)u_{2\bar{2}}$$

$$- 2\triangle(e^u + fe^{-u}) \cdot u_{2\bar{2}}$$

$$= -4(e^u - fe^{-u})\det(u_{i\bar{j}})$$

$$(5.15) \quad - 2\{(e^u + fe^{-u})|\nabla u|^2 + e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f\}u_{2\bar{2}}$$

$$\geq \frac{1}{2}(e^u - fe^{-u})^2(e^u - fe^{-u} - \triangle u) - C_3|\nabla u| - C_3|\nabla u|^2$$

$$- 2\{(e^u + fe^{-u})|\nabla u|^2 + e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f\}u_{2\bar{2}}$$

$$\geq -2\{(e^u + fe^{-u})|\nabla u|^2 + e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f\}u_{2\bar{2}}$$

$$- C_3|\nabla u| - C_3|\nabla u|^2.$$

Combining the following two terms in (5.14) and (5.15),

$$-4\{(e^u + fe^{-u})|\nabla u|^2 + e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f\}u_{1\bar{1}}$$

$$-2\{(e^u + fe^{-u})|\nabla u|^2 + e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f\}u_{2\bar{2}}$$

$$\geq \frac{1}{2}\{(e^u + fe^{-u})|\nabla u|^2 + e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f\}(e^u - fe^{-u} - 4u_{1\bar{1}})$$

$$+ (e^u + fe^{-u})(e^u - fe^{-u} - \triangle u) - (e^u - fe^{-u} - \triangle u)(C_3|\nabla u| + C_3)$$

$$- C_3|\nabla u|^3 - C_3|\nabla u|^2 - C_3|\nabla u| - C_3$$

$$\geq (e^u + fe^{-u})(e^u - fe^{-u} - \triangle u) - (e^u - fe^{-u} - \triangle u)(C_3|\nabla u| + C_3)$$

$$- C_3|\nabla u|^3 - C_3|\nabla u|^2 - C_3|\nabla u| - C_3,$$

where in the last inequality we have assumed that

$$(e^u + fe^{-u})|\nabla u|^2 + e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f \geq 0.$$

Otherwise, we could have obtained the estimate of $|\nabla u|^2$ at point q_1. Then, combining (5.14) and (5.15), we get

$$-4g'^{i\bar{j}}(\partial_i\partial_{\bar{1}}u\partial_1\partial_{\bar{j}}u + \partial_i\partial_1 u\partial_1\partial_{\bar{j}}u - \partial_i\partial_{\bar{2}}u\partial_2\partial_{\bar{j}}u)\frac{\det g'}{\det g}$$

(5.16)
$$\geq -2(e^u + fe^{-u})\frac{v'(u)}{v(u)}|\nabla u|^4 - 4(e^u - fe^{-u})\frac{v'(u)}{v(u)}u_{2\bar{2}}|\nabla u|^2$$

$$- C_3|\nabla u|^3 - C_3|\nabla u|^2 - C_3|\nabla u| - C_3.$$

Let $R = \text{Supp}\,|R_{i\bar{j}}^{1\bar{1}}|$. The fourth term is

$$2g'^{i\bar{j}}R_{i\bar{j}}^{1\bar{1}}|\nabla u|^2\frac{\det g'}{\det g} = -8u^{i\bar{j}}R_{i\bar{j}}^{1\bar{1}}|\nabla u|^2$$

(5.17)
$$\geq -8R\sum_{i,j=1}|u_{i\bar{j}}||\nabla u|^2$$

$$\geq -8R\left((\triangle u)^2 - 8\det u_{i\bar{j}}\right)^{\frac{1}{2}}|\nabla u|^2$$

$$\geq -16R(e^u - fe^{-u} - \triangle u)|\nabla u|^2 - C_3|\nabla u|^2.$$

Inserting (5.8), (5.16) and (5.17) into (5.5), we see that

(5.18)
$$P(\ln |\nabla u|^2)|\nabla u|^2 \frac{\det g'}{\det g}$$
$$\geq \{(3(e^u + fe^{-u}) - 16R)|\nabla u|^2$$
$$- C_3|\nabla u| - C_3\}(e^u - fe^{-u} - \triangle u)$$
$$+ \frac{v'(u)}{v(u)}(e^u - fe^{-u})(e^u - fe^{-u} - 4u_{2\bar{2}})|\nabla u|^2$$
$$- 2(e^u - fe^{-u})|\nabla u|^4 - C_3|\nabla u|^3$$
$$- C_3|\nabla u|^2 - C_3|\nabla u| - C_3.$$

Next, we compute

(5.19)
$$P(\ln v)\frac{\det g'}{\det g}$$
$$= 2g'^{i\bar{j}}\left(\frac{v'}{v}\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} + \frac{vv'' - v'^2}{v^2}\frac{\partial u}{\partial z_i}\frac{\partial u}{\partial \bar{z}_j}\right)\frac{\det g'}{\det g}$$
$$= \frac{v'}{v}\left\{(e^u - fe^{-u})\triangle u - 16\frac{\det(u_{i\bar{j}})}{\det g}\right\}$$
$$+ \frac{vv'' - v'^2}{v^2}g'^{1\bar{1}}|\nabla u|^2\frac{\det g'}{\det g}$$
$$= -\frac{v'(u)}{v(u)}(e^u - fe^{-u})\triangle u$$
$$+ \frac{vv'' - v'^2}{v^2}|\nabla u|^2(e^u - fe^{-u} - 4u_{2\bar{2}})$$
$$- 2\frac{v'(u)}{v(u)}\{(e^u + fe^{-u})|\nabla u|^2$$
$$+ e^{-u}\triangle f - 2e^{-u}\nabla u \cdot \nabla f\}$$
$$\geq -\frac{v'(u)}{v(u)}(e^u - fe^{-u})(e^u - fe^{-u} - \triangle u)$$
$$- 2\frac{v'(u)}{v(u)}(e^u + fe^{-u})|\nabla u|^2$$
$$+ \frac{vv'' - v'^2}{v^2}|\nabla u|^2(e^u - fe^{-u} - 4u_{2\bar{2}}) - C_3|\nabla u| - C_3.$$

From (5.18) and (5.19), we get

$$
P(\ln|\nabla u|^2 + \ln v)|\nabla u|^2 \frac{\det g'}{\det g}
$$

$$
\begin{aligned}
&= \left\{ (2(e^u + fe^{-u}) + \frac{v'(u)}{v(u)}(e^u - fe^{-u}) \right\} \\
&\quad \times (e^u - fe^{-u} - \triangle u)|\nabla u|^2 \\
&\quad + \{(e^u + fe^{-u} - 16R)|\nabla u|^2 - C_3|\nabla u| - C_3\} \\
&\quad \times (e^u - fe^{-u} - \triangle u) \\
&\quad + \left\{ \frac{vv'' - v'^2}{v^2}|\nabla u|^2 + \frac{v'(u)}{v(u)}(e^u - fe^{-u}) \right\} \\
&\quad \times (e^u - fe^{-u} - 4u_{2\bar{2}})|\nabla u|^2 \\
&\quad - 2\left\{ (e^u - fe^{-u}) + \frac{v'(u)}{v(u)}(e^u + fe^{-u}) \right\}|\nabla u|^4 \\
&\quad - C_3|\nabla u|^3 - C_3|\nabla u|^2 - C_3|\nabla u| - C_3.
\end{aligned}
$$

(5.20)

Take

$$
v(u) = e^{4\sup u - 2u} + e^{2u - 4\sup u} > 0.
$$

Then,

$$
\begin{aligned}
v'(u) &= -2e^{4\sup u - 2u} + 2e^{2u - 4\sup u} < 0, \\
v''(u) &= 4e^{4\sup u - 2u} + 4e^{2u - 4\sup u} = 4v(u) > 0.
\end{aligned}
$$

So the factor of the first term in (5.20) is

$$
(5.21) \qquad 2(e^u + fe^{-u}) + (e^u - fe^{-u})\frac{v'(u)}{v(u)} > 0.
$$

The factor of the third term in (5.20) is

$$
\begin{aligned}
\frac{vv'' - v'2}{v^2}|\nabla u|^2 + \frac{v'}{v}(e^u - fe^{-u}) &= \frac{16}{v^2}|\nabla u|^2 - \frac{e^u e^{4\sup u - 2u}}{e^{4\sup u - 2u}} \\
&> \frac{16}{e^{4\sup u - 2\inf u} + 3}|\nabla u|^2 - e^{\sup u}.
\end{aligned}
$$

(5.22)

Choose A such that

$$
(5.23) \qquad C_1^{-1} A^{-\kappa/2} > 7^{1/3},
$$

then $e^{\inf u} > \frac{1}{3} \ln 7$ and the coefficient of the fourth term in (5.20) is

$$-2\left\{(e^u - fe^{-u}) + \frac{v'}{v}(e^u + fe^{-u})\right\} \geq 2\frac{e^{4\sup u - u} - 3e^{3u - u\sup u}}{e^{4\sup u - 2u} + e^{2u - 4\sup u}}$$

(5.24)
$$\geq \frac{2e^{4\sup u - u} - 6}{e^{4\sup u - u} + 1} > 1.$$

Choose A such that

(5.25)
$$C_1^{-1}A^{-\kappa/2} > 16R + 1,$$

then $e^{\inf u} > 16R + 1$. Applying all above inequalities, we can see that at q_1

$$0 \geq P(\ln|\nabla u|^2 + \ln v)|\nabla u|^2 \frac{\det g'}{\det g}$$

(5.26)
$$\geq (|\nabla u|^2 - C_3|\nabla u| - C_3)(e^u - fe^{-u} - \triangle u)$$
$$+ \left\{\frac{16}{e^{4\sup u - 2\inf u} + 1}|\nabla u|^2 - e^{\sup u}\right\}(e^u - fe^{-u} - 4u_{2\bar{2}})|\nabla u|^2$$
$$+ |\nabla u|^4 - C_3|\nabla u|^3 - C_3|\nabla u|^2 - C_3|\nabla u| - C_3.$$

From above inequality, we can easily see that there is a constant C_4 depending only on f, S, $\sup u$ and $\inf u$ such that $|\nabla u|^2(q_1) \leq C_4$.

We have proved that if the function $\ln|\nabla u|^2 + \ln(e^{4\sup u - 2u} + e^{2u - \sup u})$ achieves its maximum at q_1, we get the bound of $|\nabla u|^2$:

(5.27)
$$|\nabla u|^2 \leq C_4 \frac{\left(e^{4\sup u - 2u(q)} + e^{2u(q) - 4\sup u}\right)}{\left(e^{4\sup u - 2u} + e^{2u - 4\sup u}\right)}$$
$$\leq C_4 \frac{\left(e^{4\sup u - 2\inf u} + e^{2\inf u - 4\sup u}\right)}{\left(e^{2\sup u} + e^{-2\sup u}\right)}.$$

6. Second-order estimate

We now give the second-order a priori estimate of u. We rewrite the equation as

(6.1)
$$\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} = F,$$

where

$$F = (e^u - fe^{-u})^2 + 2\{(e^u + fe^{-u})^2 |\nabla u|^2$$
$$+ e^{-u} \triangle f - 2e^{-u} \nabla u \cdot \nabla f\}.$$

(6.2)

Then we can see that if we suppose A is small, F has a positive lower bound. Hence, $e^u - fe^{-u} - \triangle u \geq F^{1/2}$ also has a positive lower bound. So, to get a second-order estimate of u, it sufficient to have an upper estimate of $e^u - fe^{-u} - \triangle u$. We fix a point q_2 and choose normal coordinate at that point for $g_{i\bar{j}}$.

Differentiating (6.1), we have

(6.3)
$$g^{i\bar{j}} \frac{\partial g'_{i\bar{j}}}{\partial z_k} = g^{i\bar{j}} \frac{\partial g_{i\bar{j}}}{\partial z_k} + \frac{1}{F} \frac{\partial F}{\partial z_k}.$$

By differentiating (6.3) again, we obtain

$$-g'^{i\bar{q}} g'^{p\bar{j}} \frac{\partial g'_{p\bar{q}}}{\partial \bar{z}_l} \frac{\partial g'_{i\bar{j}}}{\partial z_k} + g'^{i\bar{j}} \frac{\partial^2 g'_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}$$
$$= -g^{i\bar{q}} g^{p\bar{j}} \frac{\partial g_{p\bar{q}}}{\partial \bar{z}_l} \frac{\partial g_{i\bar{j}}}{\partial z_k} + g^{i\bar{j}} \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \frac{1}{F} \frac{\partial^2 F}{\partial z_k \partial \bar{z}_l} - \frac{1}{F^2} \frac{\partial F}{\partial z_k} \frac{\partial F}{\partial \bar{z}_l}$$

or

(6.4)
$$-4g'^{i\bar{j}} \frac{\partial^4 u}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l} = g'^{i\bar{q}} g'^{p\bar{j}} \frac{\partial g'_{p\bar{q}}}{\partial \bar{z}_l} \frac{\partial g'_{i\bar{j}}}{\partial z_k} - g'^{i\bar{j}} \frac{\partial^2 ((e^u - fe^{-u}) g_{i\bar{j}})}{\partial z_k \partial \bar{z}_l}$$
$$- g^{i\bar{q}} g^{p\bar{j}} \frac{\partial g_{p\bar{q}}}{\partial \bar{z}_l} \frac{\partial g_{i\bar{j}}}{\partial z_k} + g^{i\bar{j}} \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}$$
$$+ \frac{1}{F} \frac{\partial^2 F}{\partial z_k \partial \bar{z}_l} - \frac{1}{F^2} \frac{\partial F}{\partial z_k} \frac{\partial F}{\partial \bar{z}_l}.$$

Contracting (6.4) with g^{kl} and using the fact that the metric $g_{i\bar{j}}$ is Ricci-flat, we find

(6.5)
$$P(-\triangle u) = g^{k\bar{l}} g'^{i\bar{j}} g'^{p\bar{q}} \frac{\partial g'_{i\bar{q}}}{\partial z_k} \frac{\partial g'_{p\bar{j}}}{\partial \bar{z}_l} - \frac{1}{2} \triangle (e^u - fe^{-u}) \sum_{i=1}^{2} g'^{i\bar{i}}$$
$$+ \frac{1}{2F} \triangle F - \frac{1}{F^2} g^{k\bar{l}} \frac{\partial F}{\partial z_k} \frac{\partial F}{\partial \bar{z}_l} + 4g'^{i\bar{j}} \frac{\partial^2 g^{k\bar{l}}}{\partial z_i \partial \bar{z}_j} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_l}.$$

Multiplying the above equation by $\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}}$ and using (6.3) and (6.1), we see

$$
P(-\triangle u)\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} = g^{k\bar{l}}g'^{i\bar{j}}g'^{p\bar{q}}\left(\frac{\partial g'_{i\bar{q}}}{\partial z_k}\frac{\partial g'_{p\bar{j}}}{\partial \bar{z}_l} - \frac{\partial g'_{i\bar{j}}}{\partial z_k}\frac{\partial g'_{p\bar{q}}}{\partial \bar{z}_l}\right)\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}}
$$

(6.6)
$$
+ \frac{1}{2}\triangle F - \frac{1}{2}\triangle(e^u - fe^{-u})\sum_{i=1}^{2}g'^{i\bar{i}}\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}}
$$

$$
+ 4g'^{i\bar{j}}\frac{\partial^2 g^{k\bar{l}}}{\partial z_i\partial\bar{z}_j}\frac{\partial^2 u}{\partial z_k\partial\bar{z}_l}.
$$

Now we compute at q_2

$$
g^{k\bar{l}}g'^{i\bar{j}}g'^{p\bar{q}}\left(\frac{\partial g'_{i\bar{q}}}{\partial z_k}\frac{\partial g'_{p\bar{j}}}{\partial \bar{z}_l} - \frac{\partial g'_{i\bar{j}}}{\partial z_k}\frac{\partial g'_{p\bar{q}}}{\partial \bar{z}_l}\right)\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}}
$$

$$
= g^{k\bar{l}}\left(g'^{i\bar{j}}g'^{p\bar{q}} - g'^{i\bar{q}}g'^{p\bar{j}}\right)\frac{\partial g'_{i\bar{q}}}{\partial z_k}\frac{\partial g'_{p\bar{j}}}{\partial \bar{z}_l}\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}}
$$

$$
= g^{k\bar{l}}\left(\frac{\partial g'_{2\bar{1}}}{\partial z_k}\frac{\partial g'_{1\bar{2}}}{\partial \bar{z}_l} + \frac{\partial g'_{1\bar{2}}}{\partial z_k}\frac{\partial g'_{2\bar{1}}}{\partial \bar{z}_l} - \frac{\partial g'_{1\bar{1}}}{\partial z_k}\frac{\partial g'_{2\bar{2}}}{\partial \bar{z}_l} - \frac{\partial g'_{2\bar{2}}}{\partial z_k}\frac{\partial g'_{1\bar{1}}}{\partial \bar{z}_l}\right)
$$

$$
= 16\sum_{i\neq j}\left(\frac{\partial^3 u}{\partial z_i\partial\bar{z}_j\partial z_k}\frac{\partial^3 u}{\partial z_j\partial\bar{z}_i\partial\bar{z}_k} - \frac{\partial^3 u}{\partial z_i\partial\bar{z}_i\partial z_k}\frac{\partial^3 u}{\partial z_j\partial\bar{z}_j\partial\bar{z}_k}\right)
$$

(6.7)
$$
+ 4\sum_i\frac{\partial^3 u}{\partial z_i\partial\bar{z}_i\partial z_k}\frac{\partial(e^u - fe^{-u})}{\partial\bar{z}_k} + 4\sum_i\frac{\partial^3 u}{\partial z_i\partial\bar{z}_i\partial\bar{z}_k}\frac{\partial(e^u - fe^{-u})}{\partial z_k}
$$

$$
- 2\frac{\partial(e^u - fe^{-u})}{\partial\bar{z}_k}\frac{\partial(e^u - fe^{-u})}{\partial z_k}
$$

$$
= 16\sum_{i,j}\frac{\partial^3 u}{\partial z_i\partial\bar{z}_j\partial z_k}\frac{\partial^3 u}{\partial z_j\partial\bar{z}_i\partial\bar{z}_k} - 16\sum_{i,j}\frac{\partial^3 u}{\partial z_i\partial\bar{z}_i\partial z_k}\frac{\partial^3 u}{\partial z_j\partial\bar{z}_j\partial\bar{z}_k}
$$

$$
+ 2\triangledown(\triangle u)\cdot\triangledown(e^u - fe^{-u}) - |\triangledown(e^u - fe^{-u})|^2
$$

$$
\geq -2|\triangledown\triangle u|^2 + 2\triangledown(\triangle u)\cdot\triangledown(e^u - fe^{-u}) - C_5,
$$

where C_5 depends on f, S and u up to first-order derivation. In the following, we will use C_5 to denote a constant with similar dependence and estimate an upper bound for $e^u - fe^{-u} - \triangle u$. Assume that $e^u - fe^{-u} - \triangle u$ achieves a maximum at point q_2. By taking normal coordinate at q_2 for $g_{i\bar{j}}$, we find

(6.8)
$$
\triangledown\triangle u = \triangledown(e^u - fe^{-u}).
$$

Inserting (6.8) into (6.7) and then into (6.6), we obtain

$$
(6.9) \quad P(-\triangle u)\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \geq \frac{1}{2}\triangle F + 4g'^{i\bar{j}}\frac{\partial^2 g^{k\bar{l}}}{\partial z_i \partial \bar{z}_j}\frac{\partial^2 u}{\partial z_k \partial \bar{z}_l}
$$
$$
- \frac{1}{2}\triangle(e^u - fe^{-u})\sum_{i=1}^{2} g'^{i\bar{i}}\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} - C_5.
$$

We shall first deal with the second term in the above inequality. Using the equation, the Schwarz inequality, and the fact that the metric $g_{i\bar{j}}$ is Ricci-flat, we have

$$
(6.10) \quad
\begin{aligned}
4g'^{i\bar{j}}&\frac{\partial^2 g^{k\bar{l}}}{\partial z_i \partial \bar{z}_j}\frac{\partial^2 u}{\partial z_k \partial \bar{z}_l} \\
&= -16\left\{\frac{\partial^2 g^{k\bar{l}}}{\partial z_1 \partial \bar{z}_1}\frac{\partial^2 u}{\partial u_2 \partial \bar{z}_2} + \frac{\partial^2 g^{k\bar{l}}}{\partial z_2 \partial \bar{z}_2}\frac{\partial^2 u}{\partial u_1 \partial \bar{z}_1}\right\}\frac{\partial^2 u}{\partial z_k \partial \bar{z}_l} \\
&\quad - 16\left\{\frac{\partial^2 g^{k\bar{l}}}{\partial z_1 \partial \bar{z}_2}\frac{\partial^2 u}{\partial u_2 \partial \bar{z}_1} + \frac{\partial^2 g^{k\bar{l}}}{\partial z_2 \partial \bar{z}_1}\frac{\partial^2 u}{\partial u_1 \partial \bar{z}_2}\right\}\frac{\partial^2 u}{\partial z_k \partial \bar{z}_l} \\
&\geq -64(\max R_{i\bar{j}k\bar{l}})\sum_{ij}|u_{i\bar{j}}|^2 \\
&= -16(\max R_{i\bar{j}k\bar{l}})(\triangle u)^2 - 16(\max R_{i\bar{j}k\bar{l}})\times 8\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}, \\
&= -16(\max R_{i\bar{j}k\bar{l}})\{(\triangle u)^2 - \triangle(e^u + fe^{-u})\}, \\
&\geq -16(\max R_{i\bar{j}k\bar{l}})(\triangle u)^2 - C_5 \triangle u - C_5,
\end{aligned}
$$

since

$$
(6.11) \quad \sum_{i=1}^{2} g'^{i\bar{i}}\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} = 2(e^u - fe^{-u} - \triangle u).
$$

We can insert (6.10) and (6.11) into (6.9) and obtain

$$
(6.12) \quad
\begin{aligned}
P(-\triangle u)&\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \\
&\geq \frac{1}{2}\triangle F - \triangle(e^u - fe^{-u})(e^u - fe^{-u} - \triangle u) \\
&\quad - 16(\max R_{i\bar{j}k\bar{l}})(\triangle u)^2 - C_5 \triangle u - C_5
\end{aligned}
$$

$$\geq \frac{1}{2} \triangle F + \left\{ (e^u + fe^{-u}) \right.$$
$$\left. - 16(\max R_{i\bar{j}k\bar{l}}) \right\} (\triangle u)^2 - C_5 \triangle u - C_5.$$

By computation,

$$
\begin{aligned}
\triangle F = &\triangle(e^u - fe^{-u})^2 + 2 \triangle (e^u + fe^{-u})|\nabla u|^2 \\
&+ 2(e^u + fe^{-u}) \triangle (|\nabla u|^2) \\
&+ 2 \nabla (e^u + fe^{-u}) \cdot \nabla(|\nabla u|^2) \\
&+ 2 \triangle e^{-u} \triangle f + 2e^{-u} \triangle^2 f + 2 \nabla e^{-u} \cdot \nabla \triangle f \\
(6.13) \quad &- 4 \triangle e^{-u} \nabla u \cdot \nabla f \\
&- 4e^{-u} \triangle (\nabla u \cdot \nabla f) - 4 \nabla e^{-u} \cdot \nabla(\nabla u \cdot \nabla f) \\
\geq &+2(e^u + fe^{-u}) \triangle (|\nabla u|^2) + 2 \nabla (e^u + fe^{-u}) \cdot \nabla(|\nabla u|^2) \\
&- 4e^{-u} \triangle (\nabla u \cdot \nabla f) \\
&- 4 \nabla e^{-u} \cdot \nabla(\nabla u \cdot \nabla f) - C_5 \triangle u - C_5.
\end{aligned}
$$

Let $\Gamma = g^{i\bar{j}} g^{k\bar{l}} u_{,ik} u_{,\bar{j}\bar{l}}$, where $u_{,ik}$ indicates the covariant differentiation with respect to the given metric. As we take the normal coordinate at q_2, we have $\Gamma = g^{i\bar{j}} g^{k\bar{l}} u_{ik} u_{\bar{j}\bar{l}}$. Using (6.8) and the equation, we have

$$
\begin{aligned}
\triangle |\nabla u|^2 = &4g^{k\bar{l}} \frac{\partial^2}{\partial z_k \partial \bar{z}_l} \{ g_{i\bar{j}} u_i u_{\bar{j}} \} \\
= &4g^{k\bar{k}} g^{i\bar{i}} \left\{ \frac{\partial^3 u}{\partial z_i \partial z_k \partial \bar{z}_k} \frac{\partial u}{\partial \bar{z}_i} + \frac{\partial^3 u}{\partial \bar{z}_i \partial z_k \partial \bar{z}_k} \frac{\partial u}{\partial z_i} \right\} \\
(6.14) \quad &+ 4g^{i\bar{i}} g^{k\bar{k}} \left\{ \frac{\partial^2 u}{\partial z_i \partial z_k} \frac{\partial^2 u}{\partial \bar{z}_i \partial \bar{z}_k} + \frac{\partial^2 u}{\partial z_i \partial \bar{z}_k} \frac{\partial^2 u}{\partial \bar{z}_i \partial z_k} \right\} \\
= &\nabla \triangle u \cdot \nabla u + (\triangle u)^2 - 8 \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + 4g^{i\bar{j}} g^{k\bar{l}} u_{ik} u_{\bar{j}\bar{l}} \\
= &\nabla(e^u - fe^{-u}) \cdot \nabla u + (\triangle u)^2 - \triangle(e^u + fe^{-u}) + 4g^{i\bar{j}} g^{k\bar{l}} u_{ik} u_{\bar{j}\bar{l}} \\
\geq &(\triangle u)^2 + 4\Gamma - C_5 \triangle u - C_5.
\end{aligned}
$$

Using (6.8) and the Schwarz inequality, we also have

$$(6.15) \qquad 4e^{-u} \triangle (\nabla u \cdot \nabla f) \geq -C_5 \sum_{ij} |u_{i\bar{j}}| - C_5 \Gamma^{1/2} - C_5$$

and

$$2 \bigtriangledown (e^u + fe^{-u}) \cdot \bigtriangledown(|\bigtriangledown u|^2) - 4 \bigtriangledown e^{-u} \cdot \bigtriangledown(\bigtriangledown u \cdot \bigtriangledown f)$$

(6.16)
$$\geq -C_5 \sum_{ij} |u_{i\bar{j}}| - C_5 \Gamma^{\frac{1}{2}} - C_5.$$

Then inserting (6.14), (6.15) and (6.16) into (6.13), we see

(6.17)
$$\begin{aligned}
\triangle F &\geq 2(e^u + fe^{-u})(\triangle u)^2 + 8(e^u + fe^{-u})\Gamma - C_5 \triangle u \\
&\quad - C_5 \sum_{ij} |u_{i\bar{j}}| - C_5 \Gamma^{\frac{1}{2}} - C_5 \\
&\geq 2(e^u + fe^{-u})(\triangle u)^2 - C_5 \triangle u - C_5 \sum_{ij} |u_{i\bar{j}}| - C_5.
\end{aligned}$$

Inserting (6.17) into (6.12), we obtain

(6.18)
$$\begin{aligned}
P(- \triangle u)\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} &\geq \left\{2(e^u + fe^{-u}) - 16(\max R_{i\bar{j}k\bar{l}})\right\} (\triangle u)^2 \\
&\quad - C_5 \triangle u - C_5 \sum_{ij} |u_{i\bar{j}}| - C_5.
\end{aligned}$$

Next, we compute

(6.19)
$$\begin{aligned}
P(e^u - &fe^{-u})\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \\
&= 2g'^{k\bar{l}}\frac{\partial^2(e^u - fe^{-u})}{\partial z_k \partial \bar{z}_l}\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \\
&= \triangle(e^u - fe^{-u}) - 8 \sum_{i \neq j}\{\partial_i \partial_{\bar{i}} u \partial_j \partial_{\bar{j}}(e^u - fe^{-u}) \\
&\quad - \partial_i \partial_{\bar{j}} u \partial_j \partial_{\bar{i}}(e^u - fe^{-u})\} \\
&= \triangle(e^u - fe^{-u}) - 2(e^u + fe^{-u}) \triangle (e^u - fe^{-u}) \\
&\quad - 8(e^{-u} - fe^{-u}) \sum_{i \neq j}\{\partial_i \partial_{\bar{i}} u \partial_j u \partial_{\bar{j}} u - \partial_i \partial_{\bar{j}} u \partial_j u \partial_{\bar{i}} u\} \\
&\quad - 8e^{-u} \sum_{i \neq j}\{\partial_i \partial_{\bar{i}} u(\partial_j u \partial_{\bar{j}} f + \partial_{\bar{j}} u \partial_j f)
\end{aligned}$$

$$- \partial_i \partial_{\bar{j}} u (\partial_j u \partial_{\bar{i}} f + \partial_{\bar{i}} u \partial_j f)\}$$
$$+ 8e^{-u} \sum_{i \neq j} \{\partial_i \partial_{\bar{i}} u \partial_j \partial_{\bar{j}} f - \partial_i \partial_{\bar{j}} u \partial_j \partial_{\bar{i}} f\}$$
$$\geq -C_5 \triangle u - C_5 \sum_{ij} |u_{i\bar{j}}| - C_5.$$

Combining (6.18) and (6.19), we obtain

$$
\begin{aligned}
&P(e^u - fe^{-u} - \triangle u) \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \\
&\quad \geq \left\{ 2(e^u + fe^{-u}) - 16(\max R_{i\bar{j}k\bar{l}}) \right\} (\triangle u)^2 \\
&\qquad - C_5 \triangle u - C_5 \sum_{ij} |u_{i\bar{j}}| - C_5 \\
&\quad \geq (\triangle u)^2 - C_5 \triangle u - C_5,
\end{aligned}
$$

(6.20)

where we have chosen A such that A satisfies (5.25), from which we find $e^{\inf u} > 16R + 1$. Since we assume $e^u - fe^{-u} - \triangle u$ achieves the maximum at q_2, (6.20) implies

(6.21) $$(\triangle u)^2 - C_5 \triangle u - C \leq 0,$$

from which we obtain an upper bound estimate of $e^u - fe^{-u} - \triangle u$.

7. Estimates for general case

Multiplying the elliptic condition $e^u \omega_S + \sqrt{-1} e^{-u} \operatorname{tr}(\bar{\partial} B \wedge \partial B^* \cdot g_S^{-1}) - 2\sqrt{-1} \partial \bar{\partial} u > 0$ by pe^{-pu} and integrating, we get (4.4):

$$
\begin{aligned}
\int |\nabla e^{-p/2u}|^2 \frac{\omega_S^2}{2!} &< \frac{p}{4} \int e^{-(p-1)u} \frac{\omega_S^2}{2!} \\
&\quad + \frac{p}{4} \int e^{-(p+1)u} \frac{\sqrt{-1}}{2} \operatorname{tr} (\bar{\partial} B \wedge \partial B^* \cdot g^{-1}) \wedge \omega_S \\
&\leq \frac{p}{4} \int e^{-(p-1)u} \frac{\omega_S^2}{2!}.
\end{aligned}
$$

Note that we use $-\sqrt{-1} \operatorname{tr}(\bar{\partial} B \wedge \partial B^* \cdot g^{-1}) \wedge \omega_S \geq 0$. We can then follow the discussion in Section 3 to give the estimate $\inf u \geq -\ln C_1 - \frac{\kappa}{2} \ln A$. If A is small, we can make $\inf u > 0$ large. So the term e^u always controls the term such as $e^{-u}|\operatorname{tr} (\bar{\partial} B \wedge \partial B^* \cdot g^{-1})|$ and all the other estimates can be derived as in the special case.

8. Solution for irreducible bundles

In the last section, we have constructed a solution: $(V, \pi^* H_0, X, \omega_u)$ for reducible bundles, where $u \in C^\infty(S, \mathbb{R})$ solves the equation

$$(8.1) \qquad \sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S + \partial\bar{\partial}(e^{-u}\operatorname{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})) + \partial\bar{\partial}u \wedge \partial\bar{\partial}u = 0$$

and satisfies the elliptic condition

$$(8.2) \qquad e^u \omega_S + \sqrt{-1}e^{-u}\operatorname{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1}) - 2\sqrt{-1}\partial\bar{\partial}u > 0$$

and the normalization condition

$$\int_S e^{-u} = A.$$

In this section, we shall obtain a solution for irreducible bundles by perturbing around the above solution $(V, \pi^* H_0, X, \omega_u)$. We follow the method of Li–Yau [5].

Let D_s'' be a family of holomorphic structures on E over S, H be a Hermitian metric on E over S and ϕ be a smooth function on S. We want to look for conditions on (D_s'', H, ϕ) such that under these conditions $(V, \pi^* D_s'', \pi^* H, X, \omega_{u+\phi})$ is the solution to Strominger's system, where

$$\omega_{u+\phi} = \pi^*(e^{u+\phi}\omega) + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}.$$

Fix the metric H_0 as the reference metric on E over S. Then for any Hermitian metric H on E, we can define a smooth endomorphism h on E by

$$\langle v_1, v_2 \rangle_H = < v_1 \cdot h, v_2 >_{H_0}.$$

Under this isomorphism, we define $\mathcal{H}(E)_1$ be the space of all Hermitian metric on E where the associated endomorphism h has determinant one. Let $\mathcal{C}(\omega_S) = \{e^\phi \omega_S\}$ be the space of all Hermitian metrics on S, which are conformal to ω_S. Let $\operatorname{End}^0 E$ be the vector bundle of traceless Hermitian anti-symmetric endomorphisms of (E, H_0). Let

$$\mathcal{H}_0(S) = \left\{ \psi \frac{\omega^2}{2!} \mid \int_S \psi \frac{\omega^2}{2!} = 0 \right\}.$$

We define the operator

$$\mathbf{L} = \mathbf{L_1} \oplus \mathbf{L_2} : \mathcal{H}_1(E) \times \mathcal{C}(\omega_S) \rightarrow \Omega_{\mathbb{R}}^4(\operatorname{End}^0 E) \oplus \mathcal{H}_0(S)$$

by

$$\text{(8.3)} \qquad \mathbf{L_1}(h, e^\phi \omega_S) = e^\phi h^{-1/2} F_h h^{1/2} \wedge \omega_S,$$

$$\mathbf{L_2}(h, e^\phi \omega_S) = \sqrt{-1}\partial\bar\partial(e^{u+\phi}\omega_S) + \partial\bar\partial(e^{-u-\phi}\operatorname{tr}(\bar\partial A \wedge \partial A^* \cdot g_{\omega_S}^{-1}))$$

$$\text{(8.4)} \qquad +\partial\bar\partial(u+\phi) \wedge \partial\bar\partial(u+\phi) - \frac{1}{2}(\operatorname{tr} F_h \wedge F_h - \operatorname{tr} F_{H_0} \wedge F_{H_0}).$$

As $F_h = F_{H_0} + D_0''(D_0'h \cdot h^{-1})$ and $F_{H_0} \wedge \omega_S = 0$,

$$e^\phi h^{-1/2} F_h h^{1/2} \wedge \omega_S = e^\phi D_0''(D_0'h \cdot h^{-1}) \wedge \omega_S$$

and using the notation of paper [9], we get

$$\operatorname{tr}(e^\phi h^{-1/2} F_h h^{1/2} \wedge \omega_S) = e^\phi D_0''(D_0' \operatorname{tr} \log h) \wedge \omega_S = 0$$

as $\det h = 1$. So the image of $\mathbf{L_1}$ lies in $\Omega_{\mathbb{R}}^4(\operatorname{End}^0 E)$. As to $\mathbf{L_2}$, according to $\partial\bar\partial$-lemma on K3 surface, the image of $\mathbf{L_2}$ lies in $R(dd_c)$. For any $\sqrt{-1}\partial\bar\partial\alpha \in R(\sqrt{-1}\partial\bar\partial)$, we can write $\sqrt{-1}\partial\bar\partial\alpha = \psi\frac{\omega_S^2}{2!}$ for some function ψ such that $\int_S \psi\frac{\omega^2}{2!} = 0$. So the image of $\mathbf{L_2}$ lies in $\mathcal{H}_0(S)$. Therefore, the operator $\mathbf{L}$ is well defined.

Proposition 8.1. *If $(h, e^\phi \omega_S) \in \ker \mathbf{L}$, then $(V, \pi^* D_0'', \pi^* h, \omega_{u+\phi})$ is a solution to Strominger's system.*

Proof. According to [5], $(V, \pi^* D_0'', \pi^* h, X, \omega_{u+\phi})$ is the solution to Strominger's system if and only if $(\pi^* h, \omega_{u+\phi})$ lies in the kernel of the operator:

$$\tilde{\mathbf{L}} = \tilde{\mathbf{L}}_1 \oplus \tilde{\mathbf{L}}_2 \oplus \tilde{\mathbf{L}}_3 : \mathcal{H}(V)_1 \times \mathcal{H}(X) \to \Omega_{\mathbb{R}}^6(\operatorname{End}^0 V) \oplus R(dd_c) \oplus R(d_{\omega_0}^*)$$

defined by

$$\tilde{\mathbf{L}}_1(\tilde{h}, \tilde\omega) = \tilde{h}^{-1/2} F_{\tilde{h}} \tilde{h}^{1/2} \wedge \tilde\omega^2,$$

$$\tilde{\mathbf{L}}_2(\tilde{h}, \tilde\omega) = \sqrt{-1}\partial\bar\partial\tilde\omega - \frac{1}{2}(\operatorname{tr} F_{\tilde{h}} \wedge F_{\tilde{h}} - \operatorname{tr} R_{\tilde\omega} \wedge R_{\tilde\omega}),$$

$$\tilde{\mathbf{L}}_3(\tilde{h}, \tilde\omega) = *_{\omega_0} d(\|\Omega\|_{\tilde\omega} \tilde\omega^2).$$

We want to reduce the above operator $\tilde{\mathbf{L}}$ to vector bundle E over S. When $(\tilde{h}, \tilde{\omega}) = (\pi^* h, \omega_{u+\phi})$,

$$
\begin{aligned}
\tilde{\mathbf{L}}_1(\pi^* h, \omega_{u+\phi}) &= \pi^*(h^{-1/2} F_h h^{1/2}) \wedge (\omega_{u+\phi})^2 \\
&= \pi^*(h^{-1/2} F_h h^{1/2} \wedge (e^{u+\phi} \omega_S)) \wedge (\pi^*(e^{u+\phi} \omega_S) + \sqrt{-1}\theta \wedge \bar{\theta}) \\
&= \pi^* e^u \cdot \pi^* \mathbf{L}_1(h, e^\phi \omega_S) \wedge (\pi^*(e^{u+\phi} \omega_S) + \sqrt{-1}\theta \wedge \bar{\theta}),
\end{aligned}
$$

then $(h, e^\phi \omega_S) \in \ker \mathbf{L}_1$ if and only if $(\pi^* h, \omega_{u+\phi}) \in \ker \tilde{\mathbf{L}}_1$.

Next, we consider $\tilde{\mathbf{L}}_2$. When $(\tilde{h}, \tilde{\omega}) = (\pi^* h, \omega_{u+\phi})$, by Proposition 2.3, we have

$$
\begin{aligned}
\operatorname{tr} R_{\omega_{u+\phi}} \wedge R_{\omega_{u+\phi}} = \pi^*(&\operatorname{tr} R_{e^{(u+\phi)}\omega_S} \wedge R_{e^{(u+\phi)}\omega_S} \\
&+ 2\partial\bar{\partial}(e^{-u-\phi} \operatorname{tr}(\bar{\partial} A \wedge \partial A^* \cdot g_{\omega_S}^{-1}))).
\end{aligned}
$$

Since $\operatorname{tr} R_{\omega_S} = 0$ and

$$
R_{e^{(u+\phi)}\omega_S} = \bar{\partial}\partial(u + \phi) \cdot I + R_{\omega_S},
$$

we have

$$
\operatorname{tr} R_{e^{(u+\phi)}\omega_S} \wedge R_{e^{(u+\phi)}\omega_S} = 2\partial\bar{\partial}(u + \phi) \wedge \partial\bar{\partial}(u + \phi) + \operatorname{tr} R_{\omega_S} \wedge R_{\omega_S}.
$$

Thus,

$$
\begin{aligned}
\operatorname{tr} R_{\omega_{u+\phi}} \wedge R_{\omega_{u+\phi}} = \pi^*\{&2\partial\bar{\partial}(u + \phi) \wedge \partial\bar{\partial}(u + \phi) + \operatorname{tr}(R_{\omega_S} \wedge R_{\omega_S}) \\
&+ 2\partial\bar{\partial}(e^{-u-\phi} \operatorname{tr}(\bar{\partial} A \wedge \partial A^* \cdot g_{\omega_S}^{-1}))\}.
\end{aligned}
$$

Using the above equality, we obtain

$$
\begin{aligned}
\tilde{\mathbf{L}}_2(\pi^* h, \omega^u) = \pi^*\{&\sqrt{-1}\partial\bar{\partial}(e^{(u+\phi)}\omega_S) + \partial\bar{\partial}(e^{-u-\phi}\operatorname{tr}(\bar{\partial} A \wedge \partial A^* \cdot g_{\omega_S}^{-1})) \\
&+ \partial\bar{\partial}(u + \phi) \wedge \partial\bar{\partial}(u + \phi) - \frac{1}{2}(\operatorname{tr} F_h \wedge F_h - \operatorname{tr} F_{H_0} \wedge F_{H_0})\} \\
= \pi^*(&\mathbf{L}_2(h, e^\phi \omega_S)).
\end{aligned}
$$

So $\tilde{\mathbf{L}}_2(\pi^* h, e^u \omega) = 0$ if and only if $\mathbf{L}_2(h, \omega) = 0$.

As for $\tilde{\mathbf{L}}_3$, by Theorem 2.1, we always have $\tilde{\mathbf{L}}_3(\pi^* h, \omega_{u+\phi}) = *_0 d$ $(\|\Omega\|_{\omega_{u+\phi}} \omega_{u+\phi}^2) = 0$. $\qquad\square$

Since $(V, \pi^* D_0, \pi^* H_0, \omega_u)$ is a solution, $(I, \omega_S) \in \ker \mathbf{L}$. Now we follow [5]. Let $\mathbb{R}^{+2} = \{T = (T_1, T_2) \in \mathbb{R}^2 | T_i > 0\}$ and let I_1, I_2 and I_3 be the identity endomorphisms of L_1, L_2 and $T'S$, respectively. Then the assignment

$$T = (T_1, T_2) \in \mathbb{R}^{+2} \longmapsto h_T = T_1 I_1 \oplus T_2 I_2 \oplus T_1^{-1/2} T_2^{-1/2} I_3$$

assigns each $T \in \mathbb{R}^{+2}$ to a Hermitian endomorphism of E. Obviously, $F_{h_T} = F_{H_0}$ and we still have $(h_T, \omega_S) \in \ker \mathbf{L}$.

Pick an integer k and a large p. We endow the domain and the target of $\mathbf{L_1} \oplus \mathbf{L_2}$ with Banach space structures as follows:

$$\mathcal{H}_1(E)_{L_k^p} \times \mathcal{C}(\omega_S)_{L_k^p} \to \Omega_{\mathbb{R}}^4(\mathrm{End}^0 E)_{L_{k-2}^p} \oplus \mathcal{H}_0(S)_{L_{k-2}^p}.$$

Then, $\mathbf{L_1} \oplus \mathbf{L_2}$ becomes a smooth operator, and its linearized operator $\delta \mathbf{L_1} \oplus \delta \mathbf{L_2}$ at a solution (h_T, ω_S) becomes a linear map

$$\Omega^0(\mathrm{Her}^0 E)_{L_k^p} \oplus \{\phi \omega_S\}_{L_k^p} \to \Omega_{\mathbb{R}}^4(\mathrm{End}^0 E)_{L_{k-2}^p} \oplus \mathcal{H}_0(S)_{L_{k-2}^p}.$$

Here, we use $\mathrm{Her}^0 E$ to denote the $\mathbb{R}$-subvector bundle of $\mathrm{End}\, E$ consisting of traceless pointwise $<,>$-Hermitian symmetric endomorphisms of E and the canonical isomorphisms $T_{h_T} \mathcal{H}_1(E)_{L_k^p} \cong \Omega^0(\mathrm{Her}^0 E)_{L_k^p}$. Clearly, we have $T_{\omega_S} \mathcal{C}(\omega_S)_{L_k^p} \cong \{\phi \omega_S\}_{L_k^p}$.

To study the kernel and the cokernel of $\delta \mathbf{L_1} \oplus \delta \mathbf{L_2}$ at a trivial solution (h_T, ω_S) we need to look at the linear map

$$F(\delta h) = D_0'' D_{0,h_T}'(\delta h) \wedge \omega_S : \Omega^0(\mathrm{Her}^0 E)_{L_k^p} \to \Omega_{\mathbb{R}}^4(\mathrm{End}^0 E)_{L_{k-2}^p}.$$

Here, according to our convention, $D_{h_T} = D_{0,h_T}' \oplus D_0''$ is the Hermitian connection of (E, D_0'', h_T) for a $T = (T_1, T_2) \in \mathbb{R}^{+2}$. Since $(E, D_0'') = L_1 \oplus L_2 \oplus TS$ and $\deg L_i = 0$, the above is a linear elliptic operator of index 0 whose kernel is

$$V_0 = \left\{ M_1 \cdot I_1 \oplus M_2 \cdot I_2 \oplus - \left(\frac{M_1}{2} + \frac{M_2}{2} \right) I_3 \right\}$$

and whose cokernel is

(8.5) $$V_1 = \omega_S^2 \cdot V_0 \subset \Omega_{\mathbb{R}}^4(\mathrm{End}^0 E)_{L_{k-2}^p}.$$

Let $\mathbf{P}$ be the obvious projection

$$\mathbf{P} : \Omega_{\mathbb{R}}^4(\mathrm{End}^0 E)_{L_{k-2}^p} \to \frac{\Omega_{\mathbb{R}}^4(\mathrm{End}^0 E)_{L_{k-2}^p}}{V_1}.$$

Proposition 8.2. *Let (S, ω_S), Ω_S, H_0 and $T = (T_1, T_2) \in \mathbb{R}^{+2}$ be as before. Then, the linear operator*

$$\mathbf{P} \circ \delta\mathbf{L_1}(h_T, \omega_S) \oplus \delta\mathbf{L_2}(h_T, \omega_S) :$$

$$\Omega^0(\mathrm{Her}^0 E)_{L_k^p} \oplus \{\phi\omega_S\}_{L_k^p} \to \frac{\Omega_{\mathbb{R}}^4(\mathrm{End}^0 E)_{L_{k-2}^p}}{V_1 \oplus \mathcal{H}_0(S)_{L_{k-2}^p}}$$

is surjective.

Proof. As $F_{h_T} = F_{H_0}$ and $\delta\omega = \phi\omega_S$ for some smooth function ϕ,

$$h_T^{-1/2} F_{h_T} h_T^{1/2} \wedge \delta\omega = 0.$$

Hence,

$$\delta\mathbf{L_1}(h_T, \omega_S)(\delta h, \delta\omega) = D_0'' D_{h_T}' \delta h \wedge \omega_S$$

and

$$\mathbf{P} \circ \delta\mathbf{L_1}(h_T, \omega_S) : \Omega^0(\mathrm{Her} E)_{L_k^p} \oplus \{\phi\omega_S\}_{L_k^p} \to \frac{\Omega_{\mathbb{R}}^4(\mathrm{End}^0 E)_{L_{k-2}^p}}{V_1}$$

is surjective, where

(8.6) $$\ker \mathbf{P} \circ \delta\mathbf{L_1}(h_T, \omega_S) = V_0 \oplus \{\phi\omega_S\}_{L_k^p}.$$

On the other hand, when $\delta\omega = \phi\omega_S$,

$$\delta\mathbf{L_2}(h_T, \omega_S)(\delta h, \phi\omega_S) = \sqrt{-1}\partial\bar\partial(e^u \phi\omega_S) - \partial\bar\partial(e^{-u}\phi \,\mathrm{tr}(\bar\partial A \wedge \partial A^* \cdot g_S^{-1}))$$
$$+ 2\partial\bar\partial u \wedge \partial\bar\partial\phi - \mathrm{tr}\,\delta F_{h_T}(\delta h) \wedge F_{h_T}.$$

Because of $\mathrm{tr}\,\delta F_{h_T}(\delta h) \wedge F_{h_T} \in \mathcal{H}_0(S)$ and (8.6), we only need to prove that $\delta\mathbf{L_2}(h_T, \omega_S) : 0 \oplus \{\phi\omega_S\}_{L_k^p} \to \mathcal{H}_0(S)_{L_{k-2}^p}$ is surjective. So, we should solve the following equation:

(8.7)
$$\sqrt{-1}\partial\bar\partial(e^u \phi\omega_S) - \partial\bar\partial(e^{-u}\phi \,\mathrm{tr}(\bar\partial A \wedge \partial A^* \cdot g_S^{-1}))$$
$$+ 2\partial\bar\partial u \wedge \partial\bar\partial\phi = \psi\frac{\omega_S^2}{2!}$$

for any $\psi \in L_{k-2}^p$ such that $\int \psi = 0$. If we define the linear operator L from L_k^p to L_{k-2}^p by

$$L(\phi) = \sqrt{-1}\partial\bar\partial(e^u \phi\omega_S) - \partial\bar\partial(e^{-u}\phi \,\mathrm{tr}(\bar\partial A \wedge \partial A^* \cdot g_S^{-1})) + 2\partial\bar\partial u \wedge \partial\bar\partial\phi,$$

then L is the linearized operator of Equation (8.2). We shall prove $\ker L^* = \mathbb{R}$ and $\dim \ker L = 1$ so that $\mathcal{H}_0(S) \perp \ker L^*$. From the elliptic condition, the operator L is elliptic and Equation (8.7) has a solution $\phi \in L_k^p$ for any $\psi \in L_{k-2}^p$ such that $\int \psi = 0$. Thus we have proven that $P \circ \delta L_1 \oplus \delta L_2$ is surjective and it's kernel is $V_0 \oplus \ker L$. $\qquad\square$

Now we deform the holomorphic structure D_0''. The following proposition is due to Jun Li.

Proposition 8.3. *There is a family D_s'' of deformations of holomorphic structures of E so that its k-th order for $k < m$ Kodaira–Spencer class κ all vanish while its m-th order Kodaira–Spencer class has non-vanishing summands in $H^1(L_i^\vee \otimes TS)$ and $H^1(TS^\vee \otimes L_i)$.*

Proof. Since $\deg L_i = 0$ and TS is slope stable with respect to the Hodge class ω_S, $L_i^\vee \otimes TS$ has no global sections. Using the Serre duality, $H^2(L_i \otimes TS) = 0$ as well. Thus, to compute $H^1(L_i^\vee \otimes TS)$, we use Riemann–Roch for K3 surfaces

$$\chi(L_i^\vee \otimes TS) = \frac{1}{2} c_1(L_i^\vee \otimes TS)^2 + 2\chi(\mathcal{O}_S) - c_2(L_i^\vee \otimes TS).$$

Because $c_1(L_i^\vee \otimes TS) = -2c_1(L_i)$ and $c_2(L_i^\vee \otimes TS) = c_2(TS) + c_1(L_i)^2 = 20 + c_1(L_i)^2$, we have

$$h^1(L_i^\vee \otimes TS) = 16 - c_1(L_i)^2 \geq 16.$$

Here, the last inequality follows from $c_1(L_i) \cdot \omega_S = 0$ and the Hodge index theorem.

For the same reason, we have

$$h^1(TS^\vee \otimes L_i) \geq 16.$$

Now consider the vector bundle

$$E = L_1 \oplus L_2 \oplus TS.$$

The extension group $\operatorname{Ext}^1(E, E)$ has summands

(8.8) $$\operatorname{Ext}^1(L_i, TS) \quad \text{and} \quad \operatorname{Ext}^1(TS, L_i),$$

which are $H^1(L_i^\vee \otimes TS)$ and $H^1(TS^\vee \otimes L_i)$, respectively; hence they have positive dimension. Thus, we can find a direction $\eta \in \operatorname{Ext}^1(E, E)$ that has non-trivial components in the desired factors (8.8).

It remains to show that η can be realized as the tangent of an actual deformation. But this may not be true since the obstruction to deformation of E is the traceless part of $\mathrm{Ext}^2(E, E)$, which is isomorphic to two copies of $H^2(\mathcal{O}_S) \cong \mathbb{C}$. What is true is that there is a family of deformations of holomorphic structures of E, denoted by $\bar{\partial}_s$, so that

$$\frac{d^k}{ds^k}\bar{\partial}_s|_{s=0} = 0, \quad k < m$$

and

$$\frac{d^m}{ds^m}\bar{\partial}_s|_{s=0} \neq 0$$

has non-trivial exponents in (8.8). $\hspace{2cm}\square$

Actually, we can define the Kuranish map $K : \mathcal{U} \to \mathrm{Ext}^2(E, E)$, where $\mathcal{U}$ is some open neighborhood of origin in $\mathrm{Ext}^1(E, E)$. K is the holomorphic map and the complex analytic variety $\mathcal{X} = K^{-1}(0)$ is the parametric space of all holomorphic structures on E near D_0''. Considering the dimensions of $\mathrm{Ext}^1(E, E)$ and $\mathrm{Ext}^2(E, E)$, we can choose an element

$$\eta = \begin{pmatrix} 0 & 0 & C_{13} \\ 0 & 0 & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$$

such that $C_{i3} \neq 0$ and $C_{3j} \neq 0$ for $1 \leq i, j \leq 2$ and η belongs to the tangent cone of $\mathcal{X}$ at the point D_0''. So, there is a curve D_s'' of degree m of smooth deformation of the holomorphic structure D_0''. If we write

$$D_s'' = D_0'' + A_s, \quad A_s \in \Omega^{0,1}(\mathrm{End}^0 E),$$

then $A_0^{(k)} = 0$ for $k < m$ and $A_0^{(m)} = \eta$. We assume that C_{ij} are D_0''-harmonic. Because $\mathrm{Pic}\, S$ is discrete, we can assume further that $\mathrm{tr}\, A_s = 0$ for all s.

With the connection forms A_s, the metric H_0 and the Kähler form ω_S so chosen, we can now define operators

$$\mathbf{L_{s,1}} \oplus \mathbf{L_{s,2}} : \mathcal{H}_1(E)_{L_k^p} \times \mathcal{C}(\omega_S)_{L_k^p} \to \Omega_{\mathbb{R}}^4(\mathrm{End}^0 E)_{L_{k-2}^p} \oplus \mathcal{H}_0(S)_{L_{k-2}^p},$$

with $\mathbf{L_{s,i}}$ defined as in (8.3) and (8.4) of which the curvature form F_h is replaced by the Hermitian curvature of (E, D_s'', h):

$$F_{s,h} = D_{s,h} \circ D_{s,h}.$$

From [5], we have

$$F_{s,h} = F_s + D_s''(D_s'h \cdot h)$$

and

$$F_s = F_0 + (D_0'' + D_0')(A_s - A_s^*) - (A_s - A_s^*) \wedge (A_s - A_S^*).$$

Because $\operatorname{tr} A_s = 0$, $\det h = 1$ and $F_0 \wedge \omega_S = 0$,

$$\operatorname{tr} \mathbf{L}_{\mathbf{s},\mathbf{1}}(h, e^{\phi}\omega_S) = e^{\phi} \operatorname{tr} h^{-1/2} F_{s,h} h^{1/2} \wedge \omega_S = 0.$$

So, $\mathbf{L}_{\mathbf{s},\mathbf{1}}$ still lies in $\Omega_{\mathbb{R}}^4(\operatorname{End}^0 E)_{L_{k-2}^p}$. Let $\mathbf{P}$ be the projection from $\Omega_{\mathbb{R}}^4(\operatorname{End}^0 E)_{L_{k-2}^p} \oplus \mathcal{H}_0(S)_{L_{k-2}^p}$ to $\Omega_{\mathbb{R}}^4(\operatorname{End}^0 E)_{L_{k-2}^p}/V_1 \oplus \mathcal{H}_0(S)_{L_{k-2}^p}$. Because we have proven that the linearized operator of $\mathbf{P} \circ \mathbf{L}_{\mathbf{0},\mathbf{1}} \oplus \mathbf{L}_{\mathbf{0},\mathbf{2}}$ is surjective at (h_{T_0}, ω_S), the implicity theorem implies that for sufficiently small s, there are smooth solutions $(h_{s,T}, \omega_{s,T})$ to $\mathbf{P} \circ \mathbf{L}_{\mathbf{s},\mathbf{1}} \oplus \mathbf{L}_{\mathbf{s},\mathbf{2}} = 0$ near (h_{T_0}, ω_S). We can assume that the solutions $(h_{s,T}, \omega_{s,T})$ can be parameterized by

$$(s, T) \in [0, a) \times B_\epsilon(T_{0,1}) \times B_\epsilon(T_{0,2}),$$

where $T_0 = (T_{0,1}, T_{0,2})$. For simplicity, we denote by $F_{s,T}$ the curvature of the Hermitian vector bundle $(E, D_s'', h_{s,T})$. By our construction, it satisfies

$$\mathbf{L}_{\mathbf{s},\mathbf{1}}(H_{s,T}, \omega_{s,T}) \equiv 0 \mod V_1, \quad \mathbf{L}_{\mathbf{s},\mathbf{2}}(H_{s,T}, \omega_{s,T}) = 0.$$

Hence, to find solutions to $\mathbf{L}_{\mathbf{s},\mathbf{1}} \oplus \mathbf{L}_{\mathbf{s},\mathbf{2}} = 0$, it suffices to investigate the vanishing loci of the functional $\mathbf{r_i}(\mathbf{s}, \cdot)$ from $B_\epsilon(T_{0,i})$ to the Lie algebra $u(L_i)$ defined by

$$(8.9) \qquad \mathbf{r}_i(s, T) = \int_X \big[\mathbf{L}_{s,T}(H_{s,T}, \omega_{s,T})\big]_i \wedge \omega_S,$$

where $[\cdot]_i$ is the projection from $\Omega_{\mathbb{R}}^\bullet(\operatorname{End}^0 E)$ to $\Omega_{\mathbb{R}}^\bullet(u(L_i))$ and $u(L_i)$ is the bundle of $<,>$-Hermitian anti-symmetric endomorphisms of L_i.

We shall compute $\mathbf{r}_i^{(k)}(0, T)$ for all T and for all $k \leq 2m$. Because $\omega_{s,T} \in \mathcal{C}(\omega_S)$, we can write $\omega_{s,T} = \phi_{s,T}\omega_S$ for some positive functions $\phi_{s,T}$ on S such that $\phi_{0,T} = 1$. Then, we have

$$(8.10) \qquad \frac{d^k}{ds^k}\big|_{s=0}\omega_{s,T} = \phi_{0,T}^{(k)}\omega_S.$$

On the other hand, since $(h_{s,T}, \omega_{s,T})$ are solutions to $\mathbf{L}_{\mathbf{s},\mathbf{1}} \oplus \mathbf{L}_{\mathbf{s},\mathbf{2}} = 0$ mod V_1, there is a function $\mathbf{c}(s, T)$ taking values in V_1 with $\mathbf{c}(0, T) = 0$

so that

(8.11) $$F_{s,T} \wedge \omega_{s,T} = h_{s,T}^{-1/2} \mathbf{c}(s,T) h_{s,T}^{1/2}.$$

We can write

$$\mathbf{c}(s,T) = \text{diag} \left(M_1(s,T), M_2(s,T), -\frac{(M_1(s,T))}{2} + \frac{M_2(s,T)}{2}) I_3 \right) \cdot \omega_S^2,$$

where $M_i(s,T)$ is the function only depending on s and T.

At first, we compute $\mathbf{r}_i^{(k)}(0,T)$ for any T and $k \leq m-1$. When $k \leq m-1$, $A_0^{(k)} = 0$. Then,

(8.12) $$F_{0,T}^{(k)} = \sum_{l=0}^{k} C_k^l D_0''[D_0' h_{0,T}^{(l)} \cdot (h_{0,T}^{-1})^{(k-l)}].$$

Because $D_0' h_T = 0$, $\dot{F}_{0,T} = D_0''[D_0' \dot{h}_{0,T} \cdot h_T^{-1}]$ and

$$\frac{d}{ds}|_{s=0} \left(h_{s,T}^{-1/2} F_{s,T} h_{s,T}^{1/2} \right) = h_{0,T}^{-1/2} \dot{F}_{0,T} h_{0,T}^{1/2}.$$

We also have $F_{0,T} \wedge \omega_S = 0$. Combining these equalities with (8.10),

$$\dot{\mathbf{r}}_i(0,T) = \int_S T_i^{-1/2} [\dot{F}_{0,T}]_i T_i^{1/2} \wedge \omega_S + \int_S T^{-1/2} [F_{0,T}]_i T^{1/2} \wedge \dot{\omega}_S = 0.$$

On the other hand, taking derivative of s at $s=0$ to (8.11) and couple $\mathbf{c}(0,T) = 0$, we have

$$\dot{F}_{0,T} \wedge \omega_S = h_T^{1/2} \dot{\mathbf{c}}(0,T) h_T^{-1/2}$$

and then

$$[\dot{F}_{0,T}]_i \wedge \omega_S = [\dot{\mathbf{c}}(0,T)]_i = \dot{M}_i(0,T) \omega_S^2.$$

From

$$\int_S \dot{M}_i(0,T) \omega_S^2 = \int_S [\dot{F}_{0,T}]_i \wedge \omega_S = 0,$$

we get $\dot{M}_i(0,T) = 0$. So, we get

$$\dot{\mathbf{c}}(0,T) = 0 \quad \text{and} \quad \dot{F}_{0,T} \wedge \omega_S = (D_0'' D_0' \dot{h}_{0,T}) \cdot h_{0,T}^{-1} \wedge \omega_S = 0.$$

Thus,

$$\dot{h}_{0,T} \in V_0 \quad \text{and} \quad D_0' \dot{h}_{0,T} = 0.$$

In this way, we can prove that

$$\mathbf{r}_i^{(k)}(0,T) = 0, \quad \mathbf{c}^{(k)}(0,T) = 0, \quad F_{0,T}^{(k)} \wedge \omega_S = 0,$$

(8.13)

$$D_0' h_{0,T}^{(k)} = 0 \text{ for any } k \le m - 1.$$

When $k = m$, because of (8.13) and (8.10), we obtain

$$\frac{d^m}{ds^m}\Big|_{s=0}(h_{s,T}^{-1/2} F_{s,T} h_{s,T}^{1/2} \wedge \omega_{s,T}) = h_{0,T}^{-1/2} F_{0,T}^{(m)} h_{0,T}^{1/2} \wedge \omega_S$$

and

$$F_{0,T}^{(m)} = (D_0'' + D_0')(A_0^{(m)} - A_0^{*(m)}) + D_0''[D_0' h_{0,T}^{(m)}] \cdot h_{0,T}^{-1}.$$

Then, we get

$$\mathbf{r}_i^{(m)}(0,T) = \int_S [F_{0,T}^{(m)}]_i \wedge \omega_S = 0.$$

Because $F_{0,T}^{(k)} \wedge \omega_S = 0$, $\mathbf{c}^{(k)}(0,T) = 0$ for $k \le m - 1$, from (8.11), we get

$$F_{0,T}^{(m)} \wedge \omega_S = h_T^{1/2} \mathbf{c}^{(m)}(0,T) h_T^{-1/2}.$$

So, using the same method of case $k = 1$, we still have

$$\mathbf{c}^{(m)}(0,T) = F_{0,T}^{(m)} \wedge \omega_S = 0.$$

When $k < m$, $A_0^{(k)} = 0$ and we have proven $D_0' h_{0,T}^{(k)} = 0$. By direct computation, we see

$$F_{0,T}^{(m+k)} = (D_0'' + D_0')(A_0^{(m+k)} - A_0^{*(m+k)})$$
$$+ D_0'' \left(\frac{d^{m+k}}{ds^{m+k}}\Big|_{s=0}(D_s' h_{s,T} \cdot h_{s,T}^{-1}) \right).$$

Then, we still can get

$$\mathbf{r}_i^{(m+k)}(0,T) = 0, \quad \mathbf{c}^{(m,k)}(0,T) = 0, \quad F_{0,T}^{(m+k)} \wedge \omega_S = 0 \text{ for } k < m.$$

At last we compute $\mathbf{r}^{(2m)}(0,T)$. Directly computing, we get

$$F_{0,T}^{(2m)} = (D_0'' + D_0')(A_0^{(2m)} - A_0^{*(2m)}) - C_{2m}^m (A_0^{(m)} - A_0^{*(m)}) \wedge (A_0^{(m)} - A_0^{*(m)})$$
$$- C_{2m}^m [A_0^{(m)}, d^m/ds^m|_{s=0}(D_s' h_{s,T} \cdot h_{s,T}^{-1})]$$
$$+ D_0''(d^{2m}/ds^{2m}|_{s=0}(D_s' h_{s,T} \cdot h_{s,T}^{-1})$$

and

$$\left[A_0^{(m)}, \frac{d^m}{ds^m} |_{s=0} (D_s' h_{s,T} \cdot h_{s,T}^{-1}) \right]$$
$$= [A_0^{(m)}, [A_0^{*(m)}, h_{0,T}] h_{0,T}^{-1}] + [A_0^{(m)}, D_0' h_{0,T}^{(m)} \cdot h_{0,T}^{-1}].$$

Then   from $d^{2m}/ds^{2m}|_{s=0} (h_{s,T}^{-1/2} F_{s,T} h_{s,T}^{1/2} \wedge \omega_S) = h_{0,T}^{-1/2} F_{0,T}^{(2m)} h_{0,T}^{1/2} \wedge \omega_S,$
we see

$$\mathbf{r}_i^{(2m)}(0, T) = -C_{2m}^m \int_S [(A_0^{(m)} - A_0^{*(m)}) \wedge (A_0^{(m)} - A_0^{*(m)})]_i$$
$$- C_{2m}^m \int_S [[A_0^{(m)}, [A_0^{*(m)}, h_{0,T}] h_{0,T}^{-1}]]_i$$
$$- C_{2m}^m \int_S [[A_0^{(m)}, D_0' h_{0,T}^{(m)} \cdot h_{0,T}^{-1}]]_i.$$

The last term is zero because $D_0''^* A_0^{(m)} = 0$ and Lemma 2.3 of [5]. Using

$$A_0^{(m)} = \begin{pmatrix} 0 & 0 & C_{13} \\ 0 & 0 & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \quad \text{and} \quad H_{0,T} = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_1^{-1/2} T_2^{-1/2} I_3 \end{pmatrix},$$

one computes

$$[[A_0^{(m)}, [A_0^{*(m)}, h_{0,T}] h_{0,T}^{-1}]]_1$$
$$= (1 - T_1^{-3/2} T_2^{-1/2}) C_{13} \wedge C_{13}^* + (1 - T_1^{3/2} T_2^{1/2}) C_{31}^* \wedge C_{31}$$

and

$$-[(A_0^{(m)} - A_0^{*(m)}) \wedge (A_0^{(m)} - A_0^{*(m)})]_1 = C_{13} \wedge C_{13}^* + C_{31}^* \wedge C_{31}.$$

Therefore,

$$\mathbf{r}_1^{(2m)}(0, T) = -C_{2m}^m \int_S (T_1^{-3/2} T_2^{-1/2} C_{13} \wedge C_{13}^* + T_1^{3/2} T_2^{1/2} C_{31}^* \wedge C_{31}) \wedge \omega_S.$$

With the same reason, we have

$$\mathbf{r}_2^{(2m)}(0, T) = -C_{2m}^m \int_S (T_1^{-1/2} T_2^{-3/2} C_{23} \wedge C_{23}^* + T_1^{1/2} T_2^{3/2} C_{32}^* \wedge C_{32}) \wedge \omega_S.$$

Let $B_{i3} = \sqrt{-1} \int_S C_{i3} \wedge C_{i3}^* \wedge \omega_S$ and $B_{3i} = -\sqrt{-1} \int_S C_{3i}^* \wedge C_{3i} \wedge \omega_S$ for $i = 1, 2$. By our assumption, we have $B_{i3} > 0$ and $B_{3i} > 0$. Then,

$$\mathbf{r}_1^{(2m)}(0, T) = \sqrt{-1} C_{2m}^m \{ B_{13} T_1^{-3/2} T_2^{-1/2} - B_{31} T_1^{3/2} T_2^{1/2} \},$$
$$\mathbf{r}_2^{(2m)}(0, T) = \sqrt{-1} C_{2m}^m \{ B_{23} T_1^{-1/2} T_2^{-3/2} - B_{32} T_1^{1/2} T_2^{3/2} \}.$$

Clearly, $\mathbf{r}_i^{(2m)}(0, T_0) = 0$ if
(8.14)
$$T_0 = (T_{01}, T_{02}) = \left(\left(\frac{B_{13}}{B_{31}} \right)^{3/8} \left(\frac{B_{23}}{B_{32}} \right)^{-1/8}, \left(\frac{B_{13}}{B_{31}} \right)^{-1/8} \left(\frac{B_{23}}{B_{32}} \right)^{3/8} \right).$$

Then, we define the map $G : B_\epsilon(T_0) \to S^1(1)$ by

$$G(T) = \frac{(\mathbf{r}_1^{2m}(0, T), \mathbf{r}_2^{2m}(0, T))}{\|(\mathbf{r}_1^{2m}(0, T), \mathbf{r}_2^{2m}(0, T))\|}$$

or

(8.15) $\qquad G((T_1, T_2)) = \dfrac{(B_{13} T_1^{-1} - B_{31} T_1^2 T_2^1, B_{23} T_2^{-1} - B_{32} T_1^1 T_2^2)}{\|(B_{13} T_1^{-1} - B_{31} T_1^2 T_2^1, B_{23} T_2^{-1} - B_{32} T_1^1 T_2^2)\|}.$

Then it is easily proven that for ϵ small enough (here $\epsilon < 1$), G is homotopic to

$$G_1(T) = \frac{(B_{13} T_1^{-1}, B_{23} T_2^{-1})}{\|(B_{13} T_1^{-1}, B_{23} T_2^{-1})\|} = \frac{(B_{13} T_2, B_{23} T_1)}{\|(B_{13} T_2, B_{23} T_1)\|}.$$

Thus, $\deg G = \deg G_1 = -1$. Then, as a discussion of the proof of Theorem 4.3 in paper [5], we see that the map for ϵ small enough,

$$(\mathbf{r}_1, \mathbf{r}_2)(s, .) : B_\epsilon(T_0) \to \mathbb{R}^2, \quad s \in (0, a')$$

attains value $0 \in \mathbb{R}^2$ for all $s \in (0, a')$ in $B_\epsilon(T_0)$. So for sufficiently small s, there are solutions $(h_s, e^\phi \omega_S)$ to $\mathbf{L_s} = 0$ near (h_{T_0}, ω_S). From our definition of $\mathbf{L_{s,1}}$, we know that $h_{s,T}$ is the Hermitian–Yang–Mills solution on (E, D_s). From Proposition 8.1 for $\mathbf{L_s}$, we have obtained an irreducible solution of Strominger's system on non-Kähler manifold X. Hence, we have proven the following:

Theorem 8.4. *Let (E, H_0, S, ω_S) be as before. Fix the holomorphic structure D_0'' of E. Then there is a smooth deformation D_s'' of (E, D_0'') so that*

there are Hermitian–Yang–Mills metrics H_s on (E, D_s) and smooth function ϕ_s on S such that

$$\left(V = \pi^* E, \pi^* D_s'', \pi^* H_s, \tilde{\omega}_s = \pi^*(e^{u+\phi_s}\omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta} \right)$$

are the irreducible solutions to Strominger's system on X. Moreover, $\lim_{s\to 0} \phi_s = 0$ and $\lim_{s\to 0} H_s$ is a regular reducible Hermitian–Yang–Mills connection on $E = L_1 \oplus L_2 \oplus TS$.

Acknowledgments

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