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Admissible wavelets and inverse radon transform associated with the affine homogeneous Siegel domains of type II

JIANXUN HE AND HEPING LIU

Let $D(\Omega, \Phi)$ be the affine homogeneous Siegel domain of type II, whose Šilov boundary N is a nilpotent Lie group of step two. In this article, we develop the theory of wavelet analysis on N. By selecting a set of mutual orthogonal wavelets we give a direct sum decomposition of $L^2(D(\Omega, \Phi))$, the first component $A^0_{0,0}$ of which is the Bergman space. Moreover, we study the Radon transform on N, and obtain an inversion formula $R^{-1} = (\pi)^{-2d} LRL$ which is an extension of that by Strichartz [R. S. Strichartz, *L^p harmonic analysis and Radon transforms on the Heisenberg group*, J. Funct. Anal. **96** (1991), 350–406.]. We devise a subspace of $L^2(N)$ on which the Radon transform is a bijection. Using wavelet inverse transform, we establish an inversion formula of the Radon transform in the weak sense.

1. Introduction

The wavelet transform is a very useful analysis tool in pure and applied mathematics. The research of this subject on Euclidean space has made considerable progress [2, 4] and the references therein. It is well known that one-dimensional wavelet analysis can be explained in terms of square integrable representations associated with the affine automorphism group of the upper half plane [1, 5, 10, 14]. More precisely, the continuous (admissible) wavelets are closely related to square integrable representation of a non-unimodular group. For this we refer the reader to see [6, 8]. In this viewpoint, various authors extended the theory of wavelet analysis on real line $\mathbb{R}$ to the tube domain [17, 20] and unbounded realization of the unit ball in $\mathbb{C}^n$ [11, 21]. The Radon transform has also received considerable attention in mathematical literature due to its wide applications to partial differential equations, X-ray technology, radio astronomy and so on. For the basic theory and further results of the Radona transform, we refer the reader to the book [15] by Helgason and the references therein. A combined use of the Radon transform and the wavelet transform can be called "wavelet Radon transform" [31] which has proved to be very useful both in pure mathematics and its applications. Recently, some authors deal with the inversion formula of Radon transforms by using inverse wavelet transforms. Holschneider [16] considered the classical Radon transform on the two-dimensional plane. His results were extended by Rubin [26, 27] to the k-dimensional Radon transform on $\mathbb{R}^n$ and totally geodesic Radon transform on the sphere and hyperbolic space. Strichartz [29] defined the Radon transform on the Heisenberg group H_n and gave an inversion formula. Nessibi and Trimèche [23] obtained an inversion formula of the Radon transform on the Laguerre hypergroup $K = [0, \infty) \times \mathbb{R}$ by use of the generalized wavelet transform.

The function theory on Siegel domain $D(\Omega, \Phi)$ of type II has always exerted a strong attraction due to its important geometric background. Many classical results have been extended to this case. It is a well-known fact that the distinguished boundary or Šilov boundary N of $D(\Omega, \Phi)$ is a nilpotent Lie group of step two. The theory of harmonic analysis and other problems on N were considered in [3, 24, 30]. In this paper, we develop a theory of wavelet transform and Radon transform for the affine homogeneous Siegel domains of type II. As applications for these wavelets, we give a direct sum orthogonal decomposition for $L^2(D(\Omega, \Phi))$. Moreover, we obtain an inversion of the Radon transform by using the inverse wavelet transform. The harmonic analysis on a nilpotent Lie group developed by Ogden and Vági [24] plays an important role in this paper.

This article is organized as follows: in Section 2, we recall some basic facts relating to the group N, specifically including the Iwasawa subgroup P of the automorphism group of affine homogeneous Siegel domain $D(\Omega, \Phi)$. In the third section, we define the unitary representations of P on $L^2(N)$, and make a survey of harmonic analysis on N. We also give an irreducible decomposition of $L^2(N)$. Section 4 develops a theory of continuous wavelet analysis. In Section 5, we present the orthogonal sum decomposition for the function spaces $L^2(P)$ and $L^2(D(\Omega, \Phi))$ by using wavelet transforms. In the decomposition for $L^2(D(\Omega, \Phi))$, the first component is just the Bergman space. In Section 6, we investigate the Radon transform on N. Two function spaces $\mathscr{S}_R(N)$ and $L^2_R(N)$ are introduced on which the Radon transform R is a bijection. The inversion formula for the Radon transform $R^{-1} = (\pi)^{-2n} LRL$ holds on $\mathscr{S}_R(N)$. In Section 7, we make use of suitable wavelets to derive an inversion formula of the Radon transform on $L^2_R(N)$ in the weak sense. Finally, in Section 8 we state our results in a more explicit form in terms of Jordan algebra for the symmetric Siegel domain of type II, which is the most interested case.

2. The affine homogeneous Siegel domains of type II

We start with some notations and facts on Siegel domains of type II, especially the Iwasawa subgroup of the affine automorphism group of an affine homogeneous Siegel domain of type II. Further details can be found in [18, 22, 25, 30].

Let U be the m-dimensional Euclidean space. A regular cone Ω in U is a non-empty convex open cone which contains no entire straight line. Let V be the n-dimensional complex vector space. W denotes the complexification of U. A map Φ of $V\times V$ into W is called an Ω-positive Hermitian map if the following conditions are satisfied:

(i) $\Phi(\zeta,\eta)$ is $\mathbb{C}$-linear in ζ.

(ii) $\overline{\Phi(\zeta,\eta)}=\Phi(\eta,\zeta)$.

(iii) $\Phi(\zeta,\zeta)\in\overline{\Omega}$.

(iv) $\Phi(\zeta,\zeta)=0$ only if $\zeta=0$.

Then, the Siegel domain of type II $D=D(\Omega,\Phi)$ determined by a regular cone Ω and an Ω-positive Hermitian map Φ is defined by

$$D=\{(z,\zeta)\in W\times V:\operatorname{Im}z-\Phi(\zeta,\zeta)\in\Omega\}. \tag{2.1}$$

The Šilov boundary S of Ω is given by

$$S=\{(z,\zeta)\in W\times V:\operatorname{Im}z-\Phi(\zeta,\zeta)=0\}. \tag{2.2}$$

Let $G_a(D)$ be the affine automorphism group of D. $G_a(D)$ can be decomposed into the semi-direct product $G_a(D)=NH$ of the subgroups N and H. Here N is a simply connected nilpotent Lie group of step two with the underlying manifold $U\times V$ and the multiplication

$$(a,\alpha)(b,\beta)=(a+b+2\operatorname{Im}\Phi(\alpha,\beta),\alpha+\beta),\quad (a,\alpha),(b,\beta)\in U\times V \tag{2.3}$$

and $H=GL(W\times V)\bigcap G_a(D)$. H consists of all pairs (A,B) where A is in the automorphism group $\operatorname{Aut}(\Omega)$ of Ω, the subgroup of $GL(U)$ which leaves

Ω invariant and $B \in GL(V)$ such that

$$A\Phi(\zeta,\eta) = \Phi(B\zeta, B\eta), \quad \zeta,\eta \in V. \tag{2.4}$$

The actions of N and H on D (and S) are separately given by

$$(z,\zeta) \mapsto (a,\alpha)(z,\zeta) = (a + z + 2i\Phi(\zeta,\alpha) + i\Phi(\alpha,\alpha), \zeta + \alpha) \tag{2.5}$$

and

$$(z,\zeta) \mapsto (A,B)(z,\zeta) = (Az, B\zeta). \tag{2.6}$$

We assume that D is affine homogeneous, that is, $G_a(D)$ acts on D transitively. We define a homomorphism μ of H into $\mathrm{Aut}(\Omega)$ by $\mu(A,B) = A$. The affine homogeneity of D is characterized by the transitivity of $\mu(H)$, i.e., $G_a(D)$ acts on D transitively if and only if $\mu(H)$ acts on Ω transitively. Specifically, all symmetric Siegel domains are affine homogeneous, which will be treated more explicitly in Section 8. An example of affine homogeneous Siegel domain which is non-symmetric can be found in [22]. Let $G_a(D)^0$ be the identity component of $G_a(D)$. Then $G_a(D)^0$ also acts on D transitively. Fix a point $e \in \Omega$. Let K be the isotropy subgroup of $G_a(D)$ at the point $(ie, 0) \in D$. H^0 and K^0 are the identity components of H and K, respectively. Then $K^0 \subset H^0$ and K^0 is a maximal compact subgroup of $G_a(D)^0$ (and H^0). Therefore $H^0 = T_1K^0$ (semi-direct product) where T_1 is a maximal $\mathbb{R}$-triangular subgroup of H. The kernel of μ is in K. $\mu(H)$ acts on Ω transitively and $\mu(K)$ is the isotropy subgroup of $\mu(H)$ at the point $e \in \Omega$. Therefore $T = \mu(T_1)$ is a maximal $\mathbb{R}$-triangular subgroup of $\mathrm{Aut}(\Omega)$ and μ gives an isomorphism of T_1 and T. The elements of T_1 can be written in the form $(t, B(t))$ where $t \in T$ and $B(t) \in GL(V)$ are uniquely determined by t such that

$$t\Phi(\zeta,\eta) = \Phi(B(t)\zeta, B(t)\eta), \quad \zeta,\eta \in V. \tag{2.7}$$

Let $P = NT_1$. P is called the Iwasawa subgroup of $G_a(D)$. The action of P on D is given by

$$(z,\zeta) \mapsto (a,\alpha,t)(z,\zeta) = (a + tz + 2i\Phi(B(t)\zeta,\alpha) + i\Phi(\alpha,\alpha), B(t)\zeta + \alpha). \tag{2.8}$$

The multiplication of P is given by

$$(a,\alpha,t)(b,\beta,s) = (a + tb + 2\,\mathrm{Im}\,\Phi(\alpha, B(t)\beta), \alpha + B(t)\beta, ts). \tag{2.9}$$

Let Det denote the determinant of a linear transformation (or a matrix). From (2.7) it is not difficult to get

$$\mathrm{Det}(t)^n = |\mathrm{Det}\, B(t)|^{2m}. \tag{2.10}$$

Then, the left Haar measure of P turns out to be $\mathrm{Det}(t)^{-\frac{m+n}{m}}\, da\, d\alpha\, dm_l(t)$ where $da\, d\alpha$ denotes the Haar measure of N which coincides with the Lebesgue measure of $U \times V$ and $dm_l(t)$ is the left Haar measure of T. Obviously P acts on D simply transitively. We can identify P with D by identification of (a, α, t) and $\big(a + i(te + \Phi(\alpha, \alpha)), \alpha\big)$. We can also identify N with S by identification of (a, α) and $(a + i\Phi(\alpha, \alpha), \alpha)$.

3. The decomposition of $L^2(N)$

P has a natural unitary representation π on $L^2(N)$ (or $L^2(S)$ instead) which is defined by

$$\begin{aligned}(\pi_{(a,\alpha,t)} f)(x, \zeta) &= \mathrm{Det}(t)^{-((m+n)/2m)} f((a, \alpha, t)^{-1}(x, \zeta)) \\ &= \mathrm{Det}(t)^{-((m+n)/2m)} f(t^{-1}(x - a \\ &\quad - 2\,\mathrm{Im}\,\Phi(\alpha, \zeta), B(t)^{-1}(\zeta - \alpha)).\end{aligned} \tag{3.1}$$

This section is devoted to decompose $L^2(N)$ into the direct sum of the irreducible invariant closed subspaces under π. We shall make use of the harmonic analysis on the nilpotent Lie group N which is due to Ogden and Vági [24].

Let U' denote the (real) dual of U. The adjoint action of T on U' is given by

$$\lambda \mapsto t^{*-1}\lambda, \quad t \in T,\ \lambda \in U',$$

where t^* is the adjoint of t. Suppose Λ_ϵ is a T-orbit of U' under the adjoint action. Because T is a connected $\mathbb{R}$-triangular group, Λ_ϵ has positive Lebesgue measure if and only if the adjoint action of T on Λ_ϵ is simple. Let $\{\Lambda_\epsilon : \epsilon \in E\}$ is the set of all simple T-orbits of U' under the adjoint action. Then $\Lambda = \cup_{\epsilon \in E} \Lambda_\epsilon$ has total Lebesgue measure in U'. The parametric representation of E for symmetric case is given in Section 8. It is usually obvious for every concrete case.

For $\lambda \in U'$, we set

$$H_\lambda(\zeta, \eta) = 4\langle \lambda, \Phi(\zeta, \eta)\rangle \tag{3.2}$$

and

$$B_\lambda(\zeta,\eta)=\mathrm{Im}(H_\lambda(\zeta,\eta)). \tag{3.3}$$

The Hermitian form H_λ is non-degenerate for $\lambda\in U'$ almost everywhere. Note that

$$H_{t^*\lambda}(\zeta,\eta)=H_\lambda(B(t)\zeta,B(t)\eta). \tag{3.4}$$

It is easy to see that H_λ is non-degenerate if $\lambda\in\Lambda$. Fix a complex basis $\beta_1,\ldots,\beta_n$ for V which is compatible with the chosen Lebesgue measure $d\zeta$ of V. Let $M_\lambda(\beta)$ be the matrix defined by $\langle\lambda,\Phi(\beta_i,\beta_j)\rangle$. For $\lambda\in\Lambda$, we define

$$\rho(\lambda)=4^n|\mathrm{Det}\,M_\lambda(\beta)|. \tag{3.5}$$

Then $\rho(\lambda)$ is a positive continuous function on Λ independent of the compatible basis chosen. $\rho(\lambda)\,d\lambda$ is essentially the Plancherel measure for N. Fix a point $\lambda_\epsilon\in\Lambda_\epsilon$ for each simple T-orbit Λ_ϵ. Select a complex basis $\beta_1(\lambda_\epsilon),\ldots,\beta_n(\lambda_\epsilon)$ for V such that

$$H_{\lambda_\epsilon}(\beta_i(\lambda_\epsilon),\beta_j(\lambda_\epsilon))=\sigma_i\delta_{ij},$$

where $\sigma_i=\pm1$ and δ_{ij} is the Kronecker symbol. Suppose $\lambda=t^*\lambda_\epsilon\in\Lambda_\epsilon$. Set $\beta_j(\lambda)=B(t)^{-1}\beta_j(\lambda_\epsilon), j=1,\ldots,n$. Then $\beta(\lambda)$ is a complex basis of V such that

$$H_\lambda(\beta_i(\lambda),\beta_j(\lambda))=\sigma_i\delta_{ij}.$$

Let $V^{\mathbb{R}}$ be the underlying real space of V and J be the original complex structure of V. Then $\beta_1(\lambda),\ldots,\beta_n(\lambda),J\beta_1(\lambda),\ldots,J\beta_n(\lambda)$ is a basis of $V^{\mathbb{R}}$. Let $J_\lambda:V^{\mathbb{R}}\mapsto V^{\mathbb{R}}$ be defined in the $\beta(\lambda)$-basis by the $2n\times2n$ matrix

$$J_\lambda=\begin{pmatrix}0&-\sigma\\ \sigma&0\end{pmatrix}$$

with $\sigma=\mathrm{diag}(\sigma_1,\sigma_2,\ldots,\sigma_n)$. Then J_λ is a complex structure which commutes with J. We denote $V^{\mathbb{R}}$ equipped with the complex structure J_λ by V_λ. Let E_λ be the real span of $\beta_1(\lambda),\ldots,\beta_n(\lambda)$, then $V_\lambda=E_\lambda\oplus J_\lambda E_\lambda$. Suppose $\zeta=\xi+J_\lambda\theta$, where $\xi=\sum_{j=1}^n\xi_j\beta_j(\lambda)\in E_\lambda,\theta=\sum_{j=1}^n\theta_j\beta_j(\lambda)\in E_\lambda$. Set $\overline{\zeta}=$

$\xi - J_\lambda\theta, \zeta_j = \xi_j + i\theta_j$. We define

$$\zeta \cdot \eta = B_\lambda(J_\lambda\zeta, \overline{\eta}) + iB_\lambda(\zeta, \overline{\eta})$$

and

$$|\zeta|^2 = \zeta \cdot \overline{\zeta}.$$

Then we have $\zeta \cdot \eta = \sum_{j=1}^n \zeta_j\eta_j$ and $|\zeta|^2 = \sum_{j=1}^n |\zeta_j|^2$.

If $\tau \in E_\lambda$, then $\tau = \sum_{j=1}^n \tau_j\beta_j(\lambda)$. We also write $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ to denote the coordinates of τ under the basis $\beta_1(\lambda), \beta_2(\lambda), \ldots, \beta_n(\lambda)$. Now let $\zeta = \xi + J_\lambda\theta$, then $\xi = \sum_{j=1}^n \xi_j\beta_j(\lambda)$ and $\theta = \sum_{j=1}^n \theta_j\beta_j(\lambda)$. Letting $g(\zeta) \in L^1(V)$, we have

$$\int_V g(\zeta)\rho(\lambda)d\zeta = \int_{E_\lambda\times E_\lambda} g(\xi + J_\lambda\theta)\vartheta_\lambda(d\xi)\vartheta_\lambda(d\theta), \tag{3.6}$$

where $\vartheta_\lambda(d\xi) = d\xi_1\, d\xi_2 \cdots d\xi_n$. Let $\psi_m(t)$ be the normalized Hermite function defined by

$$\psi_m(t) = (2^{1/4}2^{-m/2}(m!)^{-1/2})h_m((2\pi)^{(1/2)}t)\exp(-\pi t^2),$$

where h_m denotes the classical Hermite polynomial. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ be a multi-index, the higher dimensional Hermite functions $\Phi_{\nu,\lambda}$ is defined by tensor products, i.e.,

$$\Phi_{\nu,\lambda}(\tau) = \prod_{j=1}^n \psi_{\nu_j}(\tau_j). \tag{3.7}$$

Obviously, $\{\Phi_{\nu,\lambda}(\tau) : \nu \in \mathbb{Z}_+^n\}$ forms an orthonormal basis for Hilbert space $\mathscr{H}_\lambda = L^2(E_\lambda, \vartheta_\lambda(d\tau))$. The Schrödinger representation of N on $\mathscr{H}_\lambda$ is defined by

$$(\pi^\lambda(x,\zeta)\phi)(\tau) = \exp(-2\pi i\langle\lambda, x\rangle)\exp(\pi i\xi\cdot\eta)\exp(-2\pi i\eta\cdot\tau)\phi(\tau - \xi), \tag{3.8}$$

where $\zeta = \xi + J_\lambda\eta$, $\tau \in E_\lambda$, $\phi \in \mathscr{H}_\lambda$. Thus $\pi^\lambda(x,\zeta)$ is an irreducible unitary representation of the group N on $\mathscr{H}_\lambda$. The Fourier transform of a function $f \in L^1(N)$ is an operator valued function defined by

$$\widehat{f}(\lambda) = \int_N f(x,\zeta)\pi^\lambda(x,\zeta)\,dx\,d\zeta. \tag{3.9}$$

We have the Plancherel formula

$$\|f\|_{L^2(N)} = \left(\int_\Lambda \|\widehat{f}(\lambda)\|^2_{\mathrm{HS}}\rho(\lambda)\,d\lambda\right)^{(1/2)}, \quad f \in L^1(N) \cap L^2(N), \tag{3.10}$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of an operator. The Plancherel formula is equivalent to

$$\langle f, g\rangle_{L^2(N)} = \int_\Lambda \mathrm{tr}(\widehat{g}(\lambda)^* \widehat{f}(\lambda))\rho(\lambda)\,d\lambda, \quad f, g \in L^1(N) \cap L^2(N), \tag{3.11}$$

which allows us to extend the Fourier transform to the tempered distributions on N by duality.

Let $f * g$ be the convolution of f and g defined by

$$f * g(x, \zeta) = \int_N f(y, \eta) g((y, \eta)^{-1}(x, \zeta))\,dy\,d\eta.$$

Set

$$\widetilde{f}(x, \zeta) = \overline{f((x, \zeta)^{-1})} = \overline{f(-x, -\zeta)}.$$

It is easy to see that

$$\widehat{f * g}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda) \tag{3.12}$$

and

$$\widehat{\widetilde{f}}(\lambda) = \widehat{f}(\lambda)^*. \tag{3.13}$$

Suppose $t \in T$, we have the following identities:

$$\begin{aligned} J_{t^*\lambda} &= B(t)^{-1} J_\lambda B(t), \\ B_{t^*\lambda}(\zeta, \eta) &= B_\lambda(B(t)\zeta, B(t)\eta), \\ \rho(t^*\lambda) &= |\mathrm{Det}\, B(t)|^2 \rho(\lambda). \end{aligned}$$

The map

$$F(\zeta) \mapsto G(\zeta) = F(B(t)\zeta)$$

gives the isometrically isomorphism from $\mathscr{H}_\lambda$ onto $\mathscr{H}_{t^*\lambda}$. We can identify $\mathscr{H}_\lambda$ with $\mathscr{H}_{t^*\lambda}$. This means that we identify $\Phi_{\nu,\lambda}$ with $\Phi_{\nu,t^*\lambda}$. Then we have

$$\pi^{t^*\lambda}(x, \zeta) = \pi^\lambda(tx, B(t)\zeta). \tag{3.14}$$

Let

$$f_t(x,\zeta) = \mathrm{Det}(t)^{-((m+n)/m)} f(t^{-1}x, B(t)^{-1}\zeta).$$

Then

$$\widehat{f_t}(\lambda) = \widehat{f}(t^*\lambda). \tag{3.15}$$

We define the operator P_ν^ϵ on $L^2(N)$ in terms of the Fourier transform by

$$\widehat{P_\nu^\epsilon f}(\lambda) = \begin{cases} \widehat{f}(\lambda)P_\nu, & \text{if } \lambda \in \Lambda_\epsilon, \\ 0, & \text{if } \lambda \notin \Lambda_\epsilon, \end{cases} \tag{3.16}$$

where P_ν denotes the orthogonal projection from $\mathscr{H}_\lambda$ to the one dimensional subspace $\mathscr{H}_{\lambda,\nu}$ spanned by $\Phi_{\nu,\lambda}$. It is clear that P_ν^ϵ is an orthogonal projection. Let H_ν^ϵ denote the range of P_ν^ϵ, i.e.,

$$H_\nu^\epsilon = \{f \in L^2(N) : \qquad \widehat{f}(\lambda) = \widehat{f}(\lambda)P_\nu \text{ and } \widehat{f}(\lambda) = 0 \text{ if } \lambda \notin \Lambda_\epsilon\}. \tag{3.17}$$

Then we have the following theorem.

Theorem 3.1. *Each H_ν^ϵ is an irreducible invariant closed subspace of $L^2(N)$ under the unitary representation π of P defined by* (3.1), *and we have the direct sum decomposition*

$$L^2(N) = \bigoplus_{\epsilon\in E,\nu\in\mathbb{Z}_+^n} H_\nu^\epsilon. \tag{3.18}$$

Proof. It is clear that the H_ν^ϵ's are mutually orthogonal closed subspaces of $L^2(N)$, and $L^2(N)$ is the direct sum of H_ν^ϵ's. We prove that H_ν^ϵ is invariant and irreducible. By (3.15), we have

$$(\pi(x,\zeta,t)\widehat{f})(\lambda) = \mathrm{Det}(t)^{\frac{m+n}{2m}} \pi^\lambda(x,\zeta)\widehat{f}(t^*\lambda).$$

Therefore H_ν^ϵ is invariant under the unitary representation π. Let A be a non-zero invariant closed subspace of H_ν^ϵ under π and $A^\perp$ the orthogonal complement of A in H_ν^ϵ. Take a function $g \in A$, not identically zero. Suppose

$f \in A^{\perp}$, then

$$\langle f, \pi(x,\zeta,t)g\rangle_{L^2(N)} = 0.$$

Since

$$\langle f, \pi(x,\zeta,t)g\rangle_{L^2(N)} = \mathrm{Det}(t)^{((m+n)/2m)} f * \widetilde{g}_t(x,\zeta),$$

by (3.12), (3.13) and (3.15),

$$\int_N \langle f, \pi(x,\zeta,t)g\rangle_{L^2(N)} \pi^{\lambda}(x,\zeta)\, dx\, d\zeta = \mathrm{Det}(t)^{((m+n)/2m)} \widehat{f}(\lambda)\widehat{g}(t^*\lambda)^*. \tag{3.19}$$

Thus for any $t \in T$, we have

$$\widehat{f}(\lambda)\widehat{g}(t^*\lambda)^* = 0, \quad a.e.\ \lambda \in \Lambda_\epsilon. \tag{3.20}$$

Suppose $\widehat{f}(\lambda) \neq 0$ for $\lambda \in \Gamma_1$, where $\Gamma_1 \subset \Lambda_\epsilon$ is a set of positive measure. Since g is not identically zero, there is a positive measure set $\Gamma_2 \subset \Lambda_\epsilon$ such that $\widehat{g}(\lambda) \neq 0$ for $\lambda \in \Gamma_2$. Let Γ_1^0 and Γ_2^0 consist of points of density of Γ_1 and Γ_2, respectively. Because the adjoint action of T on Λ_ϵ is transitive, there is $t_0 \in T$ such that $\Gamma = \Gamma_1^0 \cap (t_0^{*-1}\Gamma_2^0)$ has positive measure. Note that $\widehat{f}(\lambda) = \widehat{f}(\lambda)P_\nu$, $\widehat{g}(t^*\lambda)^* = P_\nu\widehat{g}(t^*\lambda)^*$. This implies that $\widehat{f}(\lambda)\widehat{g}(t_0^*\lambda)^* \neq 0$ for $\lambda \in \Gamma$, which contradicts (3.20). So we have $\widehat{f}(\lambda) = 0, a.e. \lambda \in \Lambda_\epsilon$. Therefore f is identically zero. This proves that H_ν^ϵ is irreducible. □

At the end of this section, we point out the following facts: the dual cone Ω^* of Ω is a simple T-orbit of U' under the adjoint action. P_0 is the orthogonal projection to the vacuum state. If $\Lambda_0 = \Omega^*$, then H_0^0 is exactly the Hardy space $H^2(N)$. All subspaces H_ν^ϵ can be explained in terms of the tangential Cauchy–Riemann operators (creation/annihilation operators) [24].

4. Admissible wavelets and wavelet transforms

Given $\nu \in \mathbb{Z}_+^n, \epsilon \in E$. The restriction of π on H_ν^ϵ is square integrable in the sense that there exists a function $\phi (\neq 0)$ in H_ν^ϵ such that

$$C_\phi = \frac{1}{\|\phi\|^2_{L^2(N)}} \int_P |\langle \phi, \pi(x,\zeta,t)\phi\rangle_{L^2(N)}|^2 \mathrm{Det}(t)^{-((m+n)/m)}\, dx\, d\zeta\, dm_l(t) < \infty. \tag{4.1}$$

(4.1) is called the admissibility condition, and ϕ is called an admissible wavelet if ϕ satisfies (4.1). Now we are going to give the characterization of the admissibility condition in terms of the Fourier transform.

We denote by AW_ν^ϵ the set of all admissible wavelets in H_ν^ϵ, i.e.,

$$AW_\nu^\epsilon = \{\phi \in H_\nu^\epsilon : \phi \text{satisfies} (4.1)\}. \tag{4.2}$$

Then we have the following theorem.

Theorem 4.1. *Suppose $\phi (\neq 0)$ in H_ν^ϵ. Then $\phi \in AW_\nu^\epsilon$ if and only if*

$$C_\phi = \int_T \|\widehat{\phi}(t^*\lambda_\epsilon)\|_{\mathrm{HS}}^2 \, dm_l(t) < \infty. \tag{4.3}$$

Proof. Suppose $\phi \in H_\nu^\epsilon$. Using (3.19) and the Plancherel formula, we get

$$\begin{aligned}
&\int_P |\langle \phi, \pi(x,\zeta,t)\phi\rangle_{L^2(N)}|^2 \mathrm{Det}(t)^{-((m+n)/m)} \, dx \, d\zeta \, dm_l(t) \\
&\quad = \int_T \left(\int_\Lambda \|\widehat{\phi}(\lambda)\widehat{\phi}(t^*\lambda)^*\|_{\mathrm{HS}}^2 \rho(\lambda) \, d\lambda \right) dm_l(t) \\
&\quad = \int_T \left(\int_\Lambda \mathrm{tr}(\widehat{\phi}(\lambda)^*\widehat{\phi}(\lambda)\widehat{\phi}(t^*\lambda)^*\widehat{\phi}(t^*\lambda))\rho(\lambda) \, d\lambda \right) dm_l(t).
\end{aligned}$$

Note that $\widehat{\phi}(\lambda)^*\widehat{\phi}(\lambda) = h(\lambda)P_\nu$, where $h(\lambda) = \|\widehat{\phi}(\lambda)\|_{\mathrm{HS}}^2$, and $h(\lambda) = 0$ if $\lambda \notin \lambda_\epsilon$. Hence

$$\begin{aligned}
&\int_P |\langle \phi, \pi(x,\zeta,t)\phi\rangle_{L^2(N)}|^2 \mathrm{Det}(t)^{-((m+n)/m)} \, dx \, d\zeta \, dm_l(t) \\
&\quad = \int_T \left(\int_\Lambda h(\lambda)h(t^*\lambda)\rho(\lambda) \, d\lambda \right) dm_l(t) \\
&\quad = \int_\Lambda \left(\int_T h(t^*\lambda) \, dm_l(t) \right) h(\lambda)\rho(\lambda) \, d\lambda \\
&\quad = \left(\int_T \|\widehat{\phi}(t^*\lambda_\epsilon)\|_{\mathrm{HS}}^2 \, dm_l(t) \right) \left(\int_\Lambda \|\widehat{\phi}(\lambda)\|_{\mathrm{HS}}^2 \rho(\lambda) \, d\lambda \right) \\
&\quad = \left(\int_T \|\widehat{\phi}(t^*\lambda_\epsilon)\|_{\mathrm{HS}}^2 \, dm_l(t) \right) \|\phi\|_{L^2(N)}^2,
\end{aligned}$$

where in the third equality we have used the left invariance of $dm_l(t)$. □

Let $\phi \in AW_\nu^\epsilon$, $f \in H_\nu^\epsilon$. We define the wavelet transform of f with respect to ϕ by

$$W_\phi f(x,\zeta,t) = \langle f, \pi(x,\zeta,t)\phi\rangle_{L^2(N)}. \tag{4.4}$$

Let $\phi, \psi \in AW_\nu^\epsilon$. We define the "inner product" of ϕ and ψ on AW_ν^ϵ by

$$\langle \phi, \psi \rangle_{AW} = \int_T \operatorname{tr}(\widehat{\psi}(t^*\lambda_\epsilon)^* \phi(t^*\lambda_\epsilon))\, dm_l(t). \tag{4.5}$$

From the theory of square-integrable representation of non-unimodular groups, which is due to Duflo and Moore [6], we have the following consequences of Theorem 4.1, which can also be proved directly.

Theorem 4.2. *Let $\phi, \psi \in AW_\nu^\epsilon, f, g \in H_\nu^\epsilon$. Then we have*

$$\langle W_\phi f, W_\psi g \rangle_{L^2(P)} = \langle \psi, \phi \rangle_{AW} \langle f, g \rangle_{L^2(N)}. \tag{4.6}$$

In particular,

$$\|W_\phi f\|_{L^2(P)} = C_\phi^{(1/2)} \|f\|_{L^2(N)}. \tag{4.7}$$

Let $\mathscr{S}(N)$ denote the Schwartz space on N, then we have the following reproducing formula.

Theorem 4.3. *Let $\phi \in AW_\nu^\epsilon, f \in H_\nu^\epsilon$. Then we have the following reproducing formula in the weak sense:*

$$\begin{aligned} f(x,\zeta) = C_\phi^{-1} \int_P & W_\phi f(a,\alpha,t)(\pi(a,\alpha,t)\phi)(x,\zeta) \\ & \operatorname{Det}(t)^{-((m+n)/m)}\, da\, d\alpha\, dm_l(t). \end{aligned} \tag{4.8}$$

Specially, if $\phi \in AW_\nu^\epsilon \cap \mathscr{S}(N), f \in H_\nu^\epsilon \cap \mathscr{S}(N)$, then the above formula (4.8) *holds for all $(x,\zeta) \in N$.*

5. The orthogonal decomposition of $L^2(D(\Omega, \Phi))$

Let $\{\psi_\delta : \delta \in \Delta\}$ be an orthonormal basis of $L^2(T, dm_l(t))$. We define the functions $\phi_{\mu,\nu,\delta}^\epsilon$ in terms of the Fourier transform by

$$\widehat{\phi}_{\mu,\nu,\delta}^\epsilon(\lambda) = \begin{cases} \psi_\delta(t) P_{\mu,\nu}, & \text{if } \lambda = t^*\lambda_\epsilon \in \Lambda_\epsilon, \\ 0, & \text{if } \lambda \notin \Lambda_\epsilon, \end{cases} \tag{5.1}$$

where $P_{\mu,\nu}$ is the partial isometric operator on $\mathscr{H}_\lambda$ defined by

$$P_{\mu,\nu} M = \langle M, \Phi_{\mu,\lambda} \rangle_{\mathscr{H}_\lambda} \Phi_{\nu,\lambda}, \quad M \in \mathscr{H}_\lambda.$$

It is easy to see that $\phi^{\epsilon}_{\mu,\nu,\delta} \in AW^{\epsilon}_{\mu}$ and $\{\phi^{\epsilon}_{\mu,\nu,\delta} : \epsilon \in E, \mu, \nu \in \mathbb{Z}^n_+, \delta \in \Delta\}$ is an orthonormal and complete set with respect to $\langle , \rangle_{AW}$. Set

$$A^{\epsilon}_{\mu,\nu,\delta} = \{W_{\phi^{\epsilon}_{\mu,\nu,\delta}} f : f \in H^{\epsilon}_{\mu}\}. \tag{5.2}$$

By Theorem 4.2, $A^{\epsilon}_{\mu,\nu,\delta} \subset L^2(P)$. We note that

$$A^{\epsilon}_{\mu,\nu,\delta} = A^{\epsilon}_{0,\nu,\delta}, \quad \mu \in \mathbb{Z}^n_+, \tag{5.3}$$

because the restrictions of π on H^{ϵ}_{μ} are equivalent for different μ. We write $A^{\epsilon}_{\nu,\delta}$ instead of $A^{\epsilon}_{\mu,\nu,\delta}$.

Theorem 5.1.

$$L^2(P) = \bigoplus_{\epsilon \in E, \nu \in \mathbb{Z}^n_+, \delta \in \Delta} A^{\epsilon}_{\nu,\delta}. \tag{5.4}$$

Proof. It is easy to see that all subspaces $A^{\epsilon}_{\nu,\delta}$'s are mutually orthogonal. Let $F(x,\zeta,t) \in L^2(P)$. Set $F_t(x,\zeta) = F(x,\zeta,t)$. For almost everywhere $t \in T$, $F_t(x,\zeta) \in L^2(N)$. By the Plancherel formula, we obtain

$$\begin{aligned}
\|F\|^2_{L^2(P)} &= \int_T \left(\int_N |F_t(x,\zeta)|^2 \, dx \, d\zeta \right) \operatorname{Det}(t)^{-((m+n)/m)} \, dm_l(t) \\
&= \int_T \left(\int_\Lambda \|\widehat{F}_t(\lambda)\|^2_{\mathrm{HS}} \rho(\lambda) \, d\lambda \right) \operatorname{Det}(t)^{-((m+n)/m)} \, dm_l(t) \\
&= \int_T \left(\int_\Lambda \sum_{\nu,\mu \in \mathbb{Z}^n_+} |\langle \widehat{F}_t(\lambda)\Phi_{\nu,\lambda}, \Phi_{\mu,\lambda}\rangle_{\mathscr{H}_\lambda}|^2 \rho(\lambda) \, d\lambda \right) \\
&\qquad \operatorname{Det}(t)^{-((m+n)/m)} \, dm_l(t) < \infty.
\end{aligned}$$

Therefore $\operatorname{Det}(t)^{-((m+n)/2m)} \langle \widehat{F}_t(\lambda)\Phi_{\nu,\lambda}, \Phi_{\mu,\lambda}\rangle_{\mathscr{H}_\lambda} \in L^2(T, dm_l(t))$ as the function of the variable t for $\nu, \mu \in \mathbb{Z}^n_+, \lambda \in \Lambda$ almost everywhere. Suppose $\lambda = t^*_\lambda \lambda_\epsilon \in \Lambda_\epsilon, t_\lambda \in T$. Set

$$\psi^\lambda_\delta(t) = \psi_\delta(t_\lambda t).$$

Obviously $\{\psi^\lambda_\delta : \delta \in \Delta\}$ is also an orthonormal basis of $L^2(T, dm_l(t))$. Therefore we have

$$\operatorname{Det}(t)^{-((m+n)/2m)} \langle \widehat{F}_t(\lambda)\Phi_{\nu,\lambda}, \Phi_{\mu,\lambda}\rangle_{\mathscr{H}_\lambda} = \sum_{\delta \in \Delta} b_\delta(\lambda, \nu, \mu) \psi^\lambda_\delta(t),$$

i.e.,

$$\mathrm{Det}(t)^{-((m+n)/2m)}\widehat{F}_t(\lambda) = \sum_{\nu,\mu\in\mathbb{Z}_+^n,\delta\in\Delta} b_\delta(\lambda,\nu,\mu)\psi_\delta^\lambda(t)P_{\nu,\mu}.$$

We define the functions $f_{\nu,\delta}^\epsilon$ in terms of the Fourier transform by

$$\widehat{f}_{\nu,\delta}^\epsilon(\lambda) = \begin{cases} \sum_{\mu\in\mathbb{Z}_+^n} b_\delta(\lambda,\nu,\mu)P_{0,\mu}, & \text{if } \lambda\in\Lambda_\epsilon, \\ 0, & \text{if } \lambda\notin\Lambda_\epsilon. \end{cases}$$

It is clear that $f_{\nu,\delta}^\epsilon \in H_0^\epsilon, \epsilon\in E$. And we have

$$\begin{aligned} &\int_N \sum_{\epsilon\in E,\nu\in\mathbb{Z}_+^n,\delta\in\Delta} (W_{\phi_{0,\nu,\delta}^\epsilon} f_{\nu,\delta}^\epsilon(x,\zeta,t))\pi^\lambda(x,\zeta)\,dx\,d\zeta \\ &\quad= \mathrm{Det}(t)^{((m+n)/2m)} \sum_{\epsilon\in E,\nu\in\mathbb{Z}_+^n,\delta\in\Delta} \widehat{f}_{\nu,\delta}^\epsilon(\lambda)\widehat{\phi}_{0,\nu,\delta}^\epsilon(t^*\lambda)^* \\ &\quad= \mathrm{Det}(t)^{((m+n)/2m)} \sum_{\epsilon\in E,\nu\in\mathbb{Z}_+^n,\delta\in\Delta} b_\delta(\lambda,\nu,\mu)\psi_\delta^\lambda(t)P_{\nu,\mu} \\ &\quad= \widehat{F}_t(\lambda). \end{aligned}$$

Therefore we obtain

$$F(x,\zeta,t) = \sum_{\epsilon\in E,\nu\in\mathbb{Z}_+^n,\delta\in\Delta} W_{\phi_{0,\nu,\delta}^\epsilon} f_{\nu,\delta}^\epsilon(x,\zeta,t).$$

The proof of Theorem 5.1 is completed. □

Let M be the characteristic function of Ω defined by

$$M(x) = \int_{\Omega^*} \mathrm{e}^{-2\pi\langle\lambda,x\rangle}\,d\lambda. \tag{5.5}$$

Similarly, the characteristic function M^* of Ω^* is defined by

$$M^*(\lambda) = \int_{\Omega} \mathrm{e}^{-2\pi\langle\lambda,x\rangle}\,dx. \tag{5.6}$$

Then we have

$$\begin{aligned} M(gx) &= \mathrm{Det}(g)^{-1}M(x), \\ M^*(g^*\lambda) &= \mathrm{Det}(g)^{-1}M^*(\lambda), \end{aligned}$$

for $g \in \mathrm{Aut}(\Omega)$. Suppose $y = te, t \in T$. Because $M(x)\,dx$ is an $\mathrm{Aut}(\Omega)$-invariant measure, there exists a constant C such that $dm_l(t) = CM(y)\,dy$. Without loss of generality, we may assume that $C = M(e)^{(m+n)/m}$.

We have identified P with D according to the bijection

$$(x, \zeta, t) \mapsto (x + i(y + \Phi(\zeta, \zeta)), \zeta),$$

where $y = te$. Then we have

$$\mathrm{Det}(t)^{-((m+n)/m)}\,dx\,d\zeta\,dm_l(t) = M(y)^{((2m+n)/m)}\,dx\,dy\,d\zeta, \tag{5.7}$$

which turns out to be the $G_a(D)$-invariant measure on D. We can regard $L^2(P)$ as $L^2(D, M(y)^{((2m+n)/m)}\,dx\,dy\,d\zeta)$, the space of square integrable functions on D with respect to $G_a(D)$-invariant measure $M(y)^{((2m+n)/m)}$ $dx\,dy\,d\zeta$. Then Theorem 5.1 gives the direct sum decomposition of $L^2(D, M$ $(y)^{((2m+n)/m)}dx\,dy\,d\zeta)$.

Suppose $(z, \zeta) \in D$. Set $x = \mathrm{Re}\, z, y = \mathrm{Im}\, z - \Phi(\zeta, \zeta)$. Then $dz\,d\zeta = dx\,dy\,d\zeta$. Let

$$L^2(D) = \left\{ F(z, \zeta) : \|F\|_{L^2(D)} = \left(\int_D |F(z, \zeta)|^2\,dz\,d\zeta \right)^{(1/2)} < \infty \right\}. \tag{5.8}$$

The Bergman space $A(D)$ is the subspace of all holomorphic functions in $L^2(D)$.

Let $\phi \in AW_0^\epsilon, f \in H_0^\epsilon$. We define the revised wavelet transform $\widetilde{W}_\phi$ of f with respect to ϕ by

$$\widetilde{W}_\phi f(x + i(te + \Phi(\zeta, \zeta)), \zeta) = C_\phi^{-(1/2)} M(e)^{((2m+n)/2m)} \mathrm{Det}(t)^{-((2m+n)/2m)} W_\phi f(x, \zeta, t).$$

Set

$$\widetilde{A}^\epsilon_{\nu,\delta} = \{\widetilde{W}_{\phi^\epsilon_{0,\nu,\delta}} f : f \in H_0^\epsilon\}. \tag{5.9}$$

The direct consequence of Theorem 5.1 is

Theorem 5.2.

$$L^2(D) = \bigoplus_{\epsilon \in E, \nu \in \mathbb{Z}_+^n, \delta \in \Delta} \widetilde{A}^\epsilon_{\nu,\delta}. \tag{5.10}$$

It is easy to see that the reproducing kernel of $\widetilde{A}^{\epsilon}_{\nu,\delta}$ is given by

$$\begin{aligned}
&K^{\epsilon}_{\nu,\delta}((x+i(te+\Phi(\zeta,\zeta)),\zeta),(x_1+i(t_1e+\Phi(\zeta_1,\zeta_1)),\zeta_1))\\
&\quad = M(\mathrm{e})^{((2m+n)/m)}\mathrm{Det}(tt_1)^{-((2m+n)/2m)}\\
&\qquad \int_N \pi(x_1,\zeta_1,t_1)\phi^{\epsilon}_{0,\nu,\delta}(a,\alpha)\overline{\pi(x,\zeta,t)\phi^{\epsilon}_{0,\nu,\delta}(a,\alpha)}\,da\,d\alpha\\
&\quad = M(\mathrm{e})^{((2m+n)/m)}\mathrm{Det}(tt_1)^{-(1/2)}\int_{\Lambda_\epsilon}\mathrm{tr}(\widehat{\phi}^{\epsilon}_{0,\nu,\delta}(t^*\lambda)^*\\
&\qquad \pi^{\lambda}(x,\zeta)^*\pi^{\lambda}(x_1,\zeta_1)\widehat{\phi}^{\epsilon}_{0,\nu,\delta}(t_1^*\lambda))\rho(\lambda)\,d\lambda.
\end{aligned}$$

We select the orthonormal basis $\{\psi_\delta : \delta\in\Delta\}$ of $L^2(T,dm_l(t))$ such that $\psi_0\in\{\psi_\delta:\delta\in\Delta\}$, where ψ_0 is defined by

$$\psi_0(t)=M(\mathrm{e})^{-((2m+n)/2m)}\mathrm{e}^{-2\pi\langle\lambda_0,te\rangle}M^*(2t^*\lambda_0)^{-(1/2)},$$

then

$$\widehat{\phi}^{0}_{0,0,0}(\lambda)=\begin{cases} M(\mathrm{e})^{-((2m+n)/2m)}\mathrm{e}^{-2\pi\langle\lambda,e\rangle}M^*(2\lambda)^{-(1/2)}P_{0,0}, & \text{if } \lambda\in\Omega^*,\\ 0, & \text{if } \lambda\notin\Omega^*.\end{cases}$$

The reproducing kernel of $\widetilde{A}^{0}_{0,0}$ is given by

$$\begin{aligned}
&K^{0}_{0,0}((x+i(te+\Phi(\zeta,\zeta)),\zeta),(x_1+i(t_1e+\Phi(\zeta_1,\zeta_1)),\zeta_1))\\
&\quad = M(\mathrm{e})^{((2m+n)/m)}\mathrm{Det}(tt_1)^{-(1/2)}\int_{\Omega^*}\mathrm{tr}(\widehat{\phi}^{0}_{0,0,0}(t^*\lambda)^*\\
&\qquad \pi^{\lambda}(x,\zeta)^*\pi^{\lambda}(x_1,\zeta_1)\widehat{\phi}^{0}_{0,0,0}(t_1^*\lambda))\rho(\lambda)\,d\lambda\\
&\quad = \int_{\Omega^*}\mathrm{e}^{-2\pi\langle\lambda,te+t_1e\rangle}M^*(2\lambda)^{-1}\\
&\qquad \langle\pi^{\lambda}(x_1,\zeta_1)\Phi_{0,\lambda},\pi^{\lambda}(x,\zeta)\Phi_{0,\lambda}\rangle_{\mathscr{H}_\lambda}\rho(\lambda)\,d\lambda. \qquad (5.11)
\end{aligned}$$

It is easy to compute that

$$\langle\pi^{\lambda}(x_1,\zeta_1)\Phi_{0,\lambda},\pi^{\lambda}(x,\zeta)\Phi_{0,\lambda}\rangle_{\mathscr{H}_\lambda}=\mathrm{e}^{-2\pi\langle\lambda,i(x_1-x)+\Phi(\zeta_1,\zeta_1)+\Phi(\zeta,\zeta)-2\Phi(\zeta,\zeta_1)\rangle}.$$

Therefore

$$K^{0}_{0,0}((z,\zeta),(z_1,\zeta_1))=\int_{\Omega^*}\mathrm{e}^{-2\pi\langle\lambda,i(\overline{z_1}-z)-2\Phi(\zeta,\zeta_1)\rangle}M^*(2\lambda)^{-1}\rho(\lambda)\,d\lambda, \qquad (5.12)$$

which is exactly the Bergman kernel [19]. So $A^{0}_{0,0}$ is nothing but the Bergman space $A(D(\Phi,\Omega))$.

6. Radon transform on N

The Radon transform on the Heisenberg group H_n was defined by Strichartz [29]. In the same way, we define the Radon transform R on N by

$$
\begin{aligned}
R(f)(x,\zeta) &= \int_V f((x,\zeta)(0,w))\,dw \\
&= \int_V f(x + 2\operatorname{Im}\Phi(\zeta,w), \zeta + w)\,dw.
\end{aligned}
\tag{6.1}
$$

We introduce the partial Fourier transforms $\mathscr{F}_\lambda$ and $\mathscr{F}$. For λ in Λ, $\mathscr{F}_\lambda$ is used to represent the Fourier transform on $\mathscr{H}_\lambda$ defined by

$$
\mathscr{F}_\lambda(\phi)(\xi) = \int_{E_\lambda} \phi(\sigma)\exp(-2\pi i\sigma\cdot\xi)\vartheta_\lambda(d\sigma), \quad \phi \in \mathscr{H}_\lambda. \tag{6.2}
$$

And $\mathscr{F}(f)$ is the Fourier transform of f with respect to $x \in U$ alone, i.e.,

$$
\mathscr{F}(f)(\lambda,\zeta) = \int_U f(x,\zeta)\exp(-2\pi i\langle\lambda,x\rangle)\,dx. \tag{6.3}
$$

By (3.9) we have

$$
\begin{aligned}
(\widehat{R(f)}(\lambda)\Phi_{\nu,\lambda})(\tau) = \int_{U\times V} & R(f)(x,\zeta)\times\exp(-2\pi i\langle\lambda,x\rangle) \\
& \exp(\pi i\xi\cdot\eta)\exp(-2\pi i\eta\cdot\tau)\Phi_{\nu,\lambda}(\tau-\xi)dxd\zeta,
\end{aligned}
$$

where $\zeta = \xi + J_\lambda\eta, \xi,\eta \in E_\lambda$. Let $w = \mu + J_\lambda\gamma, \mu,\gamma \in E_\lambda$. It follows from (3.2) and (3.3) that

$$
\begin{aligned}
2\pi i\langle\lambda, 2\operatorname{Im}\Phi(\zeta,w)\rangle &= \pi i\operatorname{Im}H_\lambda(\zeta,w) \\
&= \pi i B_\lambda(\zeta,w) \\
&= -\pi i(\gamma\cdot\xi - \eta\cdot\mu).
\end{aligned}
$$

Notice that the recursion formula for Hermite polynomials [28]

$$
\Phi_{\nu,\lambda}(-\eta) = (-1)^{|\nu|}\Phi_{\nu,\lambda}(\eta) \tag{6.4}
$$

where $|\nu| = \sum_j^n \nu_j$, by (3.6), we get

$$
\begin{aligned}
(\widehat{R(f)}(\lambda)\Phi_{\nu,\lambda})(\tau) = & \int_{UV} \left(\int_V f(x + 2\,\mathrm{Im}\,\Phi(\zeta, w), \zeta + w)\, dw \right) \\
& \exp(-2\pi i \langle \lambda, x\rangle) \exp(\pi i \xi \cdot \eta) \\
& \exp(-2\pi i \eta \cdot \tau) \Phi_{\nu,\lambda}(\tau - \xi)\, dx\, d\zeta \\
= & \, 4^{-n} |\mathrm{Det}\, M_\lambda(\beta)|^{-1} \int_V \mathscr{F}(f)(\lambda, w) \\
& \left(\int_{E_\lambda E_\lambda} \exp(-\pi i(\gamma \cdot \xi - \eta \cdot \mu)) \exp(\pi i \xi \cdot \eta) \right. \\
& \left. \exp(-2\pi i \eta \cdot \tau) \Phi_{\nu,\lambda}(\tau - \xi) \vartheta_\lambda\,(d\xi) \vartheta_\lambda\,(d\eta) \right) dw \\
= & \, 4^{-n} |\mathrm{Det} M_\lambda(\beta)|^{-1} \int_V \mathscr{F}(f)(\lambda, w) \exp(\pi i \mu \cdot \gamma) \\
& \exp(-2\pi i \gamma \cdot \tau) \left(\int_{E_\lambda} \mathscr{F}_\lambda(\Phi_{\nu,\lambda}) \left(\frac{\eta - \gamma}{2} \right) \right. \\
& \left. \exp(\pi i (\mu - \tau)(\eta - \gamma)) \vartheta_\lambda\,(d\eta) \right) dw \\
= & \, 2^{-n} |\mathrm{Det} M_\lambda(\beta)|^{-1} \int_V \mathscr{F}(f)(\lambda, w) \exp(\pi i \mu \cdot \gamma) \\
& \exp(-2\pi i \gamma \cdot \tau) \Phi_{\nu,\lambda}(\mu - \tau)\, dw \\
= & \, (-1)^{|\nu|} 2^{-n} |\mathrm{Det} M_\lambda(\beta)|^{-1} (\widehat{f}(\lambda)\Phi_{\nu,\lambda})(\tau).
\end{aligned}
$$

Thus we have

Theorem 6.1. *Let $f \in L^2(N)$. Then*

$$
\widehat{R(f)}(\lambda) = 2^{-n} |\mathrm{Det}\, M_\lambda(\beta)^{-1}| \widehat{f}(\lambda) S, \tag{6.5}
$$

where $S = \sum_{\nu \in \mathbb{Z}_+^n} (-1)^{|\nu|} P_\nu$ is a unitary operator on $\mathscr{H}_\lambda$.

It is known that $\mathrm{Det}\, M_\lambda(\beta)$ is a real homogeneous polynomial of degree n on $U' \cong R^m$ [24, p. 35]. We put $P(\lambda) = \mathrm{Det} M_\lambda(\beta)$.

Let $x = (x_1, x_2, \ldots, x_m)$, $f \in \mathscr{S}(N)$, it is obvious that

$$
\int_U \frac{\partial f}{\partial x_j}(x, \zeta) \exp(-2\pi i \langle \lambda, x\rangle)\, dx = (2\pi i \lambda_j) \mathscr{F}(f)(\lambda, \zeta). \tag{6.6}
$$

Write $D = (\frac{1}{i}\frac{\partial}{\partial x_1}, \frac{1}{i}\frac{\partial}{\partial x_2}, \ldots, \frac{1}{i}\frac{\partial}{\partial x_m}), L = P(D)$. Such type operators has been considered by Gindikin [9]. It is not difficult to see that the partial differential operator L satisfies

$$\int_U L(f)(x,\zeta)\exp(-2\pi i\langle\lambda, x\rangle)\,dx = (2\pi)^n \operatorname{Det} M_\lambda(\beta)\mathscr{F}(f)(\lambda,\zeta). \tag{6.7}$$

Therefore we have

$$\widehat{L(f)}(\lambda) = (2\pi)^n \operatorname{Det} M_\lambda(\beta)\widehat{f}(\lambda). \tag{6.8}$$

It follows that

$$\widehat{LR(f)}(\lambda) = \widehat{RL(f)}(\lambda) = \pi^n \operatorname{sgn}(\operatorname{Det} M_\lambda(\beta))\widehat{f}(\lambda)S, \tag{6.9}$$

where $\operatorname{sgn}(\cdot)$ denotes the symbol function. Then we obtain the following inversion formula of the Radon transform on N.

$$R^{-1} = \pi^{-2n} LRL. \tag{6.10}$$

This is an extension of that on the Heisenberg group [29].

For a function $f \in L^2(N)$, $R(f)$ may not belong to $L^2(N)$ [29]. We should find a dense subspace of $L^2(N)$, on which the formula (6.10) holds.

Theorem 6.2. *Let*

$$\mathscr{S}_R(N) = \Big\{ f \in \mathscr{S}(N) : \int_\Lambda \|\widehat{f}(\lambda)\|^2_{\mathrm{HS}} |\operatorname{Det} M_\lambda(\beta)|^{(2j+1)}\,d\lambda < +\infty,\ \textit{for all } j \in \mathbb{Z} \Big\}.$$

Then $\mathscr{S}_R(N)$ is a dense subspace of $L^2(N)$ and

$$R^{-1}(f) = \pi^{-2n} LRL(f) \tag{6.11}$$

for all $f \in \mathscr{S}_R(N)$.

Proof. Consider the subspace $L^2_R(N)$ of $L^2(N)$ defined by

$$L^2_R(N) = \Big\{ f \in L^2(N) : \int_\Lambda \|\widehat{f}(\lambda)\|^2_{\mathrm{HS}} |\operatorname{Det} M_\lambda(\beta)|^{(2j+1)}\,d\lambda < +\infty, \text{for all} j \in \mathbb{Z} \Big\}. \tag{6.12}$$

It is clear that $L^2_R(N)$ is a dense subspace of $L^2(N)$. By Theorem 6.1 and the Plancherel formula (3.10), we have

$$\|R^j(f)\|^2_{L^2(N)} = 4^{(-j+1)n} \int_\Lambda \|\widehat{f}(\lambda)\|^2_{\mathrm{HS}} |\mathrm{Det}\, M_\lambda(\beta)|^{(-2j+1)}\, d\lambda.$$

Thus R is a bijection from $L^2_R(N)$ onto itself. Because $\mathscr{S}_R(N) = L^2_R(N) \bigcap \mathscr{S}(N)$, LRL is well defined on $\mathscr{S}_R(N)$. Theorem 6.2 is proved. $\square$

7. An inversion formula by using wavelets

In this section, we establish an inversion formula of the Radon transform by use of the continuous wavelet, transform on N. By choosing suitable wavelets, the inversion formula of the Radon transform holds in the weak sense without the assumption of differentiability for f.

For simplicity, write $g_t(x,\zeta) = g(t^{-1}x, B(t)^{-1}\zeta)$ which deviates slightly from (3.15), we have

$$\widehat{g_t}(\lambda) = \mathrm{Det}(t)^{((n+m)/m)} \widehat{g}(t^*\lambda). \tag{7.1}$$

We define the operator $\widetilde{W}_g$ by

$$\widetilde{W}_g f = f * \widetilde{g}.$$

Then

$$\left(\widehat{\widetilde{W}_g f}\right)(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda)^*. \tag{7.2}$$

Note that

$$\mathrm{Det}\, M_{t^*\lambda}(\beta) = |\mathrm{Det}(B(t))|^2 \mathrm{Det}\, M_\lambda(\beta) = \mathrm{Det}(t)^{n/m} \mathrm{Det}\, M_\lambda(\beta). \tag{7.3}$$

Applying (7.1), (7.2) together with (6.8), we have

$$\begin{aligned}(\widehat{\widetilde{W}_{L(g)_t} f})(\lambda) &= \widehat{f}(\lambda)\widehat{L(g)_t}(\lambda)^* \\ &= \mathrm{Det}(t)^{((n+m)/m)} \widehat{f}(\lambda)\widehat{L(g)}(t^*\lambda)^* \\ &= (2\pi)^n\, \mathrm{Det}(t)^{((2n+m)/m)}\, \mathrm{Det}\, M_\lambda(\beta) \widehat{f}(\lambda)\widehat{g}(t^*\lambda)^*.\end{aligned} \tag{7.4}$$

On the other hand, by (7.2), (6.8) and (7.1) we get

$$
\begin{aligned}
(7.5)\qquad (\widehat{\widetilde{W}_{g_t}L(f)})(\lambda) &= \widehat{L(f)}(\lambda)\widehat{g_t}(\lambda)^* \\
&= (2\pi)^n \mathrm{Det} M_\lambda(\beta)\mathrm{Det}(t)^{((n+m)/m)}\widehat{f}(\lambda)\widehat{g}(t^*\lambda)^*.
\end{aligned}
$$

Consequently,

$$
(7.6)\qquad (\widetilde{W}_{L(g)_t}f)(x,\zeta) = \mathrm{Det}(t)^{n/m}(\widetilde{W}_{g_t}L(f))(x,\zeta).
$$

Theorem 7.1. *Let $g \in \mathscr{S}_R(N)\cap AW_\nu^\epsilon$, $f \in \mathscr{S}_R(N)\cap H_\nu^\epsilon$. Then*

$$
(7.7)\qquad (\widetilde{W}_{LRL(g)_t}R(f))(x,\zeta) = \pi^{2n}\mathrm{Det}(t)^{((m+3n)/2m)}(W_g f)(x,\zeta,t).
$$

Proof. By (7.6), we have

$$
(7.8)\qquad (\widetilde{W}_{LRL(g)_t}R(f))(x,\zeta) = \mathrm{Det}(t)^{n/m}(\widetilde{W}_{RL(g)_t}LR(f))(x,\zeta).
$$

Note that $\mathrm{sgn}(\mathrm{Det}\, M_{t^*\lambda}(\beta)) = \mathrm{sgn}(\mathrm{Det}\, M_\lambda(\beta))$. By taking the Fourier transform on both sides of (7.8), it follows that

$$
\begin{aligned}
(\widehat{\widetilde{W}_{LRL(g)_t}R(f)})(\lambda) &= \mathrm{Det}(t)^{n/m}(\widetilde{W}_{RL(g)_t}LR(\widehat{f}))(\lambda) \\
(7.9)\qquad &= \mathrm{Det}(t)^{n/m}\widehat{LR(f)}(\lambda)\widehat{RL(g)}_t(\lambda)^* \\
&= \mathrm{Det}(t)^{((m+2n)/m)}\widehat{LR(f)}(\lambda)\widehat{RL(g)}(t^*\lambda)^* \\
&= \pi^{2n}\mathrm{Det}(t)^{((m+2n)/m)}\widehat{f}(\lambda)\widehat{g}(t^*\lambda)^*.
\end{aligned}
$$

But by the wavelet transform

$$
\widehat{W_g f}(\lambda) = \mathrm{Det}(t)^{((m+n)/2m)}\widehat{f}(\lambda)\widehat{g}(t^*\lambda)^*,
$$

this completes the proof of (7.7). □

In Theorem 7.1, we make the assumption of differentiability for f, which is needed in (7.8). This assumption can be removed. In fact, we can deduce (7.7) from the following computation. According to (6.10) and (7.3), we have

$$
\begin{aligned}
(\widehat{\widetilde{W}_{LRL(g)_t}R(f)})(\lambda) &= \widehat{R(f)}(\lambda)\widehat{LRL(g)}_t(\lambda)^* \\
&= \mathrm{Det}(t)^{((m+n)/m)}\widehat{R(f)}(\lambda)\widehat{LRL(g)}(t^*\lambda)^* \\
&= \pi^{2n}\mathrm{Det}(t)^{((m+2n)/m)}\widehat{f}(\lambda)\widehat{g}(t^*\lambda)^*.
\end{aligned}
$$

We therefore have the following inverse Radon transform in the weak sense.

Theorem 7.2. *Let $\phi_\nu^\epsilon \in \mathscr{S}_R(N) \cap AW_\nu^\epsilon$, $f \in L_R^2(N) \cap H_\nu^\epsilon$. Then*

$$\begin{aligned} f(x,\zeta) = \frac{1}{\pi^{2n} C_{\phi_\nu^\epsilon}} \int_{N\times T} & \widetilde{W}_{LRL(\phi_\nu^\epsilon)_t} R(f)(a,\alpha) \\ & U(a,\alpha,t)\phi_\nu^\epsilon(x,\zeta) \frac{da\,d\alpha\,dm_l(t)}{\mathrm{Det}(t)^{((3m+5n)/2m)}} \end{aligned} \tag{7.10}$$

and

$$\begin{aligned} R^{-1}(f)(x,\zeta) = \frac{1}{\pi^{2n} C_{\phi_\nu^\epsilon}} \int_{N\times T} & \widetilde{W}_{LRL(\phi_\nu^\epsilon)_t} f(a,\alpha) \\ & U(a,\alpha,t)\phi_\nu^\epsilon(x,\zeta) \frac{da\,d\alpha\,dm_l(t)}{\mathrm{Det}(t)^{((3m+5n)/2m)}}. \end{aligned} \tag{7.11}$$

in the weak sense. Specifically, if $f \in \mathscr{S}_R(N) \cap H_\nu^\epsilon$, then the formula (7.10), (7.11) *hold for all $(x,\zeta) \in N$.*

It is easy to construct some wavelets which satisfy the condition in Theorem 7.2. Let $\psi \in C_0^\infty(T)$ satisfying $\mathrm{Det}(t) \geq \eta > 0$ for $t \in \operatorname{supp}\psi$. $\phi_{\nu,\mu,\delta}^\epsilon$ is defined by

$$\widehat{\phi}_{\nu,\mu,\delta}^\epsilon(\lambda) = \begin{cases} \psi(t)P_{\nu,\mu}, & \text{if } \lambda = t^*\lambda_\epsilon \in \Lambda_\epsilon, \\ 0, & \text{if } \lambda \notin \Lambda_\epsilon, \end{cases} \tag{7.12}$$

then we have $\phi_{\nu,\mu,\delta}^\epsilon \in \mathscr{S}_R(N) \cap AW_\nu^\epsilon$.

If $f \in \mathscr{S}_R(N)$ (or $f \in L_R^2(N)$), then f can be decomposed as $f = \sum_{\epsilon,\nu} f_\nu^\epsilon$ where $f_\nu^\epsilon \in \mathscr{S}_R(N) \cap H_\nu^\epsilon$ (or $f_\nu^\epsilon \in L_R^2(N) \cap H_\nu^\epsilon$). Take $\phi_\nu^\epsilon \in \mathscr{S}_R(N) \cap AW_\nu^\epsilon$. It follows from (7.11) that

$$\begin{aligned} R^{-1}(f)(x,\zeta) = \sum_{\epsilon,\nu} \frac{1}{\pi^{2n} C_{\phi_\nu^\epsilon}} \int_{N\times T} & \widetilde{W}_{LRL(\phi_\nu^\epsilon)_t} f_\nu^\epsilon(a,\alpha) \\ & U(a,\alpha,t)\phi_\nu^\epsilon(x,\zeta) \frac{da\,d\alpha\,dm_l(t)}{\mathrm{Det}(t)^{((3m+5n)/2m)}} \end{aligned} \tag{7.13}$$

holds for all $(x,\zeta) \in N$ (or in the weak sense).

8. The symmetric case

Of course, the most interested case is the symmetric Siegel domains of type II, which (together with the symmetric tube domains) are the unbounded realizations of the bounded symmetric domains. In this case, our results can be expressed in a more explicit form in terms of Jordan algebra. We shall only state the results without proof and refer the reader to [20] for some detailed calculation about the symmetric cones. A good reference on the Jordan algebras and the symmetric cones is the book [7] by J. Faraut and Korányi.

Let U be a simple Euclidean Jordan algebra with the identity e. Ω is the associated symmetric cone. Then $D = D(\Omega, \Phi)$ determined by the symmetric cone Ω and an Ω-positive Hermitian map Φ is a symmetric Siegel domain of type II. Suppose that U has the dimension m, the rank r and the degree d. $x \circ y$ denotes the Jordan product of x and y. $\mathrm{tr}(x)$ and $\det(x)$ are defined as in [7]. We also write $\Delta(x)$ instead of $\det(x)$. The inner product on U is given by $\langle x, y\rangle = \mathrm{tr}(x \circ y)$. We select K to be the isotropy subgroup of $G_a(D)$ at the point $(ie, 0)$. Because two maximal $\mathbb{R}$-triangular subgroups of a linear Lie group G are conjugate with respect to an inner automorphism of G, there exist a Jordan frame $\{c_1, \ldots, c_r\}$ and the corresponding Peirce decomposition

$$U = \bigoplus_{j \le k} U_{jk}$$

such that T has the parameterization as

$$T = \{t(u) : u \in U_+\},$$

where

$$U_+ = \left\{ u = \sum_{j=1}^{r} u_j c_j + \sum_{j<k} u_{jk} : u_j > 0, u_{jk} \in U_{jk} \right\}.$$

The left Haar measure of T is given by

$$dm_l(t(u)) = 2^r \prod_{j=1}^{r} u_j^{-d(j-1)-1}\, du. \tag{8.1}$$

We identify U' with U by the use of the inner product. Then

$$\rho(\lambda) = 4^n \Delta(\lambda)^{n/r}. \tag{8.2}$$

Let

$$E = \{\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_r) : \epsilon_j = \pm 1\}.$$

Set

$$\lambda_\epsilon = \sum_{j=1}^{r} \epsilon_j c_j.$$

All simple T-orbits of U under the adjoint action are given by

$$\Lambda_\epsilon = \{\lambda \in U : \lambda = t(u)^*\lambda_\epsilon, u \in U_+\}, \quad \epsilon \in E.$$

Specifically, $\Lambda_e = \Lambda_{(1,1,\ldots,1)} = \Omega$.

We can identify Ω with T by identification of $x = t(u)e$ and $t(u)$. Then we have

$$\Delta(x)^{-\frac{m}{r}}\, dx = d\mu_l(t(u)) \tag{8.3}$$

and

$$\mathrm{Det}(t(u)) = \Delta(x)^{(m/r)}. \tag{8.4}$$

The characteristic function $M(x)$ of Ω is given by

$$M(x) = \Gamma_\Omega\left(\frac{m}{r}\right)(2\pi)^{-m}\Delta(x)^{-(m/r)}, \tag{8.5}$$

where Γ_Ω is the gamma function of the symmetric cone Ω. Specifically,

$$M(\mathrm{e}) = \Gamma_\Omega\left(\frac{m}{r}\right)(2\pi)^{-m}. \tag{8.6}$$

Let $\Delta_j(x), j = 1, \ldots, r$, denote the principal minors. $\Delta_j^*(x) = \Delta_j(kx)$ where k is an automorphism of U such that

$$kc_j = c_{r-j+1}, \quad j = 1, \ldots, r.$$

Let $\mathbf{s} = (s_1, \ldots, s_r)$. We set

$$\Delta_{\mathbf{s}}^*(x) = \Delta_1^*(x)^{s_1-s_2} \cdots \Delta_{r-1}^*(x)^{s_{r-1}-s_r}\Delta_r^*(x)^{s_r}.$$

For the transformation $\lambda = t(u)^*\lambda_\epsilon$, we have

$$dm_l(t(u)) = |\Delta_{\underline{\mathbf{s}}}^*(\lambda)|^{-1} d\lambda, \tag{8.7}$$

where

$$\underline{\mathbf{s}} = (1 + d(r-1), 1 + d(r-2), \ldots, 1).$$

Because

$$\mathrm{Det}\, M_\lambda(\beta) = \Delta(\lambda)^{n/r},$$

we have

$$L = \Delta(D)^{n/r}. \tag{8.8}$$

Now all results in previous sections can be expressed in a more explicit form for the symmetric case. For example, Theorem 4.1 can be restated as follows.

Suppose $\phi(\neq 0)$ in H^{ϵ}_{ν}. Then $\phi \in AW^{\epsilon}_{\nu}$ if and only if

$$C_{\phi} = \int_{\Lambda_{\epsilon}} \|\widehat{\phi}(\lambda)\|^2_{\mathrm{HS}} |\Delta^*_{\underline{\mathbf{s}}}(\lambda)|^{-1}\, d\lambda < \infty. \tag{8.9}$$

A concrete example of the symmetric Siegel domains of type II is the unbounded realization of the classical domain of type one. Let $V = M_{s,r}$ be the set of all $s \times r$ complex matrices, $U = H_r$ denotes all Hermitian matrices, and write $W = M_r = M_{r,r}$. H_r is a simple Euclidean Jordan algebra with the Jordan product

$$x \circ y = \frac{1}{2}(xy + yx).$$

H_r has the dimension $m = r^2$, the rank r and the degree $d = 2$. The associated symmetric cone Ω consists of all complex Hermitian positive definite matrices in H_r. The trace function $\mathrm{tr}(x)$ and the determinant function $\det(x)$ are the usual ones. Set

$$\begin{aligned} \Phi : M_{s,r} \times M_{s,r} &\to M_r, \\ (\zeta, \eta) &\mapsto \Phi(\zeta, \eta) = \eta^* \zeta. \end{aligned}$$

Φ is an Ω-positive Hermitian map, and

$$D(\Omega, \Phi) = \left\{ (z, \zeta) \in M_r \times M_{s,r} : \quad \frac{z - z^*}{2i} - \zeta^* \zeta > 0 \right\}$$

is a symmetric Siegel domain of type II. This is the unbounded realization of the classical domain of type one. We select the Jordan frame $\{c_1, \ldots, c_r\}$ such that c_j is a diagonal matrix with 1 in the jth place and 0 in other positions. Then T consists of all $r \times r$ upper triangular matrices with positive diagonal elements. The action of T on H_r is given by $t(x) = txt^*$ and $\mathrm{Det}(t) = \det(t)^{2r}$. All results for this case are expressed in a more explicit form [12, 13]. For example, we have $L = (-i)^{sr}(\det(\frac{\partial}{\partial x_{jk}}))^s$ and

$$R^{-1}(f)(x,\zeta) = \frac{1}{\pi^{2s}C_{\phi_\nu^\epsilon}} \int_{N\times T} \widetilde{W}_{LRL(\phi_\nu^\epsilon)_t} f(a,\alpha,t)$$
$$U(a,\alpha,t)\phi_\nu^\epsilon(x,\zeta)\frac{da\,d\alpha\,dm_l(t)}{\det(t)^{3r+5s}}. \tag{8.10}$$

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Department of Mathematics
School of Mathematics and Information Sciences
Guangzhou University
Guangzhou 510006
PR China
E-mail address: h_jianxun@hotmail.com

LMAM, School of Mathematical Sciences
Peking University
Beijing 100871
PR China
E-mail address: hpliu@math.pku.edu.cn

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