Heegaard genus of the connected sum of m-small knots

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We prove that if $K_1 \subset M_1, \ldots, K_n \subset M_n$ are m-small knots in closed orientable 3-manifolds, then the Heegaard genus of $E(\#_{i=1}^n K_i)$ is strictly less than the sum of the Heegaard genera of the $E(K_i)$ $(i = 1, \ldots, n)$ if and only if there exists a proper subset I of $\{1, \ldots, n\}$ so that $\#_{i \in I} K_i$ admits a primitive meridian. This generalizes the main result of Morimoto [*On the super additivity of tunnel number of knots*, Math. Ann. 317 (2000), 489–508].

1. Introduction

Let K_i $(i = 1, \ldots, n)$ be knots in closed orientable 3-manifolds M_i, and let $K(= \#_{i=1}^n K_i) \subset M(= \#_{i=1}^n M_i)$ be their connected sum. Let $X = E(K)$ be the exterior of K and X_i be the exteriors of K_i. For $n = 2$, it is well known that $g(X) \leq g(X_1) + g(X_2)$, and by induction, we see that for any n, $g(X) \leq \Sigma_{i=1}^n g(X_i)$, where $g(\cdot)$ denotes the *Heegaard genus* (definitions are given in Section 2). In this paper, we study the phenomenon $g(X) < \Sigma_{i=1}^n g(X_i)$. We note that (for $n = 2$) the phenomena $g(X) = g(X_1) + g(X_2)$ and $g(X) \leq g(X_1) + g(X_2) - k$ (for arbitrary k) can both occur (see [13, 18] for the former, [15, 7] for the latter). An instrumental definition for our work is given in [12, Definition 4.6].

Definition 1.1. A knot exterior X admits a *primitive meridian* if there exists a minimal genus Heegaard surface $\Sigma \subset X$ that separates X into a compression body C and a handlebody V so that there exist a compressing disk $D \subset V$ and a vertical annulus $A \subset C$ with $A \cap \partial X$ a meridian, and $\partial D \subset \Sigma$ intersects $A \cap \Sigma$ transversely in one point. (For the definition of vertical annulus, see Remarks 2.2(5).)

For knots admitting primitive meridians, we have:[1]

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[1] Proposition 1.3 of [16] is stated in terms of the *1-bridge genus*. However, by [16, Proposition 2.1], it is equivalent to this proposition.

Proposition 1.2 [16, Propositions 1.3 and 2.1]. *Let K_1, K_2 be knots in closed orientable 3-manifolds. Let X_1, X_2 and X be the exteriors of the knots K_1, K_2 and $K_1 \# K_2$, respectively. If K_1 or K_2 admits a primitive meridian, then $g(X) \leq g(X_1) + g(X_2) - 1$.*

Thus, existence of primitive meridian is a sufficient condition for $g(X) < g(X_1) + g(X_2)$. A knot is called *meridionally small* (or *m-small*) if there is no essential surface in $E(K)$ with non-empty boundary so that each boundary component is a meridian. In [16, Theorem 1.6], Morimoto showed that for m-small knots in S^3, existence of primitive meridian is also necessary for $g(X) < g(X_1) + g(X_2)$ and conjectured that this holds for any two knots in S^3.

Conjecture 1.3 (Morimoto's conjecture). *Let K_1, K_2 be knots in S^3. Let X_1, X_2 and X be the exteriors of K_1, K_2 and $K_1 \# K_2$, respectively. Then, $g(X) < g(X_1) + g(X_2)$ if and only if X_1 or X_2 admits a primitive meridian.*

In this paper, we prove the following generalization of [16, Theorem 1.6].

Theorem 1.4. *Let $K_1, \ldots, K_n$ ($n \geq 2$) be m-small knots in closed orientable 3-manifolds. Let $X_1, \ldots, X_n$ and X be the exteriors of $K_1, \ldots, K_n$ and $\#_{i=1}^n K_i$, respectively. Then, $g(X) < \Sigma_{i=1}^n g(X_i)$ if and only if there exists $I \neq \emptyset$ a proper subset of $\{1, \ldots, n\}$ such that the exterior of $\#_{i \in I} K_i$ admits a primitive meridian.*

Remarks 1.5.

(1) One might hope for the following simple statement as the conclusion of Theorem 1.4: given knots $K_1, \ldots, K_n$, $g(X) < \Sigma_{i=1}^n g(X_i)$ holds if and only if for some i, K_i admits a primitive meridian. However, the following examples show that this is not the case. Let K_m ($m \in \mathbb{Z}$) be the knots introduced by Morimoto *et al.* in [18]. It is shown in [18] that for each m, $g(E(K_m)) = 2$, K_m does not admit a primitive meridian, and $g(E(K_m \# K_m)) = 4$. On the other hand, in [10, Section 5], it is shown that $E(K_m \# K_m)$ admits a primitive meridian. Hence, by Proposition 1.2, $g(E(K_m \# K_m \# K_m)) < 4 + 2 = 3g(E(K_m))$.

(2) Many of the ideas used for the proof of Theorem 1.4 can be found in [17] where Morimoto proved [17, Theorem 1.1] that if K_i ($i = 1, \ldots, n$) are m-small knots in closed orientable 3-manifolds M_i so that no M_i contains a lens space summand, then $g(X) \geq \Sigma_{i=1}^n g(X_i) - (n-1)$. In this

paper, we retrieve this result (Corollary 5.3) without the restriction on the summands of the manifolds M_i.

The "if" part of Theorem 1.4 is an easy consequence of Proposition 1.2. We now sketch the proof of the "only if" part and introduce the main tools applied. The proof is an induction on the number of summands n. Our first tool, the Swallow Follow Torus Theorem (Theorem 4.1), implies that if $K_1, \ldots, K_n$ $(n \geq 2)$ are m-small knots, then any Heegaard surface for $X(= E(\#_{i=1}^n K_i))$ weakly reduces to a swallow follow torus (say T). In particular, X does not admit a strongly irreducible Heegaard splitting. It follows from the definition of swallow follow torus that T decomposes X as $X = X_I^{(1)} \cup_T X_J$, with $I \subset \{1, \ldots, n\}$ a non-empty proper subset, $X_I = E(\#_{i \in I} K_i)$, $X_J = E(\#_{i \notin I} K_i)$ and $X_I^{(1)}$ is obtained from X_I by drilling out a simple closed curve parallel to the meridian of K_I. If $g(X) = g(X_I) + g(X_J)$, then either $g(X_I) < \Sigma_{i \in I} g(E(K_i))$ or $g(X_J) < \Sigma_{i \in J} g(E(K_i))$ (say the former). By induction, Theorem 1.4 holds for X_I, and hence for some $I' \subset I$, $E(\#_{i \in I'} K_i)$ admits a primitive meridian, and we are done. Thus, we may assume that $g(X) < g(X_I) + g(X_J)$. In that case, Corollary 5.4 implies that $g(X_I^{(1)}) = g(X_I)$.

Our second tool, the Hopf–Haken annulus theorem (Theorem 6.3), studies minimal genus Heegaard splitting of $X_I^{(1)}$. Denote the boundary of $X_I^{(1)}$ by $\partial X_I \cup T$. Note that $X_I^{(1)}$ admits an essential annulus (say A) connecting ∂X_I to T, so that $A \cap \partial X_I$ is a meridian of X_I and $A \cap T$ is a longitude of T. The Hopf–Haken annulus theorem implies that if $K_I = \#_{i \in I} K_i$ is the connected sum of m-small knots, then there exists a minimal genus Heegaard surface for $X_I^{(1)}$ (say Σ) that intersects A in a single simple closed curve that is essential in both A and the Heegaard surface. Applying this to $X_I^{(1)}$ described in the previous paragraph, since $g(X_I) = g(X_I^{(1)})$, we have that Σ is also a minimal genus Heegaard surface for X_I. It is then easy to see (Proposition 6.2) that Σ fulfills the conditions of Definition 1.1 and hence X_I admits a primitive meridian, proving Theorem 1.4. Thus, the Hopf–Haken annulus theorem connects the Swallow Follow Torus Theorem to the existence of a primitive meridian.

2. Background

Throughout this paper, we work in the category of smooth manifolds. All manifolds considered are assumed to be orientable. For a submanifold H of a manifold M, $N(H, M)$ denotes a regular neighborhood of H in M. When

M is understood from context, we often abbreviate $N(H, M)$ to $N(H)$. Let F be a surface. Then a surface in $F \times [0, 1]$ is called *vertical* if it is ambient isotopic to a surface of the form $c \times [0, 1]$, $c \subset F$ an embedded 1-manifold. Let N be a manifold embedded in a manifold M with $\dim(N) = \dim(M)$. Then, the frontier of N in M is denoted by $\mathrm{Fr}_M(N)$. For definitions of standard terms of 3-dimensional topology, we refer to [5, 6].

The following lemma is a well-known fact. Informally, it says that for manifolds with torus boundary compression implies compression; for a proof see, for example, [9, Lemma 2.7].

Lemma 2.1. *Let N be a 3-manifold with a torus boundary component T. Let S be a 2-sided surface properly embedded in N such that $S \cap T$ consists of essential simple closed curves in T. Suppose that there is a boundary compressing disk Δ for S such that Δ compresses S into T, i.e., $\Delta \cap \partial N = \partial \Delta \cap T$ is an arc, say α, and $\Delta \cap S = \partial \Delta \cap S$ is an essential arc in S, say β, such that $\alpha \cup \beta = \partial \Delta$. Then, we have either one of the following.*

(1) *S is compressible. Moreover, if S is separating in N, then the compression occurs in the same side as the boundary compression.*

(2) *S is an annulus; moreover, when N is irreducible, S is boundary parallel.*

Let S be a surface properly embedded in a 3-manifold N. Then S is called *essential* if S is incompressible and not boundary parallel. If N is irreducible and ∂N consists of tori, then by Lemma 2.1, S being essential implies S is not boundary compressible.

A 3-manifold C is called a compression body if there is a compact, connected, closed, orientable surface F such that C is obtained from $F \times [0, 1]$ by attaching 2-handles along mutually disjoint simple closed curves in $F \times \{1\}$ and capping off the 2-sphere boundary components using 3-handles. Then, $F \times \{0\}$ is denoted by $\partial_+ C$ and $\partial_- C = \partial C \setminus \partial_+ C$. A compression body is called a handlebody if $\partial_- C = \emptyset$. A compression body C is called *trivial* if C is homeomorphic to $F \times [0, 1]$ with $\partial_+ F = F \times \{0\}$ and $\partial_- F = F \times \{1\}$. A disk D embedded in a compression body C is called a *meridian disk* if ∂D is non-trivial in $\partial_+ C$. (We note that $\partial_- C$ is incompressible.) By extending the cores of the 2-handles in the definition of a compression body vertically in $F \times [0, 1]$, we obtain a union of mutually disjoint meridian disks in C, say $\mathcal{D}$, such that the manifold obtained from C by cutting along $\mathcal{D}$ is homeomorphic to a union of $\partial_- C \times [0, 1]$ and a (possibly empty) union of 3-balls corresponding to the attached 3-handles. This gives a dual description

of compression bodies, that is, a connected 3-manifold C is a compression body if there exists a closed (not necessarily connected) surface F without 2-sphere components and a (possibly empty) union of 3-balls (say $\mathcal{B}$) so that C is obtained from $(F \times [0,1]) \cup \mathcal{B}$ by attaching 1-handles along $(F \times \{0\}) \cup \partial\mathcal{B}$. We may regard each component of $\mathcal{B}$ as a 0-handle. Hence, C admits a handle decomposition $(F \times [0,1]) \cup 0\text{-handles} \cup 1\text{-handles}$. We note that $\partial_- C$ is the surface corresponding to $F \times \{1\}$.

Remarks 2.2. The following properties are known for compression bodies (the proofs are omitted or outlined below):

(1) Compression bodies are irreducible.

(2) Let $\mathcal{D}$ be a union of mutually disjoint meridian disks of a compression body C and C' a component of the manifold obtained by cutting C open along $\mathcal{D}$. Then, C' inherits a compression body structure from C, i.e., C' is a compression body such that $\partial_- C' = \partial_- C \cap C'$ and $\partial_+ C' = (C' \cap \partial_+ C) \cup \mathrm{Fr}_C(C')$.

(3) Let S be an incompressible surface in C such that $\partial S \neq \emptyset$ and $\partial S \subset \partial_+ C$. Suppose that S is not a union of meridian disks. Then S is boundary compressible into $\partial_+ C$ in C, i.e., there exists a disk Δ in C such that $\Delta \cap S = \partial\Delta \cap S = a$ is an essential arc in S and $\Delta \cap \partial C = \Delta \cap \partial_+ C = \mathrm{cl}(\partial\Delta \backslash a)$.

(4) Let A be a surface properly embedded in C such that $\partial A \subset \partial_+ C$, and each component of A is an essential annulus. Then by boundary compressing an outermost component of A (see (3) above), we obtain a meridian disk D. By a tiny isotopy, we may assume that $D \cap A = \emptyset$. This implies the following: let $A \subset C$ be as above. Then there is a meridian disk D of C such that $D \cap A = \emptyset$.

(5) Let S be an incompressible, boundary incompressible surface with non-empty boundary properly embedded in C. Then each component of S is either a meridian disk or an annulus A such that one component of ∂A is in $\partial_+ C$ and the other in $\partial_- C$. (Such an incompressible annulus is called *vertical.*)

(6) Let S be an incompressible (possibly closed) surface properly embedded in C such that $S \cap \partial_+ C = \emptyset$. Then each component of S is parallel to a subsurface of $\partial_- C$.

Let N be a 3-manifold and F_1, F_2 a partition of ∂N (possibly, $F_1 = \emptyset$ or $F_2 = \emptyset$).

Definition 2.3. We say that a decomposition $C_1 \cup_\Sigma C_2$ is a Heegaard splitting of $(N; F_1, F_2)$ if the following conditions hold:

(1) C_i $(i = 1, 2)$ is a compression body;

(2) $C_1 \cup C_2 = N$;

(3) $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = \Sigma$; and

(4) $\partial_- C_i = F_i$ $(i = 1, 2)$.

The surface Σ is called a Heegaard surface of $(N; F_1, F_2)$. If the boundary partition of N is irrelevant, we omit condition (4) in Definition 2.3 and say that Σ is a Heegaard surface of N. The genus of the Heegaard splitting is the genus of Σ, denoted by $g(\Sigma)$ or $g(C_1 \cup_\Sigma C_2)$. The minimal genus of all Heegaard splittings of $(N; F_1, F_2)$ is called the (Heegaard) genus of $(N; F_1, F_2)$ and is denoted by $g(N; F_1, F_2)$. The minimal genus of all Heegaard splittings of N is called the (Heegaard) genus of N and is denoted by $g(N)$.

Definitions 2.4.

(1) A Heegaard splitting $C_1 \cup_\Sigma C_2$ is called *reducible* if there exist meridian disks $D_1 \subset C_1$ and $D_2 \subset C_2$ such that $\partial D_1 = \partial D_2$. Otherwise, the Heegaard splitting is called *irreducible.*

(2) A Heegaard splitting $C_1 \cup_\Sigma C_2$ is called *weakly reducible* if there exist meridian disks $D_1 \subset C_1$ and $D_2 \subset C_2$ such that $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, the splitting is called *strongly irreducible.*

(3) A Heegaard splitting $C_1 \cup_\Sigma C_2$ is called *stabilized* if there exist meridian disks $D_1 \subset C_1$ and $D_2 \subset C_2$ such that ∂D_1 intersects ∂D_2 transversely in one point. Otherwise the splitting is said to be *non stabilized.*

(4) A Heegaard splitting $C_1 \cup_\Sigma C_2$ is called *trivial* if C_1 or C_2 is a trivial compression body.

Let $C_1 \cup_\Sigma C_2$ be a Heegaard splitting of $(N; F_1, F_2)$. Recall that C_1 was obtained from $(F_1 \times [0,1]) \cup$ 0-handles by attaching 1-handles and that C_2 was obtained from $\partial_+ C_2 \times [0,1]$ by attaching 2- and 3-handles. Then, by using an isotopy which pushes $\partial_+ C_2 \times [0,1]$ out of C_2, we identify $\partial_+ C_2 \times [0,1]$ with $N(\partial_+ C_1, C_1)$. This identification, together with the above handles, gives the following handle decomposition of N:

$$N = (F_1 \times [0,1]) \cup \text{0-handles} \cup \text{1-handles} \cup \text{2-handles} \cup \text{3-handles}.$$

We say that this handle decomposition is *induced* by $C_1 \cup_\Sigma C_2$. Suppose that there exists a proper subset of the 0- and 1-handles so that some subset of the 2- and 3-handles does not intersect the subset of 0- and 1-handles. Suppose further that we can rearrange the order of the handles non-trivially to obtain an increasing sequence of (not necessarily connected) submanifolds $N_1, \ldots, N_n$ of N such that the following holds:

$$N_0 = \emptyset,$$
$$N_j = N_{j-1} \cup (F_1^{(j-1)} \times [0,1]) \cup \ 0\text{-handles} \ \cup \ 1\text{-handles}$$
$$\cup \ 2\text{-handles} \ \cup \ 3\text{-handles} \quad (1 \leq j \leq n),$$

where $\{F_1^{(0)}, \ldots, F_1^{(n-1)}\}$ is a partition of the components of F_1 (with $F_1^{(j)}$ possibly empty, $0 \leq j \leq n-1$) and the handles appearing above come from a handle decomposition of N induced by $C_1 \cup_\Sigma C_2$, where each collection of 0- and 3-handles may be empty but the collections of 1- and 2-handles are never empty. Suppose further that this handle decomposition satisfies the following three conditions:

(1) N_1 is connected.

(2) At each stage j $(1 \leq j \leq n)$, let ∂_j^- denote the union of the components of ∂N_{j-1} to which the 1-handles are attached. Then, $\partial_j^- \cup (F_1^{(j-1)} \times [0,1]) \cup 0\text{-handles} \cup 1\text{-handles} \cup 2\text{-handles} \cup 3\text{-handles}$ is connected, where these handles are the handles that appear in the description of N_j.

(3) No component of ∂N_j is a 2-sphere $(j = 1, \ldots, n-1)$.

We note that if $C_1 \cup_\Sigma C_2$ is irreducible, then condition (3) holds automatically (this is a consequence of Lemma 2.10 and Proposition 2.13 of this paper). Then, for each j, let $I_j = \partial_j^- \times [0,1]$ and $R_j = I_j \cup (F_1^{(j)} \times [0,1]) \cup$ 0-handles $\cup$ 1-handles $\cup$ 2-handles $\cup$ 3-handles (handles that appear in the description of N_j), where ∂_j^- $(\subset \partial N_{j-1})$ is identified with $\partial_j^- \times \{0\}$ $(\subset I_j)$. By the above conditions, we see that this handle decomposition induces a Heegaard splitting, say $C_1^{(j)} \cup C_2^{(j)}$, of R_j.

It is clear that N can be regarded as obtained from $R_1, \ldots, R_n$ by identifying their boundaries. Hence, we have obtained a decomposition of N into submanifolds $R_1, \ldots, R_n$ and Heegaard splittings for those submanifolds as follows:

$$(C_1^{(1)} \cup C_2^{(1)}) \cup (C_1^{(2)} \cup C_2^{(2)}) \cup \cdots \cup (C_1^{(n)} \cup C_2^{(n)}).$$

This decomposition is called an *untelescoping* of $C_1 \cup_\Sigma C_2$, originally defined in a slightly different way by Scharlemann and Thompson in [24]. An untelescoping is called a *Scharlemann–Thompson untelescoping* (or S–T untelescoping) of the Heegaard splitting $C_1 \cup_\Sigma C_2$ if for each j, $C_1^{(j)} \cup C_2^{(j)}$ is strongly irreducible.

Remark 2.5. It is well known that every irreducible Heegaard splitting of a 3-manifold M with incompressible boundary (other than surface cross-interval) admits an S–T untelescoping such that each Heegaard splitting is non-trivial [23, 24]. In this case, every component of ∂R_j is incompressible in M. If, in addition, $C_1 \cup_\Sigma C_2$ is minimal genus, then by [27], each component of $\partial R_j \backslash \partial M$ is an essential surface.

2.1. Amalgamation of Heegaard splittings

Let $M^{(1)}$ and $M^{(2)}$ be compact, orientable 3-manifolds with boundary, together with partitions of the boundary components:

$$\partial M^{(1)} = F^{(1)} \cup F_1^{(1)} \cup F_2^{(1)},$$
$$\partial M^{(2)} = F^{(2)} \cup F_1^{(2)} \cup F_2^{(2)},$$

where $F^{(1)}$ and $F^{(2)}$ are non-empty and mutually homeomorphic. Let M be a manifold obtained from $M^{(1)}$ and $M^{(2)}$ by identifying $F^{(1)}$ and $F^{(2)}$. Then F denotes the image of $F^{(1)}$ (= the image of $F^{(2)}$) in M.

Suppose that we are given Heegaard splittings

$$C_1^{(1)} \cup C_2^{(1)} \quad \text{of } (M^{(1)}; F_1^{(1)}, F^{(1)} \cup F_2^{(1)})$$

and

$$C_1^{(2)} \cup C_2^{(2)} \quad \text{of } (M^{(2)}; F^{(2)} \cup F_1^{(2)}, F_2^{(2)}).$$

Recall that there are handle decompositions

$$M^{(1)} = (F_1^{(1)} \times [0, 1]) \cup \mathcal{H}_0^{(1)} \cup \mathcal{H}_1^{(1)} \cup \mathcal{H}_2^{(1)} \cup \mathcal{H}_3^{(1)},$$

and

$$M^{(2)} = (F^{(2)} \times [0, 1]) \cup (F_1^{(2)} \times [0, 1]) \cup \mathcal{H}_0^{(2)} \cup \mathcal{H}_1^{(2)} \cup \mathcal{H}_2^{(2)} \cup \mathcal{H}_3^{(2)},$$

where the $\mathcal{H}_i^{(j)}$ denotes the handles of index i induced by the Heegaard splitting $C_1^{(j)} \cup C_2^{(j)}$. Hence, we have

$$M = (F_1^{(1)} \times [0,1]) \cup \mathcal{H}_0^{(1)} \cup \mathcal{H}_1^{(1)} \cup \mathcal{H}_2^{(1)} \cup \mathcal{H}_3^{(1)}$$
$$\cup (F^{(2)} \times [0,1]) \cup (F_1^{(2)} \times [0,1]) \cup \mathcal{H}_0^{(2)} \cup \mathcal{H}_1^{(2)} \cup \mathcal{H}_2^{(2)} \cup \mathcal{H}_3^{(2)},$$

Then, via ambient isotopy, we push $M^{(2)}$ slightly into $M^{(1)}$ so that $F^{(2)} \times [0,1]$ is identified with $N(F^{(1)}, M^{(1)})$. Hence, $F^{(1)}$ ($\subset \partial M^{(1)}$) is regarded as $F^{(2)} \times \{0\} \subset \operatorname{int} M^{(2)}$. (Note that in the handle decomposition $M^{(2)} = (F^{(2)} \times [0,1]) \cup (F_1^{(2)} \times [0,1]) \cup \mathcal{H}_0^{(2)} \cup \mathcal{H}_1^{(2)} \cup \mathcal{H}_2^{(2)} \cup \mathcal{H}_3^{(2)}$, the handles of $\mathcal{H}_1^{(2)}$ are attached to $(F^{(2)} \times \{0\}) \cup (F_1^{(2)} \times \{0\}) \cup \partial \mathcal{H}_0^{(2)}$.) Here, via isotopy, we may suppose that $\mathcal{H}_1^{(2)}$ is disjoint from $\mathcal{H}_1^{(1)} \cup \mathcal{H}_2^{(1)}$ and that $\mathcal{H}_2^{(2)}$ is disjoint from $\mathcal{H}_2^{(1)}$. This implies that we can change the order of the handles so that

$$M = (F_1^{(1)} \times [0,1]) \cup (F_1^{(2)} \times [0,1]) \cup (\mathcal{H}_0^{(1)} \cup \mathcal{H}_0^{(2)})$$
$$\cup (\mathcal{H}_1^{(1)} \cup \mathcal{H}_1^{(2)}) \cup (\mathcal{H}_2^{(1)} \cup \mathcal{H}_2^{(2)}) \cup (\mathcal{H}_3^{(1)} \cup \mathcal{H}_3^{(2)}).$$

By using this handle decomposition, we can obtain a Heegaard splitting, say $C_1 \cup C_2$, of M. We say that $C_1 \cup C_2$ is obtained from $C_1^{(1)} \cup C_2^{(1)}$ and $C_1^{(2)} \cup C_2^{(2)}$ by an *amalgamation* or that $C_1 \cup C_2$ is an amalgamation of $C_1^{(1)} \cup C_2^{(1)}$ and $C_1^{(2)} \cup C_2^{(2)}$. We note that this definition of amalgamation has different appearance than that in [25]. However, it is easy to see that they yield the same surface.

By the definition of untelescoping, we immediately have the following:

Proposition 2.6. *Let $(C_1^{(1)} \cup C_2^{(1)}) \cup \cdots \cup (C_1^{(n)} \cup C_2^{(n)})$ be an untelescoping of $C_1 \cup C_2$. Then, $C_1 \cup C_2$ is obtained from $C_1^{(1)} \cup C_2^{(1)}, \ldots, C_1^{(n)} \cup C_2^{(n)}$ by a sequence of amalgamations.*

The proof of Lemma 2.7 (e.g., Remark 2.7 of [25] for the case $m = 1$) is elementary and is left to the reader.

Lemma 2.7. *Let $C_1 \cup C_2$, $C_1^{(1)} \cup C_2^{(1)}$ and $C_1^{(2)} \cup C_2^{(2)}$ be as above. Let $F_1, \ldots, F_m$ be the components of $F \subset M$. Then,*

$$g(C_1 \cup C_2) = g(C_1^{(1)} \cup C_2^{(1)}) + g(C_1^{(2)} \cup C_2^{(2)}) - \sum_{i=1}^{m} g(F_i) + (m-1).$$

Proposition 2.8. *Let M be a compact 3-manifold with partition of the boundary components:* $\partial M = (F_1^{(1)} \cup F_1^{(2)}) \cup (F_2^{(1)} \cup F_2^{(2)})$. *Suppose that there exists a minimal genus Heegaard splitting $C_1 \cup C_2$ of $(M; F_1^{(1)} \cup F_1^{(2)}, F_2^{(1)} \cup F_2^{(2)})$ such that $C_1 \cup C_2$ admits an untelescoping*

$$(C_1^{(1)} \cup C_2^{(1)}) \cup_F (C_1^{(2)} \cup C_2^{(2)})$$

with the following properties:

- $\partial_- C_1^{(1)} = F_1^{(1)}$ *and* $\partial_- C_2^{(2)} = F_2^{(2)}$, *and*
- $\partial_- C_2^{(1)} = F^{(1)} \cup F_2^{(1)}$ *and* $\partial_- C_1^{(2)} = F^{(2)} \cup F_1^{(2)}$,

where $F^{(1)}$ and $F^{(2)}$ are the surfaces that are identified in M as F. Then, $C_1^{(1)} \cup C_2^{(1)}$ and $C_1^{(2)} \cup C_2^{(2)}$ are minimal genus Heegaard splittings of $(M^{(1)}; F_1^{(1)}, F^{(1)} \cup F_2^{(1)})$ and $(M^{(2)}; F^{(2)} \cup F_1^{(2)}, F_2^{(2)})$, respectively. In particular, we have the following identity:

$$\begin{aligned}
g(M; F_1^{(1)} \cup F_1^{(2)}, F_2^{(1)} \cup F_2^{(2)}) &= g(M^{(1)}; F_1^{(1)}, F_2^{(1)} \cup F^{(1)}) \\
&\quad + g(M^{(2)}; F_1^{(2)} \cup F^{(2)}, F_2^{(2)}) \\
&\quad - \sum_{i=1}^{m} g(F_i) + (m-1),
\end{aligned}$$

where $F_1, \ldots, F_m$ are the components of F.

Proof. If either $C_1^{(1)} \cup C_2^{(1)}$ or $C_1^{(2)} \cup C_2^{(2)}$ is not a minimal genus Heegaard splitting of $(M^{(1)}; F_1^{(1)}, F_2^{(1)} \cup F^{(1)})$ or $(M^{(2)}; F_1^{(2)} \cup F^{(2)}, F_2^{(2)})$, respectively, then by amalgamating minimal genus Heegaard splittings of $(M^{(1)}; F_1^{(1)}, F^{(1)} \cup F_2^{(1)})$ and $(M^{(2)}; F_1^{(2)} \cup F^{(2)}, F_2^{(2)})$, we see, by Lemma 2.7, that we obtain a Heegaard splitting of $(M; F_1^{(1)} \cup F_1^{(2)}, F_2^{(1)} \cup F_2^{(2)})$ with genus lower than $g(C_1 \cup C_2)$, contradicting the fact that the $C_1 \cup C_2$ is a minimal genus Heegaard splitting of $(M; F_1^{(1)} \cup F_1^{(2)}, F_2^{(1)} \cup F_2^{(2)})$. Hence, $C_1^{(1)} \cup C_2^{(1)}$ and $C_1^{(2)} \cup C_2^{(2)}$ are minimal genus Heegaard splittings of $(M^{(1)}; F_1^{(1)}, F^{(1)} \cup F_2^{(1)})$ and $(M^{(2)}; F_1^{(2)} \cup F^{(2)}, F_2^{(2)})$, respectively. This together with Lemma 2.7 implies

$$\begin{aligned}
g(M; F_1^{(1)} \cup F_1^{(2)}, F_2^{(1)} \cup F_2^{(2)}) &= g(M^{(1)}; F_1^{(1)}, F_2^{(1)} \cup F^{(1)}) \\
&\quad + g(M^{(2)}; F_1^{(2)} \cup F^{(2)}, F_2^{(2)}) \\
&\quad - \sum_{i=1}^{m} g(F_i) + (m-1). \qquad \square
\end{aligned}$$

Recall that $g(M)$ denotes the genus of the minimal genus Heegaard splitting of M for all possible partitions of the components of ∂M.

Proposition 2.9. *Let N be a compact orientable 3-manifold with a minimal genus Heegaard splitting $C_1 \cup_\Sigma C_2$ admitting an untelescoping $(C_1^{(1)} \cup C_2^{(1)})$ $\cup_F (C_1^{(2)} \cup C_2^{(2)})$, where F is a connected surface. Let $M^{(1)} = C_1^{(1)} \cup C_2^{(1)}$ and $M^{(2)} = C_1^{(2)} \cup C_2^{(2)}$. Then, $C_1^{(1)} \cup C_2^{(1)}$ and $C_1^{(2)} \cup C_2^{(2)}$ are minimal genus Heegaard splittings of $M^{(1)}$ and $M^{(2)}$, respectively. In particular, we have the following equality:*

$$g(M) = g(M^{(1)}) + g(M^{(2)}) - g(F).$$

Proof. Suppose that either $C_1^{(1)} \cup C_2^{(1)}$ or $C_1^{(2)} \cup C_2^{(2)}$, say $C_1^{(1)} \cup C_2^{(1)}$, is not a minimal genus Heegaard splitting. Let $\bar{C}_1^{(1)} \cup \bar{C}_2^{(1)}$ be a minimal genus Heegaard splitting of $M^{(1)}$. Since F is connected, by exchanging $\bar{C}_1^{(1)}$ and $\bar{C}_2^{(1)}$ if necessary, we may assume that $F \subset \partial_- \bar{C}_2^{(1)}$. Hence, we can amalgamate $\bar{C}_1^{(1)} \cup \bar{C}_2^{(1)}$ and $C_1^{(2)} \cup C_2^{(2)}$ to obtain a Heegaard splitting, say $\bar{C}_1 \cup_{\bar{\Sigma}} \bar{C}_2$ of M. Here we note that $g(\bar{\Sigma}) < g(\Sigma)$ by Lemma 2.7, a contradiction. Hence, $C_1^{(1)} \cup C_2^{(1)}$ and $C_1^{(2)} \cup C_2^{(2)}$ are minimal genus Heegaard splittings. This together with Lemma 2.7 implies

$$g(M) = g(M^{(1)}) + g(M^{(2)}) - g(F). \qquad \square$$

2.2. Weak reduction of Heegaard splittings

Let M be a compact, orientable 3-manifold and $C_1 \cup_\Sigma C_2$ a Heegaard splitting of M. Let $\Delta = \Delta_1 \cup \Delta_2$ be a weakly reducing collection of disks for Σ, i.e., Δ_i $(i = 1, 2)$ is a union of mutually disjoint, non-empty meridian disks for C_i such that $\Delta_1 \cap \Delta_2 = \emptyset$. Then, $\Sigma(\Delta)$ denotes the surface obtained from Σ by compressing along Δ, i.e., $\Sigma(\Delta) = (\Sigma \backslash (N(\Delta_1, C_1) \cup N(\Delta_2, C_2))) \cup \mathrm{Fr}_{C_1} N(\Delta_1, C_1) \cup \mathrm{Fr}_{C_2} N(\Delta_2, C_2)$. Let $\widehat{\Sigma}(\Delta)$ be the surface obtained from $\Sigma(\Delta)$ by removing all the components that are contained in C_1 or C_2. We call $\widehat{\Sigma}(\Delta)$ *the surface obtained from Σ by weakly reducing along Δ* (or *the surface obtained from Σ by a weak reduction along Δ*). The next lemma is well known (for a proof, see, for example, [8, Lemma 4.1]).

Lemma 2.10. *If there is a 2-sphere component of $\widehat{\Sigma}(\Delta)$, then $C_1 \cup_\Sigma C_2$ is reducible.*

Suppose that no component of $\widehat{\Sigma}(\Delta)$ is a 2-sphere. (By Lemma 2.10, we see that when $C_1 \cup_\Sigma C_2$ is irreducible, this is always satisfied.) The following argument can be found in [1] (see also [8, Section 4]).

Let $M_1, \ldots, M_n$ be the closures of the components of $M \backslash \widehat{\Sigma}(\Delta)$, and let $M_{i,j} = M_i \cap C_j$, $(i = 1, \ldots, n,\ j = 1, 2)$. Then, we can show (for a proof, see, for example, [8, Lemma 4.3]):

Proposition 2.11. *For each j, one of the following holds*:

(1) $M_{j,2} \cap \Sigma \subset \mathrm{int}(M_{j,1} \cap \Sigma)$ *and $M_{j,1}$ is connected,*

(2) $M_{j,1} \cap \Sigma \subset \mathrm{int}(M_{j,2} \cap \Sigma)$ *and $M_{j,2}$ is connected.*

Suppose that M_j satisfies the conclusion (1) ((2) resp.) of Proposition 2.11. Note that $M_{j,1}$ ($M_{j,2}$ resp.) is a compression body ((2) of Remarks 2.2). Let $C_{j,1} = \mathrm{cl}(M_{j,1} \setminus N(\partial_+ M_{j,1}, M_{j,1}))$ ($C_{j,2} = \mathrm{cl}(M_{j,2} \backslash N(\partial_+ M_{j,2}, M_{j,2}))$ resp.) and $C_{j,2} = N(\partial_+ M_{j,1}, M_{j,1}) \cup M_{j,2}$ ($C_{j,1} = N(\partial_+ M_{j,2}, M_{j,2}) \cup M_{j,1}$ resp.). Note that $C_{j,1}$ ($C_{j,2}$ resp.) is a compression body ((2) of Remarks 2.2) and that $C_{j,2}$ ($C_{j,1}$ resp.) is also a compression body by the definition of compression body. By using this, it is easy to see:

Proposition 2.12. *Each $C_{i,j}$ is a compression body such that $C_{j,1} \cup C_{j,2}$ gives a Heegaard splitting of M_j ($j = 1, \ldots, n$) and that*

$$(C_{1,1} \cup C_{1,2}) \cup \cdots \cup (C_{n,1} \cup C_{n,2})$$

gives an untelescoping of $C_1 \cup C_2$.

2.3. Untelescoping and weak reduction

Let $C_1 \cup_\Sigma C_2$ be a Heegaard splitting of M and $(C_1^{(1)} \cup_{\Sigma^{(1)}} C_2^{(1)}) \cup \cdots \cup (C_1^{(n)} \cup_{\Sigma^{(n)}} C_2^{(n)})$ an untelescoping of $C_1 \cup_\Sigma C_2$. Let $R^{(i)} = C_1^{(i)} \cup C_2^{(i)}$.

Proposition 2.13. *Suppose that there exists an $R^{(k)}$ such that a component of $\mathrm{Fr}_M(R^{(k)})$, say F, is separating in M. Then, there exists a weak reducing collection of disks for Σ, say $\Delta = \Delta_1 \cup \Delta_2$, so that $\widehat{\Sigma}(\Delta)$ is isotopic to F.*

Proof. By exchanging superscripts if necessary, we may assume that F separates M into $R^{(1)} \cup \cdots \cup R^{(k)}$, say $M^{(1)}$, and $R^{(k+1)} \cup \cdots \cup R^{(n)}$, say $M^{(2)}$. Let $\mathcal{C}_1^{(1)} \cup \mathcal{C}_2^{(1)}$ ($\mathcal{C}_1^{(2)} \cup \mathcal{C}_2^{(2)}$ resp.) be a Heegaard splitting of $M^{(1)}$

$(M^{(2)}$ resp.)   obtained from $(C_1^{(1)} \cup C_2^{(1)}), \ldots, (C_1^{(k)} \cup C_2^{(k)})$ $((C_1^{(k+1)} \cup C_2^{(k+1)}), \ldots, (C_1^{(n)} \cup C_2^{(n)})$ resp.) by a sequence of amalgamations. Then, let

$$(\partial_- C_1^{(1)} \times [0,1]) \cup \mathcal{H}_0^{(1)} \cup \mathcal{H}_1^{(1)} \cup \mathcal{H}_2^{(1)} \cup \mathcal{H}_3^{(1)},$$
$$(\partial_- C_1^{(2)} \times [0,1]) \cup \mathcal{H}_0^{(2)} \cup \mathcal{H}_1^{(2)} \cup \mathcal{H}_2^{(2)} \cup \mathcal{H}_3^{(2)}$$

be handle decompositions of $M^{(1)}$ and $M^{(2)}$, respectively, which can be used to amalgamate $\mathcal{C}_1^{(1)} \cup \mathcal{C}_2^{(1)}$ and $\mathcal{C}_1^{(2)} \cup \mathcal{C}_2^{(2)}$, i.e., $\mathcal{H}_1^{(2)} \cap (\mathcal{H}_1^{(1)} \cup \mathcal{H}_2^{(1)}) = \emptyset$ and $\mathcal{H}_2^{(2)} \cap \mathcal{H}_2^{(1)} = \emptyset$. Hence, the handle decomposition

$$(\partial_- C_1^{(1)}) \cup ((\partial_- C_1^{(2)} \backslash F) \times [0,1]) \cup (\mathcal{H}_0^{(1)} \cup \mathcal{H}_0^{(2)}) \cup (\mathcal{H}_1^{(1)} \cup \mathcal{H}_1^{(2)})$$
$$\cup (\mathcal{H}_2^{(1)} \cup \mathcal{H}_2^{(2)}) \cup (\mathcal{H}_3^{(1)} \cup \mathcal{H}_3^{(2)})$$

corresponds to $C_1 \cup C_2$. Then take a weak reducing collection $\Delta = \Delta_1 \cup \Delta_2$ of $C_1 \cup C_2$ such that Δ_1 corresponds to the co-cores of $\mathcal{H}_1^{(2)}$ and Δ_2 corresponds to the cores of $\mathcal{H}_2^{(1)}$. It is easy to see that the weak reduction of Σ along Δ yields a surface isotopic to F, thus proving Proposition 2.13. $\square$

2.4. Minimal genus Heegaard splittings of (disk with n-holes)$\times S^1$

Let $D(n)$ be the 3-manifold (disk with n holes) $\times S^1$ with $n \geq 2$. Let T_0, $T_1, \ldots, T_n$ be the boundary components of $D(n)$. For each pair of integers $p, q (\geq 0)$ with $p + q = n + 1$ denote by $g_{p,q}$ the Heegaard genus of $D(n)$ partitioning the boundary into p and q components, that is,

$$g_{p,q} = g\left(D(n); \bigcup_{i=0}^{p-1} T_i, \bigcup_{i=p}^{n} T_i\right).$$

The next proposition is required in Section 6 and is easily deduced from [26].

Proposition 2.14. $g_{0,n+1} = g_{n+1,0} = n + 1$, *otherwise* $g_{p,q} = n$.

Moreover, by [26], we see that minimal genus Heegaard splittings are given as follows: let Q be a disk with n holes in $D(n)$, which is a cross-section of (disk with n holes) $\times S^1$. Let α_i and β_i be arcs properly embedded in Q as in figure 1. Let $\partial_0, \partial_1, \ldots, \partial_n$ be boundary components of Q such that $T_i = \partial_i \times S^1$.
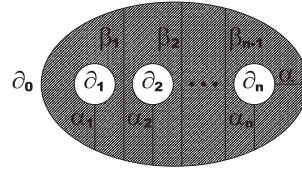
Figure 1: $D(n) \times S^1$.

Suppose that $\{p, q\} = \{0, n+1\}$. Let $C_1 = N(\partial D(n) \cup (\cup_{i=1}^n \alpha_i))$, and let $C_2 = \mathrm{cl}(D(n) \backslash C_1)$. Then, $C_1 \cup C_2$ gives a minimal genus Heegaard splitting of $(D(n); \partial D(n), \emptyset)$.

Suppose that $\{p, q\} \neq \{0, n+1\}$. Let $C_1 = N((\cup_{i=0}^{p-1} T_i) \cup (\cup_{i=1}^{p-1} \alpha_i) \cup (\cup_{i=p}^{n-1} \beta_i))$ and $C_2 = \mathrm{cl}(D(n) \backslash C_1)$. Then, $C_1 \cup C_2$ gives a minimal genus Heegaard splitting of $(D(n); \cup_{i=0}^{p-1} T_i, \cup_{i=p}^n T_i)$.

3. Haken annuli

In this section, we define Haken annuli and prove the basic facts about their behavior under amalgamation.

Definition 3.1. Let $C_1 \cup_\Sigma C_2$ be a Heegaard splitting of M. An essential annulus $A \subset M$ is called a *Haken annulus* for $C_1 \cup_\Sigma C_2$ if $A \cap \Sigma$ consists of a single simple closed curve that is essential in A.

Remarks 3.2.

(1) Note the similarity of Haken annulus to the Haken sphere [3] (see also [6, Chapter 5]) and the Haken disk [2].

(2) Let $D(n)$ be as in the end of Section 2. Observe that if $\{p, q\} \neq \{0, n+1\}$, then there is a minimal genus Heegaard surface of $(D(n); \cup_{i=0}^{p-1} T_i, \cup_{i=p}^n T_i)$ with a Haken annulus of the form $\alpha \times S^1$, where α is an essential arc as in figure 1.

Given $C_1 \cup_\Sigma C_2$, we described in Section 2 how to obtain a (not necessarily unique) handle decomposition for M of the form

$$M = (\partial_- C_1 \times [0, 1]) \cup \mathcal{H}^0 \cup \mathcal{H}^1 \cup \mathcal{H}^2 \cup \mathcal{H}^3,$$

where $\mathcal{H}^i$ are handles of index i.

We say that an annulus A properly embedded in M is *vertical* with respect to $C_1 \cup_\Sigma C_2$ if $A \cap (\cup_{i=1}^3 \mathcal{H}^i) = \emptyset$ (hence $A \subset \partial_- C \times [0,1]$), A is vertical in $\partial_- C \times [0,1]$ and $A \cap (\partial_- C_1 \times \{0\})$ is an essential simple closed curve in $\partial_- C_1 \times \{0\}$.

From the definition of Haken annulus and Heegaard splitting induced by handle decomposition above, it is easy to see that:

Lemma 3.3. *If A is vertical with respect to $C_1 \cup_\Sigma C_2$, then A is a Haken annulus for the Heegaard splitting $C_1 \cup_\Sigma C_2$.*

Conversely, we have the following:

Lemma 3.4. *Suppose that there exists a Haken annulus A for $C_1 \cup_\Sigma C_2$. Then, there exists a handle decomposition (with no 0- or 3-handles):*

$$M = (\partial_- C_1 \times [0,1]) \cup \mathcal{H}^1 \cup \mathcal{H}^2$$

induced by $C_1 \cup_\Sigma C_2$ such that $A \cap (\mathcal{H}^1 \cup \mathcal{H}^2) = \emptyset$.

Proof. Since C_1 is a compression body, there exists a disjoint collection of meridian disks for C_1 (say $\mathcal{D}_1$) such that C_1 cut open along $\mathcal{D}_1$ is $\partial_- C_1 \times [0,1]$, where each component of $\mathcal{D}_1$ can be regarded as the co-core of a 1-handle for C_1. (Since $\partial_- C_1 \neq \emptyset$, we may assume C_1 has no 0-handles.)

Note that $A \cap C_1$ is a vertical annulus in C_1. Then by applying a standard innermost loop, outermost arc arguments to $\mathcal{D}_1$ and $A \cap C_1$, we can replace the collection $\mathcal{D}_1$ (without renaming it) so that $(A \cap C_1) \cap \mathcal{D}_1 = \emptyset$. By using the same argument, we see that there exists a set of compressing disks for C_2 (say $\mathcal{D}_2$) such that $\mathcal{D}_2 \cap (A \cap C_2) = \emptyset$ and $\mathcal{D}_2$ cuts C_2 into $\partial_- C_2 \times [0,1]$ (since there exists a Haken annulus, $\partial_- C_2 \neq \emptyset$, hence we may assume no component of C_2 cut along $\mathcal{D}_2$ is a 3-ball). Each component of $\mathcal{D}_2$ can be regarded as a core of a 2-handle for C_2. Then by taking the handle decomposition for M obtained by $\mathcal{D}_1$ and $\mathcal{D}_2$, we obtain the conclusion of Lemma 3.4. $\qquad \square$

Suppose that M is the union of two compact orientable manifolds (say $M = M^{(1)} \cup M^{(2)}$) such that $M^{(1)} \cap M^{(2)} = \partial M^{(1)} \cap \partial M^{(2)} = F$, a collection of components of $\partial M^{(i)}$. Let $C_1^{(1)} \cup_{\Sigma^{(1)}} C_2^{(1)}$ and $C_1^{(2)} \cup_{\Sigma^{(2)}} C_2^{(2)}$ be Heegaard splittings of $M^{(1)}$ and $M^{(2)}$, respectively. Let $F^{(1)} \cup F_1^{(1)} \cup F_2^{(1)}$ and $F^{(2)} \cup F_1^{(2)} \cup F_2^{(2)}$ be partitions of the components of $\partial M^{(1)}$, $\partial M^{(2)}$,

respectively, such that the surfaces $F^{(1)}$ and $F^{(2)}$ are identified in M as F, and

$$\partial_- C_1^{(1)} = F_1^{(1)}, \qquad \partial_- C_2^{(1)} = F^{(1)} \cup F_2^{(1)},$$
$$\partial_- C_1^{(2)} = F^{(2)} \cup F_1^{(2)}, \qquad \partial_- C_2^{(2)} = F_2^{(2)}.$$

Let $C_1 \cup_\Sigma C_2$ be an amalgamation of $C_1^{(1)} \cup_{\Sigma^{(1)}} C_2^{(1)}$ and $C_1^{(2)} \cup_{\Sigma^{(2)}} C_2^{(2)}$.

Proposition 3.5. *Let M, $M^{(1)}$, $M^{(2)}$, $C_1 \cup_\Sigma C_2$, $C_1^{(1)} \cup_{\Sigma^{(1)}} C_2^{(1)}$ and $C_1^{(2)} \cup_{\Sigma^{(2)}} C_2^{(2)}$ be as above. Suppose that there exists a Haken annulus $A^{(1)}$ for $C_1^{(1)} \cup_{\Sigma^{(1)}} C_2^{(1)}$ such that $A^{(1)} \cap F^{(1)} = \emptyset$. Then, the image of $A^{(1)}$ is a Haken annulus for $C_1 \cup_\Sigma C_2$.*

Proof. By Lemma 3.4, there exists a handle decomposition of $M^{(1)}$ of the form

$$M^{(1)} = (\partial_- C_1^{(1)} \times [0,1]) \cup \mathcal{H}_1^{(1)} \cup \mathcal{H}_2^{(1)}$$

induced by $C_1^{(1)} \cup_{\Sigma^{(1)}} C_2^{(1)}$ such that $A^{(1)} \cap (\mathcal{H}_1^{(1)} \cup \mathcal{H}_2^{(1)}) = \emptyset$. By the definition of amalgamation, we can take a handle decomposition

$$M^{(2)} = (F^{(2)} \times [0,1]) \cup (F_1^{(2)} \times [0,1]) \cup \mathcal{H}_0^{(2)} \cup \mathcal{H}_1^{(2)} \cup \mathcal{H}_2^{(2)} \cup \mathcal{H}_3^{(2)}$$

induced by $C_1^{(2)} \cup_{\Sigma^{(2)}} C_2^{(2)}$ such that these handles give the handle decomposition

$$M = (F_1^{(1)} \times [0,1]) \cup (F_1^{(2)} \times [0,1]) \cup \mathcal{H}_0^{(2)} \cup (\mathcal{H}_1^{(1)} \cup \mathcal{H}_1^{(2)})$$
$$\cup (\mathcal{H}_2^{(1)} \cup \mathcal{H}_2^{(2)}) \cup \mathcal{H}_3^{(2)},$$

which gives the Heegaard splitting $C_1 \cup_\Sigma C_2$. Here we note that adding $F_1^{(2)} \times [0,1]$ and $\mathcal{H}_i^{(2)}$ does not affect $A^{(1)}$, since $A^{(1)} \cap F = \emptyset$. Hence we see that $A^{(1)} \cap (\cup_{i,j} \mathcal{H}_i^{(j)}) = \emptyset$, and this together with Lemma 3.3 shows that the image of $A^{(1)}$ is a Haken annulus for $C_1 \cup_\Sigma C_2$. $\square$

Proposition 3.6. *Let M, $M^{(1)}$, $M^{(2)}$, $C_1 \cup_\Sigma C_2$, $C_1^{(1)} \cup_{\Sigma^{(1)}} C_2^{(1)}$ and $C_1^{(2)} \cup_{\Sigma^{(2)}} C_2^{(2)}$ be as above. Suppose that there exists a Haken annulus $A^{(j)}$ for $C_1^{(j)} \cup_{\Sigma^{(j)}} C_2^{(j)}$ for each $j = 1, 2$ such that $A^{(1)} \cap F = A^{(2)} \cap F$ (hence the image of $A^{(1)} \cup A^{(2)}$ is an annulus properly embedded in M). Then, the image of $A^{(1)} \cup A^{(2)}$ is a Haken annulus for $C_1 \cup_\Sigma C_2$.*

Proof. By Lemma 3.4, there exists a handle decomposition $(F_1^{(1)} \times [0,1]) \cup \mathcal{H}_1^{(1)} \cup \mathcal{H}_2^{(1)}$ $((F_1^{(2)} \times [0,1]) \cup (F^{(2)} \times [0,1]) \cup \mathcal{H}_1^{(2)} \cup \mathcal{H}_2^{(2)}$ resp.) induced by $C_1^{(1)} \cup_{\Sigma^{(1)}} C_2^{(1)}$ $(C_1^{(2)} \cup_{\Sigma^{(2)}} C_2^{(2)}$ resp.) such that $A^{(1)} \cap (\mathcal{H}_1^{(1)} \cup \mathcal{H}_2^{(1)}) = \emptyset$ $(A^{(2)} \cap (\mathcal{H}_1^{(2)} \cup \mathcal{H}_2^{(2)}) = \emptyset$ resp.). By pushing $F^{(2)} \times [0,1]$ into $M^{(1)}$, we obtain a handle decomposition $M = (F_1^{(1)} \times [0,1]) \cup (\mathcal{H}_1^{(1)} \cup \mathcal{H}_2^{(1)}) \cup (F_1^{(2)} \times [0,1]) \cup (\mathcal{H}_1^{(2)} \cup \mathcal{H}_2^{(2)})$ (Section 2). Here, we may suppose that the image of $A^{(1)} \cup A^{(2)}$ is $A^{(1)}$. It is easy to see that by isotopy we may suppose, in addition to the above, that $\mathcal{H}_1^{(2)} \cap (\mathcal{H}_1^{(1)} \cup \mathcal{H}_2^{(1)}) = \emptyset$ and $\mathcal{H}_2^{(2)} \cap \mathcal{H}_2^{(1)} = \emptyset$. Hence, these handles give the Heegaard splitting $C_1 \cup_\Sigma C_2$. For these handles, we have

$$A^{(1)} \cap (\mathcal{H}_1^{(1)} \cup \mathcal{H}_1^{(2)} \cup \mathcal{H}_2^{(1)} \cup \mathcal{H}_2^{(2)}) = \emptyset.$$

Then, Lemma 3.3 establishes the conclusion of Proposition 3.6. $\qquad\square$

4. The Swallow Follow Torus Theorem

Let X be the exterior of a knot K in a closed orientable 3-manifold. Let $\gamma_1, \ldots, \gamma_p$ be mutually disjoint simple closed curves that are obtained by pushing mutually disjoint meridional curves in ∂X into $\text{int}(X)$. Then, let $X^{(p)} = \text{cl}(X \setminus \cup_{i=1}^p N(\gamma_i))$. Set $X^{(0)} = X$. A torus T in $X^{(p)}$ is called a *swallow follow torus* if it is separating and essential in $X^{(p)}$, and there exists an annulus A in $X^{(p)}$ so that $A \cap \partial X^{(p)} = \partial A \cap \partial X^{(p)}$ is a meridian curve in ∂X and $A \cap T = \partial A \cap T$ is an essential simple closed curve in T. We call this annulus a *meridionally compressing annulus* for T. Let A' be the annulus properly embedded in $X^{(p)}$ obtained from T by removing $N(A) \cap T$ and then adding two parallel copies of A. (We say that A' is obtained from T by performing surgery along A.) Since T is essential, we see that A' is also essential. The annulus A' gives a decomposition

$$X_1^{(p_1)} \cup_{A'} X_2^{(p_2)}, \quad p = p_1 + p_2,$$

where each X_i is the exterior of a knot K_i (possibly the trivial knot in S^3, in which case $X_i^{(p_i)}$ is homeomorphic to (disk with p_i holes) $\times S^1$; recall that this was denoted $D(p_i)$ at the end of Section 2). We call this decomposition the *decomposition induced by* T. Note that T can be retrieved from A' by adding a component of $\partial X \setminus \partial A'$ and applying an isotopy pushing it into $\text{int}\, X^{(p)}$. The component of $\partial X \setminus \partial A'$ follows K_1 or K_2, say K_2. This yields

the following decomposition:

$$X^{(p)} = X_1^{(p_1+1)} \cup_T X_2^{(p_2)}.$$

The purpose of this section is to prove the following:

Theorem 4.1 (the Swallow Follow Torus Theorem). *Let $K_1, \ldots, K_n$ ($n \geq 2$) be knots in closed orientable 3-manifolds. Let $X_1, \ldots, X_n$ and X be the exteriors of $K_1, \ldots, K_n$ and $\#_{i=1}^n K_i$, respectively. Suppose that no X_i is a solid torus and X is irreducible. Then,*

(1) *For any $p \geq 0$, every Heegaard surface for $X^{(p)}$ weakly reduces to exactly one surface which is a swallow follow torus, or*

(2) *For some i, X_i contains an essential meridional surface; more precisely, if a genus g Heegaard surface for $X^{(p)}$ (for some p) does not weakly reduce to a swallow follow torus, then there exists a meridional essential surface $S \subset X_i$ so that $\chi(S) \geq 4 - 2g(X^{(p)})$.*

Remark 4.2. One might hope that Theorem 4.1 holds for $n = 1$. However, this is not the case. Let K be a non-trivial 2-bridge knot in S^3. Then, K is m-small [4] and $g(X) = g(X^{(1)}) = 2$ (see, for example, [14]). If a genus 2 Heegaard surface weakly reduces, then the obtained essential surface is a 2-sphere. Applying this to $X^{(1)}$, we see that (since $X^{(1)}$ is irreducible) every minimal genus Heegaard splitting of $X^{(1)}$ is strongly irreducible; hence the conclusion of Theorem 4.1 does not hold for $X^{(1)}$.

As an immediate consequence of Theorem 4.1, we have:

Corollary 4.3. *Let K_i, K, X_i and X be as in the statement of Theorem 4.1, and suppose that each K_i is m-small. Then, for any $p \geq 0$, every Heegaard splitting of $X^{(p)}$ weakly reduces to a swallow follow torus.*

Proof of Theorem 4.1. Note that if some K_i is a composite knot then X_i contains an essential meridional annulus and conclusion (2) of Theorem 6.3 holds. We assume from now on that for each i, K_i is prime.

Claim 4.4. *Either existence and uniqueness of prime factorization holds for K or conclusion (2) holds.*

We note that prime factorization of knots in 3-manifolds is studied by Miyazaki in [11] and it is shown that neither existence nor uniqueness holds in general.

Proof of Claim 4.4. It is shown in the existence theorem of [11] that if the meridian of a knot K is non-null homotopic, then the number of prime factors of K is unique. If the meridian of K is null homotopic, then by the loop theorem (e.g., [5]), ∂X is compressible, and either X is a solid torus or it is reducible, both contrary to our assumptions. The uniqueness theorem of [11] implies that if the uniqueness of prime decomposition of K fails, then the exterior of some factor of K contains a non-separating meridional annulus; this gives conclusion (2). □

By Claim 4.4, for the remainder of the proof, we may suppose that K has a unique prime factorization.

Note that if a given Heegaard splitting weakly reduces to a swallow follow torus, then every stabilization of that Heegaard splitting can be weakly reduced to the same swallow follow torus; it therefore suffices to prove the theorem for non-stabilized Heegaard splittings. Fix p and let $C_1 \cup_\Sigma C_2$ be a non-stabilized Heegaard splitting of $X^{(p)}$.

Let $\mathcal{A} = A_1 \cup \cdots \cup A_n$ be a union of mutually disjoint annuli properly embedded in $X^{(p)}$ with the following properties:

(1) For each i, $\partial A_i \subset \partial X$,

(2) $X^{(p)}$ cut open along $\mathcal{A}$ consists of $n + 1$ components, n homeomorphic to $X_1, \ldots, X_n$ (by slightly abusing notation X_i will refer to the component homeomorphic to X_i) and one component homeomorphic to $D(p)$ (as defined in the end of Section 2), denoted X_0, and

(3) For each i, $X_i \cap \mathcal{A} = A_i$.

We say that the annuli of $\mathcal{A}$ are *not nested.* Let T_j $(j = 1, \ldots, p)$ denote the component of $\partial X^{(p)}$ corresponding to $\partial N(\gamma_j)$. (Hence $\partial X^{(p)} = \partial X \cup (\cup_{j=1}^p T_j)$.)

We divide the proof into the following cases.

Case 1. Σ is strongly irreducible. Since Σ is strongly irreducible, we can isotope it to intersect $\mathcal{A}$ in simple closed curves that are essential in both $\mathcal{A}$ and Σ. Minimize $|\mathcal{A} \cap \Sigma|$ subject to that constraint. Let $\Sigma_i = \Sigma \cap X_i$ $(i = 0, 1, \ldots, n)$. Suppose that $\Sigma_i = \emptyset$ for some i $(1 \le i \le n)$. Then, A_i is contained in a compression body C_j $(j = 1, \text{ or } 2)$. By (6) of Remarks 2.2, we see that A_i is boundary parallel in C_j. This shows that X_i is a solid torus, contradicting the assumption of Theorem 4.1. Hence, $\Sigma_i \ne \emptyset$ for each i $(i = 1, \ldots, n)$.

The proof of the following claim is similar to the proof of Claim 4 in page 247 of [9]; however, for the convenience of the reader, we include it here.

Claim 4.5.

(1) *There exists i (possibly $i = 0$) such that Σ_i is compressible in X_i, and there exist two compressing disks D_1, $D_2 \subset X_i$ that compress Σ_i into the C_1 side and C_2 side, respectively, and $D_1 \cap D_2 \neq \emptyset$.*

(2) *For each $k \neq i$, Σ_k is incompressible in X_k.*

Proof of Claim 4.5. Note that since $\Sigma \cap \mathcal{A}$ is essential in $\mathcal{A}$, any compressing disk for Σ_i is also a compressing disk for Σ.

Subclaim. For each $i = 1, 2$, there is a meridional disk D_i of C_i so that $D_i \cap \mathcal{A} = \emptyset$.

Proof. Fix i and let $D_i \subset C_i$ be a meridian disk that minimizes $|D_i \cap \mathcal{A}|$. If $D_i \cap \mathcal{A} = \emptyset$, we are done. Else, by minimality of $|D_i \cap \mathcal{A}|$ and a standard innermost disk argument, no component of $D_i \cap \mathcal{A}$ is a simple closed curve, and each outermost disk in D_i gives a boundary compression of Σ_k in X_k (for some k). By Lemma 2.1, we see that either there is a compression of Σ_k in X_k which is contained in C_i or a component of Σ_k is a boundary parallel annulus in X_k; the former is the desired conclusion of the subclaim, and the latter contradicts minimality of $|D_i \cap \mathcal{A}|$, establishing the subclaim.

Let D_1, D_2 be as in Subclaim. If $D_1 \cap D_2 = \emptyset$, then $\{D_1, D_2\}$ gives a weak reduction of Σ, contrary to our assumption. Hence, D_1 and D_2 are contained in the same component (say X_i). Finally note that if for $k \neq i$ Σ_k compresses in X_k, then the compressing disk for Σ_k and either D_1 or D_2 gives a weak reduction of Σ, a contradiction; hence, for each $k \neq i$, Σ_k is incompressible. $\square$

Since $n \geq 2$, we may suppose that Σ_1 satisfies (2), i.e., Σ_1 is incompressible in X_1. Let Σ' be the surface obtained by compressing Σ along D_1. Since $\mathcal{A}$ is incompressible and $\mathcal{A} \cap \Sigma$ is essential in $\mathcal{A}$, no component of Σ' cut open along $\mathcal{A}$ has positive Euler characteristic. Hence, $\chi(\Sigma_1) = \chi(\Sigma' \cap X_1) \geq 4 - 2g$. Suppose that each component of Σ_1 is boundary parallel in X_1. Then the minimality of $|\mathcal{A} \cap \Sigma|$ implies that each component of Σ_1 is parallel in X_1 to the annulus $X_1 \cap \partial X$. Let $X_1' = \mathrm{cl}(X_1 \backslash N(\partial X_1, X_1))$. Then, we can push $\Sigma \cap X_1'$ out of X_1'; hence, we may assume $X_1' \cap \Sigma = \emptyset$, and X_1' is contained in a compression body C_j ($j = 1$, or 2). Note that $\partial X_1'$ is incompressible in X and therefore in C_j. Hence $\partial X_1'$ is parallel to $\partial_- C_j$

((6) of Remarks 2.2). This shows that $X^{(p)}$ is homeomorphic to X_1', but this contradicts the assumption that $n \geq 2$. Hence, some component of Σ_1 is not boundary parallel in X_1. This establishes conclusion (2) of Theorem 4.1.

Case 2. Σ is weakly reducible. Take a S–T untelescoping (Section 2) of $C_1 \cup_\Sigma C_2$ to obtain a (possibly disconnected) incompressible surface S (corresponding to the union of ∂R_j in Remark 2.5 minus $\partial X^{(p)}$) with $\chi(S) \geq 6 - 2g$. Denote the connected components of S by S_j. Since $C_1 \cup_\Sigma C_2$ is non-stabilized and X is irreducible, we see, by Proposition 2.13, Lemma 2.10 and Waldhausen's classification of Heegaard surfaces of S^3 [28], that no S_j is a 2-sphere. Hence, for each j, $\chi(S_j) \geq 6 - 2g$. Since both S and $\mathcal{A}$ are incompressible, we may suppose that each component of $S \cap \mathcal{A}$ is a simple closed curve which is essential in both S and $\mathcal{A}$. Minimize $|S \cap \mathcal{A}|$ subject to this constraint.

We have the following subcases.

Subcase 2a. $S \cap \mathcal{A} = \emptyset$. Let M be the component of $X^{(p)}$ cut open along S containing ∂X. Note that in this case $\mathcal{A} \subset M$. Let $C_{M,1} \cup_{\Sigma_M} C_{M,2}$ be the Heegaard splitting of M induced from Σ. Since $\mathcal{A} \subset M$, M is not a product $\partial X \times [0,1]$. Since $C_{M,1} \cup_{\Sigma_M} C_{M,2}$ is strongly irreducible, we may suppose each component of $\Sigma_M \cap \mathcal{A}$ is essential in both Σ_M and $\mathcal{A}$; minimize $|\Sigma_M \cap \mathcal{A}|$ subject to this constraint.

Then we claim that $\Sigma_M \cap \mathcal{A} \neq \emptyset$. Suppose, for a contradiction, that $\Sigma_M \cap \mathcal{A} = \emptyset$. Then by (6) of Remarks 2.2, we have that each A_i is boundary parallel in $C_{M,1}$. Hence, each A_i is boundary parallel in $X^{(p)}$. This shows that each X_i is a solid torus, contradicting the assumption of Theorem 4.1.

By Claim 4.5, we may assume that each component of $\Sigma_M \cap (X_1 \cap M)$ $(=\Sigma_M \cap X_1)$ is incompressible in $X_1 \cap M$. Note that each component of $\mathrm{Fr}_{X^{(p)}}(X_1 \cap M)$ is either A_1 or a component of S and therefore is incompressible in X_1. This shows that each component of $\Sigma_M \cap X_1$ is incompressible in X_1.

If some component of $\Sigma_M \cap X_1$ is not boundary parallel in X_1, then we have conclusion (2) of Theorem 4.1. Suppose that each component of $\Sigma_M \cap X_1$ is a boundary parallel annulus in X_1. Pick a component of $\Sigma_M \cap X_1$ and let V be the product region that it separates from X_1; note that V is a solid torus. Since each component of S is incompressible in X, we see that no component of S is contained in V. Let $X_1' = \mathrm{cl}(X_1 \backslash N(\partial X_1, X_1))$. Since $S \cap V = \emptyset$, we can push $\Sigma_M \cap X_1'$ out of X_1'. Hence we have $\partial X_1' \subset C_{M,1}$ or $\partial X_1' \subset C_{M,2}$, say $C_{M,1}$. Since $\partial X_1'$ is incompressible in X, it is incompressible in $C_{M,1}$. Hence, $\partial X_1'$ is parallel to a component of $\partial_- C_{M,1}$

((6) of Remarks 2.2). Note that each component of $\partial_- C_{M,1}$ is either ∂X or a component of S. Since $X \neq X_1$, we see that $\partial X_1'$ is parallel to a component of S. By Proposition 2.13, there is a weak reduction of $C_1 \cup_\Sigma C_2$ which gives exactly $\partial X_1'$. This establishes conclusion (1) of Theorem 4.1.

Subcase 2b. $S \cap \mathcal{A} \neq \emptyset$. Suppose that the conclusion (2) of Theorem 4.1 does not hold; hence, for each $i \neq 0$, each component of $S \cap X_i$ is closed or a boundary parallel annulus.

Let $X_i' = \mathrm{cl}(X_i \backslash N(\partial X_i, X_i))$ $(i = 1, \ldots, n)$ and $X_0' = \mathrm{cl}(X \backslash \cup_{i=1}^n X_i')$. Note that X_0' is homeomorphic to a (disk with $n + p$ holes) $\times S^1$. Since each component of $S \cap X_i$ with non-empty boundary is a boundary parallel annulus, they can be pushed into X_0'. Hence, we have $S \cap \partial X_0' = \emptyset$, and $S \cap X_0' \neq \emptyset$. Since S is a closed incompressible surface in the Seifert fibered space X_0', it must be vertical by [6, Theorem VI.34]; that is to say, S is isotopic to a surface of the form (simple closed curve(s)) $\times S^1$. Note that any essential simple closed curve on a disk with $n + p$ holes is separating. Hence, each component of $S \cap X_0'$ is separating in X_0' and therefore also in $X^{(p)}$. Suppose that there is a component of $S \cap X_0'$, say S_0, which is not boundary parallel in $X^{(p)}$. Then S_0 separates X_0' into two components, and let X_0'' denote the closure of the component of $X_0 \backslash S_0$ such that $\partial X \subset X_0''$. Let A be a vertical annulus properly embedded in X_0'' such that one component of ∂A is contained in ∂X and the other in S_0. Note that $A \cap \partial X$ is a meridian curve and $A \cap S_0$ is an essential simple closed curve in S_0. Hence, S_0 is a swallow follow torus in $X^{(p)}$. Finally, by Proposition 2.13, we see that a weak reduction of $C_1 \cup_P C_2$ gives S_0. This establishes conclusion (1) of Theorem 4.1.

Finally suppose that each component of $S \cap X_0'$ is boundary parallel in $X^{(p)}$. Let $\widetilde{P}$ be the union of the parallel regions bounded by the components of $S \cap X_0'$. Then, let $\widetilde{X}^{(p)} = \mathrm{cl}(X^{(p)} \backslash \widetilde{P})$ and $\widetilde{C}_1 \cup_{\widetilde{\Sigma}} \widetilde{C}_2$ be the Heegaard splitting of $\widetilde{X}^{(p)}$ obtained from the S–T untelescoping of $C_1 \cup_\Sigma C_2$ by amalgamating the pieces contained in $\widetilde{X}^{(p)}$. Then, we apply the above argument to $\widetilde{C}_1 \cup_{\widetilde{\Sigma}} \widetilde{C}_2$, that is, we apply the argument of Case 1 if $\widetilde{\Sigma}$ is strongly irreducible or the argument of Case 2 is $\widetilde{\Sigma}$ is weakly reducible. Note that for $\widetilde{\Sigma}$, we do not have the case corresponding to Subcase 2b with each component of $S \cap X_0'$ being boundary parallel in $X^{(p)}$. Then, we have either conclusion (1) or conclusion (2) for $\widetilde{C}_1 \cup_{\widetilde{\Sigma}} \widetilde{C}_2$. Conclusion (2) for $\widetilde{\Sigma}$ immediately establishes conclusion (2) for $C_1 \cup_\Sigma C_2$. If we have conclusion (1) for $\widetilde{C}_1 \cup_{\widetilde{\Sigma}} \widetilde{C}_2$, then this together with Proposition 2.13 establishes conclusion (1) for $C_1 \cup_\Sigma C_2$.

This completes the proof of Theorem 4.1. $\qquad\square$

5. Numerical bounds

Theorem 5.1. *Let X be the exterior of a knot K in a closed orientable 3-manifold. Suppose that there exists a minimal genus Heegaard surface for X that weakly reduces to a swallow follow torus inducing the decomposition $K = K_1 \# K_2$; let $X_i = E(K_i)$. Then, we have*

$$g(X_1) + g(X_2) - 1 \leq g(X) \leq g(X_1) + g(X_2).$$

Remark 5.2. Recall from Section 1 that the right-hand side inequality always holds.

As a consequence of Theorem 5.1 and Corollary 4.3, we have:

Corollary 5.3. *Let $K_1, \ldots, K_n$ be m-small knots in closed orientable 3-manifolds, let $X_i = E(K_i)$, and $X = E(\#_{i=1}^n K_i)$. Suppose that no X_i is a solid torus and X is irreducible. Then, we have the following inequalities*:

$$\sum_{i=1}^n g(X_i) - (n-1) \leq g(X) \leq \sum_{i=1}^n g(X_i).$$

Proof of Theorem 5.1. By Section 4, we have a decomposition: $X = X_1^{(1)} \cup X_2$ or $X = X_1 \cup X_2^{(1)}$. Since the argument is symmetric, we may suppose that $X = X_1^{(1)} \cup X_2$.

Note that X_1 is obtained from $X_1^{(1)}$ by Dehn filling and that the core of the attached solid torus is parallel to a meridian curve on ∂X_1. Since any Heegaard surface for X_1 can be regarded as the frontier of a compression body obtained from $N(\partial X, X)$ by attaching 1-handles, this implies that the core of the attached solid torus, say l, is isotopic into every Heegaard surface of X_1, that is to say, the filling is called *good*, as in [19, Definition 4.1] (see also [20, 21]). Let $C_1' \cup_{\Sigma'} C_2'$ be a minimal genus Heegaard splitting of X_1 such that $\partial X_1 \subset C_1'$, and $l \subset C_2'$. Let $\widetilde{\Sigma}'$ be a surface obtained from Σ and $\partial N(l)$ as is shown in [19, Corollary 4.3]; note that $g(\widetilde{\Sigma}') = g(\Sigma') + 1$ (in fact, $\widetilde{\Sigma}'$ is a stabilization of Σ') and $\widetilde{\Sigma}'$ is a Heegaard surface for $X_1^{(1)}$. Thus, we get the inequality $g(X_1^{(1)}) \leq g(X_1) + 1$. Since X_1 is obtained from $X_1^{(1)}$ by Dehn filling, we immediately see that $g(X_1) \leq g(X_1^{(1)})$. Combining the two, we have

$$g(X_1) \leq g(X_1^{(1)}) \leq g(X_1) + 1.$$

On the other hand, by Proposition 2.9, we have the following equality:

$$g(X) = g(X_1^{(1)}) + g(X_2) - 1.$$

Therefore, we have

$$g(X_1) + g(X_2) - 1 \le g(X) \le g(X_1) + g(X_2).$$

□

In Section 7, we will need the following information which follows from the proof of Theorem 5.1, and so we single it out here.

Corollary 5.4. *If a minimal genus Heegaard splitting for X weakly reduces to a swallow follow torus T inducing $X = X_1^{(1)} \cup_T X_2$, then either $g(X_1) = g(X_1^{(1)})$ or $g(X_1) = g(X_1^{(1)}) - 1$, and exactly one of the following holds:*

(1) $g(X) = g(X_1) + g(X_2)$ *and* $g(X_1) = g(X_1^{(1)}) - 1$.

(2) $g(X) = g(X_1) + g(X_2) - 1$ *and* $g(X_1) = g(X_1^{(1)})$.

6. Hopf–Haken annulus theorem

Let $K_1, \ldots, K_n$ $(n \ge 1)$ be knots in closed orientable 3-manifolds. Let $X_1, \ldots, X_n$, and X be the exteriors of $K_1, \ldots, K_n$, and $K = \#_{i=1}^n K_i$, respectively.

Recall from Section 4 that for $p \ge 1$, $X^{(p)} = \mathrm{cl}(X \setminus \cup_{i=1}^p N(\gamma_i))$, where $\gamma_1, \ldots, \gamma_p$ are simple closed curves obtained by pushing mutually disjoint meridian curves in ∂X into $\mathrm{int}\, X$ (with $X^{(0)} = X$). One can think of $X^{(p)}$ as the exterior of the link obtained from K by taking connected sum with the Hopf link n times. Note that the exterior of the Hopf link has an essential annulus connecting the distinct boundary components, where one boundary component of the annulus is a meridian curve, and the other is a longitude. This shows that $X^{(p)}$ has an essential annulus as well (provided $p \ge 1$) and motivates the following definition (cf. Definition 3.1):

Definition 6.1 (Hopf–Haken annulus). A *Hopf spanning annulus* in $X^{(p)}$ is a properly embedded essential annulus A with one component of ∂A a meridian curve of ∂X and the other component a longitude of $\partial N(\gamma_i)$ for some i. If, in addition, there exists a Heegaard splitting $C_1 \cup_\Sigma C_2$ of $X^{(p)}$ such that A intersects Σ in one simple closed curve that is essential in A, then A is called a Hopf–Haken annulus for $C_1 \cup_\Sigma C_2$. In that case, Σ is said to admit a Hopf–Haken annulus.
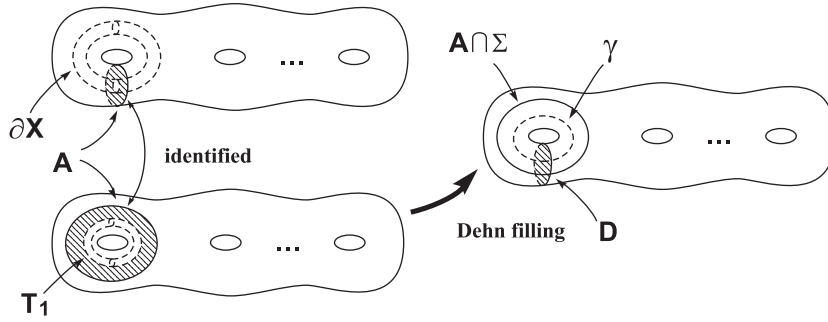
Figure 2: A primitive meridian.

By figure 2, we immediately have the following.

Proposition 6.2. *Let Y be the exterior of a knot K. Suppose that for some $p \geq 1$ there exists a Heegaard surface Σ of $Y^{(p)}$ such that $g(\Sigma) = g(Y)$ and Σ admits a Hopf–Haken annulus. Then K admits a primitive meridian.*

In this section, we prove the following theorem.

Theorem 6.3 (The Hopf–Haken annulus theorem). *Let $K_i (i = 1, \ldots, n, n \geq 1)$ be knots in closed orientable 3-manifolds. Let X_i and X be the exteriors of K_i, and $K = \#_{i=1}^n K_i$, respectively. Let $X^{(p)}$ be as above. Suppose that no X_i is a solid torus and X is irreducible. Then, one of the following holds.*

(1) *For each p (≥ 1), there exists a minimal genus Heegaard surface of $X^{(p)}$ that admits a Hopf–Haken annulus.*

(2) *For some i $(1 \leq i \leq n)$, X_i contains a meridional essential surface S such that $\chi(S) \geq 4 - 2g$, where $g = g(X^{(p)})$.*

Remark 6.4. Note that in Theorem 6.3 we cannot expect every minimal genus Heegaard surface to satisfy conclusion (1), even when conclusion (2) is not satisfied. To see this, consider the following example: let K_m be the knots constructed by Morimoto–Sakuma–Yokota in [18] and X_m their exteriors. Then, $g(X_m) = 2$ (by [18]) and $g(X_m^{(1)}) = 3$ (by [10]). There is an essential torus $T \subset X_m$ giving the decomposition $X_m^{(1)} = D(2) \cup X_m'$, where X_m' is homeomorphic to X_m and $D(2)$ is (disk with two holes) $\times S^1$. Note that $\partial D(2) = \partial X_m^{(1)} \cup T$. By Proposition 2.14 and the comment after it, $g(D(2)) = 2$, and there exists a genus two Heegaard splitting of $D(2)$

(say $C_1' \cup_{\Sigma'} C_2'$), with $\partial_- C_1' = \partial X_m^{(1)}$ and $\partial_- C_2' = T$. Let $C_1 \cup_{\Sigma} C_2$ be the Heegaard splitting of $X_m^{(1)}$ obtained by amalgamating $C_1' \cup_{\Sigma'} C_2'$ and a minimal genus Heegaard splitting of X_m'. By Lemma 2.7, $C_1 \cup_{\Sigma} C_2$ is a minimal genus Heegaard splitting of $X_m^{(1)}$. By the definition of amalgamation, we see that $\partial_- C_1 = \partial X_m^{(1)}$. We conclude that $C_1 \cup_{\Sigma} C_2$ does not admit a Hopf–Haken annulus, for if it did, then the components of $\partial X_m^{(1)}$ would be in distinct compression bodies.

Although we do not know if K_m is m-small, this example seems to indicate that replacing "there exists" with "for every" in conclusion (1) of Theorem 6.3 is not likely to be possible. Note that the only property of K_m used in the construction above is that $g(X^{(1)}) = g(X) + 1$.

As consequences of Theorem 6.3, we obtain the following.

Corollary 6.5. *Let K_i, X_i, K, X and $X^{(p)}$ be as in the statement of Theorem 6.3. If $g(X^{(p)}) = g(X)$ for some $p \geq 1$, and no X_i contains a meridional essential surface S with $\chi(S) \geq 4 - 2g(X)$, then K admits a primitive meridian.*

Corollary 6.6. *Let K_i, X_i, X and $X^{(p)}$ be as in the statement of Theorem 6.3. Suppose that no X_i contains a meridional essential surface S with $\chi(S) \geq 4 - 2g(X^{(1)})$. Then, $g(X^{(1)}) = g(X^{(1)}, \partial X, \partial X^{(1)} \setminus \partial X))$, i.e., there is a minimal genus Heegaard surface for $X^{(1)}$ separating the boundary components.*

Proof of Corollary 6.5. By Theorem 6.3, there exists a minimal genus Heegaard splitting of $X^{(p)}$ (say $C_1 \cup_{\Sigma^*} C_2$) and a Hopf–Haken annulus A^* for Σ^*. By assumption, $g(\Sigma^*) = g(X)$. Hence, by Proposition 6.2, K admits a primitive meridian. $\square$

Proof of Corollary 6.6. Our assumption implies that conclusion (2) of Theorem 6.3 cannot hold. With $p = 1$, the minimal genus Heegaard surface for $X^{(1)}$ described in conclusion (1) of Theorem 6.3 separates the boundary components, thus proving the corollary. $\square$

Proof of Theorem 6.3. Recall from the proof of Theorem 4.1 that we may assume that each K_i is a prime knot, and both existence and uniqueness of prime decomposition holds for K. We define the complexity of $X^{(p)}$ to be (n, p) with the lexicographic order (note that n is the number of prime factors). Theorem 6.3 is proved by induction on this complexity. As the first step of the induction, we prove the following.

Assertion 6.7. *Suppose that $(n, p) = (1, 1)$. Then Theorem 6.3 holds.*

Proof of Assertion 6.7. Let $C_1 \cup_\Sigma C_2$ be a minimal genus Heegaard splitting of $X^{(1)}$ and assume (without loss of generality) that $\partial X \subset \partial_- C_1$. Let A be a Hopf spanning annulus.

Case 1. $C_1 \cup_\Sigma C_2$ is strongly irreducible. In this case, we may suppose that each component of $A \cap \Sigma$ is a simple closed curve which is essential in A and Σ. Minimize $|A \cap \Sigma|$ subject to this constraint (note that $A \cap \Sigma \neq \emptyset$ by (6) of Remarks 2.2). If $|A \cap \Sigma| = 1$, then we have conclusion (1). Hence, suppose that $|A \cap \Sigma| \geq 2$. If some component of $A \cap C_j$ is compressible in C_j (for $j = 1$ or 2), then by compressing A along that disk, we obtain a disk in X with meridional boundary. This fact, together with the irreducibility of X, shows that X is a solid torus, contradicting the assumption of Theorem 6.3 (note that $X_1 = X$). Hence, every component of $A \cap C_j$ is incompressible in C_j ($j = 1, 2$). This, together with the minimality of $|A \cap \Sigma|$, implies that each component of $A \cap C_2$ is essential in C_2. Hence, there is a meridian disk of C_2 which is disjoint from $A \cap C_2$ ((4) of Remarks 2.2). Then let $\mathcal{D}$ be a maximal system of mutually disjoint and non-parallel meridian disks of C_2 such that $\mathcal{D} \cap (A \cap C_2) = \emptyset$. Let X' be the manifold obtained by cutting $X^{(1)}$ along A, and let Σ' be the image of Σ in X'. Note that X' is homeomorphic to X. Let $\mathcal{S}'$ be the surface obtained by compressing Σ' along (the image of) $\mathcal{D}$.

Claim 6.8. *$\mathcal{S}'$ is incompressible in X'.*

Proof. Suppose that there is a compressing disk D for $\mathcal{S}'$ in X'. Regard D as a disk in $X^{(1)}$. By the No Nesting Lemma [22], there exists a meridian disk D' in C_1 or C_2 with $\partial D' = \partial D$. By using a standard innermost disk argument, we may suppose that $D' \cap A = \emptyset$. Then, by isotopy, we may suppose that $D' \cap \mathcal{D} = \emptyset$. By the maximality of $\mathcal{D}$, we see that $D' \subset C_1$. Then, $D' \cup \mathcal{D}$ provides a weak reduction for Σ, a contradiction. $\square$

If there is a component, say S', of $\mathcal{S}'$ such that $\partial S' \neq \emptyset$ and S' is not boundary parallel in X', then the corresponding surface in X establishes conclusion (2) of Theorem 6.3 (note that $\chi(S') \geq 4 - 2g$). Hence, we may suppose that each component of $\mathcal{S}'$ is either closed or a boundary parallel annulus. Let A_1 and A_2 be the closures of the components of $A \backslash \Sigma$ such that $A_1 \cap \partial X \neq \emptyset$ (hence $A_1 \subset C_1$) and A_2 is adjacent to A_1 (i.e., $A_2 \subset C_2$ and $A_2 \cap A_1 \neq \emptyset$); See figure 3.

Let C_2^* be the component of $\mathrm{cl}(C_2 \backslash N(\mathcal{D}))$ which contains A_2. Note that C_2^* naturally inherits a compression body structure from C_2 ((2) of
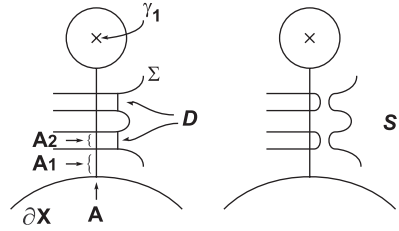
Figure 3: Σ and $\mathcal{S}'$.

Remarks 2.2). Since each component of $\mathcal{S}'$ with non-empty boundary is an annulus, we see that $\partial_+ C_2^*$ is a torus; hence either C_2^* is a solid torus or it is a trivial compression body homeomorphic to $(\text{torus}) \times [0, 1]$.

Subcase 1a. C_2^* is a solid torus; see figure 4.

Let A^* be an annulus obtained from ∂C_2^* by performing surgery along A_1. We regard A^* as an annulus properly embedded in X $(= X^{(1)} \cup N(\gamma_1))$. If A^* is not boundary parallel, then K is not prime, contradicting our assumptions. Thus, A^* is boundary parallel. If the boundary parallel region does not contain C_2^*, then X is homeomorphic to the manifold obtained from C_2^* by attaching the parallel region along an annulus corresponding to A^*. This shows that X is a solid torus, contradicting the assumptions of Theorem 6.3. Hence the parallel region contains C_2^*. This implies that A_2 is a longitudinal annulus in C_2^*. Hence, the core curve of A_2, say γ, is a core curve of C_2^*. Since γ is a core curve of C_2^*, we see that Σ is a Heegaard surface of $\text{cl}(X \backslash N(\gamma, C_2^*))$. Then, A_1 and a part of A_2 form a Hopf spanning annulus for γ, and Σ intersects this annulus in a single essential curve. Note that γ_1 is isotoped to γ along A. Hence, $\text{cl}(X \backslash N(\gamma, C_2^*))$ is homeomorphic to $X^{(1)}$. These facts establish the conclusion (1) of Theorem 6.3.
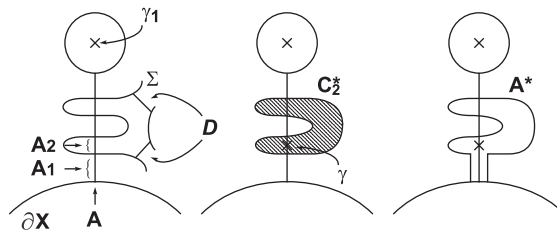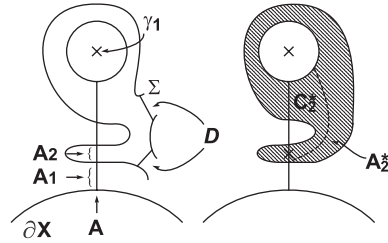

Figure 4: Subcase 1a.

Figure 5: Subcase 1b.

Subcase 1b. C_2^* is a trivial compression body. Since $\partial X \subset C_1$, $\partial_- C_2^* = T_1$ (recall from Section 4 that $T_1 = \partial N(\gamma_1)$). Note that there is a vertical annulus A_2^* in C_2^* such that $A_2^* \cap \partial_+ C_2^* = A_1 \cap \partial_+ C_2^*$. Then, $A_1 \cup A_2^*$ is a Hopf–Haken annulus for Σ (figure 5), and we obtain conclusion (1) of Theorem 6.3.

Case 2. $C_1 \cup_\Sigma C_2$ is weakly reducible. Let $(C_1^1 \cup_{\Sigma^1} C_2^1) \cup \cdots \cup (C_1^m \cup_{\Sigma^m} C_2^m)$ be a S–T untelescoping of $C_1 \cup_\Sigma C_2$ (Remark 2.5). Let $\mathcal{F} = \cup_{i=1}^r F_i$ be the collection of incompressible surfaces appearing in the untelescoping, where $F_1, \ldots, F_r$ are the connected components. Note that since $C_1 \cup_\Sigma C_2$ is minimal genus, no F_i is boundary parallel in $X^{(1)}$ (Remark 2.5). We may suppose that each component of $A \cap \mathcal{F}$ is a simple closed curve which is essential in A and $\mathcal{F}$. Minimize $|A \cap \mathcal{F}|$ subject to this constraint.

Subcase 2a. $\mathcal{F} \cap A \neq \emptyset$. Without loss of generality, we may suppose that $F_1 \cap A \neq \emptyset$. Note that $\chi(F_1) \geq 6 - 2g$. Let X' be the manifold obtained by cutting $X^{(1)}$ along A (note that $X' \cong X$) and F' the image of F_1 in X'. Since each component of $F_1 \cap A$ is essential in F_1, F' is incompressible in X'. Hence, if there exists a component of F' which is not boundary parallel in X', then we have conclusion (2) of Theorem 6.3.

Thus, we may suppose that all components of F' are boundary parallel in X'. In this case, each component of F' is an annulus. Let A', A'' be the images of A in $\partial X'$, and let A'_X, A'_T be the images of ∂X, T_1 in $\partial X'$, respectively. Note that A'_X, A'_T are annuli such that $\partial X' = A'_X \cup A' \cup A'_T \cup A''$; see figure 6.

Suppose that there exists a component of F', say A^*, such that one component of ∂A^* is contained in A' and the other in A''. Then A^* is parallel to A'_X or A'_T. We may suppose that A^* is the outermost component with that property.

Suppose that the components of ∂A^* are not identified in F_1 (see figure 7(A)). Let A^{**} be the component of F' which is adjacent to A^* and
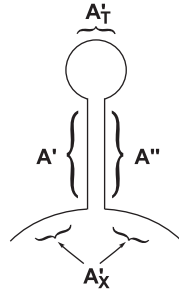
Figure 6: Subcase 2a.

is contained in the parallel region bounded by A^*. Then it is easy to see that A^{**} is parallel to an annulus in A' or A''. However, this contradicts the minimality of $|A \cap \mathcal{F}|$. Hence, the components of ∂A^* are identified in F_1. This shows that F_1 is a boundary parallel torus in $X^{(1)}$, contradicting Remark 2.5.

Now suppose that for each component A^* of F', we have that either $\partial A^* \subset A'$ or $\partial A^* \subset A''$. Let us fix A_1', a component of F'. We may suppose (without loss of generality) that $\partial A_1' \subset A'$. Let A_2' be another component of F' such that A_2' is adjacent to A_1' in F_1. Then $\partial A_2' \subset A''$. Let N_1' be the parallel region between A_1' and a subsurface of $\partial X'$. By the minimality of $|\mathcal{F} \cap A|$, we see that $N_1' \cap \partial X'$ is not contained in A'. Hence, $N_1' \cap \partial X'$ contains A'', and $A_2' \subset N_1'$; see figure 7(B). However, this implies that A_2' admits a parallel region N_2' such that $N_2' \cap \partial X'$ is contained in A'', contradicting the minimality of $|\mathcal{F} \cap A|$. Hence, there exists a component of F' which is not boundary parallel, and this establishes conclusion (2) of Theorem 6.3.

Subcase 2b. $\mathcal{F} \cap A = \emptyset$. We may assume that $\partial X \subset C_1^1$. In this case, we basically apply the arguments of Case 1 of Assertion 6.7 to the strongly
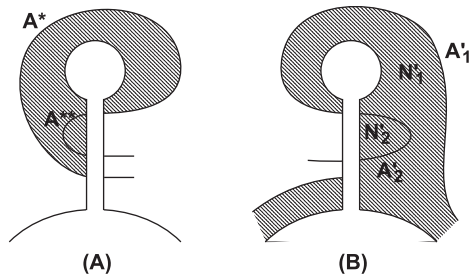

Figure 7: The annuli in Subcase 2a.

irreducible Heegaard splitting $C_1^1 \cup_{\Sigma^1} C_2^1$. Let $M = C_1^1 \cup C_2^1$ (i.e., the ambient manifold of the Heegaard splitting $C_1^1 \cup_{\Sigma^1} C_2^1$). Then, $A \subset M$. Since $C_1^1 \cup_{\Sigma^1} C_2^1$ is strongly irreducible, we may suppose that each component of $A \cap \Sigma^1$ is a simple closed curve which is essential in both A and Σ^1. Then by Remarks 2.2(6), $A \cap \Sigma^1 \neq \emptyset$ and each component of $A \cap C_i^1$ is incompressible in C_i^1 ($i = 1, 2$). If $|A \cap \Sigma^1| = 1$, then by Proposition 3.5, we have conclusion (1) of Theorem 6.3. Hence, we may suppose that $|A \cap \Sigma^1| \geq 2$.

Let A_1, A_2 be the closures of the components of $A \backslash \Sigma^1$ such that $A_1 \cap \partial X \neq \emptyset$ (hence $A_1 \subset C_1^1$) and $A_2 \subset C_2^1$ with $A_2 \cap A_1 \neq \emptyset$. Then by arguments of Case 1 of Assertion 6.7, we can show that there exists a union of mutually disjoint meridian disks of C_2^1 (say $\mathcal{D}$) which is non-empty and $\mathcal{D} \cap (A \cap C_2^1) = \emptyset$. Let M' be the manifold obtained from M by cutting along A, and let Σ' be the image of Σ^1 in M'. Let $\mathcal{S}'$ be the surface obtained by compressing Σ' along $\mathcal{D}$. Then by using the argument of the proof of Claim 6.8 in Case 1 of Assertion 6.7, we can show that $\mathcal{S}'$ is incompressible in M'.

If there is a component of $\mathcal{S}'$ which is neither closed nor boundary parallel in M', then it is easy to see that the surface establishes conclusion (2) of Theorem 6.3. Therefore, we may suppose that each component of $\mathcal{S}'$ is either closed or boundary parallel in M'.

Let C_2^* be the component of $\mathrm{cl}(C_2^1 \backslash N(\mathcal{D}))$ which contains A_2. Then, C_2^* is a compression body and $\partial_+ C_2^*$ is a torus (for a detailed argument, see Case 1 of Assertion 6.7). Hence, we have the following three subcases:

Subcase A. C_2^* is a solid torus,

Subcase B. C_2^* is a trivial compression body homeomorphic to (torus) $\times [0, 1]$ with $\partial_- C_2^* = T_1$, or

Subcase C. C_2^* is a trivial compression body homeomorphic to (torus) $\times [0, 1]$ with $\partial_- C_2^* = F_k \subset \mathcal{F}$.

For Subcases A and B, we apply the argument of Subcases 1a and 1b in Case 1 of Assertion 6.7 to show that there is a Hopf–Haken annulus for Σ^1 in M with, possibly, the position of γ_1 changed. Then by Proposition 3.5, we have conclusion (1) of Theorem 6.3. We may therefore suppose that Subcase C holds; see figure 8.

Let $\widetilde{A}$ be the annulus obtained from $\partial_+ C_2^*$ by performing surgery along A_1. Regard $\widetilde{A}$ as an annulus in $X (= X^{(1)} \cup N(\gamma_1))$. If $\widetilde{A}$ is not boundary parallel, then we have a contradiction to K being prime. Thus $\widetilde{A}$ is boundary parallel. Let $\widetilde{N}$ be the parallel region. Then, $\widetilde{N}$ is a solid torus and
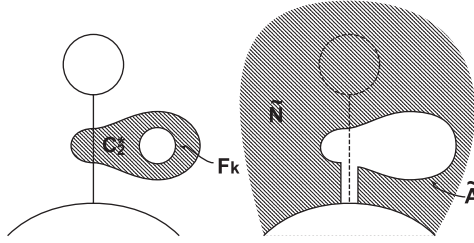
Figure 8: Subcase C.

$\widetilde{N} \cap \partial X = \partial \widetilde{N} \cap \partial X$ is a longitudinal annulus, that is, the annulus wraps around $\widetilde{N}$ once.

Suppose that $\widetilde{N} \supset F_k$. Since $\partial_+ C_2^*$ and F_k are parallel, this implies that F_k bounds a solid torus in X which is isotopic to $\widetilde{N}$. Since F_k is incompressible in $X^{(1)}$, γ_1 must be contained in this solid torus. Note that A connects γ_1 to ∂X. However, this implies that $A \cap F_k \neq \emptyset$, contradicting the condition of Subcase 2b. Hence, we have $\widetilde{N} \cap F_k = \emptyset$ (see the right side of figure 8). This shows that $\partial_+ C_2^*$ (hence also F_k which is parallel to it) is boundary parallel in X. By Remark 2.5, F_k is not boundary parallel in $X^{(1)}$ and, therefore, γ_1 must be contained in the parallel region between F_k and ∂X. Then recall that $A \cap F_k = \emptyset$ (the condition of Subcase 2b), and ∂A consists of γ_1 and a meridian curve of ∂X. This shows that $M \cong D(2)$ (with $D(2) \cong$ (disk with two holes) $\times S^1$, as defined in the end of Section 2).

As conclusions of the above, we have the following:

(1) F_k is a boundary parallel torus in X, and

(2) M is homeomorphic to a (disk with two holes) $\times S^1$, where $\partial M = \partial X \cup T_1 \cup F_k$.

Let $X^* = \mathrm{cl}(X^{(1)} \backslash M)$ ($\cong X$). Note that $M \cap X^* = F_k$. By Proposition 2.13, there is a weakly reducing collection of disks Δ for Σ such that $\widehat{\Sigma}(\Delta) = F_k$. Then by Proposition 2.9, we see that $g(X^{(1)}) = g(M) + g(X^*) - 1$. By (2) of Remarks 3.2, there exists a minimal genus Heegaard splitting $\bar{C}_1' \cup_{\bar{\Sigma}'} \bar{C}_2'$ of M such that $\bar{\Sigma}' \cap A$ consists of a single simple closed curve that is essential in A. Take the amalgamation of $\bar{C}_1' \cup_{\bar{\Sigma}'} \bar{C}_2'$ and a minimal genus Heegaard splitting of X^*. Then by Lemma 2.7, and Proposition 3.5, we see that conclusion (1) of Theorem 6.3 holds.

This completes the proof of Assertion 6.7. □

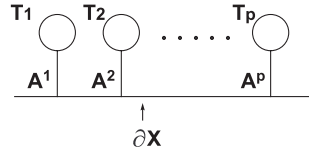**Assertion 6.9.** *Suppose that $n = 1$. Then, Theorem 6.3 holds.*

Figure 9: Hopf spanning annuli.

Proof of Assertion 6.9. Assertion 6.9 is proved by inducting on p. Assertion 6.7 gives the first step of the induction. Hence, we suppose that $p \geq 2$. Let $C_1 \cup_{\Sigma^{(p)}} C_2$ be a minimal genus Heegaard splitting for $X^{(p)}$ and assume that $\partial X \subset \partial_- C_1$. Recall (from Section 4) that $\partial X^{(p)} = \partial X \cup T_1 \cup \cdots \cup T_p$. Let $A^1, \ldots, A^p$ be mutually disjoint Hopf spanning annuli between ∂X and $T_1, \ldots, T_p$, respectively, and $A = \cup_{j=1}^p A^j$; see figure 9. By (6) of Remarks 2.2, we see that for each j, $A^j \cap \Sigma^{(p)} \neq \emptyset$.

Case 1. $C_1 \cup_{\Sigma^{(p)}} C_2$ is strongly irreducible. In this case, we may suppose that $A \cap \Sigma^{(p)}$ consists of simple closed curves that are essential in A and $\Sigma^{(p)}$. Subject to that constraint, we minimize $|A \cap \Sigma^{(p)}|$. If $|A^j \cap \Sigma^{(p)}| = 1$ for some j, then we have conclusion (1) of Theorem 6.3. Hence, suppose that for each j, $|A^j \cap \Sigma^{(p)}| \geq 2$. The following argument is quite similar to that of Step 1 in the proof of Assertion 6.7; hence we will only sketch it here.

Let $\mathcal{D}$ be a maximal collection of meridian disks for C_2 so that $\mathcal{D} \cap (A \cap C_2) = \emptyset$; by (4) of Remarks 2.2, $\mathcal{D} \neq \emptyset$. Let X' be the manifold obtained by cutting $X^{(p)}$ along A, and let Σ' be the image of $\Sigma^{(p)}$ in X'. Let $\mathcal{S}'$ be the surface obtained from Σ' by compressing along $\mathcal{D}$. Then by the same argument as in the proof of Claim 6.8 of the proof of Assertion 6.7, we see that $\mathcal{S}'$ is incompressible in X'.

If there exists a component of $\mathcal{S}'$ with non-empty boundary that is not boundary parallel in X', then we have conclusion (2) of Theorem 6.3. Thus, we may assume that each component of $\mathcal{S}'$ is either closed or boundary parallel. (Hence each component of $\mathcal{S}'$ with non-empty boundary is a boundary parallel annulus.) Let A_1, A_2 be the closures of the components of $A^1 \setminus \Sigma^{(p)}$ such that $A_1 \cap \partial X \neq \emptyset$ (hence $A_1 \subset C_1$) and that $A_2 \subset C_2$ with $A_2 \cap A_1 \neq \emptyset$. Let C_2^* be the component of $\mathrm{cl}(C_2 \setminus N(\mathcal{D}))$ which contains A_2. Then C_2^* is either a solid torus or homeomorphic to (torus) $\times [0, 1]$.

Subcase 1a. C_2^* is a solid torus. In this subcase, the argument of Subcase 1a of the proof of Assertion 6.7 can be applied to show that we can isotope γ_1 along A^1 to a core curve of A_2 to show that there exists a minimal genus Heegaard splitting of $X^{(p)}$ that intersects a Hopf spanning annulus in a single essential simple closed curve (conclusion (1) of Theorem 6.3).

Subcase 1b. C_2^* is homeomorphic to (torus) $\times$ $[0,1]$. In this case, the argument of Subcase 1.b of the proof of Assertion 6.7 applies to show that conclusion (1) of Theorem 6.3 holds.

Case 2. $C_1 \cup_{\Sigma^{(p)}} C_2$ is weakly reducible. Let

$$(C_1^1 \cup_{\Sigma^1} C_2^1) \cup \cdots \cup (C_1^m \cup_{\Sigma^m} C_2^m)$$

be a S–T untelescoping of $C_1 \cup_{\Sigma^{(p)}} C_2$, where $\partial X \subset \partial_- C_1^1$. Let $\mathcal{F}$ be the union of the essential surface appearing in the above untelescoping (Remark 2.5) and $\{F_i\}_{i=1}^r$ its connected components. We may suppose that each component of $\mathcal{F} \cap A$ is a simple closed curve that is essential in both $\mathcal{F}$ and A, and $|\mathcal{F} \cap A|$ is minimal subject to this constraint. Then we have the following subcases.

Subcase 2a. $\mathcal{F} \cap A \neq \emptyset$. Without loss of generality, we may suppose that $F_1 \cap A^1 \neq \emptyset$. Let F' be the image of F_1 in X' (X' as in Case 1 of Assertion 6.9). Since each component of $F_1 \cap A$ is essential in F_1, F' is incompressible in X'. Hence, if there exists a component of F' which is not boundary parallel, then we have conclusion (2) of Theorem 6.3. We suppose that each component of F' is boundary parallel in X' (see figure 10).

This implies that F' can be isotoped to be contained in $N(\partial X', X')$. Here, we note that $X^{(p)}$ admits the decomposition

$$X^{(p)} = X^* \cup_T D(p+1),$$

where $X^* \cong X$, $D(p+1)$ is a (disk with $p+1$ holes) $\times$ S^1 (see Section 2), and a meridian of X^* is identified with $\{*\} \times S^1$, where $*$ is a point on the boundary of the disk.

Since F' is contained in $N(\partial X', X')$, we may suppose that F_1 is contained in int$(D(p+1))$. Then by [6, VI.34, VI.18], we see that F_1 is vertical, i.e., F_1 is isotopic to a torus of the form (simple closed curve) $\times$ S^1 in $D(p+1)$.
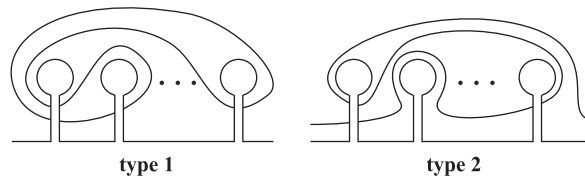

type 1   type 2

Figure 10: F_1 in Subcase 2a.

Hence, F_1 gives the following decomposition:

$$X^{(p)} = X^{(p_1)} \cup_{F_1} D(p_2),$$

where $p_1 + p_2 = p + 1$. We say that F_1 is of type 1 if $\partial X \subset \partial X^{(p_1)}$ and of type 2 if $\partial X \subset \partial D(p_2)$, see figure 10.

Suppose that F_1 is of type 1. Here, we note that $p_2 \geq 2$ (Remark 2.5); hence $p_1 < p$ and we may apply the inductive hypothesis on $X^{(p_1)}$, that is, there is a minimal genus Heegaard splitting $C_1'' \cup_{\Sigma''} C_2''$ of $X^{(p_1)}$ and a Hopf spanning annulus $A_1'' \subset X^{(p_1)}$ between ∂X and F_1 so that $\Sigma'' \cap A_1''$ is a single simple closed curve which is essential in A_1''. On the other hand, by (2) of Remarks 3.2, there is a minimal genus Heegaard splitting of $D(p_2)$ and an essential annulus $A_2'' \subset D(p_2)$ such that the intersection of the Heegaard surface and A_2'' is a single simple closed curve which is essential in A_2'', and $A_2'' \cap F_1 = A_1'' \cap F_1$. Then by applying Lemma 2.7 and Propositions 2.9 and 3.6 to these Heegaard splittings, we see that conclusion (1) of Theorem 6.3 holds.

Suppose that F_1 is of type 2. By Remark 2.5, we see that $\partial D(p_2)$ contains at least one T_i. By changing subscripts if necessary, we may suppose that $T_1 \subset \partial D(p_2)$. We take a minimal genus Heegaard splitting of $D(p_2)$ and an essential annulus $A_2'' \subset D(p_2)$ such that A_2'' intersects the Heegaard surface in a single simple closed curve which is essential in A_2'' and that A_2'' connects ∂X to T_1 (Remarks 3.2(2)). Then, take an amalgamation of this Heegaard splitting and a minimal genus Heegaard splitting of $X^{(p_1)}$. By Lemma 2.7 and Propositions 2.9 and 3.5, we see that we have conclusion (1) of Theorem 6.3.

Subcase 2b. $\mathcal{F} \cap A = \emptyset$. We apply arguments similar to that of Case 1 of Assertion 6.9 to the strongly irreducible Heegaard splitting $C_1^1 \cup_{\Sigma^1} C_2^1$. Let $M = C_1^1 \cup C_2^1$. By the argument of Case 1 of Assertion 6.9 of the proof of Assertion 6.7, we see that after isotopy each component $A \cap C_2^1$ is essential in C_2^1. Let $\mathcal{D}$ be a maximal collection of meridian disks for C_2^1 such that $\mathcal{D} \cap (A \cap C_2^1) = \emptyset$ (as above, $\mathcal{D} \neq \emptyset$). Let M' be the manifold obtained by cutting M along A, and let Σ' be the image of Σ^1 in M'. Let $\mathcal{S}'$ be the surface obtained by compressing Σ' along $\mathcal{D}$. Then by using the argument of Claim 6.8 in the proof of Assertion 6.7, we can show that $\mathcal{S}'$ is incompressible in M'.

If there is a component of $\mathcal{S}'$ with non-empty boundary that is not boundary parallel, then we have conclusion (2) of Theorem 6.3. Suppose then that each component of $\mathcal{S}'$ is either closed or boundary parallel in X'. As before, let A_1, A_2 be the closures of components of $A^1 \backslash \Sigma^1$ such that

$A_1 \cap \partial X \neq \emptyset$ and that $A_2 \subset C_2^1$ with $A_2 \cap A_1 \neq \emptyset$. Let C_2^* be the component of $\mathrm{cl}(C_2^1 \backslash N(\mathcal{D}))$ containing A_2. As in the proof of Assertion 6.7, we have the following subcases.

Subcase A. C_2^* is a solid torus.

Subcase B. C_2^* is a trivial compression body homeomorphic to (torus) $\times[0,1]$ with $\partial_- C_2^* = T_i$.

Subcase C. C_2^* is a trivial compression body homeomorphic to (torus) $\times[0,1]$ with $\partial_- C_2^* = F_k \subset \mathcal{F}$.

For Subcases A and B, we apply the arguments of Subcases 1a and 1b in Case 1 of Assertion 6.9 and using Proposition 3.5 conclude that we have conclusion (1) of Theorem 6.3.

Suppose Subcase C holds (figure 8). Let $\widetilde{A}$ be an annulus obtained from $\partial_+ C_2^*$ by performing surgery along A_1. Regard $\widetilde{A}$ as an annulus in X $(=X^{(p)} \cup (\cup_{i=1}^p N(\gamma_i)))$. If $\widetilde{A}$ is not boundary parallel, then we have a contradiction to our assumption that $n = 1$. Therefore, we may suppose that $\widetilde{A}$ is boundary parallel in X. Then by using the argument of Subcase C of Subcase 2b of the proof of Assertion 6.7, we see that the following hold. (Note that $A \subset M \cup (\cup_{i=1}^p N(\gamma_i))$.)

(1) F_k is boundary parallel in X, and

(2) M is homeomorphic to $D(p+1)$, where $\partial M = \partial X \cup F_k \cup (T_1 \cup \cdots \cup T_p)$.

This gives the following decomposition:

$$X^{(p)} = X \cup_{F_k} D(p+1).$$

Then by the inductive hypothesis, Lemma 2.7 and Remark 3.2 Propositions 2.9, and 3.5, we see that conclusion (1) of Theorem 6.3 holds (for detailed arguments, see the last paragraph of Subcase 2a of this proof).

This completes the proof of Assertion 6.9. □

We now complete the proof of Theorem 6.3. Recall that the proof is carried out by induction on (n,p) with the lexicographic ordering. By Assertion 6.9, we may assume that $n \geq 2$. Let $C_1 \cup_\Sigma C_2$ be a minimal genus Heegaard splitting of $X^{(p)}$. By Theorem 4.1, we either have conclusion (2) of Theorem 6.3, or $C_1 \cup_\Sigma C_2$ weakly reduces to a swallow follow torus T,

which gives the following decomposition of $X^{(p)}$:

$$X^{(p)} = \mathcal{X}_1^{(p_1+1)} \cup_T \mathcal{X}_2^{(p_2)},$$

where $\partial X \subset \partial \mathcal{X}_1^{(p_1+1)}$, $p_1 + p_2 = p$ and $\mathcal{X}_1 = E(\mathcal{K}_1)$, $\mathcal{X}_2 = E(\mathcal{K}_2)$ (possibly $\mathcal{K}_1$ or $\mathcal{K}_2$ being the trivial knot in S^3) and $K = \mathcal{K}_1 \# \mathcal{K}_2$.

Case 1. $\mathcal{K}_1$ is the trivial knot in S^3. In this case, $\mathcal{X}_1^{(p_1+1)}$ is (disk with $p_1 + 1$ holes) $\times S^1$, $p_1 \geq 1$ (Remark 2.5). By changing the subscripts if necessary, we may suppose that $T_1 \subset \partial \mathcal{X}_1^{(p_1+1)}$. By (2) of Remarks 3.2, there exists a minimal genus Heegaard splitting $C_1^1 \cup_{\Sigma^1} C_2^1$ of $\mathcal{X}_1^{(p_1+1)}$ and an essential annulus A in $\mathcal{X}_1^{(p_1+1)}$ such that A connects ∂X to T_1, $A \cap \partial X$ is a meridian and $\Sigma^1 \cap A$ is a single simple closed curve that is essential in A. Then by Lemma 2.7 and Propositions 2.9 and 3.5, we see that conclusion (1) of Theorem 6.3 holds.

Case 2. $\mathcal{K}_2$ is the trivial knot in S^3. In this case, $\mathcal{X}_2^{(p_2)}$ is (disk with p_2 holes)$\times S^1$, $p_2 \geq 2$ (Remark 2.5). By changing the subscripts if necessary, we may suppose that $T_1 \subset \partial \mathcal{X}_2^{(p_2)}$. Since $p_2 \geq 2$, we have that $p_1 \leq p - 1$, and this allows us to apply the inductive hypothesis to $\mathcal{X}_1^{(p_1+1)}$. If conclusion (2) holds for $\mathcal{X}_1^{(p_1+1)}$, then conclusion (2) holds for $X^{(p)}$ (note that $K = \mathcal{K}_1$, and $g(\mathcal{X}_1^{(p_1+1)}) \leq g(X^{(p)})$). Hence we may suppose that conclusion (1) holds for $\mathcal{X}_1^{(p_1+1)}$, i.e., there exists a minimal genus Heegaard splitting (say $C_1^1 \cup_{\Sigma^1} C_2^1$) of $\mathcal{X}_1^{(p_1+1)}$ admitting a Hopf–Haken annulus A^* so that $A^* \cap T \neq \emptyset$. By (2) of Remarks 3.2, there exists a minimal genus Heegaard splitting $C_1^2 \cup_{\Sigma^2} C_2^2$ of $\mathcal{X}_2^{(p_2)}$ and an essential annulus A in $\mathcal{X}_2^{(p_2)}$ such that A connects T_1 to T, $A \cap T = A^* \cap T$ and $A \cap \Sigma^2$ is a simple closed curve which is essential in A. Then by Lemma 2.7 and Propositions 2.9 and 3.6, we see that $C_1^1 \cup_{\Sigma^1} C_2^1$ and $C_1^2 \cup_{\Sigma^2} C_2^2$ are amalgamated to give a minimal genus Heegaard splitting of $X^{(p)}$ admitting a Hopf–Haken annulus; hence conclusion (1) of Theorem 6.3 holds.

Case 3. Both $\mathcal{K}_1$ and $\mathcal{K}_2$ are non-trivial knots. Then we have the following subcases:

Subcase 3a. $p_2 = 0$. Since $p_1 + p_2 = p > 0$, this implies that $p_1 > 0$. Note that T_1 is contained in $\mathcal{X}_1^{(p_1+1)}$. Recall that we used lexicographic order on (n, p) and since $\mathcal{K}_2$ is a non-trivial knot, the number of prime factors of $\mathcal{K}_1$ is strictly less than that of K. We may therefore apply the inductive

hypothesis to $\mathcal{X}_1^{(p_1+1)}$. If conclusion (2) of Theorem 6.3 holds for $\mathcal{X}_1^{(p_1+1)}$, then conclusion (2) holds for $X^{(p)}$. Suppose that conclusion (1) holds for $\mathcal{X}_1^{(p_1+1)}$. Then we obtain a minimal genus Heegaard splitting $C_1^1 \cup_{\Sigma^1} C_2^1$ of $\mathcal{X}_1^{(p_1+1)}$ and an essential annulus A connecting ∂X to T_1 such that $A \cap \Sigma^1$ is a single simple closed curve which is essential in A. Then by Lemma 2.7 and Propositions 2.9 and 3.5, we see that conclusion (1) of Theorem 6.3 holds.

Subcase 3b. $p_2 > 0$. We may assume that T_1 is contained in $\mathcal{X}_2^{(p_2)}$. Apply the inductive hypothesis to both $\mathcal{X}_1^{(p_1+1)}$ and $\mathcal{X}_2^{(p_2)}$. If conclusion (2) of Theorem 6.3 holds for $\mathcal{X}_1^{(p_1+1)}$ or $\mathcal{X}_2^{(p_2)}$, then conclusion (2) holds for $X^{(p)}$. Thus, we may assume that conclusion (1) holds for both $\mathcal{X}_1^{(p_1+1)}$ and $\mathcal{X}_2^{(p_2)}$. Then we obtain minimal genus Heegaard splittings $C_1^1 \cup_{\Sigma^1} C_2^1$ and $C_1^2 \cup_{\Sigma^2} C_2^2$, respectively, and Hopf–Haken annuli $A_1 \subset \mathcal{X}_1^{(p_1+1)}$ and $A_2 \subset \mathcal{X}_2^{(p_2)}$ for Σ^1 and Σ^2, with A_1 connecting ∂X to T and A_2 connecting T to T_1, and $A_1 \cap T = A_2 \cap T$. Then by Lemma 2.7 and Propositions 2.9 and 3.6, we see that conclusion (1) of Theorem 6.3 holds.

This completes the proof of Theorem 6.3. □

7. Morimoto's conjecture for m-small knots

Proof of Theorem 1.4. For the "if" part of Theorem 1.4, recall that $g(E(\#_{i \in I} K_i)) \leq \Sigma_{i \in I} g(E(K_i))$ and $g(E(\#_{i \notin I} K_i)) \leq \Sigma_{i \notin I} g(E(K_i))$ hold in general. If $E(\#_{i \in I} K_i)$ admits a primitive meridian, then by Proposition 1.2 we have that $g(X) < g(E(\#_{i \in I} K_i)) + g(E(\#_{i \notin I} K_i))$. Combining these inequalities shows that $g(X) < \Sigma_{i=1}^{n} g(E(K_i))$ as required.

Assume for a contradiction that the "only if" part of Theorem 1.4 is false and let $\{K_i | i = 1, \ldots, n\}$ be a collection of m-small knots that gives a counterexample, which minimizes n among all such counterexamples. (Note that we may assume the uniqueness of prime decomposition of $\#_{i=1}^{n} K_i$ by Claim 4.4 of the proof of Theorem 4.1.) Let $X_i = E(K_i)$ and $X = E(\#_{i=1}^{n} K_i)$; this and all other notation is kept as in the previous sections. Let $\Sigma \subset X$ be a minimal genus Heegaard surface. By Corollary 4.3, Σ weakly reduces to a swallow follow torus, say T, giving the decomposition:

$$X = X_I^{(1)} \cup_T X_J,$$

where $I \cup J = \{1, \ldots, n\}$, $I \neq \emptyset$, $J \neq \emptyset$, $I \cap J = \emptyset$, $X_I = E(\#_{i \in I} K_i)$ and $X_J = E(\#_{i \in J} K_i)$.

By Corollary 5.4, we know that either $g(X_I) = g(X_I^{(1)})$ or $g(X_I) = g(X_I^{(1)}) - 1$. If $g(X_I) = g(X_I^{(1)})$, then by Corollary 6.5, X_I admits a

primitive meridian and we have a contradiction to our assumption that $\{K_i\}$ forms a counterexample. We may therefore assume that $g(X_I^{(1)}) - 1 = g(X_I)$. By Corollary 5.4, $g(X) = g(X_I) + g(X_J)$. Therefore, by assumption, we have that $g(X_I) + g(X_J) = g(X) < \Sigma_{i=1}^n g(X_i) = \Sigma_{i \in I} g(X_i) + \Sigma_{i \in J} g(X_i)$. Therefore, either $g(X_I) < \Sigma_{i \in I} g(X_i)$ or $g(X_J) < \Sigma_{i \in J} g(X_i)$ must hold (say the former); by the assumption of minimality on n, X_I cannot give a counterexample; therefore for some $\emptyset \neq I' \subset I$, the exterior of $\#_{i \in I'} K_i$ admits a primitive meridian, contradicting the assumption that the collection $\{K_i | i = 1, \ldots, n\}$ provides a counterexample.

This completes the proof of Theorem 1.4. $\square$

Acknowledgments

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