Wave 0-trace and length spectrum on convex co-compact hyperbolic manifolds

COLIN GUILLARMOU AND FRÉDÉRIC NAUD

For convex co-compact hyperbolic quotients $\Gamma\backslash\mathbb{H}^{n+1}$, we obtain a formula relating the 0-trace of the wave operator with the resonances and some conformal invariants of the boundary, generalizing a formula of Guillopé and Zworski in dimension 2. Then, by writing this 0-trace with the length spectrum, we prove precise asymptotics of the number of closed geodesics with an effective, exponentially small error term when the dimension of the limit set of Γ is greater than $n/2$.

1. Introduction

The purpose of this note is to establish the relation between the renormalized trace (called 0-trace) of the wave operator, the resonances, some conformal invariants of the boundary and the length spectrum on a convex co-compact quotient of the hyperbolic space $\mathbb{H}^{n+1}$. From the work of Patterson and Perry [32] and Bunke and Olbrich [7] on Selberg zeta function, we derive a trace formula similar to the one given by Guillopé and Zworski [18, 19] on surfaces. Then, we compute the 0-trace of the wave operator in terms of the primitive geodesics to obtain asymptotic expansions for the counting function of the set of closed geodesics with precise error terms related to the spectrum of the Laplacian, improving asymptotics previously obtained by Perry [34].

Actually, a Selberg formula relating resonances and length spectrum has already been obtained by Perry [35] in this setting, but Perry did not use explicitly the 0-trace of the wave operator. The first motivation for studying this 0-trace is that our formula could be extended to more general settings like asymptotically hyperbolic manifolds. Another interest of this formula comes from the conformal invariants of the boundary, which appear in the expansion of the 0-trace of the wave operator and are related to the divisors of Selberg's zeta function at certain values.

Let us first recall standard notations and definitions. A discrete group Γ of (orientation preserving) isometries of the $n + 1$-dimensional real hyperbolic space $\mathbb{H}^{n+1}$ is called convex co-compact if it admits a finite-sided fundamental domain whose intersection with $\partial\mathbb{H}^{n+1}$ does not touch the limit set of Γ. If we require that Γ has no elliptic elements, then $X = \Gamma\backslash\mathbb{H}^{n+1}$ is a hyperbolic manifold of infinite volume. The non-wandering set of the geodesic flow on the unit tangent bundle is then a compact set whose Hausdorff dimension is $2\delta + 1$, where δ is the dimension of the limit set, see [41]. The dimension δ is also the topological entropy of the geodesic flow on its non-wandering set, see the work of Sullivan [38, 39]. Let $\mathcal{P}$ be the set of primitive closed geodesics γ on X, and if $\gamma \in \mathcal{P}$, let $l(\gamma)$ denote its length.

A wide class of convex co-compact groups are given by Schottky groups of isometries, and every convex co-compact surface is in fact obtained by a quotient of $\mathbb{H}^2$ by a fuchsian Schottky group. In higher dimensions, as pointed out by Maskit [26], the set of Schottky groups does not exhaust the set of convex co-compact groups. Indeed, a 3-manifold is Schottky if and only if its fundamental group is a free group of finite type, and thus all convex co-compact manifolds obtained by a quotient of $\mathbb{H}^3$ by quasifuchsian groups cannot be Schottky.

We also recall that on a convex co-compact hyperbolic quotient $X = \Gamma\backslash\mathbb{H}^{n+1}$ equipped with its hyperbolic metric g, the Laplacian Δ_X has for continuous spectrum the half line $[n^2/4, +\infty)$ and a finite set $\sigma_{pp}(\Delta_X)$ of eigenvalues included in $\left(0, n^2/4\right)$. The modified resolvent of the Laplacian

$$R_X(s) := (\Delta_X - s(n-s))^{-1}$$

is meromorphic on $\left\{\Re(s) > n/2\right\}$ with a finite number of poles related to $\sigma_{pp}(\Delta_X)$, and extends to $s \in \mathbb{C}$ (as a map from $L^2_{\text{comp}}(X)$ to $L^2_{\text{loc}}(X)$) with poles of finite multiplicity called resonances (see [17, 27]), the multiplicity of a resonance s_0 being defined by

$$(1.1) \qquad\qquad m_{s_0} := \operatorname{rank}(\operatorname{Res}_{s_0} R_X(s)).$$

We will denote by

$$\mathcal{R} := \left\{s_0 \in \mathbb{C}; R(s) \text{ has a pole at } s = s_0\right\}$$

the set of resonances of Δ_X. We can also add that X can be compactified in a compact manifold with boundary $\bar{X}$ such that, for x a boundary defining function of $\partial\bar{X}$ in $\bar{X}$ (i.e., $\partial\bar{X} = \{x = 0\}$ and $dx|_{\partial\bar{X}} \neq 0$), the metric $x^2 g$ on

X extends smoothly to $\bar{X}$. In view of the non-uniqueness of the boundary-defining function, the boundary carries a natural conformal class of metric $[h_0]$ associated to g by taking the conformal class of $h_0 := x^2 g|_{\partial \bar{X}}$. Graham *et al.* [11] have defined some natural conformally covariant powers of the Laplacian on conformal manifolds $(M, [h_0])$ and those were identified by Graham and Zworski [14], with poles of the scattering operator on asymptotically hyperbolic Einstein manifolds with conformal infinity $(M, [h_0])$ by using scattering theory. We will denote by P_k the k-th conformal power of the Laplacian on the conformal infinity $(\partial \bar{X}, [h_0])$. Note that in even dimension of $\partial \bar{X}$, the existence of natural P_k for $k > n/2$ does not hold [10], and the definition of what we call P_k will be detailed later in that case.

On a convex co-compact hyperbolic manifold, although the trace of the wave operator is not well defined, there is a natural renormalization called 0-trace. Thus, following Joshi and Sá Barreto [25] or Guillopé and Zworski [18, 19], we can define the 0-trace of the wave operator as a distribution on $\mathbb{R}^*$ noted

$$0\text{-tr}\left(\cos\left(t\sqrt{\Delta_X - \frac{n^2}{4}}\right)\right).$$

The definition of this renormalized trace will be detailed in the next section.

We recall that the set of closed geodesics $\mathcal{P}$ is in one-to-one correspondence with primitive conjugacy classes of hyperbolic isometries in Γ. Given an hyperbolic element $h \in \Gamma$ associated to a closed geodesic γ, then there exists $\alpha \in \text{Isom}(\mathbb{H}^{n+1})$ such that for all $(x, y) \in \mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$,

$$\alpha^{-1} \circ h \circ \alpha(x, y) = e^{l(\gamma)}(O_\gamma(x), y),$$

where $O_\gamma \in SO_n(\mathbb{R})$. We will denote by $\alpha_1(\gamma), \ldots, \alpha_n(\gamma)$ the eigenvalues of O_γ, and we set[1]

$$G_\gamma(k) = \det\left(I - e^{-kl(\gamma)}O_\gamma^k\right) = \prod_{i=1}^{n}\left(1 - e^{-kl(\gamma)}\alpha_i(\gamma)^k\right).$$

We can now state the wave 0-trace formula.

---

[1]If P_γ is the Poincaré linear map associated to the primitive periodic orbit γ of the geodesic flow on the unit tangent bundle, then

$$|\det\left(I - P_\gamma^m\right)|^{1/2} = e^{(n/2)ml(\gamma)}G_\gamma(m).$$

Theorem 1.1. *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold, $(P_k)_{k \in \mathbb{N}}$ the k-th conformal Laplacian on the boundary $\partial \bar{X}$, then we have as distributions on $\mathbb{R}^*$*

$$0\text{-tr}\left(\cos\left(t\sqrt{\Delta_X - \frac{n^2}{4}}\right)\right) = \frac{1}{2}\sum_{s \in \mathcal{R}} m_s e^{-(n/2-s)|t|} + \frac{1}{2}\sum_{k \in \mathbb{N}} d_k e^{-k|t|}$$
$$- 2^{-n-1}A(X)\frac{\cosh t/2}{(\sinh |t|/2)^{n+1}},$$

where

$$d_k := \dim \ker P_k,$$

$$A(X) = \begin{cases} 0 & \text{if } n+1 \text{ is even} \\ \chi(X) \text{ the Euler characteristic of } X & \text{if } n+1 \text{ is odd} \end{cases}.$$

We also have

$$0\text{-tr}\left(\cos\left(t\sqrt{\Delta_X - \frac{n^2}{4}}\right)\right) = \sum_{\gamma}\sum_{m=1}^{\infty} \frac{l(\gamma)e^{-(n/2)ml(\gamma)}}{2G_\gamma(m)}\delta(|t| - ml(\gamma))$$
$$+ B(X)\frac{\cosh t/2}{(\sinh |t|/2)^{n+1}},$$

where the sum runs over the primitive closed geodesics of X and

$$B(X) = \begin{cases} n!!2^{-(3(n+1)/2)}(-\pi)^{-(n+1)/2}\,0\text{-vol}(X) & \text{if } n+1 \text{ is even} \\ 0 & \text{if } n+1 \text{ is odd}. \end{cases}$$

Remark 1.2. The first formula is different from Euclidean cases in view of these additional terms d_k coming from the boundary. Of course, we expect that this one could be extended to compact perturbations of hyperbolic convex co-compact manifolds and possibly to asymptotically Einstein manifolds, at least when $n+1$ is even. Observe also that it corresponds to Guillopé–Zworski trace formula in [18] in the sense that in dimension 2, we have $d_k = 0$ for all k (see [6, Proposition 4.3], where in this case d_k must be interpreted as the dimension of the kernel of the residue of the scattering matrix at $n/2 + k$).

As a corollary of the trace formula, we can study the asymptotic behaviour of the primitive geodesics counting function. We recall that $\mathcal{P}$ denotes the set of primitive closed geodesics, the set $\{l(\gamma); \gamma \in \mathcal{P}\}$ is

often referred as the length spectrum of the manifold $X = \Gamma \backslash \mathbb{H}^{n+1}$, it is a countable subset of $\mathbb{R}^+$ which accumulates only at $+\infty$ (for more details about the length spectrum of hyperbolic manifolds, we refer the reader to Buser [8] or Hejhal [22]). The basic counting function $N(T)$ for the closed geodesics is defined as usual for $T \geq 0$ by

$$N(T) := \#\{\gamma \in \mathcal{P} \; ; \; l(\gamma) \leq T\},$$

In this paper, we show the following:

Theorem 1.3. *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact manifold as above such that $\delta > n/2$. As $T \to +\infty$, we have*

$$N(T) = \mathrm{li}(e^{\delta T}) + \sum_{\beta_n(\delta) < \alpha_i < \delta} \mathrm{li}(e^{\alpha_i T}) + O\left(\frac{e^{\beta_n(\delta)T}}{T}\right),$$

where $\mathrm{li}(x) = \int_2^x dt/\log(t)$ and $\beta_n(\delta) = n/(n+1)(\frac{1}{2} + \delta)$. The coefficients α_i are in bijection with $\sigma_{pp}(\Delta_X)$ by $\alpha_i(n - \alpha_i) = \lambda_i \in \sigma_{pp}(\Delta_X)$.

Remarks 1.4. The leading term $N(T) \sim e^{\delta T}/\delta T$ has been obtained by Perry in [34] without any assumption on the dimension δ. If we let $\delta \to 1$, then we essentially recover the error term $O(e^{(3/4)T}/T)$ known for finite area surfaces; see the work of Huber, Hejhal, Randol and Sarnak [22, 23, 36, 37]. For 3-manifolds, Theorem 1.3 can be applied, for example, to the case of strictly quasifuchsian groups Γ (i.e., the limit set is a strict quasicircle) for which we know since Bowen [4] that $\delta > 1 = n/2$. In the particular case of surfaces ($n = 1$), the second author [30] has obtained unconditional exponentially small error terms using transfer operator techniques and thermodynamical formalism. We point out that this asymptotic expansion also holds for geometrically finite surfaces with cusps [31]. Very recently, a non-trivial extension of these techniques has been obtained by Stoyanov [40]. His result implies an exponentially small error term for the counting function $N(T)$ for an arbitrary convex co-compact manifold without any assumptions on δ. However, because of the techniques employed, his error term is not explicit and has no spectral interpretation.

2. 0-Trace formula

2.1. 0-Renormalization

Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold equipped with the hyperbolic metric g and let $\bar{X}$ be its compactification as a smooth manifold with boundary. On such a manifold and after having chosen a boundary defining function x, we can define the 0-integral of a smooth function f on $\bar{X}$ by the formula

$$\int_X^0 f := FP_{\epsilon \to 0} \int_{x(m) > \epsilon} f(m) \, d\operatorname{vol}_g(m),$$

this depends of course on the function x, and this definition can be extended for functions for which the finite part exists. However, it is shown, for example, by Graham [12] that the 0-volume of X, defined as the 0-integral of the function 1, is independent of the choice of x (here X is Einstein) if the dimension $n + 1$ of X is even. For an operator A on X, which has a Schwartz kernel $A(m, m')$ which is smooth when restricted to the diagonal of $X \times X$, one can also define its 0-trace by

$$0\text{-tr}\, A := \int_X^0 A(w, w)$$

when the 0-integral exists (see [3, 18, 25, 32] for details and examples). Note that those renormalizations are naturally related to the 0-calculus and 0-structure defined by Mazzeo and Melrose [27–29] (i.e., related to the "geometric operators" on conformally compact manifolds).

Following Joshi and Sá Barreto [25] (see also Guillopé and Zworski [18, 19]), we can define its 0-trace as a distribution on $\mathbb{R}^*$

$$(2.1) \qquad u(t) := 0\text{-tr}\left(\cos\left(t\sqrt{\Delta_X - \frac{n^2}{4}} \right) \right),$$

which means that for all $\varphi \in C_0^\infty(\mathbb{R}^*)$,

$$\langle u, \varphi \rangle := 0\text{-tr}\left\langle \cos\left(\bullet\sqrt{\Delta_X - \frac{n^2}{4}} \right), \varphi \right\rangle.$$

2.2. Resonances, scattering poles and conformal operators

If s is not a resonance and not in $\frac{1}{2}\mathbb{Z}$, the generalized eigenfunctions of Δ_X for the "eigenvalue" $s(n-s)$ have the following behaviour on $\bar{X}$

$$E(s) = x^s F_1(s) + x^{n-s} F_2(s), \quad F_i(s) \in C^\infty(\bar{X})$$

and one can show (see [14], for example) that it is unique if one requires that $F_2(s)|_{\partial\bar{X}} = f_0$ for a fixed function $f_0 \in C^\infty(\partial\bar{X})$. Thus, one can define the scattering operator as the operator on $C^\infty(\partial\bar{X})$

$$S(s) : f_0 \longrightarrow F_1(s)|_{\partial\bar{X}}.$$

It turns out that (see [32, 33]) for $h_0 := (x^2 g)|_{\partial\bar{X}}$, the operator

$$\widetilde{S}(s) := 2^{2s-n} \frac{\Gamma(s-n/2)}{\Gamma(n/2-s)}(1+\Delta_{h_0})^{-(s/2)+n/4} S(s)(1+\Delta_{h_0})^{-(s/2)+n/4}$$

has also a meromorphic extension to $\mathbb{C}$, with poles (called scattering poles) of finite multiplicity in $\mathbb{C}$, the multiplicity of a pole s_0 being defined here by

$$(2.2) \qquad \nu_{s_0} := -\mathrm{tr}(\mathrm{Res}_{s_0}(\widetilde{S}'(s)\widetilde{S}^{-1}(s))).$$

The set of the scattering poles contained in $\{\Re(s) < n/2\}$ will be denoted by $\mathcal{S}$, whereas the set of resonances will be denoted by $\mathcal{R}$. In [15], we have given a formula relating the multiplicities (1.1) and (2.2) on an general asymptotically hyperbolic manifold. We recall this result in the present case and extend it slightly (see the following paragraph for the precise definition of what exactly is P_k in this setting):

Proposition 2.1. *Let* $X = \Gamma\backslash\mathbb{H}^{n+1}$ *be a convex co-compact hyperbolic manifold with conformal infinity* $(\partial\bar{X}, [h_0])$, *and let* $\Re(s_0) < n/2$, *then we have the relation*

$$(2.3) \qquad \nu_{s_0} = m_{s_0} - m_{n-s_0} + \mathbb{1}_{n/2-\mathbb{N}}(s_0)\dim\ker P_{n/2-s_0}$$

with $\mathbb{1}_{n/2-\mathbb{N}}$ *the characteristic function of* $n/2 - \mathbb{N}$, *and* P_k *for* $k \in \mathbb{N}$ *is the k-th conformal Laplacian on the conformal manifold* $(\partial\bar{X}, [h_0])$ *defined by Graham and Zworski [14].*

Proof. In [15], we dealt with all cases except when $s_0 \in \{s; s(n-s) \in \sigma_{pp}$ $(\Delta_g)\} \cap (n/2 - \mathbb{N})$. In fact, we can deal with these special points using the

perturbation method of Borthwick and Perry [2]. Indeed, since the resolvent and scattering operator are meromorphic with poles of finite multiplicity in $\{\Re(s) < \frac{n}{2}\}$, one can remark from [2] that if $s_0 \in (n-k)/2$, it is possible to add a sufficiently small non-negative compactly supported potential V on X such that m_{n-s_0}, m_{s_0} and ν_{s_0} remain invariant and the eigenvalue $s_0(n-s_0)$ is pushed a little so that $s_0(n-s_0) \notin \sigma_{pp}(\Delta_g + V)$. The formula being now satisfied at s_0, we have the result since $P_{k/2}$ when k is even depends only on the k first derivatives $(\partial_x^j(x^2 g)|_{x=0})_{j=1...,k}$ if g is the hyperbolic metric on X (see again [14]). $\square$

Now we say a few words about the conformal powers of the Laplacian on the compact conformal manifold $(\partial \bar{X}, [h_0])$. By [11], for each conformal representative h in a conformal class $[h_0]$ on $\partial \bar{X}$, the k-th conformal Laplacian P_k associated to h is, for all $k \in \mathbb{N}$ if n is odd (resp. for $k \leq n/2$ if n is even), a well-defined natural differential operator of order $2k$ with principal symbol

$$\sigma_0(P_k) = \sigma_0(\Delta_h^k),$$

which satisfies a covariant rule when the conformal representative is changed, that is,

$$(2.4) \qquad\qquad \hat{P}_k = e^{-(n/2+k)\omega} P_k e^{(n/2-k)\omega}$$

if $\hat{h} = e^{2\omega} h$ (for $\omega \in C^\infty(\partial \bar{X})$) and $\hat{P}_k$ the operator associated to $\hat{h}$. The first example is the conformal Laplacian $P_1 = \Delta_h + (n-2)/[4(n-1)]R$, where R is the scalar curvature of h on $\partial \bar{X}$ and P_2 is the so-called Paneitz operator. If $n = 1$ or if n is even and $k > n/2$, these conformal Laplacians P_k are not naturally well defined in general but for locally conformally flat manifolds, it is also proved in [11] that the k-th conformal Laplacian is well defined canonically for all k and $n > 2$, including when n is even. With the restriction $k \leq n/2$ if n is even, Graham and Zworski proved that $c_k P_k$ is equal to $c_k p_k$ with $c_k = (-1)^{k+1}(2^{2k} k!(k-1)!)^{-1}$ and p_k the residue of the scattering operator at $n/2 + k$ (modulo smoothing operators on $\partial \bar{X}$ involving only the possible L^2 eigenvalues of Δ_X), where we recall that we are in the framework of asymptotically hyperbolic Einstein manifolds. Actually X is hyperbolic, thus the conformal infinity $(\partial \bar{X}, [h_0])$ is locally conformally flat and, according to Robin Graham [13], the operator $c_k^{-1} p_k$ is the canonical P_k constructed in [11] on locally conformally flat manifolds for $n > 2$ even and all $k \in \mathbb{N}$. In any case and by convention, P_k will be called k-th conformal power of the Laplacian and defined as the differential operator $P_k := c_k^{-1} p_k$,

and $d_k =: \dim \ker P_k$ is a finite number which is conformally invariant on the conformal infinity since P_k is Fredholm and satisfies a covariant rule.

There is a special case where the operators P_k are computable, this is, when the conformal infinity has a conformal representative with constant curvatures K. Note that by dilation of the metric and the covariant rule (2.4), this amounts to consider the cases $K = 0, -1, 1$.

Proposition 2.2. *Let $(M, [h_0])$ be a conformal connected manifold of dimension $n > 2$. If a conformal representative $h \in [h_0]$ has constant sectional curvatures equal to $K = -1, 0$ or 1, then the k-th conformal Laplacian P_k for h is*

$$(2.5) \qquad P_k = \prod_{j=1}^{k} \left(\Delta_h + \left(\frac{n}{2} - j \right) \left(\frac{n}{2} + j - 1 \right) K \right)$$

and

$$d_k = \sharp \left\{ j \in \mathbb{N}; j \le k, -K \left(\frac{n}{2} - j \right) \left(\frac{n}{2} + j - 1 \right) \in \sigma(\Delta_h) \right\}.$$

Proof. The flat case is well known (see [14]). Since the local expression of P_k on a manifold with constant curvature $K = 1$ is clearly the same as for the sphere S^n, the result is obtained, for example, by calculating the residue of the scattering operator at $n/2 + k$ on the hyperbolic space, which is $c_k P_k$ on the sphere according to [14, Theorem 1]. Using, for example, [16, Appendix], the scattering matrix on $\mathbb{H}^{n+1}$ is

$$S(s) = 2^{n-2s} \frac{\Gamma(n/2 - s)}{\Gamma(s - n/2)} \frac{\Gamma\left(\sqrt{\Delta_{S^n} + ((n-1)/2)^2} - n/2 + 1/2 + s \right)}{\Gamma\left(\sqrt{\Delta_{S^n} + ((n-1)/2)^2} + n/2 + 1/2 - s \right)},$$

and the residue is easily seen to be $c_k P_k$ with P_k in (2.5). To deal with the case of negative constant curvature, one can use the expression of the scattering operator on the cylindrical manifold studied in [32, Appendix B], and we find the same result. The formula for d_k is a straightforward consequence of the expression of P_k. $\square$

As a consequence, $d_k = 1$ if $K = 0$, $d_k \le d_{n/2}$ for all k if $K < 0$ and $d_k = 0$ if $k < n/2$ and $K > 0$. The expression of P_k on the sphere was obtained previously by Thomas Branson [5] (see also Beckner [1] when $k = n/2$). In a recent preprint, Rod Gover [9] extends Proposition 2.5 to the case of a conformal class which contains an Einstein representative. Note that this

Proposition and consequences will not be used for what follows but they might be of independent interest.

Let us consider the counting function for resonances $\mathcal{N}(R)$ and for scattering poles $\mathcal{N}_s(R)$ defined by

$$\mathcal{N}(R) := \sum_{|s| \leq R,\ s \in \mathcal{R}} m_s, \quad \mathcal{N}_s(R) := \sum_{|s| \leq R,\ s \in \mathcal{S}} \nu_s.$$

We clearly have

$$\mathcal{N}_s(R) \geq \mathcal{N}(R).$$

Patterson and Perry [32] proved the upper bounds $\mathcal{N}_s(R) = O(R^{n+1})$ which implies in view of (2.3) (note that the term m_{n-s_0} in (2.3) are non-zero only for finite number of terms if $s_0 \in n/2 - \mathbb{N}$).

Lemma 2.3. *Using the above notations, we have as $R \to +\infty$,*

$$\mathcal{N}(R) = O(R^{n+1}), \quad \sum_{k=1}^{R} d_k = O(R^{n+1}).$$

Note that those results are optimal in general in the sense that for some examples (take $\mathbb{H}^{n+1}$ with $n+1$ even for the first bound and $\mathbb{H}^{n+1}$ with $n+1$ odd for second bound) this is an asymptotic.

2.3. The hyperbolic space model

We begin by a simple calculation for the wave kernel on the hyperbolic space $\mathbb{H}^{n+1}$ which shows the relation between resonances and 0-trace. Recall that the set of resonances $\mathcal{R}_0$ for the Laplacian on $\mathbb{H}^{n+1}$ is given by

$$\mathcal{R}_0 = \begin{cases} \{-k \text{ with multiplicity } h_n(k); k \in \mathbb{N}_0\} & \text{if } n+1 \text{ is even} \\ \emptyset & \text{if } n+1 \text{ is odd}, \end{cases}$$

where $h_n(k)$ is the dimension of the space of spherical harmonics of degree k on S^{n+1}

$$h_n(k) := (2k+n)\frac{(k+1)(k+2)\cdots(k+n-1)}{n!}, \quad h_n(0) = 1.$$

Let us write for simplicity

$$R_0(s) := (\Delta_{\mathbb{H}^{n+1}} - s(n-s))^{-1}, \quad U_0(t) := \cos\left(t\sqrt{\Delta_{\mathbb{H}^{n+1}} - \frac{n^2}{4}}\right)$$

and $R_0(s; w, w'), U_0(t; w, w')$ their respective kernel. Then, we choose a boundary defining function x_0 of $\mathbb{H}^{n+1}$ and define the distribution on $\mathbb{R}^*$

$$u_0(t) := 0\text{-tr}(U_0(t)),$$

where the 0-trace is taken with respect to x_0.

Lemma 2.4. *On $\mathbb{H}^{n+1}$, we have the following formula for $t > 0$*

$$u_0(t) = \begin{cases} \dfrac{1}{2}\left(\displaystyle\sum_{k=0}^{\infty} h_n(k)e^{-t(n/2+k)}\right) & \text{if } n+1 \text{ is even} \\ 0 & \text{if } n+1 \text{ is odd.} \end{cases}$$

Proof. We use the formula of the wave kernel on $\mathbb{H}^{n+1}$ [21]. In odd dimension, the result is clear since the wave kernel vanishes on the diagonal

$$(2.6) \qquad\qquad\qquad U_0(t; w, w) = 0.$$

In even dimension, it suffices to use the formula

$$(2.7) \qquad U_0(t; w, w) = (-\pi)^{-(n+1)/2} n!! 2^{-(3(n+1))/2} \frac{\cosh t/2}{(\sinh t/2)^{n+1}}$$

and check that

$$(2.8) \qquad\qquad \frac{\cosh t/2}{(\sinh t/2)^{n+1}} = 2^n \left(\sum_{k=0}^{\infty} h_n(k)e^{-t(n/2+k)}\right),$$

then

$$(2.9)$$

$$u_0(t) = \begin{cases} (-\pi)^{-(n+1)/2} n!! 2^{-(n+3)/2} \\ \quad \times \left(\displaystyle\sum_{k=0}^{\infty} h_n(k)e^{-t(n/2+k)}\right) 0\text{-vol}(\mathbb{H}^{n+1}) & \text{if } n+1 \text{ is even} \\ 0 & \text{if } n+1 \text{ is odd,} \end{cases}$$

and using that

$$0\text{-vol}(\mathbb{H}^{n+1}) = \frac{(-2\pi)^{(n+1)/2}}{n!!}$$

when $n+1$ is even [12], this gives the result. $\qquad\qquad\square$

2.4. Proof of Theorem 1.1

We will now study the general case of a convex co-compact quotient of $\mathbb{H}^{n+1}$. Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be the quotient of the hyperbolic space by a convex co-compact group of isometries. We first define the distribution on $\mathbb{R}^*$

$$u_c(t) := 0\text{-tr}\left(P_X \cos\left(t\sqrt{\Delta_X - \frac{n^2}{4}} \right) \right)$$

with P_X the projector on the continuous part of Δ_X. Note that in Theorem 1.1, $u(t)$ is the sum of $u_c(t)$ with the contribution of the discrete spectrum, that is,

$$\sum_{\substack{s \in \mathcal{R} \\ \Re(s) > n/2}} m_s \cosh\left(t\left(\frac{n}{2} - s \right) \right).$$

From [19, Equation 4.13],[2] we have for $\varphi \in C_0^\infty(\mathbb{R}^*)$,

(2.10) $$\int u_c(t)\varphi(t)dt = (4\pi)^{-1} \int_{\mathbb{R}} \hat{\varphi}(z)\theta(z)dz,$$

where

$$\theta(z) := 2iz\left(0\text{-tr}\left(R_X\left(\frac{n}{2} + iz \right) - R_X\left(\frac{n}{2} - iz \right) \right) \right)$$

will be shown to be a tempered distribution on $\mathbb{R}$. We recall that the Selberg zeta function $Z(s)$ of $X = \Gamma \backslash \mathbb{H}^{n+1}$ is defined by the Euler product (which converges for $\Re(s) > \delta$),

$$Z(s) = \prod_{\gamma \in \mathcal{P}} \prod_{k_1, \ldots, k_n \in \mathbb{N}} \left(1 - \alpha_1(\gamma)^{k_1} \ldots \alpha_n(\gamma)^{k_n} e^{-(s+|k|)l(\gamma)} \right),$$

where $|k| = k_1 + \cdots + k_n$.

Let F be a fundamental domain of Γ in $\mathbb{H}^{n+1}$. Then, from the Patterson–Perry formula [32, Equation 6.7], we have

(2.11) $$\theta(z) = \theta_1(z) + \theta_2(z)$$
$$\theta_1(z) := \frac{Z'(n/2 + iz)}{Z(n/2 + iz)} + \frac{Z'(n/2 - iz)}{Z(n/2 - iz)}$$

---

[2]We believe there is a sign typo.

$$\theta_2(z) := FP_{\epsilon \to 0} \int_{\{x > \epsilon\} \cap F} 2iz \left[R_0 \left(\frac{n}{2} + iz; w, w' \right) \right.$$

$$\left. - R_0 \left(\frac{n}{2} - iz; w, w' \right) \right]_{w=w'} d\operatorname{vol}(w).$$

Using the factorization of the Selberg zeta function in a Hadamard product by Patterson and Perry [32, Theorem 1.9] (they also use Bunke–Olbrich results [7]), we observe that

$$\theta_1(z) = -i \left\{ P(z) - \chi(X) \sum_{k=0}^{\infty} h_n(k) \left(\frac{1}{z - i(n/2 + k)} - \frac{1}{z + i(n/2 + k)} \right. \right.$$

$$\left. \left. + Q(z, -k) \right) + \sum_{s \in \mathcal{S}} \nu_s \left(\frac{1}{z - i(n/2 - s)} - \frac{1}{z + i(n/2 - s)} + Q(z, s) \right) \right\}$$

with $P(z), Q(z, s)$ some polynomials in z of degree less or equal to n and χ the Euler characteristic of X. This proves, using the arguments of Lemma 4.7 of [18], that θ_1 is a tempered distribution on $\mathbb{R}$, and hence we can define its Fourier transform $\hat{\theta}_1$ which is also a tempered distribution on $\mathbb{R}$. Now we differentiate the last equation $n + 1$ times and obtain

$$(-i\partial_z)^{n+1}\theta_1(z) = i^n(n+1)! \left\{ -\chi(X) \sum_{k=0}^{\infty} h_n(k) \left(\frac{1}{(z - i(n/2 + k))^{n+2}} \right. \right.$$

$$\left. - \frac{1}{(z + i(n/2 + k))^{n+2}} \right) + \sum_{s \in \mathcal{S}} \nu_s \left(\frac{1}{(z - i(n/2 - s))^{n+2}} \right.$$

$$\left. \left. - \frac{1}{(z + i(n/2 - s))^{n+2}} \right) \right\}.$$

We combine this formula with the following

$$\mathcal{F}_{t \to z}^{-1}(t^{n+1}e^{i\zeta|t|}) = (2\pi)^{-1}(n+1)!i^n \left(\frac{1}{(z - \zeta)^{n+2}} - \frac{1}{(z + \zeta)^{n+2}} \right)$$

for $\Im(\zeta) \geq 0$ to conclude that
(2.12)

$$t^{n+1}\hat{\theta}_1(t) = -2\pi t^{n+1} \left\{ \chi(X) \left(\sum_{k=0}^{\infty} h_n(k)e^{-(n/2+k)|t|} \right) - \sum_{s \in \mathcal{S}} \nu_s e^{-(n/2-s)|t|} \right\}$$

in view of the identity

$$t^{n+1}\hat{\theta}_1(t) = \mathcal{F}_{z \to t} \left((-i\partial_z)^{n+1}\theta_1(z) \right).$$

To study θ_2, we use that [32]

$$\theta_2(z) = \pi^{-n/2}\frac{\Gamma(n/2)}{\Gamma(n)}\frac{\Gamma(n/2+iz)\Gamma(n/2-iz)}{\Gamma(iz)\Gamma(-iz)}0\text{-vol}(X),$$

which is a tempered distribution for $z \in \mathbb{R}$ since it is bounded by a polynomial (note that the 0-volume of X could depend of the defining function x when $n+1$ is odd). We deduce that θ is a tempered distribution on $\mathbb{R}$ and, by (2.10), its Fourier transform is a tempered distribution on $\mathbb{R}$ which, when restricted to $\mathbb{R}^*$, is $4\pi u_c$. To find the Fourier transform of θ_2, we remark by using again [18, Equation 4.13] that this is equivalent to calculate

$$FP_{\epsilon \to 0}\int_{F \cap \{x > \epsilon\}}U_0(t; w, w)d\operatorname{vol}(w)$$

$$= 4\pi\, 0\text{-vol}(X)\begin{cases} 0 & \text{if } n+1 \text{ is odd} \\[2mm] \dfrac{n!!2^{-(3(n+1))/2}}{(-\pi)^{(n+1)/2}}\dfrac{\cosh t/2}{(\sinh t/2)^{n+1}} & \text{if } n+1 \text{ is even,} \end{cases}$$

where we used (2.6) and (2.7). But from Epstein formula for the 0-volume of X in [32]

$$0\text{-vol}(X) = \frac{(-2\pi)^{(n+1)/2}}{n!!}\chi(X),$$

and (2.8), we deduce that

$$(2.13) \qquad \hat{\theta}_2(t) = \begin{cases} 0 & \text{if } n+1 \text{ is odd} \\[3mm] 2\pi\chi(X)\left(\displaystyle\sum_{k=0}^{\infty}h_n(k)e^{-(n/2+k)|t|}\right) & \text{if } n+1 \text{ is even.} \end{cases}$$

To achieve the proof of the formula relating 0-trace and resonances, it suffices to add (2.12) with (2.13), use (2.10) and we obtain, as distribution on $\mathbb{R}^*$,

$$u(t) = \frac{1}{2}\sum_{s \in \mathcal{S}}\nu_s e^{-(n/2-s)|t|} + \sum_{\substack{s \in \mathcal{R} \\ \Re(s) > n/2}}m_s\cosh\left(t\left(\frac{n}{2}-s\right)\right)$$

if $n+1$ is even and

$$u(t) = \frac{1}{2}\sum_{s \in \mathcal{S}}\nu_s e^{-(n/2-s)|t|} - 2^{-n-1}\chi(X)\frac{\cosh t/2}{(\sinh|t|/2)^{n+1}}$$
$$+ \sum_{\substack{s \in \mathcal{R} \\ \Re(s) > n/2}}m_s\cosh\left(t\left(\frac{n}{2}-s\right)\right)$$

if $n+1$ is odd. Now it suffices to use formula (2.3) relating ν_{s_0} and m_{s_0}, m_{n-s_0}. Note that we do not analyze the singularity at $t = 0$ (involving distributions supported at $t = 0$), this would require better results about the factorization of $Z(s)$ as a product over its zeros. The analysis of $u(t)$ at $t = 0$ is done in a general setting by Joshi and Sá Barreto [25, Proposition 4.3], and one could actually get good informations of this singularity in our case if $\delta < n/2$.

To obtain the part with the length spectrum, we remark that for $\Re(s) > \delta$,

$$Z(s) = \exp\left(-\sum_{\gamma} \sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{-sml(\gamma)}}{G_\gamma(m)}\right).$$

As before $s = n/2 + iz$, so when the sum converges (this is the case if $\Im(z) = 0 < n/2 - \delta$),

$$\theta_1(z) = \sum_{\gamma} \sum_{m=1}^{\infty} l(\gamma) G_\gamma(m)^{-1} (e^{-(n/2+iz)ml(\gamma)} + e^{-(n/2-iz)ml(\gamma)}),$$

and thus

$$\hat{\theta}_1(t) = \sum_{\gamma} \sum_{m=1}^{\infty} 2\pi l(\gamma) G_\gamma(m)^{-1} e^{-n/2ml(\gamma)} (\delta(t + ml(\gamma)) + \delta(t - ml(\gamma))),$$

and we are done at least when $\delta < n/2$ since there are no L^2-eigenvalues in this case. The other cases are treated by the arguments of Perry [35, Lemma 2.3] using a contour deformation which makes the term

$$-\sum_{\substack{s \in \mathcal{R} \\ \Re(s) > n/2}} m_s \cosh\left(t\left(\frac{n}{2} - s\right)\right)$$

appear in addition, and this term cancels the term $0\text{-tr}((1 - P_X)\cos(t\sqrt{\Delta_X - n^2/4}))$.

3. Asymptotics of the counting function for prime geodesics

In this section, we prove Theorem 1.3. The proof is based directly on the trace formula. The standard method using zeta functions and contour deformation could be applied, but a good estimate of the growth of $|Z'(s)/Z(s)|$ in strips parallel to the imaginary axis is lacking and the error term obtained

might be marginally large. In the special case of Schottky manifolds, Guillopé *et al.* [20] have shown that $|Z(s)| = O(e^{C|\Im(s)|^\delta})$. In that direction, it could be interesting in that case to use the method of regularization, contour deformation and then finite differences and compare the error term obtained with the remainder of Theorem 1.3.

Let us define for $x \geq 1$ the following counting functions.

$$\Pi_0(x) := \#\{\gamma \in \mathcal{P} \ : \ e^{l(\gamma)} \leq x\}, \quad \Pi(x) := \sum_{\substack{\gamma \in \mathcal{P}, k \in \mathbb{N}, \\ e^{kl(\gamma)} \leq x}} \frac{1}{k},$$

$$\Psi(x) := \sum_{\substack{\gamma \in \mathcal{P}, k \in \mathbb{N}, \\ e^{kl(\gamma)} \leq x}} \frac{l(\gamma)}{G_\gamma(k)}.$$

We have obviously $N(T) = \Pi_0(e^T)$ and $\Pi(x) = \sum_{k=1}^{+\infty} \frac{1}{k}\Pi_0(x^{1/k})$. Using the asymptotic formula of Perry [34], we have

$$\Pi_0(x) = O\left(\frac{x^\delta}{\log(x)}\right),$$

which is enough to deduce

$$\Pi(x) = \Pi_0(x) + O(x^{\delta/2}).$$

The counting function $\Psi(x)$ is related to $\Pi(x)$ by remarking that

$$\int_2^x \frac{d\Psi(u)}{\log(u)} = \Pi(x) + \Phi(x) + O(1),$$

where the remainder $\Phi(x) = \sum_{kl(\gamma) \leq \log(x)} \frac{1}{k}(G_\gamma(k)^{-1} - 1)$ can be estimated by

$$|\Phi(x)| \leq C_X \sum_{kl(\gamma) \leq \log(x)} \frac{1}{k} e^{-kl(\gamma)} = O\left(\int_2^x \frac{d\Pi(u)}{u}\right).$$

Since we have $\Pi(x) = O(x^\delta)$, a straightforward Stieltjes integration by parts shows that

$$\Phi(x) = O(x^{\delta-1}).$$

In a nutshell, we have

(3.1) $$\Pi_0(x) = \int_2^x \frac{d\Psi(u)}{\log(u)} + O\left(x^{\max(\delta/2, \delta-1)}\right).$$

Our goal is now to get precise asymptotics of $\Psi(x)$. For this purpose, we need to introduce the following family of test functions. Let $l(X)$ denote the length of the shortest closed geodesic on X. In the following of the proof, x, y will be large real parameters satisfying

$$y > 0, \quad 0 < l(X) < x < x + y \quad \text{and} \quad y = O(x^\alpha),$$

where $\alpha < 1$ will be chosen *a posteriori* to optimize the error term. Let $\varphi_{x,y}$ be a $C_0^\infty(\mathbb{R})$ positive *even* test function such that

$$\varphi_{x,y}(u) = \begin{cases} 0 & \text{if } u \in \left[0, \dfrac{l(X)}{2}\right] \\ 1 & \text{if } u \in [l(X), \log(x)] \\ 0 & \text{if } u \in [\log(x+y), +\infty). \end{cases}$$

Clearly, such a function does exist, and it can be chosen such that for all $k \in \mathbb{N}$, there exists a constant $C_k > 0$ such that

(3.2) $$\sup_{\log(x) \leq u \leq \log(x+y)} |\varphi_{x,y}^{(k)}(u)| \leq C_k \left(\frac{x}{y}\right)^k.$$

Set $h_{x,y}(t) := e^{(n/2)|t|}\varphi_{x,y}(t)$. Testing the trace formula on $h_{x,y}$, we obtain the relation

(3.3)
$$\sum_{s \in \mathcal{R}} m_s \int_0^\infty e^{st}\varphi_{x,y}(t)dt + \sum_{k \in \mathbb{N}} d_k \int_0^{+\infty} e^{-kt}h_{x,y}(t)dt$$
$$= \sum_{0 \leq kl(\gamma) \leq \log(x+y)} \frac{l(\gamma)}{G_\gamma(k)}\varphi_{x,y}(kl(\gamma))$$
$$+ \left(2^{-n}A(X) + 2B(X)\right)\int_0^{+\infty} \frac{\cosh(t/2)}{(\sinh(t/2))^{n+1}}h_{x,y}(t)dt.$$

Obviously, we have

$$\int_0^{+\infty} \frac{\cosh(t/2)}{(\sinh(t/2))^{n+1}}h_{x,y}(t)dt = O(\log(x)),$$

and since $t \mapsto \sum_{k \in \mathbb{N}} d_k e^{-kt}$ is uniformly convergent and bounded on $[\frac{l(X)}{2}, \infty)$, we can write

$$\sum_{k \in \mathbb{N}} d_k \int_0^{+\infty} e^{-kt}h_{x,y}(t)dt = O\left(\log(x)x^{n/2}\right).$$

Since we assume that $\delta > n/2$, we know that the point spectrum of the Laplacian is a non-empty finite subset of $(0, n^2/4)$ whose bottom is the simple eigenvalue $\delta(n - \delta)$. Let

$$\frac{n}{2} < \alpha_0 \leq \alpha_1 \leq \cdots < \alpha_p = \delta$$

be the corresponding resonances with respect to the modified spectral parameter $s(n - s)$. We have the partition

$$\mathcal{R} = \{\alpha_0, \ldots, \delta\} \cup \left\{ s \in \mathcal{R} \; : \; \Re(s) \leq \frac{n}{2} \right\}.$$

In the following, we will denote by $\mathcal{R}^+ = \{ s \in \mathcal{R} \; : \; \Re(s) \leq n/2 \}$. For all $s \in \mathcal{R}$, we set

$$\psi_{x,y}(s) = \int_0^{+\infty} e^{st} \varphi_{x,y}(t) dt.$$

On the spectral side of formula (3.3), we have

$$\sum_{s \in \mathcal{R}} m_s \psi_{x,y}(s) = \sum_{s \in \mathcal{R}^+} m_s \psi_{x,y}(s) + \sum_{k=0}^{p} \psi_{x,y}(\alpha_k).$$

For all $0 \leq k \leq p$, we get directly

$$\psi_{x,y}(\alpha_k) = \frac{x^{\alpha_k}}{\alpha_k} + O\left(\frac{y}{x} x^{\alpha_k}\right) = \frac{x^{\alpha_k}}{\alpha_k} + O\left(\frac{y}{x^{\delta-1}}\right).$$

It remains to estimate carefully the spectral sum $\sum_{s \in \mathcal{R}^+} m_s \psi_{x,y}(s)$. If $s = 0$, then we certainly have $\psi_{x,y}(0) = O(x^{n/2})$. If $s \neq 0$, then integrating by parts k times yields (we use the fact that $\Re(s) \leq n/2$ and the estimate (3.2)),

$$(3.4) \qquad \psi_{x,y}(s) = \frac{(-1)^k}{s^k} \int_0^{+\infty} e^{st} \varphi_{x,y}^{(k)}(t) dt = O_k\left(\frac{x^{n/2}}{|s|^k} \left(\frac{x}{y}\right)^{k-1}\right).$$

Writing

$$(3.5) \qquad \sum_{s \in \mathcal{R}^+} m_s \psi_{x,y}(s) = \sum_{|s| \leq x/y} m_s \psi_{x,y}(s) + \sum_{|s| \geq x/y} m_s \psi_{x,y}(s),$$

and using the estimate (3.4) for $k = 1$ and $k = n + 2$, we get

$$\left| \sum_{s \in \mathcal{R}^+} m_s \psi_{x,y}(s) \right| \leq x^{n/2} \int_1^{x/y} \frac{d\mathcal{N}(u)}{u} + x^{n/2} \left(\frac{x}{y} \right)^{n+1} \int_{x/y}^{+\infty} \frac{d\mathcal{N}(u)}{u^{n+2}}$$
$$+ O(x^{n/2}).$$

A Stieltjes integration by parts combined with Lemma 2.3 shows that

$$\int_1^{x/y} \frac{d\mathcal{N}(u)}{u} = O\left(\left(\frac{x}{y} \right)^n \right),$$

and a similar argument yields

$$\int_{x/y}^{+\infty} \frac{d\mathcal{N}(u)}{u^{n+2}} = O\left(\frac{y}{x} \right).$$

Gathering all our previous estimates, we get for all x, y large (and satisfying the previously defined a priori properties),

$$\sum_{0 \leq kl(\gamma) \leq \log(x+y)} \frac{l(\gamma)}{G_\gamma(k)} \varphi_{x,y}(kl(\gamma)) = \frac{x^\delta}{\delta} + \sum_{k=0}^{p-1} \frac{x^{\alpha_k}}{\alpha_k} + O\left(\frac{y}{x^{1-\delta}} + \frac{x^{3n/2}}{y^n} \right)$$
$$+ O((\log x) x^{n/2}).$$

Subtracting the above formula from that of $x + y$ instead of x and using the positivity of the left side, we observe that

$$\sum_{x \leq e^{kl(\gamma)} \leq x+y} \frac{l(\gamma)}{G_\gamma(k)} \varphi_{x,y}(kl(\gamma)) \leq \sum_{i=0}^p \frac{(x+y)^{\alpha_i} - x^{\alpha_i}}{\alpha_i} + O\left(\frac{y}{x^{1-\delta}} + \frac{x^{3n/2}}{y^n} \right)$$
$$+ O((\log x) x^{n/2}) = O\left(\frac{y}{x^{1-\delta}} \right)$$
$$+ O\left(\frac{y}{x^{1-\delta}} + \frac{x^{3n/2}}{y^n} \right) + O((\log x) x^{n/2}).$$

In other words, the above standard argument shows that we can drop the terms over $e^{kl(\gamma)}$ without changing the asymptotics. We have thus obtained

$$\Psi(x) = \frac{x^\delta}{\delta} + \sum_{k=0}^{p-1} \frac{x^{\alpha_k}}{\alpha_k} + O\left(\frac{y}{x^{1-\delta}} + \frac{x^{3n/2}}{y^n} \right) + O((\log x) x^{n/2}).$$

We recall that all the preceding estimates are valid for all $y = O(x^\alpha)$ with $\alpha < 1$. A straightforward computation shows that the choice of

$$y = n^{1/(n+1)} x^{(3/(n+1))(n/2) + (1-\delta)/(n+1)}$$

minimizes the global error term which becomes $O(x^{\beta_n(\delta)})$, where $\beta_n(\delta) = n/(n+1)\left(\frac{1}{2} + \delta\right)$. Remark that because $n/2 < \delta$, we have indeed the exponent $3/(n+1)n/2 + (1-\delta)/(n+1) < 1$. Going back to formula (3.1), the final asymptotics follow by a direct Stieltjes integration by parts.   □

Acknowledgments

References

[1] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality*, Ann. Math. **138** (1993), 213–242.

[2] D. Borthwick and P. Perry, *Scattering poles for asymptotically hyperbolic manifolds*, Trans. A.M.S. **354** (2002), 1215–1231.

[3] D. Borthwick, C. Judge and P. Perry, *Selberg's zeta function and the spectral geometry of geometrically finite hyperbolic surfaces.* Preprint.

[4] R. Bowen, *Hausdorff dimension of quasicircles*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 11–25.

[5] T. Branson, The Functional Determinant, Lecture Notes Series, **4**, Seoul National University, 1993.

[6] U. Bunke and M. Olbrich, *Fuchsian groups of the second kind and representations carried by the limit set*, Invent. Math. **127** (1997), 127–154.

[7] U. Bunke and M. Olbrich, *Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group*, Ann. Math. **149** (1999), 627–689.

[8] P. Buser, Geometry and Spectra of Compact Riemann Surfaces, Progress in Mathematics, Birkhauser Boston, 1992.

[9] R. Gover, *Laplacian operators and Q curvature on conformally Einstein manifolds*, ArXiv:math.DG/0506037.

[10] R. Gover and K. Hirachi, *Conformally invariant powers of the Laplacian—A complete non-existence theorem*, J. Amer. Math. Soc. **17** (2004), no. 2, 389–405.

[11] C. R. Graham, R. Jenne, L. J. Manson and G. A. J. Sparling, *Conformally invariant powers of the Laplacian. I. Existence*, J. London Math. Soc. **46** (1992), no. 2, 557–565.

[12] C. R. Graham, *Volume and area renormalizations for conformally compact Einstein metrics*, Rend. Circ. Mat. Palermo, (Suppl.), **63** Ser. II (2000), 31–42.

[13] C. R. Graham, Private communication.

[14] C. R. Graham and M. Zworski, *Scattering matrix in conformal geometry*, Invent. Math. **152** (2003), 89–118.

[15] C. Guillarmou, *Resonances and scattering poles on asymptotically hyperbolic manifolds*, Math. Res. Lett. **12** (2005), 103–119.

[16] L. Guillopé and M. Zworski, *Upper bounds on the number of resonances for non-compact Riemann surfaces*, J. Funct. Anal. **129** (1995), no. 2, 364–389.

[17] L. Guillopé and M. Zworski, *Polynomial bounds on the number of resonances for some complete spaces of constant negative curvature near infinity*, Asymp. Anal. **11** (1995), 1–22.

[18] L. Guillopé and M. Zworski, *Scattering asymptotics for Riemann surfaces*, Ann. Math. **145** (1997), 597–660.

[19] L. Guillopé and M. Zworski, *The wave trace for Riemann surfaces*, G.A.F.A. **9** (1999), 1156–1168.

[20] L. Guillopé, K. Lin and M. Zworski, *The Selberg zeta function for convex co-compact Schottky groups*, Comm. Math. Phys. **245** (2004), no. 1, 149–176.

[21] S. Helgason, Groups and Geometric Analysis, Academic Press, 1984.

[22] D. Hejhal, The Selberg Trace Formula for $PSL(2,\mathbb{R})$, Lecture Notes in Mathematics, **548**, Springer-Verlag, Berlin, 1976.

[23] H. Huber, *Zur analytischen theorie hyperbolisher raumformen und bewegungsgruppen 2*, Math. Ann. **142** (1961), 385–398.

[24] M. Joshi and A. Sá Barreto, *Inverse scattering on asymptotically hyperbolic manifolds*, Acta Math. **184** (2000), 41–86.

[25] M. Joshi and A. Sá Barreto, *The wave group on asymptotically hyperbolic manifolds*, J. Funct. Anal. **184** (2001), no. 2, 291–312.

[26] B. Maskit, *A characterization of Schottky groups*, J. Anal. Math. **19** (1967), 227–230.

[27] R. Mazzeo and R. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987), 260–310.

[28] R. Mazzeo, *The Hodge cohomology of a conformally compact metric*, J. Diff. Geom. **28** (1988), 309–339.

[29] R. Mazzeo, *Elliptic theory of differential edge operators. I*, Commun. Partial Diff. Equations **16** (1991), 1615–1664.

[30] F. Naud, *Expanding maps on Cantor sets and analytic continuation of zeta functions*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 1, 116–153.

[31] F. Naud, *Precise asymptotics of the length spectrum for finite geometry Riemann surfaces*, Int. Math. Res. Notices (2005), no. 5, 299–310.

[32] S. Patterson and P. Perry, *The divisor of Selberg's zeta function for Kleinian groups. Appendix A by Charles Epstein*, Duke Math. J. **106** (2001), 321–391.

[33] P. Perry, *The Laplace operator on a hyperbolic manifold II, Eisenstein series and the scattering matrix*, J. Reine Angew. Math. **398** (1989), 67–91.

[34] P. Perry, *Asymptotics of the length spectrum for hyperbolic manifolds of infinite volume*, GAFA, Geom. Funct. Anal. **11** (2001), 132–141.

[35] P. Perry, *A Poisson formula and lower bounds for resonances on hyperbolic manifolds*, Int. Math. Res. Notices **34** (2003), 1837–1851.

[36] B. Randol, *On the asymptotic distribution of closed geodesics on compact Riemann surfaces*, Trans. Amer. Math. Soc. **233** (1977), 241–247.

[37] P. Sarnak, *Prime geodesic theorems*, Phd Thesis, Stanford University, 1980.

[38] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 171–202.

[39] D. Sullivan, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups.*, Acta Math. **153** (1984), nos. 3, 4, 259–277.

[40] L. Stoyanov, *Ruelle zeta functions and spectra of transfer operators for some axiom A flows,* Preprint (2005).

[41] M. Zworski, *Dimension of the limit set and the density of resonances for convex co-compact hyperbolic surfaces*, Invent. Math. **136** (1999), no. 2, 353–409.

Department of mathematics
Purdue University
150 N. University Street
West-Lafayette
IN 47907
USA
E-mail address: cguillar@math.unice.fr

Department of Mathematics
University of California Berkeley
Berkeley
CA 94720-3840
USA
E-mail address: frederic.naud@univ-avignon.fr