An explicit formula for the Webster pseudo-Ricci curvature on real hypersurfaces and its application for characterizing balls in $\mathbb{C}^n$

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In the paper, we provide an explicit formula for computing the Webster pseudo Ricci curvature, we also apply this formula to obtain a theorem on characterizing balls by using area and pseudo scalar curvature.

1. Introduction and main results

Let M be a $(2n + 1)$-dimensional CR manifold with CR dimension n. We say that (M, θ) is a strictly pseudo-convex pseudo-Hermitian manifold in the sense of Webster [29] if θ is a real one-form (contact form) on M, and at each point of M, there is a neighborhood with a local basis $\theta^1, \ldots, \theta^n$ for the holomorphic cotangent space $T^{1,0}$ so that

$$(1.1) \qquad d\theta = i\, h_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}},$$

where $[h_{\alpha\overline{\beta}}]$ is a positive definite $n \times n$ matrix, determined by the Levi-form L_θ on M. Here,

$$(1.2) \qquad L_\theta(w, \overline{v}) = -i\, d\theta(w, \overline{v}), \quad w, v \in T_{1,0}(M).$$

Let $R_{\alpha\overline{\beta}}$ be the Webster pseudo-Ricci curvature and let $R = h^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$ be the pseudo-scalar curvature. It is known that the contact form θ is neither unique nor CR invariant, but lies in a conformal class ($\theta_f = e^f \theta$ for some smooth function f). There are many fundamental works done on CR manifolds by different authors. We refer to the book of Baouendi *et al.* [1], Beals *et al.* [2], Chang and Li [5], Chen and Shaw [6], Folland and Stein [10], Huang [14], Webster [28, 29] and several papers of Jerison and Lee which will be mentioned later on. Here, we will address a few major problems on the Webster pseudo-Ricci curvature, which are related to the problems we are interested in this paper.

The CR Yamabe problem

Find a contact form θ_f so that the Webster pseudo-scalar curvature R_{θ_f} with respect to θ_f is a given constant. The problem can be solved if the variational equation

$$(1.3) \qquad \lambda(M) = \inf_\theta \lambda(\theta) = \inf_\theta \frac{\int_M R_\theta \, \theta \wedge (d\theta)^n}{(\int_M \theta \wedge (d\theta)^n)^{n/(n+1)}},$$

has a minimum. Much fundamental work has been done on the problem by Jerison and Lee in [16, 17, 18], Gamara and Yacoub [11, 12]. In [16], Jerison and Lee proved that $\lambda(M) \leq \lambda(S^{2n+1})$ and solved the CR Yamabe problem for those M with $\lambda(M) < \lambda(S^{2n+1}) = n(n+1)$. In particular, they proved in [16] that if $\lambda(M) = \lambda(S^{2n+1})$, then M is locally spherical. It suffices to consider $M = \partial D$, where D is a smoothly bounded strictly pseudo-convex domain in $\mathbb{C}^{n+1}$. In addition, it was proved by Chern and Ji [8] that if D is simply connected and local spherical then D must global spherical, or D is biholomorphical to the unit ball in $\mathbb{C}^{n+1}$. In this case, one can easily construct a contact form θ with constant pseudo-scalar curvature (see formula in Theorem 1.1 below). It was proved by Huang and Ji in [15] that if $M = \partial D$ is locally spherical, then D is biholomorphically equivalent to the unit ball when M is algebraic; and a counterexample was constructed by Burns and Schnider [3] that the algebraic condition cannot be replaced by real analyticity. For the general case when $\lambda(M) = \lambda(S^{2n+1})$, the CR Yamabe problem was solved by Gamara and Yacoub [12] and Gamara [11]. When $M = S^{2n+1}$, the contact form θ with $\lambda(\theta) = n(n+1)$ was characterized by Jerison and Lee [17].

The existence of pseudo-Hermitian Einstein metric

(M, θ) is said to be pseudo-Einstein if

$$(1.4) \qquad R_{\alpha\bar{\beta}} = \left(\frac{1}{n}\right) R \, h_{\alpha\bar{\beta}}.$$

This problem was solved by Lee in his series of excellent works [20, 21], where he gave a few characterizations for such manifolds. In particular, if there is a non-vanishing holomorphic $(n+1)$-form on M, then M is pseudo-Einstein, which includes the boundary of strictly pseudo-convex domain in $\mathbb{C}^{n+1}$. In fact, the last result was obtained earlier by Luk in [27]. He showed that the boundary of any smoothly bounded strictly pseudo-convex domain

$D = \{\rho(z) < 0\}$ with the contact form $\theta = (-i/2)(\partial \rho - \overline{\partial}\rho)$ generated by the potential function ρ of the Fefferman metric is pseudo-Einstein.

Let $u \in C^2(D)$, and let $H(u)$ denote the complex hessian matrix of u. For any positive function $f(z) \in C^\infty(\overline{D})$. The boundary value problem for the Fefferman equation:

$$(1.5) \qquad J(\rho) = -\det \begin{bmatrix} \rho & \rho_{\overline{j}} \\ \rho_i & \rho_{i\overline{j}} \end{bmatrix} \equiv f(z) \text{ in} D, \quad \rho = 0 \text{ on } \partial D,$$

and with $U = -\log(-\rho)$ being strictly plurisubharmonic in D has been studied by many mathematicians. When D is a smoothly bounded strictly pseudo-convex domain in $\mathbb{C}^n$, the formal existence of such a solution was first given by Fefferman in [9] when $f \equiv 1$ on D; existence and uniqueness was proved by Cheng and Yau in [7] with $\rho \in C^{n+1+3/2}(\overline{D})$ for $f \in C^\infty(\overline{D})$. Lee and Melrose in [22] gave an asymptotic expansion for ρ; in particular, they showed $\rho \in C^{n+3-\epsilon}(\overline{D})$ for any $\epsilon > 0$.

Main results

The best lower bound for the first positive eigenvalue of the sub-Laplace $\Delta_b = \mathrm{Re}\,(\Box_b)$ where $\Box_b$, is Kohn's Laplacian acting on functions, was given by Greenleaf [13] and Li and Luk in [26]. In the study of the eigenvalue problems, computability of the Webster pseudo-Ricci curvature is very important. This leads us to one of the main purposes of the present paper: to give an explicit formula for the Webster pseudo-Ricci curvatures and pseudo-scalar curvature of ∂D for a major class of contact forms. In other words, we will prove the following theorem.

Theorem 1.1. *Let M be a smooth strictly pseudo-convex hypersurface in $\mathbb{C}^{n+1}$. Let U be a neighborhood of M, and let $\theta = (-i/2)(\partial \rho - \overline{\partial}\rho)$, where $\rho \in C^3(U)$ is a defining function for M with $J(\rho) > 0$ in U. Then, for any $w, v \in T_{1,0}(M)$, the Webster pseudo-Ricci curvature is given by the formula:*

$$(1.6) \qquad \mathrm{Ric}(w, \overline{v}) = -\sum_{k,j=1}^{n+1} \frac{\partial^2 \log J(\rho)}{\partial z^k \partial \overline{z}^j} w^k \overline{v}^j + (n+1)\frac{\det H(\rho)}{J(\rho)} L_\theta(w, \overline{v}).$$

In particular, if $\log J(\rho)$ is pluriharmonic near ∂D, then

$$\mathrm{Ric}(w, \overline{v}) = (n+1)\frac{\det H(\rho)}{J(\rho)} L_\theta(w, \overline{v}) \quad \text{and} \quad R = n(n+1)\frac{\det H(\rho)}{J(\rho)}.$$
(1.7)

The characterization for balls in $\mathbb{C}^{n+1}$ is always an interesting subject [8, 15, 19, 23, 29]. Formula (1.7) in Theorem 1.1 and the main theorem in [23] on characterizing D to be a ball in $\mathbb{C}^{n+1}$ lead us to the second main purpose of this paper by using the pseudo-scalar curvature to characterize a strictly pseudo-convex domain to be a ball. In order to state our result, we let ρ_0 be the unique plurisubharmoic solution of

$$(1.8) \qquad \det H(\rho_0) = J(\rho) \quad \text{in } D, \qquad \rho_0(z) = 0 \quad \text{on } \partial D,$$

and let $\theta_0 = 1/2i(\partial\rho_0 - \bar{\partial}\rho_0)$. The existence and uinqueness of such a smooth ρ_0 was given by Caffarelli *et al.* in [4]. We denote the areas of ∂D with respect to θ and θ_0 as $\text{Area}_\theta(\partial D)$ $(:= \int_{\partial D} \theta \wedge (d\theta)^n)$ and $\text{Area}_{\theta_0}(\partial D)$ $(:= \int_{\partial D} \theta_0 \wedge (d\theta_0)^n)$. Let us first look at the case when D is biholomorphic to a ball. If $\phi : D \to B_{n+1}$ is a biholomorphic mapping, then $\rho(z) = |\phi(z)|^2 - 1$ solves the boundary value problem (1.5) with $J(\rho) = |\det \phi'(z)|^2$ and $\log J(\rho)$ being pluriharmonic in D. In this case, the solution ρ_0 of (1.8) agrees with ρ. Moreover, using formula (1.6), one can compute easily that the pseudo-scalar curvature $R_\theta = n(n+1)$, where $\theta = 1/2i(\partial\rho - \bar{\partial}\rho)$. Thus, we have

$$(1.9) \qquad\qquad \frac{R_\theta}{\text{Area}_\theta(\partial D)} = \frac{n(n+1)}{\text{Area}_{\theta_0}(\partial D)}.$$

Conversely, we have the following theorem.

Theorem 1.2. *Let D be a bounded strictly pseudo-convex domain in $\mathbb{C}^{n+1}$ with smooth boundary. Let $\rho \in C^3(\overline{D})$ be a defining function for D with $J(\rho) \in C^2(\overline{D})$ being positive and $\log J(\rho)$ being pluriharmonic in D. Let $M = \partial D$ and $\theta = (-i/2)(\partial\rho - \bar{\partial}\rho)$. Let $\theta_0 = 1/2i(\partial\rho_0 - \bar{\partial}\rho_0)$, with ρ_0 being the unique plurisubharmonic solution for (1.8). Then*

(i) *If the Webster pseudo-scalar curvature R_θ satisfies*

$$(1.10) \qquad\qquad \frac{R_\theta}{\text{Area}_\theta(\partial D)} \geq \frac{n(n+1)}{\text{Area}_{\theta_0}(\partial D)},$$

then D must be biholomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$.

(ii) *If $J(\rho) \equiv 1$ on D and the Webster pseudo-scalar curvature R_θ satisfies (1.10), then D must be biholomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$ under a constant Jacobian biholomorphic map.*

The paper is organized as follows. In Section 2, we will prove Theorem 1.1. Moreover, we provide some details for computing quantities related to the Webster pseudo-Ricci curvature of a general contact form. The proof of Theorem 1.2 will be given in Section 3.

2. Pseudo-Ricci curvatures on hypersurfaces

Let M be a smooth hypersurface (of real dimension $(2n+1)$) in $\mathbb{C}^{n+1}$. Let $\rho \in C^2(\mathbb{C}^{n+1})$ be a defining function for M: $M = \{\rho = 0\}$ and $\partial\rho \neq 0$ on M. Let $D = \{z \in \mathbb{C}^{n+1} : \rho(z) < 0\}$. Let u be a C^2 function on D, and we let $H(u) = [\partial^2 u/\partial z^i \partial \overline{z}^j]_{(n+1)\times(n+1)}$ be the complex Hessian matrix of u. We use some notations in [24] and [25]. If $H(u)$ is invertible, then we let $[u^{k\overline{j}}]$ be the inverse of $H(u)^t$ so that

$$(2.1) \qquad \sum_{j=1}^{n+1} u^{k\overline{j}} u_{p\overline{j}} = \sum_{k=1}^{n+1} u^{\ell\overline{k}} u_{\ell\overline{p}} = \delta_{kp}.$$

Let $u_k = \partial u/\partial z^k$, $u_{\overline{j}} = \partial u/\partial \overline{z}^j$ and

$$(2.2) \quad |\partial u|_u^2 = u^{k\overline{j}} u_k u_{\overline{j}}, \qquad u^j = \sum_{k=1}^{n+1} u^{j\overline{k}} u_{\overline{k}} \qquad \text{and} \qquad u^{\overline{j}} = \sum_{k=1}^{n+1} u^{k\overline{j}} u_k.$$

Lemma 2.1. *Let D be a strictly pseudo-convex domain in $\mathbb{C}^{n+1}$ with smooth boundary M. Let ρ be a smooth defining function for D so that $H(\rho)$ is positive definite on M. Let*

$$(2.3) \qquad \theta = \frac{1}{2i}[\partial\rho - \overline{\partial}\rho], \qquad \theta^k = dz^k - \frac{\rho^k}{|\partial\rho|_\rho^2}\partial\rho = dz^k - h^k\partial\rho.$$

Then,

$$(2.4) \qquad \theta = -i\partial\rho = i\overline{\partial}\rho, \qquad d\theta = i\sum_{k,\ell=1}^{n+1} \rho_{k\overline{\ell}}\theta^k \wedge \theta^{\overline{\ell}}.$$

Proof. Since

$$\rho_j h^j = \frac{\rho_j \rho^j}{|\partial\rho|_\rho^2} = \frac{\rho_j \rho^{j\overline{k}}\rho_{\overline{k}}}{|\partial\rho|_\rho^2} = 1, \qquad \sum_{k=1}^{n+1} \rho_k \theta^k = \partial\rho - \partial\rho = 0 \quad \text{on } M.$$

Notice that since $d\rho = 0$ on ∂D, we have $\partial \rho = -\overline{\partial}\rho$ and $\theta = -i\partial\rho = i\overline{\partial}\rho$ on M. Therefore,

$$
\begin{aligned}
d\theta &= i\partial\overline{\partial}\rho(z) \\
&= i\sum_{j,k=1}^{n+1} \rho_{k\overline{j}} \, dz^k \wedge d\overline{z}^j \\
&= i\sum_{j,k=1}^{n+1} \rho_{k\overline{j}} \, \theta^k \wedge d\overline{z}^j + i\sum_{j,k=1}^{n+1} \rho_{k\overline{j}} h^k \partial\rho \wedge d\overline{z}^j \\
&= i\sum_{j,k=1}^{n+1} \rho_{k\overline{j}} \, \theta^k \wedge \theta^{\overline{j}} + i\sum_{j,k=1}^{n+1} \rho_{k\overline{j}} \, h^{\overline{j}} \, \theta^k \wedge \overline{\partial}\rho + i\sum_{j,k=1}^{n+1} \rho_{k\overline{j}} \, h^k \, \partial\rho \wedge \theta^{\overline{j}}.
\end{aligned}
$$

Then on M

$$
\begin{aligned}
\sum_{k=1}^{n+1}\sum_{j=1}^{n+1} \rho_{k\overline{j}} h^{\overline{j}}\theta^k &= \frac{1}{|\partial\rho|^2}\sum_{k=1}^{n+1}\sum_{j=1}^{n+1} \rho_{k\overline{j}}\sum_{\ell=1}^{n+1}\rho^{\ell\overline{j}}\rho_\ell\theta^k \\
&= \frac{1}{|\partial\rho|^2}\sum_{k=1}^{n+1}\sum_{\ell=1}^{n+1}\delta_{k\ell}\rho_\ell\theta^k \\
&= \frac{\sum_{k=1}^{n+1}\rho_k\theta^k}{|\partial\rho|_\rho^2} \\
&= 0.
\end{aligned}
$$

We have proved that

$$
(2.5) \qquad\qquad d\theta = i\sum_{j,k=1}^{n+1}\rho_{k\overline{j}}\,\theta^k \wedge \theta^{\overline{j}}
$$

and the proof of the lemma follows. $\square$

Let $M_1 = \{z \in M : \rho_{n+1}(z) \neq 0\}$. Since

$$
(2.6) \qquad\qquad \sum_{j=1}^{n+1}\rho_j\theta^j = 0
$$

we can write

$$
\theta^{n+1} = -\sum_{k=1}^{n}\frac{\rho_k}{\rho_{n+1}}\theta^k.
$$

Then,

$$d\theta = i \sum_{j,k=1}^{n} \left[\rho_{k\bar{j}} - \frac{\rho_{n+1\bar{j}}\,\rho_k}{\rho_{n+1}} - \frac{\rho_{k\overline{n+1}}\,\rho_{\bar{j}}}{\rho_{\overline{n+1}}} + \rho_{n+1\overline{n+1}}\frac{\rho_k \rho_{\bar{j}}}{|\rho_{n+1}|^2} \right] \theta^k \wedge \theta^{\bar{j}}$$

$$(2.7) \qquad := i \sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}}\, \theta^\alpha \wedge \theta^{\bar{\beta}}.$$

Let

$$(2.8) \qquad Y_j = \frac{\partial}{\partial z^j} - \frac{\rho_j}{\rho_{n+1}}\frac{\partial}{\partial z^{n+1}}, \qquad Y = i \sum_{j=1}^{n+1} \left(h^j \frac{\partial}{\partial z^j} - h^{\bar{j}} \frac{\partial}{\partial \bar{z}^j} \right).$$

Then,

$$\sum_{p=1}^{n} h_{\beta\bar{p}}h^{\bar{p}} = \sum_{p=1}^{n} \left[\rho_{\beta\bar{p}} - \frac{\rho_{n+1\bar{p}}\rho_\beta}{\rho_{n+1}} - \frac{\rho_{\beta\overline{n+1}}\rho_{\bar{p}}}{\rho_{\overline{n+1}}} + \rho_{n+1\overline{n+1}}\frac{\rho_\beta \rho_{\bar{p}}}{|\rho_{n+1}|^2} \right] h^{\bar{p}}$$

$$= \frac{\rho_\beta}{|\partial\rho|_\rho^2} - \rho_{\beta\overline{n+1}}h^{\overline{n+1}} - \frac{\rho_\beta}{|\partial\rho|_\rho^2} + \frac{\rho_{n+1\overline{n+1}}\rho_\beta h^{\overline{n+1}}}{\rho_{n+1}}$$

$$\quad + \sum_{p=1}^{n} \left[-\frac{\rho_{\beta\overline{n+1}}}{\rho_{\overline{n+1}}} + \rho_{n+1\overline{n+1}}\frac{\rho_\beta}{|\rho_{n+1}|^2} \right] \rho_{\bar{p}}h^{\bar{p}}$$

$$= -\rho_{\beta\overline{n+1}}h^{\overline{n+1}} + \frac{\rho_{n+1\overline{n+1}}\rho_\beta h^{\overline{n+1}}}{\rho_{n+1}}$$

$$\quad + \left[-\frac{\rho_{\beta\overline{n+1}}\rho_\beta}{\rho_{\overline{n+1}}} + \rho_{n+1\overline{n+1}}\frac{\rho_\beta}{|\rho_{n+1}|^2} \right] (1 - \rho_{\overline{n+1}}h^{\overline{n+1}})$$

$$= -\frac{\rho_{\beta\overline{n+1}}}{\rho_{\overline{n+1}}} + \rho_{n+1\overline{n+1}}\frac{\rho_\beta}{|\rho_{n+1}|^2}$$

$$= -Y_\beta \log \rho_{\overline{n+1}}.$$

Moreover, for any $f \in C^1(M_1)$, a standard computation [29] shows that

$$(2.9) \qquad df = \sum_{j=1}^{n+1} f_j\, dz^j + f_{\bar{j}}\, d\bar{z}^j = \sum_{j=1}^{n}(Y_j f\, \theta^j + Y_{\bar{j}}f\, \theta^{\bar{j}}) + Y(f)\theta.$$

Notice that $\theta^\alpha = dz^\alpha - ih^\alpha\theta$. We have

$$d\theta^\alpha = -i\,dh^\alpha \wedge \theta - ih^\alpha\,d\theta$$
$$= -i\,dh^\alpha \wedge \theta + h^\alpha h_{\gamma\bar\beta}\theta^\gamma \wedge \theta^{\bar\beta}$$
$$= \sum_{\gamma=1}^n \theta^\gamma \wedge \Big(h^\alpha \sum_{\beta=1}^n h_{\gamma\bar\beta}\theta^{\bar\beta} - iY_\gamma h^\alpha\theta\Big) + \theta \wedge \Big(\sum_{\beta=1}^n i\overline{Y}_\beta h^\alpha\theta^{\bar\beta}\Big)$$

$$(2.10) \qquad = \sum_{\gamma=1}^n \theta^\gamma \wedge \omega_\gamma'^\alpha + \theta \wedge \tau^\alpha,$$

where

$$(2.11) \qquad \omega_\gamma'^\alpha = h^\alpha\sum_{\beta=1}^n h_{\gamma\bar\beta}\theta^{\bar\beta} - i\,Y_\gamma h^\alpha\theta, \qquad \tau^\alpha = i\sum_{\beta=1}^n Y_{\bar\beta}h^\alpha\theta^{\bar\beta}$$

and

$$(2.12) \qquad \tau_\alpha = h_{\alpha\bar\gamma}\tau^{\bar\gamma} = -i\sum_{\beta=1}^n\sum_{\gamma=1}^n h_{\alpha\bar\gamma}Y_\beta h^{\bar\gamma}\theta^\beta = \sum_{\beta=1}^n \mathcal{A}_{\alpha\beta}\theta^\beta$$

with

$$(2.13) \qquad \mathcal{A}_{\alpha\beta} = (-i)\sum_{\gamma=1}^n h_{\alpha\bar\gamma}Y_\beta\,h^{\bar\gamma}.$$

The torsion of M with respect to θ (see (1.20) in Webster [28]) is defined as follows:

$$(2.14) \qquad \mathrm{Tor}(z^\alpha Y_\alpha, w^\beta Y_\beta) = i(\mathcal{A}_{\bar\alpha\bar\beta}z^{\bar\alpha}w^{\bar\beta} - \mathcal{A}_{\alpha\beta}z^\alpha w^\beta).$$

Using the fact $d\theta = ih_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta}$ on M_1, one has

$$0 = dd\theta$$
$$= dh_{\alpha\bar\beta} \wedge \theta^\alpha \wedge \theta^{\bar\beta} + h_{\alpha\bar\beta}\,d\theta^\alpha \wedge \theta^{\bar\beta} - h_{\alpha\bar\beta}\theta^\alpha \wedge d\theta^{\bar\beta}$$
$$= dh_{\alpha\bar\beta} \wedge \theta^\alpha \wedge \theta^{\bar\beta} + h_{\alpha\bar\beta}[(\theta^\gamma \wedge \omega_\gamma'^\alpha + \theta\tau^\alpha) \wedge \theta^{\bar\beta} - \theta^\alpha \wedge (\theta^{\bar\gamma} \wedge \omega'^{\bar\beta}_{\bar\gamma} + \theta \wedge \tau^{\bar\beta})]$$
$$= (dh_{\alpha\bar\beta} - h_{\gamma\bar\beta}\omega'^\gamma_\alpha - h_{\alpha\bar\gamma}\omega'^{\bar\gamma}_{\bar\beta})\theta^\alpha \wedge \theta^{\bar\beta} + \theta \wedge \tau_{\bar\beta} \wedge \theta^{\bar\beta} - \theta \wedge \tau_\alpha \wedge \theta^\alpha.$$

This implies that

$$(2.15) \qquad \tau_\alpha \wedge \theta^\alpha = 0 \qquad \text{or} \qquad \mathcal{A}_{\alpha\beta} = \mathcal{A}_{\beta\alpha}$$

and

(2.16) $dh_{\alpha\overline{\beta}} - \omega'_{\alpha\overline{\beta}} - \omega'_{\overline{\beta}\alpha} = dh_{\alpha\overline{\beta}} - h_{\gamma\overline{\beta}}\omega'^{\gamma}_{\alpha} - h_{\alpha\overline{\gamma}}\omega'^{\overline{\gamma}}_{\overline{\beta}} = \mathcal{A}_{\alpha\overline{\beta}\gamma}\theta^{\gamma} + \mathcal{B}_{\alpha\overline{\beta}\overline{\gamma}}\theta^{\overline{\gamma}},$

where

$$\mathcal{A}_{\alpha\overline{\beta}\gamma} = \mathcal{A}_{\gamma\overline{\beta}\alpha}, \qquad \mathcal{B}_{\alpha\overline{\beta}\overline{\gamma}} = \mathcal{A}_{\overline{\beta}\alpha\overline{\gamma}}.$$

Then,

$$\omega'_{\alpha\overline{\beta}} = h_{\gamma\overline{\beta}}\omega'^{\gamma}_{\alpha}$$

$$= h_{\gamma\overline{\beta}}\left(h^{\gamma}\sum_{k=1}^{n}h_{\alpha\overline{k}}\theta^{\overline{k}} - iY_{\alpha}h^{\gamma}\theta\right)$$

(2.17) $$= h_{\overline{\beta}}\sum_{k=1}^{n}h_{\alpha\overline{k}}\theta^{\overline{k}} - ih_{\gamma\overline{\beta}}Y_{\alpha}h^{\gamma}\,\theta$$

and

(2.18) $$\mathcal{A}_{\alpha\overline{\beta}\gamma} = Y_{\gamma}h_{\alpha\overline{\beta}} - h_{\alpha}h_{\gamma\overline{\beta}}.$$

If we let

(2.19) $$\begin{aligned}\omega_{\alpha\overline{\beta}} &= \omega'_{\alpha\overline{\beta}} + (Y_{\gamma}h_{\alpha\overline{\beta}} - h_{\alpha}h_{\gamma\overline{\beta}})\theta^{\gamma} \\ &= (Y_{\gamma}h_{\alpha\overline{\beta}} - h_{\alpha}h_{\gamma\overline{\beta}})\theta^{\gamma} + h_{\overline{\beta}}h_{\alpha\overline{\gamma}}\theta^{\overline{\gamma}} - ih_{\gamma\overline{\beta}}Y_{\alpha}h^{\gamma}\theta\end{aligned}$$

and

(2.20) $$\omega_{\overline{\beta}\alpha} = (Y_{\overline{\gamma}}h_{\alpha\overline{\beta}} - h_{\overline{\beta}}h_{\alpha\overline{\gamma}})\theta^{\overline{\gamma}} + h_{\alpha}h_{\gamma\overline{\beta}}\theta^{\gamma} + ih_{\alpha\overline{\gamma}}Y_{\overline{\beta}}h^{\overline{\gamma}}\theta,$$

then

(2.21) $$dh_{\alpha\overline{\beta}} - h_{\gamma\overline{\beta}}\omega^{\gamma}_{\alpha} - h_{\alpha\overline{\gamma}}\omega^{\overline{\gamma}}_{\overline{\beta}} = 0$$

and

(2.22) $$Y_{\gamma}h_{\alpha\overline{\beta}} - h_{\alpha}h_{\gamma\overline{\beta}} = Y_{\alpha}h_{\gamma\overline{\beta}} - h_{\gamma}h_{\alpha\overline{\beta}}.$$

Note. We should note that the purpose of changing ω'^{β}_{α} in (2.16) to ω^{β}_{α} in (2.21) is to modify the connection so that it is compatible with the metric induced by the Levi form (1.2).

Let

$$(2.23) \qquad \Omega_\beta^\alpha = d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha - i\theta_\beta \wedge \tau^\alpha + i\tau_\beta \wedge \theta^\alpha.$$

Then,

$$
\begin{aligned}
\Omega_{\alpha\overline{\beta}} &= h_{\gamma\overline{\beta}}\Omega_\alpha^\gamma \\
&= h_{\gamma\overline{\beta}}[d\omega_\alpha^\gamma - \omega_\alpha^k \wedge \omega_k^\gamma - i\theta_\alpha \wedge \tau^\gamma + i\tau_\alpha \wedge \theta^\gamma] \\
&= -dh_{\gamma\overline{\beta}} \wedge \omega_\alpha^\gamma + d\omega_{\alpha\overline{\beta}} - \omega_\alpha^k \wedge \omega_{k\overline{\beta}} - i\theta_\alpha \wedge \tau_{\overline{\beta}} + i\tau_\alpha \wedge \theta_{\overline{\beta}} \\
&= -[h_{k\overline{\beta}}\omega_\gamma^k + h_{\gamma\overline{k}}\omega_{\overline{\beta}}^{\overline{k}}] \wedge \omega_\alpha^\gamma + d\omega_{\alpha\overline{\beta}} - \omega_\alpha^k \wedge \omega_{k\overline{\beta}} - i\theta_\alpha \wedge \tau_{\overline{\beta}} + i\tau_\alpha \wedge \theta_{\overline{\beta}} \\
&= -[\omega_{\gamma\overline{\beta}} + \omega_{\overline{\beta}\gamma}] \wedge \omega_\alpha^\gamma + d\omega_{\alpha\overline{\beta}} - \omega_\alpha^k \wedge \omega_{k\overline{\beta}} - i\theta_\alpha \wedge \tau_{\overline{\beta}} + i\tau_\alpha \wedge \theta_{\overline{\beta}}
\end{aligned}
$$

$$(2.24) \quad = d\omega_{\alpha\overline{\beta}} - \omega_{\overline{\beta}\gamma} \wedge \omega_\alpha^\gamma - i\theta_\alpha \wedge \tau_{\overline{\beta}} + i\tau_\alpha \wedge \theta_{\overline{\beta}}.$$

We compute

$$
\begin{aligned}
&d\omega_{\alpha\overline{\beta}}(\mathrm{mod}\,\theta, \theta^p \wedge \theta^q, \theta^{\overline{p}} \wedge \theta^{\overline{q}}) \\
&= [-Y_{\overline{\ell}}Y_\gamma h_{\alpha\overline{\beta}} + Y_{\overline{\ell}}(h_\alpha h_{\gamma\overline{\beta}}) + Y_\gamma(h_{\overline{\beta}}h_{\alpha\overline{\ell}})]\theta^\gamma \wedge \theta^{\overline{\ell}} + (Y_\gamma h_{\alpha\overline{\beta}} - h_\alpha h_{\gamma\overline{\beta}})\,d\theta^\gamma \\
&\quad + h_{\overline{\beta}}h_{\alpha\overline{\gamma}}d\theta^{\overline{\gamma}} + h_{j\overline{\beta}}Y_\alpha h^j h_{\gamma\overline{\ell}}\theta^\gamma \wedge \theta^{\overline{\ell}} \\
&= [-Y_{\overline{\ell}}Y_\gamma h_{\alpha\overline{\beta}} + Y_{\overline{\ell}}(h_\alpha h_{\gamma\overline{\beta}}) + Y_\gamma(h_{\overline{\beta}}h_{\alpha\overline{\ell}}) + h_{\gamma\overline{\ell}}h_{j\overline{\beta}}Y_\alpha h^j]\theta^\gamma \wedge \theta^{\overline{\ell}} \\
&\quad + (Y_\gamma h_{\alpha\overline{\beta}} - h_\alpha h_{\gamma\overline{\beta}})h^\gamma \sum_{k=1}^n h_{j\overline{k}}\theta^j \wedge \theta^{\overline{k}} + h_{\overline{\beta}}h_{\alpha\overline{\gamma}}h^{\overline{\gamma}}\sum_{k=1}^n h_{k\overline{j}}\theta^{\overline{j}} \wedge \theta^k \\
&= \Big[-Y_{\overline{\ell}}Y_\gamma h_{\alpha\overline{\beta}} + Y_{\overline{\ell}}(h_\alpha h_{\gamma\overline{\beta}}) + Y_\gamma(h_{\overline{\beta}}h_{\alpha\overline{\ell}}) + h_{\gamma\overline{\ell}}h_{j\overline{\beta}}Y_\alpha h^j \\
&\quad + (Y_j h_{\alpha\overline{\beta}} - h_\alpha h_{j\overline{\beta}})h^j h_{\gamma\overline{\ell}} - h_{\overline{\beta}}h_{\alpha\overline{k}}h^{\overline{k}}h_{\gamma\overline{\ell}} \Big]\theta^\gamma \wedge \theta^{\overline{\ell}} \\
&= \Big[-Y_{\overline{\ell}}Y_\gamma h_{\alpha\overline{\beta}} + Y_{\overline{\ell}}(h_\alpha h_{\gamma\overline{\beta}}) + Y_\gamma(h_{\overline{\beta}}h_{\alpha\overline{\ell}}) + h_{\gamma\overline{\ell}}h_{j\overline{\beta}}Y_\alpha h^j \\
&\quad + h_{\gamma\overline{\ell}}h^j Y_j h_{\alpha\overline{\beta}} - 2h_{\overline{\beta}}h_\alpha h_{\gamma\overline{\ell}} \Big]\theta^\gamma \wedge \theta^{\overline{\ell}}
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_\alpha^\gamma &= h^{\gamma\overline{\ell}}\omega_{\alpha\overline{\ell}} \\
&= h^{\gamma\overline{\ell}}(Y_j h_{\alpha\overline{\ell}} - h_\alpha h_{j\overline{\ell}})\theta^j + h^{\gamma\overline{\ell}}h_{\overline{\ell}}h_{\alpha\overline{j}}\theta^{\overline{j}}\,\mathrm{mod}\,\theta \\
&= (h^{\gamma\overline{\ell}}Y_j h_{\alpha\overline{\ell}} - h_\alpha \delta_{j\gamma})\theta^j + h^\gamma h_{\alpha\overline{j}}\theta^{\overline{j}}\,\mathrm{mod}\,\theta.
\end{aligned}
$$

By (2.20), we have

$$(2.25) \qquad \omega_{\overline{\beta}\gamma} = (Y_{\overline{p}}h_{\gamma\overline{\beta}} - h_{\overline{\beta}}h_{p\overline{\gamma}})\theta^{\overline{p}} + h_\gamma h_{p\overline{\beta}}\theta^p\,\mathrm{mod}\,\theta.$$

Thus,

$$\omega_{\overline{\beta}\gamma} \wedge \omega_\alpha^\gamma \bmod \theta, \ \theta^p \wedge \theta^q, \theta^{\overline{p}} \wedge \theta^{\overline{q}}$$

$$= [(Y_{\overline{p}}h_{\gamma\overline{\beta}} - h_{\overline{\beta}}h_{\gamma\overline{p}})\theta^{\overline{p}} + h_\gamma h_{p\overline{\beta}}\theta^p] \wedge [(h^{\gamma\overline{\ell}}Y_j h_{\alpha\overline{\ell}} - h_\alpha \delta_{j\gamma})\theta^j + h^\gamma h_{\alpha\overline{j}}\theta^{\overline{j}}]$$

$$= h_\gamma h_{p\overline{\beta}}h^\gamma h_{\alpha\overline{j}}\theta^p \wedge \theta^{\overline{j}} - (Y_{\overline{p}}h_{\gamma\overline{\beta}} - h_{\overline{\beta}}h_{\gamma\overline{p}})(h^{\gamma\overline{\ell}}Y_j h_{\alpha\overline{\ell}} - h_\alpha \delta_{j\gamma})\theta^j \wedge \theta^{\overline{p}}$$

$$= [h_\gamma h^\gamma h_{p\overline{\beta}}h_{\alpha\overline{j}} - (Y_{\overline{j}}h_{\gamma\overline{\beta}} - h_{\overline{\beta}}h_{\gamma\overline{j}})(h^{\gamma\overline{\ell}}Y_p h_{\alpha\overline{\ell}} - h_\alpha \delta_{p\gamma})]\theta^p \wedge \theta^{\overline{j}}$$

$$= \Big[h_\gamma h^\gamma h_{p\overline{\beta}}h_{\alpha\overline{j}} - h^{\gamma\overline{\ell}}Y_j h_{\gamma\overline{\beta}}Y_p h_{\alpha\overline{\ell}} + h_\alpha \delta_{p\gamma}Y_{\overline{j}}h_{\gamma\overline{\beta}}$$

$$\quad + h_{\overline{\beta}}h_{\gamma\overline{j}}h^{\gamma\overline{\ell}}Y_p h_{\alpha\overline{\ell}} - h_{\overline{\beta}}h_\alpha h_{\gamma\overline{j}}\delta_{p\gamma}\Big]\theta^p \wedge \theta^{\overline{j}}$$

$$= [h_\gamma h^\gamma h_{p\overline{\beta}}h_{\alpha\overline{j}} - h^{\gamma\overline{\ell}}Y_{\overline{j}}h_{\gamma\overline{\beta}}Y_p h_{\alpha\overline{\ell}} + h_\alpha Y_{\overline{j}}h_{p\overline{\beta}} + h_{\overline{\beta}}Y_p h_{\alpha\overline{j}} - h_{\overline{\beta}}h_\alpha h_{p\overline{j}}]\theta^p \wedge \theta^{\overline{j}}$$

$$= [h_j h^j h_{\gamma\overline{\beta}}h_{\alpha\overline{\ell}} - h^{j\overline{q}}Y_{\overline{\ell}}h_{j\overline{\beta}}Y_\gamma h_{\alpha\overline{q}} + h_\alpha Y_{\overline{\ell}}h_{\gamma\overline{\beta}} + h_{\overline{\beta}}Y_\gamma h_{\alpha\overline{\ell}} - h_{\overline{\beta}}h_\alpha h_{\gamma\overline{\ell}}]\theta^\gamma \wedge \theta^{\overline{\ell}}.$$

Therefore,

$$(2.26) \qquad\qquad \Omega_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}\gamma\overline{\ell}}\theta^\gamma \wedge \theta^{\overline{\ell}} + \theta \wedge \lambda_{\alpha\overline{\beta}}$$

where

$$R_{\alpha\overline{\beta}\gamma\overline{\ell}} = -Y_{\overline{\ell}}Y_\gamma h_{\alpha\overline{\beta}} + Y_{\overline{\ell}}(h_\alpha h_{\gamma\overline{\beta}}) + Y_\gamma(h_{\overline{\beta}}h_{\alpha\overline{\ell}}) + h_{\gamma\overline{\ell}}h_{j\overline{\beta}}Y_\alpha h^j$$

$$\qquad + h_{\gamma\overline{\ell}}h^j Y_j h_{\alpha\overline{\beta}} - 2h_{\overline{\beta}}h_\alpha h_{\gamma\overline{\ell}} - h_j h^j h_{\gamma\overline{\beta}}h_{\alpha\overline{\ell}}$$

$$(2.27) \qquad + h^{j\overline{q}}Y_{\overline{\ell}}h_{j\overline{\beta}}Y_\gamma h_{\alpha\overline{q}} - h_\alpha Y_{\overline{\ell}}h_{\gamma\overline{\beta}} - h_{\overline{\beta}}Y_\gamma h_{\alpha\overline{\ell}} + h_{\overline{\beta}}h_\alpha h_{\gamma\overline{\ell}}.$$

Let

$$(2.28) \qquad\qquad g(z) = \log \det(h_{\alpha\overline{\beta}}), \quad z \in M_1.$$

Then, for $1 \leq \gamma, \ell \leq n$, we have

$$R_{\gamma\overline{\ell}} = h^{\alpha\overline{\beta}}R_{\alpha\overline{\beta}\gamma\overline{\ell}}$$

$$= -h^{\alpha\overline{\beta}}Y_{\overline{\ell}}Y_\gamma h_{\alpha\overline{\beta}} + Y_{\overline{\ell}}(h_\gamma) + h^{\overline{\beta}}Y_{\overline{\ell}}(h_{\gamma\overline{\beta}}) + Y_\gamma(h_{\overline{\ell}}) + h^\alpha Y_\gamma(h_{\alpha\overline{\ell}})$$

$$\quad + h_{\gamma\overline{\ell}}Y_\alpha h^\alpha + h_{\gamma\overline{\ell}}h^j Y_j g - 2h^\alpha h_\alpha h_{\gamma\overline{\ell}} - h_j h^j h_{\gamma\overline{\ell}} + h^{\alpha\overline{\beta}}h^{j\overline{q}}Y_{\overline{\ell}}h_{j\overline{\beta}}Y_\gamma h_{\alpha\overline{q}}$$

$$\quad - h^{\overline{\beta}}Y_{\overline{\ell}}h_{\gamma\overline{\beta}} - h^\alpha Y_\gamma h_{\alpha\overline{\ell}} + h^\alpha h_\alpha h_{\gamma\overline{\ell}}$$

$$= -h^{\alpha\overline{\beta}}Y_{\overline{\ell}}Y_\gamma h_{\alpha\overline{\beta}} + h^{\alpha\overline{\beta}}h^{j\overline{q}}Y_{\overline{\ell}}h_{j\overline{\beta}}Y_\gamma h_{\alpha\overline{q}} + Y_{\overline{\ell}}(h_\gamma) + Y_\gamma(h_{\overline{\ell}})$$

$$\quad + [Y_\alpha h^\alpha + h^j Y_j g - 2h^\alpha h_\alpha]h_{\gamma\overline{\ell}}$$

$$
\begin{aligned}
&= -Y_{\bar{\ell}}Y_{\gamma}g(z) + Y_{\bar{\ell}}(h_{\gamma}) + Y_{\gamma}(h_{\bar{\ell}}) + [Y_{\alpha}h^{\alpha} + h^{j}Y_{j}g - 2h^{\alpha}h_{\alpha}]h_{\gamma\bar{\ell}} \\
&= -Y_{\bar{\ell}}Y_{\gamma}g(z) - Y_{\bar{\ell}}Y_{\gamma}(\log \rho_{\overline{n+1}}) - Y_{\gamma}Y_{\bar{\ell}}(\log \rho_{n+1}) \\
&\quad + [Y_{\alpha}h^{\alpha} + h^{j}Y_{j}g - 2h^{\alpha}h_{\alpha}]h_{\gamma\bar{\ell}} \\
&= -Y_{\bar{\ell}}Y_{\gamma}g(z) - Y_{\bar{\ell}}Y_{\gamma}(\log |\rho_{n+1}|^{2}) + [Y_{\bar{\ell}}, Y_{\gamma}](\log \rho_{n+1}) \\
&\quad + [Y_{\alpha}h^{\alpha} + h^{j}Y_{j}g - 2h^{\alpha}h_{\alpha}]h_{\gamma\bar{\ell}}.
\end{aligned}
$$

Since

$$
[Y_{\bar{\ell}}, Y_{\gamma}] = -Y_{\bar{\ell}}\left(\frac{\rho_{\gamma}}{\rho_{n+1}}\right)\partial_{n+1} + Y_{\gamma}\left(\frac{\rho_{\bar{\ell}}}{\rho_{\overline{n+1}}}\right)\partial_{\overline{n+1}} = h_{\gamma\bar{\ell}}\left[\frac{\partial_{\overline{n+1}}}{\rho_{\overline{n+1}}} - \frac{\partial_{n+1}}{\rho_{n+1}}\right],
$$

(2.29)

we get

(2.30) $\qquad [Y_{\bar{\ell}}, Y_{\gamma}](\log \rho_{n+1}) = \left(\frac{\rho_{n+1\,\overline{n+1}}}{|\rho_{n+1}|^{2}} - \frac{\rho_{n+1\,n+1}}{\rho_{n+1}^{2}}\right)h_{\gamma\bar{\ell}}.$

Lemma 2.2. *With the notation above, one has the following identity*

$$
g(z) + \log |\rho_{n+1}|^{2} = \log \det[h_{\alpha\bar{\beta}}] + \log |\rho_{n+1}|^{2} = \log[J(\rho) + \rho \det H(\rho)].
$$

(2.31)

Proof. Let $\bar{\partial}\rho = (\partial_{\bar{1}}\rho, \ldots, \partial_{\overline{n+1}}\rho)$ be viewed as a $1 \times (n+1)$ matrix. With the notation $|\partial\rho|_{\rho}^{2} = \sum_{i,j=1}^{n+1}\rho^{i\bar{j}}\rho_{i}\rho_{\bar{j}}$, we have

$$
\begin{aligned}
J(\rho) &= -\det\begin{bmatrix} \rho & \bar{\partial}\rho \\ (\bar{\partial}\rho)^{*} & H(\rho) \end{bmatrix} \\
&= -\rho \det\left(H(\rho) - \frac{1}{\rho}(\bar{\partial}\rho)^{*}\bar{\partial}\rho\right) \\
&= -\rho \det H(\rho)\left(1 - \frac{1}{\rho}|\partial\rho|_{\rho}^{2}\right).
\end{aligned}
$$

Therefore,

$$
J(\rho) + \rho \det H(\rho) = \det H(\rho)|\partial\rho|_{\rho}^{2}.
$$

For $z^{0} \in M_{1}$, we discuss it in two cases:

(a) if $\rho_{j}(z^{0}) = 0$ for all $1 \le j \le n$, then

$$
\det H(\rho)|\partial\rho|_{\rho}^{2}(z_{0}) = |\rho_{n+1}|^{2}\det[\rho_{\alpha\bar{\beta}}](z_{0}) = |\rho_{n+1}|^{2}\det[h_{\alpha\bar{\beta}}](z_{0}),
$$

and (2.31) holds.

(b) if $\rho_j(z^0) \neq 0$ for some $1 \leq j \leq n$, then we let

$$z_j = w_j, \quad 1 \leq j \leq n; \qquad z_{n+1} = w_{n+1} - \sum_{\alpha=1}^{n} \frac{\rho_\alpha(z^0)}{\rho_{n+1}(z^0)} \sum_{j=1}^{n+1} \rho_j(z^0) w_j$$

and let $r(w) = \rho(z(w))$ and $z^0 = z(w^0)$. Since $\det(\partial z/\partial w)(w^0) = 1$ and $\partial r/\partial w_\alpha(w^0) = 0$ for all $1 \leq \alpha \leq n$, one can easily see by the previous case that

$$\det H(\rho)|\partial\rho|_\rho^2(z^0) = \det H(r)|\partial r|_r^2(w^0) = |r_{n+1}(w^0)|^2 \det[r_{\alpha\bar\beta}](w^0).$$

A simple computation shows that $|r_{n+1}(w^0)|^2 = |\rho_{n+1}(z^0)|^2$ and for $1 \leq \alpha$, $\beta \leq n$

$$\frac{\partial^2 r}{\partial w_\alpha \partial \overline{w}_\beta}(w^0) = \rho_{\alpha\bar\beta}(z^0) - \frac{\rho_{n+1\bar\beta}\rho_\alpha}{\rho_{n+1}} - \frac{\rho_{\alpha\overline{n+1}}}{\rho_{\overline{n+1}}}\rho_{\bar\beta} + \rho_{n+1\overline{n+1}}$$
$$\frac{\rho_\alpha\rho_{\bar\beta}}{|\rho_{n+1}|^2}(z^0) = h_{\alpha\bar\beta}(z^0).$$

Therefore, $\det H(\rho)(z^0)|\partial\rho|_\rho^2(z^0) = |\rho_{n+1}(z^0)|^2 \det[h_{\alpha\bar\beta}(z^0)]$, and the proof is complete. $\qquad\square$

Combining all the above, particularly (2.31) and that $Y_{\bar\ell}, Y_\gamma$ are tangential to M_1, we have, on M_1

$$R_{\gamma\bar\ell} = -Y_{\bar\ell}Y_\gamma(g + \log|\rho_{n+1}|^2) + \left(Y_\alpha(h^\alpha) + h^\alpha Y_\alpha(g) - 2h^\alpha h_\alpha \right.$$
$$\left. + \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2} - \frac{\rho_{n+1\,n+1}}{\rho_{n+1}^2} \right) h_{\gamma\bar\ell}$$
$$= -Y_{\bar\ell}Y_\gamma(\log J(\rho)) + \left(Y_\alpha(h^\alpha) + h^\alpha Y_\alpha g(z) - 2h^\alpha h_\alpha \right.$$
$$(2.32) \qquad \left. + \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2} - \frac{\rho_{n+1\,n+1}}{\rho_{n+1}^2} \right) h_{\gamma\bar\ell},$$

where α is summing from 1 to n.

The main result for this section is the following proof of Theorem 1.1.

The proof of Theorem 1.1. We first assume that $H(\rho)$ is positive definite on M. Since

$$(2.33) \qquad \sum_{\alpha=1}^{n}\sum_{k=1}^{n+1} \rho^{\alpha\bar{k}} Y_\alpha(\rho_{\bar{k}}) = \sum_{\alpha=1}^{n}\sum_{k=1}^{n+1} \rho^{\alpha\bar{k}}\left(\rho_{\alpha\bar{k}} - \frac{\rho_\alpha}{\rho_{n+1}}\rho_{n+1\bar{k}}\right) = n,$$

and

$$(2.34) \qquad Y_\gamma(\rho^{\alpha\bar{k}}) = -\sum_{p,q=1}^{n+1} \rho^{\alpha\bar{q}}\rho^{p\bar{k}} Y_\gamma(\rho_{p\bar{q}}),$$

we have, by $h^\alpha = \rho^\alpha/|\partial\rho|_\rho^2$,

$$|\partial\rho|_\rho^2 \sum_{\alpha=1}^{n} Y_\alpha(h^\alpha) = \rho^{\alpha\bar{k}} Y_\alpha(\rho_{\bar{k}}) + \rho_{\bar{k}} Y_\alpha(\rho^{\alpha\bar{k}}) - h^\alpha Y_\alpha(|\partial\rho|_\rho^2)$$

$$= n - \rho_{\bar{k}}\rho^{\alpha\bar{q}}\rho^{p\bar{k}} Y_\alpha(\rho_{p\bar{q}}) - h^\alpha Y_\alpha(|\partial\rho|_\rho^2)$$

$$(2.35) \qquad = n - |\partial\rho|_\rho^2 \rho^{\alpha\bar{q}}h^p Y_\alpha(\rho_{p\bar{q}}) - h^\alpha Y_\alpha(|\partial\rho|_\rho^2).$$

Since $J(\rho) = \det H(\rho)(|\partial\rho|_\rho^2 - \rho)$, we have

$$\rho^{\alpha\bar{q}}h^p Y_\alpha(\rho_{p\bar{q}}) = \rho^{\alpha\bar{q}}h^p \rho_{\alpha p\bar{q}} - \rho^{\alpha\bar{q}}h^p \frac{\rho_\alpha}{\rho_{n+1}}\rho_{n+1 p\bar{q}}$$

$$= h^p \partial_p \log \det H(\rho) - \rho^{n+1\bar{q}}h^p \rho_{n+1 p\bar{q}} - \frac{\rho^{\alpha\bar{q}}\rho_\alpha}{\rho_{n+1}}h^p \rho_{n+1 p\bar{q}}$$

$$= h^p \partial_p \log \det H(\rho) - h^p \frac{\rho_{n+1 p\bar{q}}}{\rho_{n+1}}(\rho^{n+1\bar{q}}\rho_{n+1} + \rho^{\alpha\bar{q}}\rho_\alpha)$$

$$= h^p \partial_p \log \det H(\rho) - h^p \frac{\rho_{n+1 p\bar{q}}}{\rho_{n+1}}\rho^{\bar{q}}$$

$$= h^p \partial_p \log J(\rho) - h^p \partial_p \log(|\partial\rho|_\rho^2 - \rho) - h^p \frac{\rho_{n+1 p\bar{q}}}{\rho_{n+1}}\rho^{\bar{q}},$$

where we sum p and q from 1 to $n+1$ and α from 1 to n. Since $\sum_{k=1}^{n+1} \rho_k h^k = 1$, $Y_\alpha \log |\partial\rho|_\rho^2 = Y_\alpha \log(|\partial\rho|_\rho^2 - \rho)$ and

$$\frac{1}{\rho_{n+1}}\partial_{n+1}\log(|\partial\rho|_\rho^2 - \rho) = \frac{1}{\rho_{n+1}}\frac{\partial}{\partial z_{n+1}}\log |\partial\rho|_\rho^2 - \frac{1}{|\partial\rho|_\rho^2} \quad \text{on } M_1,$$

we have

$$\sum_{\alpha=1}^{n} Y_\alpha(h^\alpha) = \frac{n}{|\partial\rho|_\rho^2} - h^p\partial_p \log J(\rho) + h^p\partial_p \log(|\partial\rho|_\rho^2 - \rho) + \frac{h^p\rho^{\overline{q}}\rho_{n+1p\overline{q}}}{\rho_{n+1}}$$
$$- h^\alpha Y_\alpha(\log|\partial\rho|_\rho^2)$$

$$= \frac{n}{|\partial\rho|_\rho^2} - h^p\partial_p \log J(\rho) + h^{n+1}\partial_{n+1}\log(|\partial\rho|_\rho^2 - \rho) + \frac{h^p\rho^{\overline{q}}\rho_{n+1p\overline{q}}}{\rho_{n+1}}$$
$$+ \frac{h^\alpha\rho_\alpha}{\rho_{n+1}}\partial_{n+1}\log(|\partial\rho|_\rho^2 - \rho)$$

$$= \frac{n}{|\partial\rho|_\rho^2} - h^p\partial_p \log J(\rho) + \frac{\sum_{p=1}^{n+1}\rho_p h^p}{\rho_{n+1}}\partial_{n+1}\log(|\partial\rho|_\rho^2) - \frac{1}{|\partial\rho|_\rho^2}$$
$$+ \frac{\partial_{n+1}(\rho^p\rho^{\overline{q}}\rho_{p\overline{q}})}{|\partial\rho|_\rho^2\,\rho_{n+1}} - \frac{\partial_{n+1}(\rho^p)\,\rho^{\overline{q}}\rho_{p\overline{q}}}{|\partial\rho|_\rho^2\rho_{n+1}} - \frac{\rho^p\partial_{n+1}(\rho^{\overline{q}})\rho_{p\overline{q}}}{|\partial\rho|_\rho^2\,\rho_{n+1}}$$

$$= \frac{n-1}{|\partial\rho|_\rho^2} - h^p\partial_p \log J(\rho) + \frac{2}{\rho_{n+1}}\partial_{n+1}\log(|\partial\rho|_\rho^2) - \frac{\partial_{n+1}(\rho^p)\,\rho_p}{|\partial\rho|_\rho^2\,\rho_{n+1}}$$
$$- \frac{\rho_{\overline{q}}\partial_{n+1}(\rho^{\overline{q}})}{|\partial\rho|_\rho^2\,\rho_{n+1}}$$

$$= \frac{n-1}{|\partial\rho|_\rho^2} - h^p\partial_p \log J(\rho) + \frac{2}{\rho_{n+1}}\partial_{n+1}\log(|\partial\rho|_\rho^2)$$
$$- \frac{\partial_{n+1}(\rho^p\rho_p) - \rho^p\rho_{n+1p}}{|\partial\rho|_\rho^2\,\rho_{n+1}} - \frac{\partial_{n+1}(\rho_{\overline{q}}\rho^{\overline{q}}) - \rho^{\overline{q}}\rho_{n+1\overline{q}}}{|\partial\rho|_\rho^2\,\rho_{n+1}}$$

$$= \frac{n-1}{|\partial\rho|_\rho^2} - h^p\partial_p \log J(\rho) + \frac{2}{\rho_{n+1}}\partial_{n+1}\log(|\partial\rho|_\rho^2) - 2\frac{\partial_{n+1}\log|\partial\rho|_\rho^2}{\rho_{n+1}}$$
$$+ \frac{\rho^p\rho_{n+1p} + \rho^{\overline{q}}\rho_{n+1\overline{q}}}{|\partial\rho|_\rho^2\,\rho_{n+1}}$$

$$= \frac{n-1}{|\partial\rho|_\rho^2} - h^p\partial_p \log J(\rho) + \frac{h^p\partial_{n+1}(\rho_p) + h^{\overline{q}}\partial_{n+1}(\rho_{\overline{q}})}{\rho_{n+1}}$$

Since

$$h^\alpha Y_\alpha g = h^\alpha Y_\alpha(-\log|\rho_{n+1}|^2 + \log J(\rho))$$
$$= -h^\alpha\frac{\rho_{\alpha n+1}}{\rho_{n+1}} - h^\alpha\frac{\rho_{\alpha\overline{n+1}}}{\rho_{\overline{n+1}}} + \frac{h^\alpha\rho_\alpha}{\rho_{n+1}}\left(\frac{\rho_{n+1n+1}}{\rho_{n+1}} + \frac{\rho_{n+1\overline{n+1}}}{\rho_{\overline{n+1}}}\right)$$
$$+ h^\alpha Y_\alpha \log J(\rho)$$

and

$$(2.36) \qquad -h^p \partial_p \log J(\rho) + \sum_{\alpha=1}^{n} h^\alpha Y_\alpha \log J(\rho) = -\frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho),$$

we have

$$\sum_{\alpha=1}^{n} Y_\alpha(h^\alpha) + h^\alpha Y_\alpha g(z) + \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2} - \frac{\rho_{n+1n+1}}{\rho_{n+1}^2}$$

$$= \frac{n-1}{|\partial\rho|_\rho^2} - \frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho) + \frac{h^p \partial_{n+1}(\rho_p) + h^{\overline{q}} \partial_{n+1}(\rho_{\overline{q}})}{\rho_{n+1}} - h^\alpha \frac{\rho_{\alpha n+1}}{\rho_{n+1}}$$

$$- h^\alpha \frac{\rho_{\alpha\overline{n+1}}}{\rho_{\overline{n+1}}} + \frac{h^\alpha \rho_\alpha}{\rho_{n+1}} \left(\frac{\rho_{n+1n+1}}{\rho_{n+1}} + \frac{\rho_{n+1\overline{n+1}}}{\rho_{\overline{n+1}}} \right) + \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2} - \frac{\rho_{n+1n+1}}{\rho_{n+1}^2}$$

$$= \frac{n-1}{|\partial\rho|_\rho^2} - \frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho) + \frac{h^{n+1} \rho_{n+1n+1}}{\rho_{n+1}} + \frac{h^{\overline{n+1}} \rho_{n+1\overline{n+1}}}{\rho_{n+1}}$$

$$+ \left(-\frac{h^\alpha}{\rho_{\overline{n+1}}} \rho_{\alpha\overline{n+1}} + \frac{h^{\overline\alpha}}{\rho_{n+1}} \rho_{n+1\overline\alpha} \right) + \frac{h^\alpha \rho_\alpha}{\rho_{n+1}} \left(\frac{\rho_{n+1n+1}}{\rho_{n+1}} + \frac{\rho_{n+1\overline{n+1}}}{\rho_{\overline{n+1}}} \right)$$

$$+ \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2} - \frac{\rho_{n+1n+1}}{\rho_{n+1}^2}$$

$$= \frac{n-1}{|\partial\rho|_\rho^2} - \frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho) + \frac{\rho_{n+1n+1}}{\rho_{n+1}^2} + \frac{\rho_{\overline{n+1}} h^{\overline{n+1}} \rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2}$$

$$+ \left(-\frac{h^\alpha}{\rho_{\overline{n+1}}} \rho_{\alpha\overline{n+1}} + \frac{h^{\overline\alpha}}{\rho_{n+1}} \rho_{n+1\overline\alpha} \right) + \frac{h^\alpha \rho_\alpha}{\rho_{n+1}} \left(\frac{\rho_{n+1\overline{n+1}}}{\rho_{\overline{n+1}}} \right) + \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2}$$

$$- \frac{\rho_{n+1n+1}}{\rho_{n+1}^2}$$

$$= \frac{n-1}{|\partial\rho|_\rho^2} - \frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho) + 2\frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2} + E(\rho),$$

where

$$E(\rho) := \sum_{\alpha=1}^{n} \left(-\frac{h^\alpha}{\rho_{\overline{n+1}}} \rho_{\alpha\overline{n+1}} + \frac{h^{\overline\alpha}}{\rho_{n+1}} \rho_{n+1\overline\alpha} \right) + (h^\alpha \rho_\alpha - h^{\overline\alpha} \rho_{\overline\alpha}) \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2}$$

(2.37)

is a pure imaginary number.

For $1 \le p, q \le n+1$, we let

$$B_{p\overline{q}} := \sum_{\alpha,\beta=1}^{n} h_{\alpha\overline{\beta}}\rho^{\alpha\overline{q}}\rho^{p\overline{\beta}}.$$

Then,

$$B_{p\overline{q}} = \sum_{\alpha,\beta=1}^{n}\left(\rho_{\alpha\overline{\beta}} - \frac{\rho_\alpha \rho_{n+1\overline{\beta}}}{\rho_{n+1}} - \frac{\rho_{\alpha\overline{n+1}}\rho_{\overline{\beta}}}{\rho_{\overline{n+1}}} + \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2}\rho_\alpha\rho_{\overline{\beta}}\right)\rho^{\alpha\overline{q}}\rho^{p\overline{\beta}}$$

$$= \sum_{\beta=1}^{n}\rho^{p\overline{\beta}}(\delta_{q\beta} - \rho_{n+1\overline{\beta}}\rho^{n+1\overline{q}}) - \sum_{\alpha=1}^{n}\frac{\rho_\alpha}{\rho_{n+1}}(\delta_{pn+1} - \rho_{n+1\overline{n+1}}\rho^{p\overline{n+1}})\rho^{\alpha\overline{q}}$$

$$- \sum_{\beta=1}^{n}\frac{\rho_{\overline{\beta}}}{\rho_{\overline{n+1}}}(\delta_{n+1q} - \rho^{n+1\overline{q}}\rho_{n+1\overline{n+1}})\rho^{p\overline{\beta}} + \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2}\rho_\alpha\rho_{\overline{\beta}}\rho^{\alpha\overline{q}}\rho^{p\overline{\beta}}$$

and

$$\sum_{\alpha=1}^{n} h^\alpha h_\alpha |\partial\rho|_\rho^4 = \sum_{\alpha,\beta=1}^{n} h_{\alpha\overline{\beta}}h^\alpha h^{\overline{\beta}}|\partial\rho|_\rho^4$$

$$= \sum_{p,q=1}^{n+1} B_{p\overline{q}}\rho_p\rho_{\overline{q}}$$

$$= \sum_{p,q=1}^{n+1}\rho_p\rho_{\overline{q}}\sum_{\beta=1}^{n}\rho^{p\overline{\beta}}(\delta_{q\beta} - \rho_{n+1\overline{\beta}}\rho^{n+1\overline{q}})$$

$$- \sum_{p,q=1}^{n+1}\rho_p\rho_{\overline{q}}\sum_{\alpha=1}^{n}\frac{\rho_\alpha}{\rho_{n+1}}(\delta_{pn+1} - \rho_{n+1\overline{n+1}}\rho^{p\overline{n+1}})\rho^{\alpha\overline{q}}$$

$$- \sum_{p,q=1}^{n+1}\rho_p\rho_{\overline{q}}\sum_{\beta=1}^{n}\frac{\rho_{\overline{\beta}}}{\rho_{\overline{n+1}}}(\delta_{n+1q} - \rho^{n+1\overline{q}}\rho_{n+1\overline{n+1}})\rho^{p\overline{\beta}}$$

$$+ \sum_{p,q=1}^{n+1}\rho_p\rho_{\overline{q}}\frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2}\rho_\alpha\rho_{\overline{\beta}}\rho^{\alpha\overline{q}}\rho^{p\overline{\beta}}$$

$$= \sum_{\beta=1}^{n}\rho^{\overline{\beta}}\rho_{\overline{\beta}} - \rho^{n+1}\rho_{n+1} + \rho_{n+1\overline{n+1}}\rho^{n+1}\rho^{\overline{n+1}}$$

$$- \sum_{\alpha=1}^{n}\rho^\alpha\rho_\alpha + \sum_{\alpha=1}^{n}\rho^\alpha\rho_\alpha\frac{\rho^{\overline{n+1}}}{\rho_{n+1}}\rho_{n+1\overline{n+1}} - \sum_{\beta=1}^{n}\rho^{\overline{\beta}}\rho_{\overline{\beta}}$$

$$+ \sum_{\beta=1}^{n}\rho^{\overline{\beta}}\rho_{\overline{\beta}}\frac{\rho^{n+1}}{\rho_{\overline{n+1}}}\rho_{n+1\overline{n+1}} + \sum_{\alpha,\beta=1}^{n}\rho^\alpha\rho_\alpha\rho^{\overline{\beta}}\rho_{\overline{\beta}}\frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2}$$

$$= -|\partial\rho|_\rho^2 + \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2}\rho_{n+1}\rho^{n+1}\rho_{\overline{n+1}}\rho^{\overline{n+1}}$$

$$+ \left[\sum_{\alpha=1}^n \rho^\alpha \rho_\alpha \rho^{\overline{n+1}}\rho_{\overline{n+1}} + \sum_{\beta=1}^n \rho^{\overline{\beta}}\rho_{\overline{\beta}}\rho^{n+1}\rho_{n+1}\right]\frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2}$$

$$+ \sum_{\alpha,\beta=1}^n \rho^\alpha \rho_\alpha \rho^{\overline{\beta}}\rho_{\overline{\beta}}\frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2}$$

$$= -|\partial\rho|_\rho^2 + |\partial\rho|^4 \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2}.$$

Therefore,

$$Y^\alpha(h_\alpha) + h^\alpha Y_\alpha g(z) - 2h^\alpha h_\alpha + \frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2} - \frac{\rho_{n+1n+1}}{\rho_{n+1}^2}$$

$$= \frac{n-1}{|\partial\rho|_\rho^2} - \frac{1}{\rho_{n+1}}\partial_{n+1}\log J(\rho) + 2\frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2} + \frac{2}{|\partial\rho|_\rho^2} - 2\frac{\rho_{n+1\overline{n+1}}}{|\rho_{n+1}|^2} + E(\rho)$$

$$= \frac{n+1}{|\partial\rho|_\rho^2} - \frac{1}{\rho_{n+1}}\partial_{n+1}\log J(\rho) + E(\rho).$$

Notice that $R_{\alpha\overline{\beta}} = \overline{R_{\beta\overline{\alpha}}}$, $h_{\alpha\overline{\beta}} = \overline{h_{\beta\overline{\alpha}}}$ and $\overline{E(\rho)} = -E(\rho)$ (by (2.37), $E(\rho)$ is pure imaginary). By (2.32), we have

$$(2.38) \qquad R_{\gamma\overline{\ell}} = \frac{1}{2}(R_{\gamma\overline{\ell}} + \overline{R_{\ell\overline{\gamma}}})$$

$$= -\frac{1}{2}(Y_{\overline{\ell}}Y_\gamma + Y_\gamma Y_{\overline{\ell}})\log J(\rho)$$

$$+ \left(\frac{n+1}{|\partial\rho|_\rho^2} - \frac{1}{2}\left(\frac{1}{\rho_{n+1}}\partial_{n+1} + \frac{1}{\rho_{\overline{n+1}}}\partial_{\overline{n+1}}\right)\log J(\rho)\right)h_{\gamma\overline{\ell}}.$$

Since

$$Y_{\overline{\ell}}Y_\gamma = \frac{\partial^2}{\partial\overline{z}^\ell \partial z^\gamma} - \frac{\partial}{\partial\overline{z}^\ell}\left(\frac{\rho_\gamma}{\rho_{n+1}}\partial_{n+1}\right) - \frac{\rho_{\overline{\ell}}}{\rho_{\overline{n+1}}}\frac{\partial^2}{\partial\overline{z}^{n+1}\partial z^\gamma}$$

$$+ \frac{\rho_{\overline{\ell}}}{\rho_{\overline{n+1}}}\partial_{\overline{n+1}}\left(\frac{\rho_\gamma}{\rho_{n+1}}\partial_{n+1}\right)$$

$$
= \frac{\partial^2}{\partial \overline{z}^\ell \partial z^\gamma} + \left[-\frac{\rho_{\gamma\overline{\ell}}}{\rho_{n+1}} + \frac{\rho_\gamma}{\rho_{n+1}^2} \rho_{n+1\overline{\ell}} \right] \partial_{n+1} - \frac{\rho_\gamma}{\rho_{n+1}} \frac{\partial^2}{\partial z^{n+1} \partial \overline{z}^\ell}
$$

$$
- \frac{\rho_{\overline{\ell}}}{\rho_{\overline{n+1}}} \frac{\partial^2}{\partial \overline{z}^{n+1} \partial z^\gamma} + \frac{\rho_{\overline{\ell}}}{\rho_{\overline{n+1}}} \frac{\rho_\gamma \overline{n+1}}{\rho_{n+1}} \partial_{n+1} - \frac{\rho_{\overline{\ell}}}{\rho_{\overline{n+1}}} \frac{\rho_\gamma \rho_{n+1}\overline{n+1}}{\rho_{n+1}^2} \partial_{n+1}
$$

$$
+ \frac{\rho_{\overline{\ell}}}{\rho_{\overline{n+1}}} \frac{\rho_\gamma}{\rho_{n+1}} \frac{\partial^2}{\partial z^{n+1} \partial \overline{z}^{n+1}}
$$

$$
= -\frac{h_{\gamma\overline{\ell}}}{\rho_{n+1}} \partial_{n+1} + \frac{\partial^2}{\partial \overline{z}^\ell \partial z^\gamma} - \frac{\rho_\gamma}{\rho_{n+1}} \frac{\partial^2}{\partial z^{n+1} \partial \overline{z}^\ell} - \frac{\rho_{\overline{\ell}}}{\rho_{\overline{n+1}}} \frac{\partial^2}{\partial \overline{z}^{n+1} \partial z^\gamma}
$$

$$
+ \frac{\rho_\gamma \rho_{\overline{\ell}}}{|\rho_{n+1}|^2} \frac{\partial^2}{\partial z^{n+1} \partial \overline{z}^{n+1}}
$$

we have

$$
(2.39) \qquad Y_{\overline{\ell}} Y_\gamma = -\frac{h_{\gamma\overline{\ell}}}{\rho_{n+1}} \partial_{n+1} + \mathcal{D}^\rho_{\gamma\overline{\ell}},
$$

where

$$
\mathcal{D}^\rho_{\gamma\overline{\ell}} = \frac{\partial^2}{\partial \overline{z}^\ell \partial z^\gamma} - \frac{\rho_\gamma}{\rho_{n+1}} \frac{\partial^2}{\partial z^{n+1} \partial \overline{z}^\ell} - \frac{\rho_{\overline{\ell}}}{\rho_{\overline{n+1}}} \frac{\partial^2}{\partial \overline{z}^{n+1} \partial z^\gamma} + \frac{\rho_\gamma \rho_{\overline{\ell}}}{|\rho_{n+1}|^2} \frac{\partial^2}{\partial z^{n+1} \partial \overline{z}^{n+1}}.
$$
$$(2.40)$$

Then,

$$
(2.41) \qquad h_{\alpha\overline{\beta}} = \mathcal{D}^\rho_{\alpha\overline{\beta}}(\rho), \quad \overline{\mathcal{D}^\rho_{\beta\overline{\alpha}}} = \mathcal{D}^\rho_{\alpha\overline{\beta}}
$$

and

$$
(2.42) \qquad Y_{\overline{\beta}} Y_\alpha + Y_\alpha Y_{\overline{\beta}} = -\frac{h_{\alpha\overline{\beta}}}{\rho_{n+1}} \partial_{n+1} - \frac{h_{\alpha\overline{\beta}}}{\rho_{\overline{n+1}}} \partial_{\overline{n+1}} + 2\mathcal{D}^\rho_{\alpha\overline{\beta}}.
$$

By (2.38) and (2.42), we have

$$
(2.43) \qquad R_{\alpha\overline{\beta}} = -\mathcal{D}^\rho_{\alpha\overline{\beta}} \log J(\rho) + (n+1) \frac{\det H(\rho)}{J(\rho)} \mathcal{D}^\rho_{\alpha\overline{\beta}}(\rho)
$$

since $J(\rho) = |\partial \rho|_\rho^2 (\det H(\rho) - \rho)$.

In order to prove (1.6), for any $z_0 \in M_1$, we let

$$
w = \sum_{j=1}^{n+1} w^j \frac{\partial}{\partial z^j}, \quad v = \sum_{j=1}^{n+1} v^j \frac{\partial}{\partial z^j} \in (T_{1,0})_{z_0}(M).
$$

Then,

$$
(2.44) \qquad \sum_{j=1}^{n+1} w^j \rho_j(z_0) = \sum_{j=1}^{n+1} v^j \rho_j(z_0) = 0.
$$

It is easy to see that

$$
(2.45) \qquad w = \sum_{j=1}^{n} w^j Y_j, \quad v = \sum_{j=1}^{n} v^j Y_j.
$$

Thus,

$$
\mathrm{Ric}(w, \overline{v})(z_0) = \sum_{\alpha,\beta=1}^{n} R_{\alpha\overline{\beta}} w^\alpha \overline{v}^\beta
$$

$$
(2.46) \qquad = - \sum_{\alpha,\beta=1}^{n} w^\alpha \overline{v}^\beta \mathcal{D}^\rho_{\alpha\overline{\beta}} \log J(\rho) + (n+1) \frac{\det H(\rho)}{J(\rho)} \sum_{\alpha,\beta=1}^{n} h_{\alpha\overline{\beta}} w^\alpha \overline{v}^\beta.
$$

Applying (2.45), we have

$$
\sum_{\alpha,\beta=1}^{n} w^\alpha \overline{v}^\beta \mathcal{D}^\rho_{\alpha\overline{\beta}} = \sum_{\alpha,\beta=1}^{n} w^\alpha \overline{v}^\beta \frac{\partial^2}{\partial z^\alpha \partial \overline{z}^\beta} - \sum_{\alpha,\beta=1}^{n} \frac{\rho_{\overline{\beta}}}{\rho_{\overline{n+1}}} w^\alpha \overline{v}^\beta \frac{\partial^2}{\partial z^\alpha \partial \overline{z}^{n+1}}
$$

$$
- \sum_{\alpha,\beta=1}^{n} \frac{\rho_\alpha}{\rho_{n+1}} w^\alpha \overline{v}^\beta \frac{\partial^2}{\partial z^{n+1} \partial \overline{z}^\beta}
$$

$$
+ \sum_{\alpha,\beta=1}^{n} \frac{\rho_\alpha \rho_{\overline{\beta}}}{|\rho_{n+1}|^2} w^\alpha \overline{v}^\beta \frac{\partial^2}{\partial z^{n+1} \partial \overline{z}^{n+1}}
$$

$$
= \sum_{\alpha,\beta=1}^{n} w^\alpha \overline{v}^\beta \frac{\partial^2}{\partial z^\alpha \partial \overline{z}^\beta} + \sum_{\alpha=1}^{n} w^\alpha \overline{v}^{n+1} \frac{\partial^2}{\partial z^\alpha \partial \overline{z}^{n+1}}
$$

$$
+ \sum_{\beta=1}^{n} w^{n+1} \overline{v}^\beta \frac{\partial^2}{\partial z^{n+1} \partial \overline{z}^\beta} + w^{n+1} \overline{v}^{n+1} \frac{\partial^2}{\partial z^{n+1} \partial \overline{z}^{n+1}}
$$

$$
(2.47) \qquad = \sum_{k,j=1}^{n+1} w^k \overline{v}^j \frac{\partial^2}{\partial z^k \partial \overline{z}^j}.
$$

Similarly,

$$(2.48) \qquad \sum_{\alpha,\beta=1}^{n} h_{\alpha\bar\beta} w^\alpha \bar v^\beta = \sum_{k,j=1}^{n+1} \frac{\partial^2 \rho(z_0)}{\partial z^k \partial \bar z^j} w^k \bar v^j.$$

Combining (2.46), (2.47) and (2.48), we have proved formula (1.6).

In particular, if $\log J(\rho)$ is pluriharmonic near ∂D, then $\mathcal{D}^\rho_{\alpha\bar\beta} \log J(\rho) = 0$, and

$$(2.49) \qquad R_{\gamma\bar\ell}(z_0) = (n+1)\frac{\det H(\rho)}{J(\rho)} h_{\gamma\bar\ell}(z_0).$$

This with (2.46) and (2.48) gives the formula (1.7). Therefore, Theorem 1.1 is proved when $H(\rho)$ is positive definite at $z_0 \in M_1$.

Next, we consider the case when $H(\rho)$ may not be positive definite. Instead of ρ, we use $r(z) = \rho + c\rho^2$ with $c > 0$ being chosen so that $H(r)$ is positive definite on M. Moreover, the contact form θ remains the same if we use r replacing ρ. Then (1.6) holds by replacing ρ by r. We will show the right side is the same for ρ and for r. We know $J(r) = J(\rho)$ on M. However, $J(r)$ will be different in D. Notice that

$$J(r) = |r_{n+1}|^2 \det(h_{\alpha\bar\beta}(r)) - r(z) \det(H(r))$$

and

$$\begin{aligned}
h_{\alpha\bar\beta}(r) &= r_{\alpha\bar\beta} - \frac{r_{\alpha\overline{n+1}}}{r_{\overline{n+1}}} r_{\bar\beta} - \frac{r_{n+1\bar\beta}}{r_{n+1}} r_\alpha + r_{n+1\overline{n+1}} \frac{r_\alpha r_{\bar\beta}}{|r_{n+1}|^2} \\
&= (1+2c\rho)\rho_{\alpha\bar\beta} + 2c\rho_\alpha\rho_{\bar\beta} - \frac{(1+2c\rho)\rho_{\alpha\overline{n+1}} + 2c\rho_\alpha\rho_{\overline{n+1}}}{(1+2c\rho)\rho_{\overline{n+1}}}(1+2c\rho)\rho_{\bar\beta} \\
&\quad - \frac{(1+2c\rho)\rho_{n+1\bar\beta} + 2c\rho_{n+1}\rho_{\bar\beta}}{(1+2c\rho)\rho_{n+1}}(1+2c\rho)\rho_\alpha \\
&\quad + ((1+2c\rho)\rho_{n+1\overline{n+1}} + 2c|\rho_{n+1}|^2)\frac{(1+2c\rho)^2\rho_\alpha\rho_{\bar\beta}}{(1+2c\rho)^2|\rho_{n+1}|^2} \\
&= (1+2c\rho)\rho_{\alpha\bar\beta} + 2c\rho_\alpha\rho_{\bar\beta} - \frac{(1+2c\rho)\rho_{\alpha\overline{n+1}} + 2c\rho_\alpha\rho_{\overline{n+1}}}{\rho_{\overline{n+1}}}\rho_{\bar\beta} \\
&\quad - \frac{(1+2c\rho)\rho_{n+1\bar\beta} + 2c\rho_{n+1}\rho_{\bar\beta}}{\rho_{n+1}}\rho_\alpha \\
&\quad + ((1+2c\rho)\rho_{n+1\overline{n+1}} + 2c|\rho_{n+1}|^2)\frac{\rho_\alpha\rho_{\bar\beta}}{|\rho_{n+1}|^2} \\
&= (1+2c\rho)h_{\alpha\bar\beta}(\rho).
\end{aligned}$$

Thus,

$$
\begin{aligned}
J(r) &= (1 + 2c\rho)^{n+2}|\rho_{n+1}|^2 \det(h_{\alpha\bar{\beta}}(\rho)) - \rho(1 + c\rho) \det H(\rho + c\rho^2) \\
&= (1 + 2c\rho)^{n+2} J(\rho) + (1 + 2c\rho)^{n+2}\rho(z) \det(H(\rho)) \\
&\quad - \rho(1 + c\rho) \det H(\rho + c\rho^2) \\
&= (1 + 2c\rho)^{n+2} J(\rho) + \rho[\det H(\rho) - \det H(\rho + c\rho^2)] + O(\rho^2) \\
&= J(\rho) + \rho[2(n + 2)cJ(\rho) + \det H(\rho) - \det H(\rho + c\rho^2)] + O(\rho^2).
\end{aligned}
$$

It is easy to see from the definition that on M we have

$$
\mathcal{D}^r_{\alpha\bar{\beta}} = \mathcal{D}^\rho_{\alpha\bar{\beta}}
$$

and

$$
\begin{aligned}
\mathcal{D}^r_{\alpha\bar{\beta}} \log J(r) &= \mathcal{D}^\rho_{\alpha\bar{\beta}} \log J(\rho) \\
&\quad + \left(2(n + 2)c + \frac{\det H(\rho) - \det H(\rho + c\rho^2)}{J(\rho)} \right) \mathcal{D}^\rho_{\alpha\bar{\beta}}(\rho) \\
R_{\alpha\bar{\beta}}(r)(z_0) &= -\mathcal{D}^r_{\alpha\bar{\beta}} \log J(r) + (n + 1)\frac{\det H(r)}{J(r)}(z)h_{\alpha\bar{\beta}} \\
&= -\mathcal{D}^\rho_{\alpha\bar{\beta}} \log J(\rho) - 2(n + 2)ch_{\alpha\bar{\beta}} + \frac{\det H(r) - \det H(\rho)}{J(\rho)(z_0)}h_{\alpha\bar{\beta}}(z) \\
&\quad + (n + 1)\frac{\det H(r)}{J(\rho)}h_{\alpha\bar{\beta}}(z) \\
&= -\mathcal{D}^\rho_{\alpha\bar{\beta}} \log J(\rho) + (n + 1)\frac{\det H(\rho)}{J(\rho)}h_{\alpha\bar{\beta}}(z) \\
&\quad + \frac{(n + 2)}{J(\rho)}[\det H(r) - \det H(\rho) - 2c\,J(\rho)(z_0)]h_{\alpha\bar{\beta}}(z_0) \\
&= -\mathcal{D}^\rho_{\alpha\bar{\beta}} \log J(\rho) + (n + 1)\frac{\det H(\rho)}{J(\rho)}h_{\alpha\bar{\beta}}(z_0),
\end{aligned}
$$

provided

(2.50) $\phi(c) = \det H(\rho + c\rho^2) - \det H(\rho) - 2cJ(\rho) \equiv 0.$

Let $z_0 \in M_1$. Then, we will prove that (2.50) holds in the following two cases:

(a) If $\det H(\rho)(z_0) \neq 0$, then

$$
\det H(r)(z_0) = \det(H(\rho) + 2c\partial\rho \otimes \bar{\partial}\rho) = \det H(\rho)(1 + 2c|\partial\rho|^2_\rho),
$$

where $|\partial\rho|^2_\rho = \rho^{i\bar{j}}\rho_i\rho_{\bar{j}}(z_0)$ (it may not be positive). Thus,

$$\phi'(c) = \det H(\rho)(z_0)2|\partial\rho|^2_\rho - 2J(\rho) = 2(J(\rho) - J(\rho)) = 0.$$

Thus,

$$\phi(c) = \phi(0) = 0$$

(b) If $\det H(\rho)(z_0) = 0$, then $\det H(\rho + \epsilon\rho^2)(z_0) \neq 0$ for any $\epsilon > 0$. Thus,

$$\phi(c) = \det H(\rho + \epsilon\rho^2 + (c - \epsilon)\rho^2) - 2cJ(\rho) - \det H(\rho).$$

Thus,

$$\phi'(c) = 2J(\rho + \epsilon\rho^2) - 2J(\rho), \quad \text{for any small } \epsilon > 0.$$

Let $\epsilon \to 0^+$, we have $\phi'(c) = 0$. Thus, $\phi(c) = \phi(0) = 0$. The proof of the theorem is complete. $\qquad\square$

Furthermore, we will show that the pseudo-scalar curvatures are the same for those ρ with $J(\rho) = 1$ on M and for ρ^0 with $J(\rho^0) = 1$ in a neighborhood of M.

Corollary 2.3. *Let D be a strictly pseudo-convex domain in $\mathbb{C}^{n+1}$ with smooth boundary M. Let ρ^0 be a defining function of D with $J(\rho^0) \equiv 1$ in a neighborhood U of M, and let $\rho \in C^3(\overline{D})$ be any defining function for D with $J(\rho) > 0$ on $\overline{D} \cap U$ and $\theta = (\partial\rho - \bar{\partial}\rho)/(2i)$. Then, for $v, w \in T_{1,0}(M)$, we have two conclusions:*

(a) *If $J(\rho) = 1$ on M or $J(\rho) = 1 + O(\rho^0)$, then*

(2.51) $$\mathrm{Ric}(w, \bar{v}) = (n + 1)\det H(\rho^0)L_\theta(w, \bar{v}).$$

(b) *If $\log J(\rho)$ is pluriharmonic near M, then*

(2.52) $$Ric(w, \bar{v}) = (n + 1)\frac{\det H(\rho)}{J(\rho)}L_\theta(w, \bar{v}).$$

Proof. We can write $\rho = a(z)\rho^0$ with $a \neq 0$ on M. Since $1 = J(\rho) = J(a\rho^0) = a^{n+2}J(\rho^0)$ on M, we have $a \equiv 1$ on M. Thus, we can write $a(z) = 1 + b\rho^0(z)$ and

(2.53) $$\rho(z) = \rho^0(z) + b(z)\rho^0(z)^2.$$

It is not difficult to see that the last argument of the proof of the last theorem remains true when c is a function (or since $\theta(\rho) = \theta(\rho^0)$ on M, they must

have the same Ricci curvature). Therefore,

$$R_{\alpha\bar{\beta}}(\rho) = (n+1)\det H(\rho^0)h_{\alpha\bar{\beta}}.$$

This with (2.46) and (2.48) gives (2.51). Moreover, (2.52) follows directly from (1.6) which we have proved. Therefore, the proof of the corollary is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we recall a theorem proved by Li (see Theorem 1.1 in [23]). Let

$$(3.1) \qquad\qquad U(z) = -\log(-\rho(z)), \quad z \in D \subset \mathbb{C}^{n+1}$$

be strictly plurisubharmonic in D. Let

$$(3.2) \quad |\partial U|_U^2 = \sum_{i,j=1}^{n+1} U^{i\bar{j}} U_i U_{\bar{j}}, \quad H(U) = [U_{i\bar{j}}]_{n\times n}, \quad [U^{i\bar{j}}] = (H(U)^t)^{-1}.$$

Then,

$$(3.3) \qquad\qquad \det H(\rho) = \det H(-\mathrm{e}^{-U})$$
$$= \mathrm{e}^{-(n+1)U} \det(H(U) - (\bar{\partial}U)^*(\bar{\partial}U))$$
$$= \mathrm{e}^{-(n+1)U} \det(H(U))(1 - |\partial U|_U^2)$$

and

$$(3.4) \quad J(\rho)\mathrm{e}^{(n+2)U} = -\det \begin{bmatrix} -1 & \bar{\partial}U \\ (\bar{\partial}U)^* & H(U) - (\bar{\partial}U)^*(\bar{\partial}U) \end{bmatrix} = \det H(U).$$

The following theorem is proved in [23]:

Theorem 3.1. *Let D be a bounded strictly pseudo-convex domain in $\mathbb{C}^{n+1}$ with smooth boundary. Let $\rho \in C^3(\overline{D})$ be a defining function for D with $J(\rho) > 0$ and $\log J(\rho)$ being pluriharmonic in D. Then the following conclusions hold:*

(i) *The function $\det H(\rho)/J(\rho)$ attains its minimum over $\overline{D}$ at some point in ∂D.*

(ii) *If*

$$(3.5) \qquad \frac{\det H(\rho)}{J(\rho)} \geq \frac{\det H(\rho)(z_0)}{J(\rho)(z_0)}, \quad on \ \partial D,$$

for some $z_0 \in \{z \in D : \rho(z) = \min\{\rho(w) : w \in \overline{D}\}\}$, then D is biholomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$.

(iii) *In addition to the condition (3.5), if we assume that $J(\rho) \equiv 1$ on D, then D is biholomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$ with a constant Jacobian biholomorphic map.*

We are ready to prove Theorem 1.2.

Proof. Let $\rho_0(z)$ be the plurisubharmonic solution of the Monge–Ampère equation (given in [4]):

$$(3.6) \qquad \det H(\rho_0) = J(\rho) \quad in \ D, \quad \rho_0 = 0 \text{ on } \partial D.$$

Since $\log J(\rho)$ is pluriharmonic, by formula (2.43) or (1.6), we have

$$(3.7) \qquad R_\theta = n(n+1)\frac{\det H(\rho)}{J(\rho)}, \quad on \ \partial D.$$

Combining this with assumption (1.9) ($\text{Area}_\theta(\partial D) = \int_{\partial D} \theta \wedge (d\theta)^n$), we have

$$(3.8) \qquad \frac{\det H(\rho)}{J(\rho)} \geq \frac{\int_{\partial D} \theta \wedge (d\theta)^n}{\int_{\partial D} \theta_0 \wedge (d\theta_0)^n}, \quad on \ \partial D.$$

We claim that

$$(3.9) \qquad \frac{\det H(\rho)}{J(\rho)} = \text{constant} \quad on \ D.$$

In fact, since $\log J(\rho)$ is pluriharmonic, by main theorem in [23] stated in Theorem 3.1, we have that $\det H(\rho)/J(\rho)$ attains its minimum over $\overline{D}$ at some point in ∂D. Thus, (3.8) holds for all $z \in \overline{D}$. Suppose that (3.9) fails, then there exist $z_0 \in D$ and $\delta > 0$ such that

$$(3.10) \qquad \frac{\det H(\rho)}{J(\rho)} > \frac{\int_{\partial D} \theta \wedge (d\theta)^n}{\int_{\partial D} \theta_0 \wedge (d\theta_0)^n} \quad in \ B(z_0, \delta).$$

Let

$$(3.11) \qquad \theta_s = \frac{1}{2i} \sum_{j=1}^{n+1} (\bar{z}^j \, dz^j - z^j \, d\bar{z}^j).$$

Then, by Stokes' theorem,

$$\int_{\partial D} \theta_0 \wedge (d\theta_0)^n = \int_D (d\theta_0)^{n+1} = \int_D \det H(\rho_0)(d\theta_s)^{n+1} = \int_D J(\rho)(d\theta_s)^{n+1}.$$
(3.12)

By the fact that (3.8) holds for all $z \in D$ and Stokes theorem

$$\begin{aligned}
C &= \int_{\partial D} \theta \wedge (d\theta)^n \\
&= \int_D (d\theta)^{n+1} \\
&= \int_D \det H(\rho)(d\theta_s)^{n+1} \\
&= \int_D \frac{\det H(\rho)}{J(\rho)} J(\rho)(d\theta_s)^{n+1} \\
&> \frac{C}{\int_{\partial D} \theta_0 \wedge (d\theta_0)^n} \int_D J(\rho)(z)(d\theta_s)^{n+1} \\
&= \frac{C}{\int_{\partial D} \theta_0 \wedge (d\theta_0)^n} \int_{\partial D} \theta_0 \wedge (d\theta_0)^n \\
&= C.
\end{aligned}$$

This is a contradiction. Therefore,

$$(3.13) \qquad \frac{\det H(\rho)}{J(\rho)} = \text{constant} = \frac{\int_{\partial D} \theta \wedge (d\theta)^n}{\int_{\partial D} \theta_0 \wedge (d\theta_0)^n}.$$

By Theorem 3.1 and (3.13), there is a biholomorphic map ϕ from D onto B_{n+1}, the unit ball in $\mathbb{C}^{n+1}$.

In addition, $J(\rho) \equiv 1$, then Theorem 3.1 and (3.13) imply that there is a biholomorphic map $\phi : D \to B_{n+1}$ so that $\det \phi'(z) \equiv c$, a constant. This completes the proof of Theorem 1.2. $\qquad \square$

Acknowledgment

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