COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 14, Number 4, 625–632, 2006

A remark on lower bound of Milnor number and characterization of homogeneous hypersurface singularities

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Let $f : (\mathcal{C}^{n+1}, 0) \to (\mathcal{C}, 0)$ be a holomorphic germ defining an isolated hypersurface singularity V at the origin. Let μ and ν and p_g be the Milnor number, multiplicity and geometric genus of $(V, 0)$, respectively. We conjecture that $\mu \geq (\nu - 1)^{n+1}$ and the equality holds if and only if f is a semi-homogeneous function. We prove that this inequality holds for $n = 1$, and also for $n = 2$ or 3 with additional assumption that f is a quasihomogeneous function. For $n = 1$, if V has at most two irreducible branches at the origin, or if f is a quasi-homogeneous function, then $\mu = (\nu - 1)^2$ if and only if f is a homogeneous polynomial. For $n = 2$, if f is a quasi-homogeneous function, then $\mu = (\nu - 1)^3$ iff $6p_g = \nu(\nu - 1)(\nu - 2)$ iff f is a homogeneous polynomial after biholomorphic change of variables. For $n = 3$, if f is a quasi-homogeneous function, then $\mu = (\nu - 1)^4$ iff $24p_g = \nu(\nu - 1)(\nu - 2)(\nu - 3)$ iff f is a homogeneous polynomial after biholomorphic change of variables.

1. Introduction

Since the fundamental work of Milnor [1] on isolated hypersurface singularities, a principal tool in the study of topology of isolated singularities has been the Milnor fibration of the singularity. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ defining an isolated hypersurface singularity so that for local coordinates $(z_0, z_1, \ldots, z_n)$ the partials $\partial f/\partial z_i$ do not simultaneously vanish in a punctured neighborhood of zero. Milnor associates to f a fibration, defined for $\epsilon > 0$ and $\delta > 0$ sufficiently small,

$$f^{-1}(D_\delta^*) \cap B_\epsilon \longrightarrow D_\delta^*,$$

where B_ϵ denotes a ball of radius ϵ about 0 in $\mathbb{C}^{n+1}$, D_δ denotes the disk with radius δ about 0 in $\mathbb{C}$ and $D_\delta^* = D_\delta \backslash \{0\}$. This fibration has fiber $V_t = f^{-1}(t) \cap B_\epsilon$, which is the Milnor fiber of the singularity $V_0 = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$. Then, Milnor proves:

Theorem 1.1 [1]. *If f has an isolated singularity at 0, then for $\epsilon > 0$ and $\delta > |t| > 0$ sufficiently small*:

(1) *The Milnor fibration is a smooth fiber bundle, with the diffeomorphism type of the fiber V_t independent of ϵ and t.*

(2) *The Milnor fiber V_t is homotopy equivalent to the bouquet of spheres of real dimension n, in particular, the Milnor fiber is $(n-1)$-connected.*

(3) *The number of such spheres (which are the "vanishing cycles") is called the Milnor number and can be computed by the formula*

$$\mu(f) = \mu(V_0) = \frac{\dim \mathbb{C}\{z_0, \ldots, z_n\}}{(\partial f/\partial z_0, \partial f/\partial z_1, \ldots, \partial f/\partial z_n)}.$$

Milnor number is an important tool to study the topology of singularity. For example, Lê and Ramanujam [2] have proved the following important theorem.

Theorem 1.2 [2, 3]. *Let $f_t : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a family of germs of holomorphic maps, smoothly depending on the parameter $t \in \mathbb{R}^p$. Suppose that for any t, the Milnor number μ_t of the germs f_t is finite and μ_t does not depend on t. Suppose also that $n \neq 2$. Then all the germs f_t are topologically equivalent.*

Let $G^{(i)}$ be the Grassmannian of i-planes in $\mathbb{C}^{n+1}$. Teissier [4] proves that there exists a Zariski-open dense $U^{(i)} \subseteq G^{(i)}$ such that $\mu(f/H) = \mu^{(i)}(f)$ for all $H \in U^{(i)}$. Notice that $\mu^{(n+1)}(f) = \mu(f)$, $\mu^{(1)}(f) = \nu - 1$, where ν is the multiplicity of f at 0, and $\mu^{(0}(f) = 1$. Set $\mu^*(f) = (\mu^{(n+1)}(f), \ldots, \mu^{(1)}(f), \mu^{(0)}(f))$.

Let $\lambda : \mathcal{V} \to T$ be the germ of a flat deformation of the two-dimension isolated hypersurface singularity $(V, 0)$. We take T to be reduced. In [5], Teissier introduced, for all dimensions, various notions of simultaneous resolution of λ. Namely, let V_t denote $\lambda^{-1}(t)$, the fiber above t in T.

Definition 1.3. The map germ $\prod : \mathcal{M} \to \mathcal{V}$ is a very weak simultaneous resolution of λ if for all sufficiently small representatives of λ, the germ $\prod$ has a representative, also denoted $\prod$, such that

(i) $\prod$ is a proper modification map;

(ii) $\lambda \circ \prod : \mathcal{M} \to T$ is a flat map;

(iii) $\prod_t : M_t \to V_t$ is a resolution of V_t for all t.

Definition 1.4. With the notation in Definition 1.3, consider V to have dimension 2. Let $\mathcal{A}$ denote the exceptional set in $\mathcal{M}$. $\prod$ is a weak simultaneous resolution if additionally the map induced by restriction $\widetilde{\lambda \circ \prod} : \mathcal{A} \to T$ is simple, i.e., a locally trivial deformation.

Definition 1.5. With the notations in Definition 1.3 and Definition 1.4, let $\mathcal{L}$ denote the singular locus of $\mathcal{V}$. Consider $\prod^{-1}(\mathcal{L})$ as nonreduced analytic space (with $\mathcal{A}$ as its underlying reduced space). $\prod$ is a strong simultaneous resolution if in addition to (i), (ii) and (iii) in Definition 1.3, the map induced by restriction $\widetilde{\lambda \circ \prod} : \prod^{-1}(\mathcal{L}) \to T$ is simple.

The following beautiful theorem was proved by Laufer [6].

Theorem 1.6 [6]**.** *Let $\lambda : \mathcal{V} \to T$, with T reduced, be a (flat) family of isolated hypersurface two-dimensional singularities. Suppose that $\mu^*(V_t)$ is constant as a function of t. Then λ has a strong simultaneous resolution.*

It is clear that Milnor number is an important numerical invariant which measures the complexity of the singularity. Therefore, it is desirable to give a lower bound of the Milnor number.

Conjecture 1.7. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ defining an isolated hypersurface singularity $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ at the origin. Let μ and ν be the Milnor number and multiplicity of $(V, 0)$, respectively. Then*

$$\mu \geq (\nu - 1)^{n+1}, \tag{1.1}$$

and the equality in (1.1) *holds if and only if f is a semi-homogeneous function* (*i.e., $f = f_\nu + g$, where f_ν is a nondegenerate homogeneous polynomial of degree ν and g consists of terms of degree at least $\nu + 1$*)*. Suppose that f is a quasihomogeneous function, i.e., $f \in (\partial f/\partial z_0, \ldots, \partial f/\partial z_n)$. Then the equality in* (1.1) *holds if and only if f is a homogeneous polynomial* (*after a biholomorphic change of coordinates*)*.*

The purpose of this paper is to prove the following theorems.

Theorem 1.8. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ defining an isolated plane curve singularity $V = \{z \in \mathbb{C}^2 : f(z) = 0\}$ at the origin. Let μ and ν be the Milnor number and multiplicity of $(V, 0)$, respectively. Then*

$$\mu \geq (\nu - 1)^2. \tag{1.2}$$

Furthermore, if V has at most two irreducible branches at the origin, or if f is a quasi-homogeneous function, then the equality in (1.2) holds if and only if f is a homogeneous polynomial (after a biholomorphic change of coordinates).

Theorem 1.9. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ defining an isolated hypersurface singularity $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ at the origin. Let μ, ν and $\tau = \dim \mathbb{C}\{z_0, \ldots, z_n\}/(f, \partial f/\partial z_0, \ldots, \partial f/\partial z_n)$ be the Milnor number, multiplicity and Tjurina number of $(V, 0)$, respectively. Suppose $\mu = \tau$ and n is either 2 or 3. Then*

$$\mu \geq (\nu - 1)^{n+1}, \tag{1.3}$$

and the equality in (1.3) holds if and only if f is a homogeneous polynomial (after a biholomorphic change of coordinates).

2. Plane curve singularities

The purpose of this section is to prove Theorem 1.8 in the previous section for plane curve singularities. Let $(V, 0) \subseteq (\mathbb{C}^2, 0)$ be a singularity with r irreducible components. It is well known that (see p. 574 of [7])

$$\mu = \sum \nu_i(\nu_i - 1) - r + 1, \tag{2.1}$$

where ν_i runs through the multiplicities of the strict preimages of V at all infinitely near points of $0 \in V$. In particular,

$$\mu \geq \nu(\nu - 1) - r + 1 = (\nu - 1)^2 + \nu - r. \tag{2.2}$$

Since multiplicity of the singularity is bigger than or equal to the number of irreducible components of the singularity (i.e., $\nu - r \geq 0$), (2.2) implies

$$\mu \geq (\nu - 1)^2, \tag{2.3}$$

and the equality in (2.3) occurs if and only if $(V, 0)$ can be resolved by one quadratic transformation and $\nu = r$. Observe that $\nu = r$ means that each irreducible component of $(V, 0)$ is smooth. These irreducible components intersect transversely because $(V, 0)$ can be resolved by one quadratic transformation. Therefore, by the hypothesis of Theorem 1.8, equality of (2.3) occurs if and only if f is a homogeneous polynomial after a biholomorphic change of coordinates.

3. Surface singularities

In this section, we shall prove Theorem 1.9 for surface singularities. Since $\mu(f) = \tau(f)$, by a theorem of Saito [8], f is a weighted homogeneous polynomial after a biholomorphic change of coordinates. Xu–Yau's theorem [9] asserts that

$$\mu \geq 6p_g + \nu - 1, \tag{3.1}$$

and the equality in (3.1) holds if and only if f is a homogeneous polynomial. Observe that (3.1) can be rewritten as

$$\mu \geq 6p_g - \nu(\nu-1)(\nu-2) + (\nu-1)^3. \tag{3.2}$$

We claim that $6p_g \geq \nu(\nu-1)(\nu-2)$. To see this, let us recall a beautiful theorem of Merle and Teissier [10]. Let $f(z_0, \ldots, z_n)$ be a germ of analytic functions at the origin such that $f(0) = 0$. Suppose f has an isolated critical point at the origin. f can be developed in a convergent Taylor series $f(z_0, \ldots, z_n) = \sum a_\lambda z^\lambda$, where $z^\lambda = z_0^{\lambda_0} \cdots z_n^{\lambda_n}$. Recall that Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda + (\mathbb{R}_+)^{n+1}\}$ for λ such that $a_\lambda \neq 0$. Finally, let $\Gamma_-(f)$, the Newton polyhedron of f, be the cone over $\Gamma(f)$ with cone point at 0. For any closed face Δ of $\Gamma(f)$, we associate the polynomial $f_\Delta(x) = \sum_{\lambda \in \Delta} a_\lambda x^\lambda$. We say that f is nondegenerate if f_Δ has no critical point in $(\mathbb{C}^*)^{n+1}$ for any $\Delta \in \Gamma(f)$, where $C^* = \mathbb{C} - \{0\}$. We say that a point p of the integral lattice $\mathbb{Z}^{n+1}$ in $\mathbb{R}^{n+1}$ is positive if all the coordinates of p are positive.

Theorem 3.1 [10]. *Let $(V, 0)$ be a isolated hypersurface singularity defined by a nondegenerate holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Then the geometric genus $p_g = \#\{p \in \mathbb{Z}_+^{n+1} \cap \Gamma_-(f) : p$ is positive$\}$.*

Now the claim $6p_g \geq \nu(\nu-1)(\nu-2)$ follows from Merle–Teissier theorem because $\Gamma_-(f)$ contains the tetrahedron with vertices $(0,0,0)$, $(\nu,0,0)$, $(0,\nu,0)$ and $(0,0,\nu)$ which contains $\frac{1}{6}\nu(\nu-1)(\nu-2)$ positive integral points.

It is also clear that $6p_g = \nu(\nu-1)(\nu-2)$ if and only if $\Gamma_-(f)$ coincide with this tetrahedron, i.e., f is a homogeneous polynomial. Therefore, (3.2) implies

$$\mu \geq (\nu-1)^3, \tag{3.3}$$

and equality in (3.3) holds if and only if $6p_g = \nu(\nu-1)(\nu-2)$, which holds if and only if f is a homogeneous polynomial.

Corollary 3.2. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ defining on isolated hypersurface singularity $V = \{z \in \mathbb{C}^3 : f(z) = 0\}$ at the origin. Let μ, ν and τ be the Milnor number, multiplicity and Tjurina number of $(V, 0)$, respectively. Suppose $\mu = \tau$. Then*

$$\mu \geq (\nu-1)^3, \tag{3.4}$$

$$6p_g \geq \nu(\nu-1)(\nu-2), \tag{3.5}$$

and equality in (3.4) holds if and only if equality in (3.5) holds, which holds if and only if f is a homogeneous polynomial after biholomorphic change of coordinates.

4. Three-dimensional singularities

The same method in the previous section can be used to prove Theorem 1.9 for three-dimensional singularities. Instead of using Xu–Yau's theorem, we use Lin–Yau's theorem [11, 12] which asserts that

$$\begin{aligned} \mu &\geq 4!p_g + 2\nu^3 - 5\nu^2 + 2\nu + 1 \\ &= 4!p_g + (\nu-1)^4 - \nu(\nu-1)(\nu-2)(\nu-3). \end{aligned} \tag{4.1}$$

Corollary 4.1. *Let $f : (\mathbb{C}^4, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ defining an isolated hypersurface singularity $V = \{z \in \mathbb{C}^4 : f(z) = 0\}$ at the origin. Let μ, ν and τ be the Milnor number, multiplicity and Tjurina number of $(V, 0)$, respectively. Suppose $\mu = \tau$. Then*

$$\mu \geq (\nu-1)^4, \tag{4.2}$$

$$24p_g \geq \nu(\nu-1)(\nu-2)(\nu-3), \tag{4.3}$$

and equality in (4.2) holds if and only if equality in (4.3) holds, which holds if and only if f is a homogeneous polynomial after biholomorphic change of coordinates.

Acknowledgment

Stephen S.-T. Yau is supported in part by NSA, Zi-Jiang Professor of East China Normal University, and Hong Kong RGC. Hing-Sun Luk is supported in part by Hong Kong RGC.

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Received July 11, 2006