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Changing sign solutions of a conformally invariant fourth-order equation in the Euclidean space

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We prove the existence of infinitely many solutions for the critical equation $\Delta^2 u = |u|^{2^\sharp-2}u$ in $\mathbb{R}^n$, where Δ^2 denotes the bilaplacian for the euclidean metric. These solutions are non-equivalent in the sense that we cannot pass from one to another by translation and rescaling. Moreover, infinitely many of them must change sign.

Fourth-order equations of critical Sobolev growth have been an intensive target of investigations in the last years, particularly because of the applications of the fourth-order Paneitz operator to conformal geometry and also because of the parallel that exists between fourth-order equations of critical growth and their second-order analogues. References for the Paneitz operator are Branson [2] and Paneitz [7]. We consider in this paper the following fourth-order equation

$$\Delta^2 u = |u|^{2^\sharp-2}u \tag{1}$$

on $\mathbb{R}^n$, $n \geq 5$, where $2^\sharp = 2n/(n-4)$ is the critical exponent for the Sobolev embedding of H_2^2-spaces (consisting of functions in L^2 with two derivatives in L^2) into L^p-spaces, and $\Delta^2 = \Delta_\xi^2$ is the bilaplacian operator with respect to the Euclidean metric ξ. In [6], Lin proved that the only smooth positive solutions of (1) are the functions given by

$$u_{\lambda,a}(x) = \alpha_n \left(\frac{\lambda}{1+\lambda^2|x-a|^2}\right)^{(n-4)/2}, \tag{2}$$

where $\alpha_n = (n(n-4)(n^2-4))^{(n-4)/8}$, $\lambda > 0$ and $a \in \mathbb{R}^n$. The result extends to non-trivial non-negative solutions of (1) when they belong to the Beppo–Levi space $\mathcal{D}_2^2(\mathbb{R}^n)$. Following standard terminology, we say that two solutions u and v of an equation such as (1) are equivalent if they are related by an equation such as

$$v(x) = \lambda^{-(n-4)/2} u\left(\frac{x-a}{\lambda}\right) \tag{3}$$

for some $\lambda > 0$ and $a \in \mathbb{R}^n$. Thanks to the above mentioned result of Lin [6], two smooth positive solutions of (1) are always equivalent. Indeed,

$$u_{\lambda,a}(x) = \lambda^{(n-4)/2} u_{1,0}(\lambda(x-a)).$$

Moreover, it is easily checked that equivalent solutions have the same energy in the sense that

$$\int_{\mathbb{R}^n} (\Delta v)^2\, dx = \int_{\mathbb{R}^n} (\Delta u)^2\, dx$$

if u and v are related by (3). The energy of the $u_{\lambda,a}$'s in (2) is precisely the quantum of energy of a bubble in the blow-up study of positive solutions of Paneitz-type equations. We refer to Hebey and Robert [4] for more details.

The purpose of this paper is to prove the following theorem. Such a theorem is the analogue of Ding's result [3] when passing from the second-order critical equation $\Delta u = |u|^{4/(n-2)}u$ to the fourth-order critical Equation (1) we consider in this paper.

Theorem. *There exists a sequence $(u_k)_{k=1}^{\infty}$ of solutions of* (1) *whose energy tends to $+\infty$ as $k \to +\infty$, namely such that*

$$\int_{\mathbb{R}^n} (\Delta u_k)^2\, dx \longrightarrow +\infty$$

as $k \to +\infty$. In particular, there exist infinitely many non-equivalent solutions of equation (1)*. These solutions u_k necessarily change sign when k is large.*

We prove the theorem in the rest of the paper following Ding's approach [3] when proving the existence of infinitely many non-equivalent solutions of the second order critical equation $\Delta u = |u|^{4/(n-2)}u$. Specific technical difficulties are attached to the fourth-order case.

Proof of the theorem. The Paneitz operator P_h^n on the unit n-sphere (S^n, h) reads as

$$P_h^n u = \Delta_h^2 u + c_n \Delta_h u + d_n u,$$

where $c_n = (n^2 - 2n - 4)/2$ and $d_n = (n(n-4)(n^2-4))/16$ (see Paneitz [7] and Branson [2] for the definition of P_h^n). We let $\Phi : S^n \backslash \{N\} \to \mathbb{R}^n$ be the stereographic projection of north pole N in S^n. Then, as is well known,

$$(\Phi^{-1})^* h = \phi^{4/(n-4)}\xi, \tag{4}$$

where

$$\phi(x) = 4^{(n-4)/4}(1+|x|^2)^{-(n-4)/2}. \tag{5}$$

We let $u \in C^2(\mathbb{R}^n)$ be a solution of (1) and let $\hat{u}$: $S^n \to \mathbb{R}$ be given by

$$\hat{u} = (u\phi^{-1}) \circ \Phi. \tag{6}$$

By the conformal properties of P_h^n

$$\phi^{2^\sharp-1}(P_h^n \hat{u}) \circ \Phi^{-1} = P_\xi^n u = \Delta_\xi^2 u = |u|^{2^\sharp-2}u = \phi^{2^\sharp-1}(|\hat{u}|^{2^\sharp-2}\hat{u}) \circ \Phi^{-1}.$$

Therefore, $\hat{u}$ is a solution of

$$P_h^n \hat{u} = |\hat{u}|^{2^\sharp-2}\hat{u}. \tag{7}$$

Moreover, it is easily checked that

$$\int_{\mathbb{R}^n} |u|^{2^\sharp}\, dx = \int_{S^n} |\hat{u}|^{2^\sharp}\, dv_h. \tag{8}$$

Conversely, if $\hat{u}$ is a solution of (7), then u: $\mathbb{R}^n \to \mathbb{R}$ given by $u = (\hat{u} \circ \Phi^{-1})\phi$ is a solution of (1) satisfying (8). As a remark, if $\hat{u} \in H_2^2(S^n)$ is a solution of (7), then $\hat{u} \in L^p(S^n)$ for all p, and $\hat{u}$ is actually in $C^4(S^n)$. We claim now that

$$\int_{\mathbb{R}^n} (\Delta_\xi u)^2\, dx < +\infty \tag{9}$$

In order to prove (9), we let $\tilde{\xi}$ be the Riemannian metric on $\mathbb{R}^n$ given by $\tilde{\xi} = \phi^{4/(n-4)}\xi$. Then, if g is a Riemannian metric on $\mathbb{R}^n$, we let L_g be the conformal Laplacian with respect to g given by

$$L_g u = \Delta_g u + \frac{n-2}{4(n-1)} S_g u,$$

where S_g is the scalar curvature of g. By the conformal properties of L_g,

$$\begin{aligned}\Delta_\xi u &= L_\xi u\\ &= \phi^{(n+2)/(n-4)} L_{\tilde{\xi}} \left(u\phi^{-(n-2)/(n-4)}\right)\\ &= \phi^{(n+2)/(n-4)} \left(\Delta_{\tilde{\xi}}(u\phi^{-(n-2)/(n-4)}) + \frac{n(n-2)}{4} u\phi^{-(n-2)/(n-4)}\right).\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\int_{\mathbb{R}^n} (\Delta_\xi u)^2\, dv_\xi = \int_{\mathbb{R}^n} \phi^{4/(n-4)} \Big(&\Delta_{\tilde{\xi}}(u\phi^{-(n-2)/(n-4)})\\ &+\frac{n(n-2)}{4} u\phi^{-(n-2)/(n-4)}\Big)^2\, dv_{\tilde{\xi}},\end{aligned}$$

and we can also write that

$$\begin{aligned}\Delta_{\tilde{\xi}}(u\phi^{-(n-2)/(n-4)}) &= \Delta_{\tilde{\xi}}\left((\hat{u}\circ\Phi^{-1})\phi^{-2/(n-4)}\right)\\ &= \Delta_{\tilde{\xi}}(\hat{u}\circ\Phi^{-1})\phi^{-2/(n-4)} + \Delta_{\tilde{\xi}}(\phi^{-2/(n-4)})(\hat{u}\circ\Phi^{-1})\\ &\quad - 2\langle\nabla(\hat{u}\circ\Phi^{-1}); \nabla\phi^{-2/(n-4)}\rangle_{\tilde{\xi}},\end{aligned}$$

where $\langle\cdot;\cdot\rangle_{\tilde{\xi}}$ is the scalar product with respect to $\tilde{\xi}$. It follows that

$$\begin{aligned}\int_{\mathbb{R}^n} (\Delta_\xi u)^2\, dv_\xi &\le 4\int_{S^n} (\Delta_h \hat{u})^2\, dv_h + C_1(I_1+I_2+I_3)\\ &\le C_2 + C_1(I_1+I_2+I_3),\end{aligned}$$

where $C_1, C_2 > 0$ are positive constants, and

$$\begin{aligned}I_1 &= \int_{S^n} \left(\Delta_h(\phi^{-2/(n-4)}\circ\Phi)\right)^2 (\phi^{4/(n-4)}\circ\Phi)\, dv_h\\ I_2 &= \int_{S^n} \left(\phi^{4/(n-4)}\circ\Phi\right) \left|\nabla\left(\phi^{-2/(n-4)}\circ\Phi\right)\right|_h^2 dv_h\\ I_3 &= \int_{S^n} (\phi^{-2}\circ\Phi)(u\circ\Phi)^2\, dv_h.\end{aligned}$$

Thanks once again to the conformal invariance of the conformal Laplacian, we can write that

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} \left(\Delta_{\tilde{\xi}}(\phi^{-2/(n-4)})\right)^2 \phi^{4/(n-4)}\, dv_{\tilde{\xi}} \\ &= \int_{\mathbb{R}^n} \phi^{(2n+4)/(n-4)} \left(\phi^{-(n+2)/(n-4)}\Delta_\xi \phi - \frac{n(n-2)}{4}\phi^{-2/(n-4)}\right)^2 dx \\ &\le C_3 \int_{\mathbb{R}^n} (\Delta_\xi \phi)^2\, dx + C_4 \int_{\mathbb{R}^n} \phi^{2^\sharp}\, dx < +\infty, \end{aligned}$$

where C_3, $C_4 > 0$ are positive constants. In a similar way, we can write that

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^n} \phi^{4/(n-4)} \left|\nabla \phi^{-2/(n-4)}\right|_{\tilde{\xi}}^2 dv_{\tilde{\xi}} \\ &= \int_{\mathbb{R}^n} \phi^{2^\sharp} \left|\nabla \phi^{-2/(n-4)}\right|_{\xi}^2 dx < +\infty. \end{aligned}$$

At last, by (6), we also have that

$$\begin{aligned} |I_3| &\le C_5 \int_{\mathbb{R}^n} dv_{\tilde{\xi}} \\ &= C_5 \int_{\mathbb{R}^n} \phi^{2^\sharp}\, dx < +\infty, \end{aligned}$$

where $C_5 > 0$ is a positive constant. Hence, (9) is true. In a similar way, we claim that we also have that

$$\int_{\mathbb{R}^n} |\nabla u|^{2^\star}\, dx < +\infty, \tag{10}$$

where $2^\star = 2n/(n-2)$ is the critical Sobolev exponent for the embedding of H_1^2-spaces (consisting of functions in L^2 with one derivative in L^2) into L^p-spaces. Another possible equation for 2^* is $2^* = 2 \times 1^\sharp$. In order to prove (10), we note that, by (6),

$$\left|\nabla(\hat{u} \circ \Phi^{-1})\right|_\xi = \phi^{2/(n-4)} \left|\nabla \hat{u}\right|_h .$$

Then, we write that

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla u|^{2^\star}\, dx &\le C_6 \int_{\mathbb{R}^n} \left|\nabla(\hat{u} \circ \Phi^{-1})\right|_\xi^{2^\star} \phi^{2^\star}\, dx + C_7 \int_{\mathbb{R}^n} |\nabla \phi|_\xi^{2^\star}\, dx \\ &\le C_8 \int_{\mathbb{R}^n} \phi^{2^\sharp}\, dx + C_6 \int_{\mathbb{R}^n} |\nabla \phi|_\xi^{2^\star}\, dx < +\infty, \end{aligned}$$

where $C_6, C_7, C_8 > 0$ are positive constants. This proves (10).

Now we consider $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_0(1)$ and $\eta \equiv 0$ in $\mathbb{R}^n \backslash B_0(2)$, where $B_0(r)$ stands for the open Euclidean ball of centre 0 and radius r in $\mathbb{R}^n$. For $R > 0$, we set

$$\eta_R(x) = \eta\left(\frac{x}{R}\right)$$

and let u be a solution of (1). Multiplying (1) by $\eta_r u$ and integrating by parts over $\mathbb{R}^n$, we get that

$$\int_{\mathbb{R}^n} \eta_R |u|^{2^\sharp}\, dx = \int_{\mathbb{R}^n} \Delta_\xi(\eta_R u)\Delta u\, dx = I_1(R) + I_2(R) - 2I_3(R), \tag{11}$$

where

$$\begin{aligned} I_1(R) &= \int_{B_0(2R)} \eta_R(\Delta_\xi u)^2\, dx \\ I_2(R) &= \int_{A_R} (\Delta_\xi \eta_R) u (\Delta_\xi u)\, dx \\ I_3(R) &= \int_{A_R} \langle \nabla \eta_R; \nabla u\rangle_\xi (\Delta_\xi u)\, dx, \end{aligned}$$

and where A_R is the annulus $A_R = B_0(2R)\backslash B_0(R)$. Clearly, thanks to (9), we have that

$$I_1(R) \longrightarrow \int_{\mathbb{R}^n} (\Delta_\xi u)^2\, dx,$$

as $R \to +\infty$. We also have that

$$\int_{\mathbb{R}^n} \eta_R |u|^{2^\sharp}\, dx \longrightarrow \int_{\mathbb{R}^n} |u|^{2^\sharp}\, dx,$$

as $R \to +\infty$. Independently, letting $V(R) = \mathrm{Vol}_\xi(A_R)$, by help of Hölder's inequality, and noting that $V(R) \le CR^n$, we can write that

$$\begin{aligned} |I_2(R)| &= \left| \int_{A_R} (\Delta_\xi \eta_R) u (\Delta_\xi u)\, dx \right| \\ &\le CR^{-2} \|u\|_{2^\sharp} \left(\int_{A_R} (\Delta_\xi u)^{2n/(n+4)}\, dx \right)^{(n+4)/2n} \end{aligned}$$

$$\leq CR^{-2}\|u\|_{2^\sharp}\left(\int_{A_R}(\Delta_\xi u)^2dx\right)^{1/2}V(R)^{2/n}$$
$$\leq C\|u\|_{2^\sharp}\left(\int_{A_R}(\Delta_\xi u)^2\,dx\right)^{1/2}.$$

Hence, $I_2(R)\to 0$ as $R\to+\infty$. In a similar way, by (10), we can write that

$$|I_3(R)|\leq CR^{-1}\|\nabla u\|_{2^\star}\left(\int_{A_R}|\Delta_\xi u|^{2n/(n+2)}\,dx\right)^{(n+2)/2n}$$
$$\leq CR^{-1}\|\nabla u\|_{2^\star}\left(\int_{A_R}(\Delta_\xi u)^2\,dx\right)^{1/2}V(R)^{1/n}$$
$$\leq C\|\nabla u\|_{2^\star}\left(\int_{A_R}(\Delta u)^2\,dx\right)^{1/2}.$$

Hence, we also have that $I_3(R)\to 0$ as $R\to+\infty$. Passing to the limit as $R\to+\infty$ in (11), we get that if $\hat{u}$ is a solution of (7), then $u\colon\mathbb{R}^n\to\mathbb{R}$ given by $u=(\hat{u}\circ\Phi^{-1})\phi$ is a solution of (1) such that

$$\int_{\mathbb{R}^n}(\Delta_\xi u)^2\,dx=\int_{\mathbb{R}^n}|u|^{2^\sharp}\,dx$$
$$=\int_{S^n}|\hat{u}|^{2^\sharp}\,dv_h<+\infty.$$

In view of this result, and in order to prove the theorem, it suffices to prove that there exists a sequence $(\hat{u}_k)_k$ of solution of (7) such that

$$\int_{S^n}|\hat{u}_k|^{2^\sharp}\,dv_h\longrightarrow+\infty,$$

as $k\to+\infty$. Let J be the functional associated to (7) given by

$$J(u)=\frac{1}{2}\int_{S^n}\big((\Delta_h u)^2+c_n|\nabla u|_h^2+d_nu^2\big)dv_h-\frac{1}{2^\sharp}\int_{S^n}|u|^{2^\sharp}\,dv_h.$$

Let also G be a closed subgroup of the isometry group $\mathrm{Isom}_h(S^n)$ of (S^n,h). For $q=1,2$, and $p>1$, we let

$$H^p_{q,G}(S^n)=\big\{u\in H^p_q(S^n)\text{ s.t. }u(g\cdot x)=u(x)$$
$$\text{for all }\ g\in G\ \text{ and }\ \text{a.a. }x\in S^n\big\},$$

where $H_q^p(S^n)$ is the Sobolev space of functions in L^p with q derivatives in L^p. We denote by $O_G^x = \{g \cdot x / g \in G\}$ the orbit of x under G and let

$$k = \min_{x \in S^n} \dim O_G^x .$$

The composition of a continuous embedding and of a compact embedding is compact. Moreover, we know from the general result in Hebey and Vaugon [5] that if $k \geq 1$, then the embedding $H_{1,G}^p(S^n) \subset L^q(S^n)$ is continuous for all $1 < q \leq p_G^\star$ and compact for all $1 < q < p_G^\star$, where $p_G^\star = +\infty$ if $n - k \leq p$, and $p_G^\star = (n-k)p/(n-k-p)$ if $n - k > p$. Noting that $(2^\star)_G^\star > 2^\sharp$ when $k \geq 1$, the sequence

$$H_{2,G}^2(S^n) \subset H_{1,G}^{2^\star}(S^n) \subset L^{(2^\star)_G^\star}(S^n)$$

then gives that the embedding $H_{2,G}^2(S^n) \subset L^{2^\sharp}(S^n)$ is compact when $k \geq 1$. In what follows, we let G be such that $k \geq 1$ and such that $H_{2,G}^2(S^n)$ is infinite dimensional. For instance, as in Ding [3], we can let $G = O(n_1) \times O(n_2)$, where n_1, n_2 are such that $n_1 + n_2 = n + 1$ and $n_1, n_2 \geq 2$. In this example, $k = \min(n_1, n_2) - 1$. We claim now that there exists a sequence $(\hat{u}_m)_m$ of critical points of J restricted to $H_{2,G}^2(S^n)$ such that

$$\int_{S^n} \hat{u}_m^{2^\sharp} \, dv_h \longrightarrow +\infty, \tag{12}$$

as $m \to +\infty$. In order to prove this claim, we first let $\| \cdot \|$ be the norm on $H_2^2(S^n)$ be given by

$$\|u\|^2 = \int_{S^n} \left((\Delta_h u)^2 + c_n |\nabla u|_h^2 + d_n u^2 \right) dv_h .$$

For J as above, it is easily seen that J is even, that $J(0) = 0$ and that

(A1) there exist $\rho, \alpha > 0$ such that $J > 0$ in $B_0(\rho) \backslash \{0\}$ and $J \geq \alpha$ on $S_0(\rho)$, and

(A2) J satisfies the Palais–Smale condition,

where $B_0(\rho)$ is the ball of centre 0 and radius ρ in $H_2^2(S^n)$, and $S_0(\rho)$ is the sphere of centre 0 and radius ρ in $H_2^2(S^n)$. We can also prove that for any finite dimensional subspace $E \subset H_{2,G}^2(S^n)$,

(A3) $E \bigcap \{J \geq 0\}$ is bounded.

Indeed, since E is finite dimensional, there exists $C > 0$ such that for any $u \in E$, $\|u\| \leq C \|u\|_{2^\sharp}$. Let $E = \operatorname{span}\{f_1, \ldots, f_N\}$, where the f_i's are an orthonormal basis for E, and $u = \sum_{i=1}^{N} \alpha_i f_i$ be such that $\|u\| = 1$. Then, for $R > 0$,

$$\begin{aligned} J(Ru) &= \frac{R^2}{2} - \frac{R^{2^\sharp}}{2^\sharp}\|u\|_{2^\sharp}^{2^\sharp} \\ &\leq \frac{R^2}{2}\left(1 - \frac{2R^{2^\sharp-2}}{2^\sharp C^{2^\sharp}}\right) \end{aligned}$$

and (A3) follows. Now, by (A1)–(A3) we can apply Theorem 2.13 of Ambrosetti and Rabinowitz [1] and we get the existence of an increasing sequence $(\alpha_m)_m$ of critical values for J given by

$$\alpha_m = \sup_{h\in\Gamma^\star} \inf_{u\in S\cap E_{m-1}^\perp} J\big(h(u)\big), \tag{13}$$

where $S = S_0(1)$, $E_m = \mathrm{span}\big\{f_1, \ldots, f_m\big\}$, $E_m^\perp$ is the orthogonal complement of E_m, $(f_i)_{i\geq 1}$ is an orthonormal basis of $H_{2,G}^2(S^n)$ and $\Gamma^\star$ is the space of odd homeomorphisms of $H_{2,G}^2(S^n)$ onto $H_{2,G}^2(S^n)$ such that $J(h(B)) \geq 0$, where B is the ball of centre 0 and radius 1 in $H_{2,G}^2(S^n)$. Then, in order to prove that there exists a sequence $(\hat{u}_m)_m$ of critical points of J restricted to $H_{2,G}^2(S^n)$ such that (12) is true, it suffices to prove that

$$\alpha_m \longrightarrow +\infty \tag{14}$$

as $m \to +\infty$. We define

$$T = \big\{u \in H_{2,G}^2(S^n) \text{ s.t. } 2^\sharp\|u\|^2 = 2\|u\|_{2^\sharp}^{2^\sharp}\big\}$$

and let

$$\beta_m = \inf_{u\in T\cap E_m^\perp} \|u\|.$$

Then,

$$\beta_m \longrightarrow +\infty \tag{15}$$

as $m \to +\infty$. Indeed, if it is not the case, there exists $(u_m)_m$ such that $u_m \in E_m^\perp$ for all m, $u_m \in T$ for all m, the u_m's are bounded in $H_2^2(S^n)$ and $u_m \rightharpoonup 0$ in $H_{2,G}^2(S^n)$ since $u_m \in E_m^\perp$. The compactness of the embedding $H_{2,G}^2(S^n) \subset L^{2^\sharp}(S^n)$ then implies that (up to a subsequence) $u_m \to 0$ in $L^{2^\sharp}(S^n)$. It follows that $u_m \to 0$ in $H_2^2(S^n)$ since $u_m \in T$ for all m. On the other hand, by the Sobolev inequality corresponding to the embedding

$H_2^2(S^n) \subset L^{2^\sharp}(S^n)$, and still since $u_m \in T$ for all m, there exists $C > 0$ such that $\|u_m\| \geq C$ for all m. A contradiction, and (15), is proved. For $u \in E_m^\perp$, we let

$$h_m(u) = \frac{1}{2}\beta_m u.$$

Following Ambrosetti and Rabinowitz [1], it is easily seen that h_m extends to $\tilde{h}_m \in \Gamma^\star$. Given $u \in H_{2,G}^2(S^n)\backslash\{0\}$, we let $\beta(u) \in \mathbb{R}$ be such that $\beta(u)u \in T$. Then, if $u \in S \cap E_m^\perp$,

$$\begin{aligned} J\big(h_m(u)\big) &= \frac{1}{2}\left(\frac{\beta_m}{2}\right)^2\left(1 - \left(\frac{\beta_m}{2\beta(u)}\right)^{2^\sharp-2}\right) \\ &\geq \frac{1}{2}\left(\frac{\beta_m}{2}\right)^2\left(1 - \left(\frac{1}{2}\right)^{2^\sharp-2}\right) \end{aligned}$$

and we get with (13) and (15) that (14) holds. In particular, there exists a sequence $(\hat{u}_m)_m$ of critical points of J restricted to $H_{2,G}^2(S^n)$ such that (12) holds. The $\hat{u}_m$'s are solutions of (7) when restricted to $H_{2,G}^2(S^n)$ in the sense that for any m and any $\varphi \in H_{2,G}^2(S^n)$,

$$\begin{aligned} &\int_{S^n} \left((\Delta_h \hat{u}_m)(\Delta_h \varphi) + c_n \langle \nabla \hat{u}_m, \nabla \varphi \rangle_h + d_n \hat{u}_m \varphi\right) dv_h \\ &\quad = \int_{S^n} |\hat{u}_m|^{2^\sharp-2} \hat{u}_m \varphi \, dv_h \end{aligned}$$

Let φ be any smooth function on S^n or any function in $H_2^2(S^n)$. Let also φ_G be given by the equation

$$\varphi_G(x) = \int_G \varphi\left(\sigma(x)\right) d\mu(\sigma),$$

where $d\mu$ is the Haar measure on G. Clearly, φ_G is smooth and G-invariant if φ is smooth or $\varphi_G \in H_{2,G}^2(S^n)$ if $\varphi \in H_2^2(S^n)$. Then we can write that

$$\int_{S^n} \left((\Delta_h \hat{u}_m)(\Delta_h \varphi_G) + c_n \langle \nabla \hat{u}_m, \nabla \varphi_G \rangle_h + d_n \hat{u}_m \varphi_G\right) dv_h$$

$$= \int_G \left(\int_{S^n} \Big((\Delta_h \hat{u}_m) \left(\Delta_h(\varphi \circ \sigma) \right) \right.$$
$$\left. + c_n \langle \nabla \hat{u}_m, \nabla(\varphi \circ \sigma) \rangle_h + d_n \hat{u}_m (\varphi \circ \sigma) \Big) dv_h \right) d\mu(\sigma)$$
$$= |G| \int_{S^n} \left((\Delta_h \hat{u}_m)(\Delta_h \varphi) + c_n \langle \nabla \hat{u}_m, \nabla \varphi \rangle_h + d_n \hat{u}_m \varphi \right) dv_h,$$

where $|G|$ is the volume of G with respect to $d\mu$, and that

$$\int_{S^n} |\hat{u}_m|^{2^\sharp - 2} \hat{u}_m \varphi_G \, dv_h$$
$$= \int_G \left(\int_{S^n} |\hat{u}_m|^{2^\sharp - 2} \hat{u}_m (\varphi \circ \sigma) dv_h \right) d\mu(\sigma)$$
$$= |G| \int_{S^n} |\hat{u}_m|^{2^\sharp - 2} \hat{u}_m \varphi \, dv_h.$$

It follows that

$$\int_{S^n} \left((\Delta_h \hat{u}_m)(\Delta_h \varphi) + c_n \langle \nabla \hat{u}_m, \nabla \varphi \rangle_h + d_n \hat{u}_m \varphi \right) dv_h = \int_{S^n} |\hat{u}_m|^{2^\sharp - 2} \hat{u}_m \varphi \, dv_h,$$

for all $\varphi \in H_2^2(S^n)$ and all m. In particular, for any m, $\hat{u}_m$ is a solution of (7). The u_m's associated to the $\hat{u}_m$'s have to change sign for $m \gg 1$ according to the remark on equivalent solutions as given earlier and the fact that

$$\int_{\mathbb{R}} (\Delta u_m)^2 dx = \int_{S^n} |\hat{u}_m|^{2^\sharp} \, dv_h \longrightarrow +\infty.$$

This ends the proof of the theorem. □

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