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Ricci flow on locally homogeneous closed 4-manifolds

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We discuss the Ricci flow on homogeneous 4-manifolds. After classifying these manifolds, we note that there are families of initial metrics such that we can diagonalize them and the Ricci flow preserves the diagonalization. We analyze the long time behavior of these families. We find that if a solution exists for all time, then the flow exhibits a type III singularity in the sense of Hamilton.

1. Introduction.

It is well-known that there are eight maximal, simply connected geometries (X, G) with compact quotient in dimension three ([17], p.474). In Thurston's geometrization conjecture, any closed three-manifold can be cut into pieces each of which admits one of these geometries. To explore the relation between the Ricci flow and the model geometries, the first two named authors analyze the long time behavior of the Ricci flow on locally homogeneous three-manifolds in [8]. In later work ([11]), using the notion of quasi-convergence, Knopf and McLeod identify the equivalence classes of all such flows except the case $X = \widehat{SL}(2, \mathbb{R})$.

Ricci flow has proven to be very successful in studying the geometric and topological properties of three manifolds ([4], [15], [16]), and there are indications ([3], [5], [7]) that it could be useful for the study of such properties in four dimensions. In order to further explore its possible use in dimension 4, we study the Ricci flow on locally homogeneous four-manifolds in this paper. We find that unlike in the case of three dimensions ([14], [8]), some of the families of locally homogeneous metrics can not be diagonalized because even if one diagonalizes the initial metric, the flow destroys the diagonalization of the metric at later times. The analysis of the ODE system given by Ricci flow for locally homogeneous manifolds is considerably simplified if the flow preserves the diagonalization. In this paper, we identify some families of initial metrics such that the Ricci flow preserves their diagonalization. For these families, we find that the behavior of the flow is very close to that seen in dimension three ([8]): either (a) the volume-normalized Ricci flow converges to a metric of constant sectional curvature or constant holomorphic bisectional curvature ($\mathbb{C}P^2$ and $\mathbb{C}H^2$); or (b) as $t \to +\infty$, the Ricci flow collapses to a lower dimensional flat manifold with the curvature decaying at the rate $\frac{1}{t}$; or (c) the Ricci flow approaches, either in finite time or in infinite time, a direct product of lower dimensional geometries with constant sectional curvature.

After describing locally homogeneous geometric structures in dimension 4 in Section 1, we consider in Section 2 the case that the homogeneous space X is a Lie group . We identify families of initial metrics whose diagonalization is preserved by the Ricci flow, and then we discuss the long time behavior of the Ricci flow for those families. In Section 3, we discuss the long time behavior of the Ricci flow for the remaining cases. Since the Ricci flow on closed manifolds preserves the isometry group, for any locally homogeneous closed 4-manifolds, we discuss the Ricci flow on their universal covering spaces.

2. Compact locally homogeneous 4-geometries.

We identify a class of four dimensional homogeneous geometries by specifying a simply connected four manifold M, a Lie group G that acts transitively on M, and the minimal isotropy group I of the action. We only consider those (M, G, I) in which M is the universal cover of a closed manifold M_q. Such a class we call a **compact four dimensional homogeneous geometry**. For each (M, G, I), there is a collection of Riemannian metrics on M for which G is the isometry group. These are the lifts of the locally homogeneous metrics on M_q.

2.1. List of compact four dimensional homogeneous geometries.

Let H^n be the simply-connected hyperbolic n-manifold and S^n be the simply-connected round n-sphere. We denote the group of isometries of H^n by $H(n)$. We summarize the compact three dimensional homogeneous geometries in the following table.

Manifold M^3	Lie group G	Isotropy group I
$\mathbb{R}^3$	$\mathbb{R}^3$	$\{0\}$
S^3	$SU(2)$	$\{e\}$
$\widehat{SL}(2,\mathbb{R})$	$\widehat{SL}(2,\mathbb{R})$	$\{e\}$
Nil^3	Nil^3	$\{e\}$
$\widehat{Sol}^3$	$\widehat{Sol}^3$	$\{e\}$
$\mathbb{R}^3$	$E(2)$	$\{e\}$
$S^2 \times \mathbb{R}$	$SO(3) \times \mathbb{R}$	$SO(2) \times \{0\}$
$H^2 \times \mathbb{R}$	$H(2) \times \mathbb{R}$	$SO(2) \times \{0\}$
H^3	$H(3)$	$SO(3)$

Here, $\widehat{SL}(2,\mathbb{R})$ is the universal cover of the special linear group $SL(2,\mathbb{R})$; its lie algebra sl_2 has a basis X_1, X_2, X_3 such that the Lie bracket is given by

$$[X_1, X_2] = -X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

Nil^3 is the 3-dimensional Heisenberg group consisting of matrices of the form

$$\begin{bmatrix} 1 & c_1 & c_2 \\ 0 & 1 & c_3 \\ 0 & 0 & 1 \end{bmatrix};$$

its Lie algebra n_3 has a basis X_1, X_2, X_3 such that the Lie bracket is given by

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = 0, \quad [X_3, X_1] = 0.$$

$\widehat{Sol}^3$ is the simply-connected solvable Lie group whose Lie algebra sol_3 has a basis X_1, X_2, X_3 satisfying

$$[X_1, X_2] = 0, \quad [X_2, X_3] = -X_2, \quad [X_3, X_1] = -X_1.$$

$E(2)$ is also a solvable Lie group whose Lie algebra $L(E_2)$ has a basis X_1, X_2, X_3 satisfying

$$[X_1, X_2] = 0, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = -X_2.$$

The Lie algebra $su(2)$ of $SU(2)$ can be described by

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

The compact four dimensional homogeneous geometries have been classified by Ishihara [9]. We list them in the following table (see [18]).

Manifold M^4	Lie group G	Isotropy group I
Nil^4	Nil^4	$\{e\}$
$Sol^4_{m,n}$	$Sol^4_{m,n}$	$\{e\}$
Sol^4_1	Sol^4_1	$\{e\}$
Sol^4_0	Sol^4_0	$\{e\}$
$\widehat{SL}(2,\mathbb{R}) \times \mathbb{R}$	$\widehat{SL}(2,\mathbb{R}) \times \mathbb{R}$	$\{e\}$
$Nil^3 \times \mathbb{R}$	$Nil^3 \times \mathbb{R}$	$\{e\}$
$S^3 \times \mathbb{R}$	$SU(2) \times \mathbb{R}$	$\{e\}$
$\mathbb{R}^4$	$E(2) \times \mathbb{R}$	$\{e\}$
$\mathbb{R}^4$	$\mathbb{R}^4$	$\{0\}$
$S^2 \times S^2$	$SO(3) \times SO(3)$	$SO(2) \times SO(2)$
$S^2 \times \mathbb{R}^2$	$SO(3) \times \mathbb{R}^2$	$SO(2) \times \{0\}$
$S^2 \times H^2$	$SO(3) \times H(2)$	$SO(2) \times SO(2)$
$H^2 \times \mathbb{R}^2$	$H(2) \times \mathbb{R}^2$	$SO(2) \times \{0\}$
$H^2 \times H^2$	$H(2) \times H(2)$	$SO(2) \times SO(2)$
$\mathbb{C}P^2$	$SU(3)$	$U(2)$
$\mathbb{C}H^2$	$SU(1,2)$	$U(2)$
$H^3 \times \mathbb{R}$	$H(3) \times \mathbb{R}$	$SO(3) \times \{0\}$
S^4	$SO(5)$	$SO(4)$
H^4	$H(4)$	$SO(4)$

Nil^4, $Sol^4_{m,n}$, Sol^4_1 and Sol^4_0 are simply connected 4-dimensional Lie groups; we describe their Lie algebras in Section 2.2. Note that $Sol^4_{m,n}$ includes $\widehat{Sol^3} \times \mathbb{R}$. $\mathbb{C}H^2$ is complex hyperbolic space which has Kähler symmetric space structure (see [12], pp. 282–285).

Note that there is another locally homogeneous space $M = F^4$ listed in [18]. This is not a compact homogeneous geometry because it does not have compact quotients. The isometry group G contains a discrete subgroup Γ such that F^4/Γ has finite volume.

One can find a more detailed description of four dimensional locally homogeneous geometries in Part II of [6].

The Ricci flow study for those classes with trivial isotropy group requires substantial new analysis; we group these in a category labelled A. We describe these classes in Section 2.2. Those classes with non-trivial isotropy group are grouped in category B (Section 2.3).

2.2. Four dimensional unimodular Lie groups.

Recall that a Lie group G is called **co-compact** if G contains a discrete subgroup Γ such that G/Γ is compact. Each Lie group in (A) is co-compact. A co-compact Lie group has unimodular Lie algebra ([14] Lemma 6.2). Instead of studying Ricci flow on spaces in (A), we broaden the discussion to Ricci flow on 4-dimensional unimodular Lie groups.

According to the classification of the 4-dimensional unimodular Lie algebras ([13]), for each such algebra, there is some basis X_1, X_2, X_3, X_4 such that the Lie bracket takes the form indicated below. We adopt the notation[1] in [13].

A1. Class $U1[(1,1,1)]$.

$$[X_2, X_3] = 0, \qquad [X_3, X_1] = 0, \qquad [X_1, X_2] = 0,$$
$$[X_1, X_4] = 0, \qquad [X_2, X_4] = 0, \qquad [X_3, X_4] = 0.$$

This corresponds to $(M, G, I) = (\mathbb{R}^4, \mathbb{R}^4, \{0\})$ where G acts on M by translation.

A2. Class $U1[1,1,1]$.

$$[X_2, X_3] = 0, \qquad [X_3, X_1] = 0, \qquad [X_1, X_2] = 0,$$
$$[X_1, X_4] = X_1, \qquad [X_2, X_4] = kX_2, \qquad [X_3, X_4] = -(k+1)X_3,$$

where, without loss of generality, we assume $k \geq -\frac{1}{2}$ since otherwise we can interchange X_2 and X_3. Only the following special cases correspond to compact homogeneous geometries.

(A2i) if $k = 0$, the Lie algebra is isomorphic to the direct sum $sol_3 \oplus \mathbb{R}$. This corresponds to $(M, G, I) = (\widehat{Sol^3} \times \mathbb{R}, \widehat{Sol^3} \times \mathbb{R}, \{e\})$.

(A2ii) if $k = 1$, the corresponding geometry is $(M, G, I) = (Sol_0^4, Sol_0^4, \{e\})$; this can be seen by choosing $e_1 = X_1, e_2 = X_2, e_3 = X_3$, and $e_4 = -X_4$ on p.273 in [18].

(A2iii) if there is a number $\alpha > 0$ such that the exponentials of α, $\beta \doteq k\alpha$ and $\gamma \doteq -(k+1)\alpha$ are roots of $\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$ for some $m, n \in \mathbb{N}$ and $m \neq n$, then one has $(M, G, I) = (Sol_{m,n}^4, Sol_{m,n}^4, \{e\})$ for the geometry. This can be seen by choosing $e_1 = \alpha X_1, e_2 = \alpha X_2, e_3 = \alpha X_3$, and $e_4 = -\alpha X_4$ on p. 274 and p. 270 in [18].

[1] For example, U stands for unimodular and the various integers 1,2,3 refer to certain characteristics of the Lie algebra structure; see [13] for details.

A3. Class $U1[Z, \bar{Z}, 1]$.

$$[X_2, X_3] = 0, \quad [X_3, X_1] = 0, \quad [X_1, X_2] = 0,$$
$$[X_1, X_4] = kX_1 + X_2, \quad [X_2, X_4] = -X_1 + kX_2, \quad [X_3, X_4] = -2kX_3,$$

where k is a real number. If $k = 0$, this corresponds to the geometry $(M, G, I) = (\mathbb{R}^4, E(2) \times \mathbb{R}, \{e\})$. Other values of k do not correspond to compact homogeneous geometries.

A4. Class $U1[2, 1]$ with $\mu = 0$.

$$[X_2, X_3] = 0, \quad [X_3, X_1] = 0, \quad [X_1, X_2] = 0,$$
$$[X_1, X_4] = X_2, \quad [X_2, X_4] = 0, \quad [X_3, X_4] = 0.$$

This Lie algebra is isomorphic to the direct sum $n_3 \oplus \mathbb{R}$ where n_3 is the Lie algebra of Nil^3. Hence, in this case $(M, G, I) = (Nil^3 \times \mathbb{R}, Nil^3 \times \mathbb{R}, \{e\})$.

A5. Class $U1[2, 1]$ with $\mu = 1$.

$$[X_2, X_3] = 0, \quad [X_3, X_1] = 0, \quad [X_1, X_2] = 0,$$
$$[X_1, X_4] = -\frac{1}{2}X_1 + X_2, \quad [X_2, X_4] = -\frac{1}{2}X_2, \quad [X_3, X_4] = X_3.$$

This does not correspond to any of the compact homogeneous geometries.

A6. Class $U1[3]$.

$$[X_2, X_3] = 0, \quad [X_3, X_1] = 0, \quad [X_1, X_2] = 0,$$
$$[X_1, X_4] = X_2, \quad [X_2, X_4] = X_3, \quad [X_3, X_4] = 0.$$

This corresponds to the geometry $(M, G, I) = (Nil^4, Nil^4, \{e\})$ which can be seen by choosing $e_1 = X_1, e_2 = X_2, e_3 = X_3$, and $e_4 = -X_4$ on p. 274 in [18].

A7. Class $U3I0$.

$$[X_1, X_4] = 0, \quad [X_2, X_4] = 0, \quad [X_3, X_4] = 0,$$
$$[X_2, X_3] = X_4, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = -X_3.$$

This corresponds to the geometry $(M, G, I) = (Sol_1^4, Sol_1^4, \{e\})$ which can be seen by choosing $e_1 = X_1, e_2 = X_2 + X_3, e_3 = X_2 - X_3, e_4 = -2X_4$ on p. 272 in [18].

A8. Class $U3I2$.

$$[X_1, X_4] = 0, \quad [X_2, X_4] = 0, \quad [X_3, X_4] = 0,$$
$$[X_2, X_3] = -X_4, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = X_3.$$

This does not correspond to any of the compact homogeneous geometries.

A9. Class $U3S1$.

$$[X_1, X_4] = 0, \quad [X_2, X_4] = 0, \quad [X_3, X_4] = 0,$$
$$[X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = -X_3.$$

This Lie algebra is isomorphic to the direct sum $sl_2 \oplus \mathbb{R}$. This corresponds to the geometry $(M, G, I) = (\widehat{SL}(2, \mathbb{R}) \times \mathbb{R}, \widehat{SL}(2, \mathbb{R}) \times \mathbb{R}, \{e\})$.

A10. Class $U3S3$.

$$[X_1, X_4] = 0, \quad [X_2, X_4] = 0, \quad [X_3, X_4] = 0,$$
$$[X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = X_3.$$

This Lie algebra is isomorphic to the direct sum $su(2) \oplus \mathbb{R}$. This corresponds to the geometry $(M, G, I) = (S^3 \times \mathbb{R}, SU(2) \times \mathbb{R}, \{e\})$.

2.3. Compact four dimensional homogeneous geometries with non-trivial isotropy group.

Now we list the compact 4-dimensional homogeneous geometries (M^4, G, I) for which dimension of G is bigger than 4. Recall $H(n)$ is the isometry group of the simply-connected hyperbolic n-manifolds H^n.

B1. $(M, G, I) = (H^3 \times \mathbb{R}, H(3) \times \mathbb{R}, SO(3))$

B2. $(M, G, I) = (S^2 \times \mathbb{R}^2, SO(3) \times \mathbb{R}^2, SO(2) \times \{0\})$

B3. $(M, G, I) = (H^2 \times \mathbb{R}^2, H(2) \times \mathbb{R}^2, SO(2) \times \{0\})$

B4. $(M, G, I) = (S^2 \times S^2, SO(3) \times SO(3), SO(2) \times SO(2))$

B5. $(M, G, I) = (S^2 \times H^2, SO(3) \times H(2), SO(2) \times SO(2))$

B6. $(M, G, I) = H^2 \times H^2, H(2) \times H(2), SO(2) \times SO(2))$

B7. $(M, G, I) = (\mathbb{C}P^2, SU(3), U(2))$

B8. $(M, G, I) = (\mathbb{C}H^2, SU(1, 2), U(2))$

B9. $(M, G, I) = (S^4, SO(5), SO(4))$

B10. $(M, G, I) = (H^4, H(4), SO(4))$

3. The Ricci flow on 4-dimensional unimodular Lie groups.

Recall that our strategy is to analyze the Ricci flow on a simply connected manifold M that is the universal cover of a closed manifold M_q. For a fixed class (M, G, I) and a fixed initial homogeneous metric g_0 compatible with the class, let $g(t)$ be the homogeneous solution of the Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t)) \qquad g(0) = g_0.$$

On the closed manifold M_q, we consider the volume-normalized Ricci flow $\tilde{g}_N(\tilde{t})$ as the solution of

$$\frac{\partial}{\partial \tilde{t}} \tilde{g}_N(\tilde{t}) = -2\,\mathrm{Ric}(\tilde{g}_N(\tilde{t})) + \frac{\tilde{r}_N}{2} \tilde{g}_N \qquad \tilde{g}_N(0) = g_0$$

where $\tilde{r}_N$ is the scalar curvature of $\tilde{g}_N$. Note that averaging of the scalar curvature is not needed for homogeneous metrics. We also note that the Ricci flow equation for homogeneous metrics reduces to a system of ordinary differential equations.

For each class (Ai) listed in Section 2.2, we describe the families of initial metrics which are diagonal and remain diagonal under the Ricci flow and then study their long time behavior. To address the diagonalization issue, we use the following strategy: Fix a homogeneous metric h and let $\{X_i\}$ be any basis of left-invariant vectors on the Lie group G with the bracket structure

$$[X_i, X_j] = C^k{}_{ij} X_k.$$

For those classes (e.g., A4, A9 and A10) in which the Lie group G is a product group $G_1 \times \mathbb{R}$ with $\dim(G_1) = 3$, we choose X_1, X_2, X_3 as left invariant vector fields on G_1 and $X_4 = \frac{\partial}{\partial u}$ on $\mathbb{R}$.

Let $\{Y_i\}$ be any basis orthogonal with respect to h; that is,

$$h(Y_i, Y_j) = \lambda_i \delta_{ij}.$$

Let the transformation from $\{X_i\}$ to $\{Y_i\}$ be given by

$$Y_i = \Lambda^k{}_i X_k.$$

Computing the bracket structure for $\{Y_i\}$, we get

$$[Y_i, Y_j] = [\Lambda^k{}_i X_k, \Lambda^l{}_j X_l] = \Lambda^k{}_i \Lambda^l{}_j C^m_{kl} X_m = \Lambda^k{}_i \Lambda^l{}_j C^m_{kl} (\Lambda^{-1})^n{}_m Y_n.$$

Thus,

$$[Y_i, Y_j] = \tilde{C}^n{}_{ij} Y_n$$

where

$$\tilde{C}^n{}_{ij} = \Lambda^k{}_i \Lambda^l{}_j (\Lambda^{-1})^n{}_m C^m_{kl}.$$

Now, compute the Ricci curvature of h using the orthonormal basis $\{\bar{Y}_i\}$ defined by $\bar{Y}_i = \frac{1}{\sqrt{\lambda_i}} Y_i$. For this, we use the following Ricci curvature formula for unimodular Lie groups from Corollary 7.33 p. 184 [1],

$$\mathrm{Ric}(W,W) = -\frac{1}{2}\sum_i \big|[W,\bar{Y}_i]\big|^2 - \frac{1}{2}\sum_i \langle [W,[W,\bar{Y}_i]], \bar{Y}_i\rangle + \frac{1}{2}\sum_{i<j} \langle [\bar{Y}_i,\bar{Y}_j], W\rangle^2. \tag{1}$$

Finally, check if any positive values of the parameters λ_i produce a diagonal Ricci tensor. Only for these values does the metric remain diagonal under the Ricci flow. We follow this strategy and use the same notation in the rest of this section.

Remark In our search for families of initial metrics which remain diagonal under the Ricci flow, we have chosen special Y_i so that the Lie brackets $[Y_i, Y_j]$ are simple. Presumably, there are other families of initial metrics and other bases Y_i for which the diagonalization is preserved by the Ricci flow. This possibility has not yet been explored. Our calculations do show that there are (M, G, I) and bases Y_i such that the property that the initial metric has components $(g_0)_{a4} = 0$, $a = 1, 2, 3$, is preserved under Ricci flow.

To study the decay of the curvature tensor, we use the following sectional curvature formula for Lie groups from Theorem 7.30 p. 183 [1]. For the Lie algebra $\mathfrak{g}$ of G, define the operator $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$2\langle U(X,Y), Z\rangle = \langle [Z,X], Y\rangle + \langle X, [Z,Y]\rangle \text{ for all } Z \in \mathfrak{g}; \tag{2}$$

then the curvature is given by

$$\begin{aligned}\langle R(X,Y)X,Y\rangle = -\frac{3}{4}|[X,Y]|^2 - \frac{1}{2}\langle [X,[X,Y]],Y\rangle - \frac{1}{2}\langle [Y,[Y,X]],X\rangle \\ + |U(X,Y)|^2 - \langle U(X,X), U(Y,Y)\rangle.\end{aligned} \tag{3}$$

3.A1. U1[(1,1,1)].

For $(M, G, I) = (\mathbb{R}^4, \mathbb{R}^4, \{0\})$, G acts on M by translation $h(x) = h + x$ for $h \in G$. Any homogeneous metric g_0 on M must be of the form

$$g_0 = \lambda_1 dx^1 \otimes dx^1 + +\lambda_2 dx^2 \otimes dx^2 + \lambda_3 dx^3 \otimes dx^3 + \lambda_4 dx^4 \otimes dx^4$$

for some constants $\lambda_i > 0$. The metric g_0 is flat; hence $g(t) \equiv g_0$ for $-\infty < t < \infty$.

3.A2. $U1[1,1,1]$.

For $U1[1,1,1]$, we use $Y_i = \Lambda^k{}_i X_k$ with

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & a_3 & 1 & 0 \\ a_4 & a_5 & a_6 & 1 \end{bmatrix}$$

to diagonalize the initial metric g_0.

Proposition 3.1. *For the class $U1[1,1,1]$ suppose the initial metric g_0 is diagonal in the basis Y_i. Then*

(i) if $k \neq 1, -\frac{1}{2}$, the Ricci flow solution $g(t)$ remains diagonal in the basis Y_i if and only if $a_1 = a_2 = a_3 = 0$;

(ii) if $k = 1$, the Ricci flow solution $g(t)$ remains diagonal in the basis Y_i if and only if $a_2 = a_3 = 0$;

(iii) if $k = -\frac{1}{2}$, the Ricci flow solution $g(t)$ remains diagonal in the basis Y_i if and only if $a_1 = a_2 = 0$;

Proof. We compute

$$\begin{aligned} &[Y_2, Y_3] = 0, && [Y_3, Y_1] = 0, && [Y_1, Y_2] = 0, \\ &[Y_1, Y_4] = Y_1, && [Y_2, Y_4] = kY_2 + \alpha Y_1, && [Y_3, Y_4] = -(k+1)Y_3 + \beta Y_2 + \gamma Y_1. \end{aligned}$$

where

$$\alpha = (1-k)a_1, \quad \beta = (1+2k)a_3, \quad \text{and} \quad \gamma = (k+2)a_2 - (1+2k)a_1 a_3.$$

Let $W = w_1\bar{Y}_1 + w_2\bar{Y}_2 + w_3\bar{Y}_3 + w_4\bar{Y}_4$. We compute $[W, \bar{Y}_i]$ first and then use (1) with $h = g_0$ to compute the coefficients of $w_i w_j$ in $\mathrm{Ric}(W,W)$. We find that the off-diagonal components of the Ricci tensor in the basis $\{\bar{Y}_i\}$ are given by

$$\begin{aligned} \mathrm{Ric}(\bar{Y}_1, \bar{Y}_2) &= \frac{\big(\beta\gamma\lambda_2 + (k-1)\alpha\lambda_3\big)\sqrt{\lambda_1}}{2\sqrt{\lambda_2}\lambda_3\lambda_4} \\ \mathrm{Ric}(\bar{Y}_1, \bar{Y}_3) &= -\frac{(2+k)\gamma\sqrt{\lambda_1}}{2\sqrt{\lambda_3}\lambda_4} \\ \mathrm{Ric}(\bar{Y}_2, \bar{Y}_3) &= -\frac{\alpha\gamma\lambda_1 + (1+2k)\beta\lambda_2}{2\sqrt{\lambda_2}\sqrt{\lambda_3}\lambda_4} \\ \mathrm{Ric}(\bar{Y}_1, \bar{Y}_4) &= \mathrm{Ric}(\bar{Y}_2, \bar{Y}_4) = \mathrm{Ric}(\bar{Y}_3, \bar{Y}_4) = 0 \end{aligned}$$

In order for these off-diagonal components to be zero, we must have $\alpha = \beta = \gamma = 0$. The proposition follows. □

Now we discuss the long time behavior for the families of the initial metrics in Proposition 3.1 We start with (A2iv) and later show that the other three cases are covered by the same analysis.

(A2iv) In this case, $k \neq 0, 1, -\frac{1}{2}$. We have

$$\begin{aligned} Y_1 &= X_1, & Y_2 &= X_2, \\ Y_3 &= X_3, & Y_4 &= X_4 + a_4 X_1 + a_5 X_2 + a_6 X_3. \end{aligned}$$

and

$$\begin{aligned} [Y_2, Y_3] &= 0, & [Y_3, Y_1] &= 0, & [Y_1, Y_2] &= 0, \\ [Y_1, Y_4] &= Y_1, & [Y_2, Y_4] &= kY_2, & [Y_3, Y_4] &= -(k+1)Y_3. \end{aligned}$$

The bases Y_i and X_i both satisfy the same Lie bracket relations so either can be used in the Ricci flow analysis. We use X_i.

Let θ_i be the frame of 1-forms dual to X_i. Assume the Ricci flow solution takes the special form

$$g(t) = A(t)(\theta_1)^2 + B(t)(\theta_2)^2 + C(t)(\theta_3)^2 + D(t)(\theta_4)^2 \tag{3.5}$$

with

$$g_0 = \lambda_1(\theta_1)^2 + \lambda_2(\theta_2)^2 + \lambda_3(\theta_3)^2 + \lambda_4(\theta_4)^2.$$

Then, $\bar{X}_1 = \frac{1}{\sqrt{A}} X_1, \cdots, \bar{X}_4 = \frac{1}{\sqrt{D}} X_4$ is an orthonormal frame with respect to the metric g. Let $W = w_1\bar{X}_1 + w_2\bar{X}_2 + w_3\bar{X}_3 + w_4\bar{X}_4$ and then compute

$$\begin{aligned} [W, \bar{X}_1] &= -\frac{1}{\sqrt{D}} w_4 \bar{X}_1 & [W, \bar{X}_2] &= -\frac{k}{\sqrt{D}} w_4 \bar{X}_2 \\ [W, \bar{X}_3] &= \frac{k+1}{\sqrt{D}} w_4 \bar{X}_3 & [W, \bar{X}_4] &= \frac{1}{\sqrt{D}} w_1 \bar{X}_1 + \frac{k}{\sqrt{D}} w_2 \bar{X}_2 - \frac{k+1}{\sqrt{D}} w_3 \bar{X}_3. \end{aligned}$$

We have from (1) with $h = g$

$$\operatorname{Ric}(W, W) = 0 \cdot w_1^2 + 0 \cdot w_2^2 + 0 \cdot w_3^2 - \frac{2(k^2+k+1)}{D} \cdot w_4^2.$$

So

$$\begin{aligned} &\operatorname{Ric}(X_1, X_1) = \operatorname{Ric}(X_2, X_2) = \operatorname{Ric}(X_3, X_3) = 0, \\ &\operatorname{Ric}(X_4, X_4) = D \cdot \operatorname{Ric}(\bar{X}_4, \bar{X}_4) = -2(k^2+k+1), \end{aligned}$$

and the Ricci flow is

$$\frac{dA}{dt} = 0, \qquad \frac{dB}{dt} = 0,$$
$$\frac{dC}{dt} = 0, \qquad \frac{dD}{dt} = 4(k^2 + k + 1).$$

The solution is given by

$$A(t) = \lambda_1, \quad B(t) = \lambda_2, \quad C(t) = \lambda_3, \quad \text{and} \quad D(t) = \lambda_4 + 4(k^2 + k + 1)t.$$

Hence for the subfamily in Proposition 3.1, the Ricci flow does not move in three directions and expands in the fourth direction at a speed linear in t.

Next, we compute the curvature decay of $g(t)$. From (2) we find

$$U(X_1, X_1) = -\frac{A}{D}X_4 \qquad U(X_2, X_2) = -\frac{kB}{D}X_4$$
$$U(X_3, X_3) = \frac{(k+1)C}{D}X_4 \quad U(X_4, X_4) = 0 \qquad U(X_1, X_2) = 0$$
$$U(X_1, X_3) = 0 \qquad U(X_2, X_3) = 0 \qquad U(X_1, X_4) = \frac{1}{2}X_1$$
$$U(X_2, X_4) = \frac{k}{2}X_2 \qquad U(X_3, X_4) = -\frac{k+1}{2}X_3.$$

From (3) with $h = g$ we find the sectional curvatures

$$K(X_1, X_2) = -\frac{k}{D}, \quad K(X_1, X_3) = \frac{k+1}{D}, \quad K(X_2, X_3) = \frac{k(k+1)}{D},$$
$$K(X_1, X_4) = -\frac{1}{D}, \quad K(X_2, X_4) = -\frac{k^2}{D}, \quad K(X_3, X_4) = -\frac{(k+1)^2}{D}.$$

These curvatures of the solution $g(t)$ decay at the rate $1/t$.

Recall on closed manifolds the volume-normalized solution $\tilde{g}_N(\tilde{t})$ relates to $g(t)$ by scaling, where $\tilde{t}$ is a function of t with $\lim_{t\to\infty} \tilde{t} = \infty$. From (3.5), it is easy to see that the behavior of $\tilde{g}_N(\tilde{t})$ as $\tilde{t} \to \infty$ is the same as the behavior of $g_N(t)$ as $t \to \infty$ where

$$g_N(t) \doteq \left(\frac{\lambda_1\lambda_2\lambda_3\lambda_4}{A(t)B(t)C(t)D(t)}\right)^{1/4} g(t).$$

In discussing the long time behavior of volume normalized Ricci flow solutions, from this point on, we work with $g_N(t)$. For convenience, we call this the volume-normalized solution.

Let us pick a point $p \in M_q$. It is clear that the volume-normalized solution $(M_q, g_N(t), p)$ collapses to a line in the the pointed Gromov–Hausdorff topology.

(3.A2i) This is a special case of (A2iv) if we allow $k = 0$, so the analysis in (A2iv) applies. Note that since the Lie algebra is the direct sum $sol_3 \oplus \mathbb{R}$, we can get the same conclusions from the analysis in [8](pp. 733-735).

(3.A2ii) In this case, $k = 1$. We have

$$\begin{aligned} Y_1 &= X_1, & Y_2 &= X_2 + a_1 X_1, \\ Y_3 &= X_3, & Y_4 &= X_4 + a_4 X_1 + a_5 X_2 + a_6 X_3. \end{aligned}$$

and

$$\begin{aligned} [Y_2, Y_3] &= 0, & [Y_3, Y_1] &= 0, & [Y_1, Y_2] &= 0, \\ [Y_1, Y_4] &= Y_1, & [Y_2, Y_4] &= Y_2, & [Y_3, Y_4] &= -2Y_3. \end{aligned}$$

Note that this is the Lie algebra structure in (A2iv) if we allow $k = 1$. Hence, the analysis of (A2iv) applies with the same conclusion.

(3.A2iii) In this case, $k = -\frac{1}{2}$. We have

$$\begin{aligned} Y_1 &= X_1, & Y_2 &= X_2, \\ Y_3 &= a_3 X_2 + X_3, & Y_4 &= X_4 + a_4 X_1 + a_5 X_2 + a_6 X_3. \end{aligned}$$

and

$$\begin{aligned} [Y_2, Y_3] &= 0, & [Y_3, Y_1] &= 0, & [Y_1, Y_2] &= 0, \\ [Y_1, Y_4] &= Y_1, & [Y_2, Y_4] &= -\frac{1}{2} Y_2, & [Y_3, Y_4] &= -\frac{1}{2} Y_3. \end{aligned}$$

Note that this is the Lie algebra structure in (A2iv) if we allow $k = -\frac{1}{2}$. Hence, the analysis of (A2iv) applies with the same conclusion.

3.A3. $U1[Z, \bar{Z}, 1]$.

For $U1[Z, \bar{Z}, 1]$, we use $Y_i = \Lambda^k{}_i X_k$ with

$$\Lambda = \begin{bmatrix} 1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & 0 \\ 0 & 0 & 1 & 0 \\ a_4 & a_5 & a_6 & 1 \end{bmatrix}$$

to diagonalize the initial metric g_0.

Proposition 3.2. *For the class $U1[Z, \bar{Z}, 1]$, suppose the initial metric g_0 is diagonal in the basis Y_i. Then, the Ricci flow solution $g(t)$ remains diagonal in the basis Y_i if and only if $a_1 = a_2 = a_3 = 0$.*

Proof. We compute

$$[Y_2, Y_3] = 0, \qquad [Y_3, Y_1] = 0, \qquad [Y_1, Y_2] = 0, \qquad [Y_3, Y_4] = -2kY_3$$
$$[Y_1, Y_4] = (k-\alpha)Y_1 + (\alpha^2+1)Y_2 + \beta Y_3, \quad [Y_2, Y_4] = -Y_1 + (k+\alpha)Y_2 + \gamma Y_3.$$

where

$$\alpha = a_2, \quad \beta = a_2a_3 - a_1a_2^2 - a_1 - 3ka_3, \quad \text{and} \quad \gamma = a_3 - 3ka_1 - a_1a_2.$$

We compute the off-diagonal components of the Ricci tensor in the basis $\{\bar{Y}_i\}$ using (1) as in Section 2.A2 and get

$$\begin{aligned}
\operatorname{Ric}(\bar{Y}_1, \bar{Y}_2) &= -\frac{2\alpha\lambda_1 + 2\alpha(1+\alpha^2)\lambda_2 + \beta\gamma\lambda_3}{2\sqrt{\lambda_1\lambda_2}\lambda_4} \\
\operatorname{Ric}(\bar{Y}_1, \bar{Y}_3) &= -\frac{\big(\gamma\lambda_1 + (\alpha-3k)\beta\lambda_2\big)\sqrt{\lambda_3}}{2\sqrt{\lambda_1}\lambda_2\lambda_4} \\
\operatorname{Ric}(\bar{Y}_2, \bar{Y}_3) &= \frac{\big((\alpha+3k)\gamma\lambda_1 + (1+\alpha^2)\beta\lambda_2\big)\sqrt{\lambda_3}}{2\lambda_1\sqrt{\lambda_2}\lambda_4} \\
\operatorname{Ric}(\bar{Y}_1, \bar{Y}_4) &= \operatorname{Ric}(\bar{Y}_2, \bar{Y}_4) = \operatorname{Ric}(\bar{Y}_3, \bar{Y}_4) = 0.
\end{aligned}$$

In order for these off-diagonal components to be zero, we must have $\alpha = \beta = \gamma = 0$ and the proposition follows. □

If $\alpha = \beta = \gamma = 0$, the Y_i and X_i both satisfy the same Lie bracket relations. As in Section 2.A2, we use X_i in carrying out the analysis of the long time behavior of the Ricci flow solution for the family in Proposition 3.2. Proceeding as in Section 2.A2, we find

$$\begin{aligned}
&[W, \bar{X}_1] = -\frac{k}{\sqrt{D}}w_4\bar{X}_1 - \sqrt{\frac{B}{AD}}w_4\bar{X}_2 \\
&[W, \bar{X}_2] = \sqrt{\frac{A}{BD}}w_4\bar{X}_1 - \frac{k}{\sqrt{D}}w_4\bar{X}_2, \qquad [W, \bar{X}_3] = \frac{2k}{\sqrt{D}}w_4\bar{X}_3 \\
&[W, \bar{X}_4] = (\frac{k}{\sqrt{D}}w_1 - \sqrt{\frac{A}{BD}}w_2)\bar{X}_1 + (\sqrt{\frac{B}{AD}}w_1 + \frac{k}{\sqrt{D}}w_2)\bar{X}_2 - \frac{2k}{\sqrt{D}}w_3\bar{X}_3.
\end{aligned}$$

We have from (1)

$$\operatorname{Ric}(W, W) = \frac{A^2 - B^2}{2ABD}w_1^2 - \frac{A^2 - B^2}{2ABD}w_2^2 + 0 \cdot w_3^2 - \frac{(A-B)^2 + 12k^2AB}{2ABD}w_4^2,$$

so the Ricci flow is

$$\frac{dA}{dt} = -\frac{A^2 - B^2}{BD}, \qquad \frac{dB}{dt} = -\frac{B^2 - A^2}{AD},$$
$$\frac{dC}{dt} = 0, \qquad \frac{dD}{dt} = \frac{(A-B)^2 + 12k^2 AB}{AB}.$$

Clearly, $C(t) = \lambda_3$.

If $\lambda_1 = \lambda_2$, then

$$A(t) = \lambda_1, \quad B(t) = \lambda_2, \quad \text{and} \quad D(t) = \lambda_4 + 12k^2 t.$$

If $\lambda_1 \neq \lambda_2$, we may assume $\lambda_2 > \lambda_1$ without loss of generality by the symmetry of A and B in this system. A simple computation gives

$$\frac{1}{A}\frac{dA}{dt} + \frac{1}{B}\frac{dB}{dt} = 0,$$

so the product $AB = \lambda_1 \lambda_2$ for all t. Another computation gives

$$\frac{d}{dt}[B - A] = -\frac{(A+B)^2}{ABD}(B - A),$$

so $B - A$ is positive and is decreasing in t. From the equation for $\frac{dD}{dt}$, it follows easily that

$$12k^2 \leq \frac{dD}{dt} \leq 12k^2 + \frac{\lambda_2}{\lambda_1},$$

and so

$$\lambda_4 + 12k^2 t \leq D(t) \leq \lambda_4 + (12k^2 + \frac{\lambda_2}{\lambda_1})t.$$

Since,

$$\frac{(A+B)^2}{ABD} \geq \frac{4}{D} \geq \frac{4}{\lambda_4 + (12k^2 + \frac{\lambda_2}{\lambda_1})t},$$

it follows that

$$\frac{1}{B-A}\frac{d}{dt}[B - A] \leq -\frac{4}{\lambda_4 + (12k^2 + \frac{\lambda_2}{\lambda_1})t}.$$

Integrating the inequality, we get

$$B - A \leq (\lambda_2 - \lambda_1)\left(1 + (\frac{12k^2}{\lambda_4} + \frac{\lambda_2}{\lambda_1 \lambda_4})t\right)^{-\frac{4}{12k^2 + \frac{\lambda_2}{\lambda_1}}}.$$

Hence for the family in Proposition 3.2, When $K \neq 0$ the long-time behavior of the solution $g(t)$ as $t \to +\infty$ is (see [8] for $K = 0$)

$$A(t) \to \sqrt{\lambda_1\lambda_2}, \quad B(t) \to \sqrt{\lambda_1\lambda_2}, \quad C(t) = \lambda_3, \quad D(t) \to \infty \text{ linearly.}$$

Next, we compute the curvature decay of $g(t)$. From (2), we find

$$\begin{aligned}
&U(X_1, X_1) = -\frac{kA}{D}X_4 && U(X_2, X_2) = -\frac{kB}{D}X_4 \quad U(X_3, X_3) = \frac{2kC}{D}X_4\\
&U(X_4, X_4) = 0 && U(X_1, X_2) = \frac{A-B}{2D}X_4\, U(X_1, X_3) = 0\\
&U(X_2, X_3) = 0 && U(X_1, X_4) = \frac{k}{2}X_1 - \frac{A}{2B}X_2\\
&U(X_2, X_4) = \frac{B}{2A}X_1 + \frac{k}{2}X_2\, U(X_3, X_4) = -kX_3.
\end{aligned}$$

From (3), with $h = g$, we find the sectional curvatures

$$\begin{aligned}
&K(X_1, X_2) = \frac{\frac{A}{B} + \frac{B}{A} - 2 - 4k^2}{4D} && K(X_1, X_3) = \frac{2k^2}{D}\\
&K(X_2, X_3) = \frac{2k^2}{D} && K(X_1, X_4) = \frac{\frac{A}{B} - 3\frac{B}{A} + 2 - 4k^2}{4D}\\
&K(X_2, X_4) = \frac{-3\frac{A}{B} + \frac{B}{A} + 2 - 4k^2}{D} && K(X_3, X_4) = -\frac{4k^2}{D}.
\end{aligned}$$

Hence for the family in Proposition 3.2, the curvatures of the solution $g(t)$ decay at the rate $1/t$. Let us pick a point $p \in M_q$. It is clear that the volume-normalized solution $(M_q, g_N(t), p)$ collapses to a line in the pointed Gromov–Hausdorff topology.

3.A4. $U1[2,1]$.

For $U1[2,1]$, we use $Y_i = \Lambda^k{}_i X_k$ with

$$\Lambda = \begin{bmatrix} 1 & a_2 & a_3 & 0\\ 0 & 1 & 0 & 0\\ 0 & a_1 & 1 & 0\\ a_4 & a_5 & a_6 & 1 \end{bmatrix}$$

to diagonalize the initial metric g_0.

Proposition 3.3. *For the class $U1[2,1]$ suppose the initial metric g_0 is diagonal in the basis Y_i. Then the Ricci flow solution $g(t)$ remains diagonal in the basis Y_i for all $t \geq 0$.*

Proof. We compute

$$[Y_2, Y_3] = 0, \qquad [Y_3, Y_1] = 0, \qquad [Y_1, Y_2] = 0,$$
$$[Y_1, Y_4] = Y_2, \qquad [Y_2, Y_4] = 0, \qquad [Y_3, Y_4] = 0.$$

This bracket structure is identical to that of the basis $\{X_i\}$.

We compute the off-diagonal components of the Ricci tensor in the basis $\{\bar{Y}_i\}$ using (1) as in Section 2.A2 and find $\mathrm{Ric}(\bar{Y}_i, \bar{Y}_j) = 0$ for all $i < j$. The proposition is proved. □

Since Y_i and X_i both satisfy the same Lie bracket relations, as in Section 2.A2, we use X_i and can carry out the analysis of the long time behavior of the Ricci flow solution for the family in Proposition 3.3 Proceeding as in Section 2.A2, we find the Ricci tensor

$$\mathrm{Ric}(W, W) = -\frac{B}{2AD} w_1^2 + \frac{B}{2AD} w_2^2 + 0 \cdot w_3^2 - \frac{B}{2AD} w_4^2.$$

Hence, the Ricci flow is

$$\frac{dA}{dt} = \frac{B}{D}, \qquad \frac{dB}{dt} = -\frac{B^2}{AD},$$
$$\frac{dC}{dt} = 0, \qquad \frac{dD}{dt} = \frac{B}{A}.$$

From $\frac{d}{dt}(\frac{A}{D}) = \frac{d}{dt}(AB) = 0$, we get

$$A = \lambda_1 (1 + \frac{3\lambda_2}{\lambda_1 \lambda_4} t)^{1/3} \qquad B = \lambda_2 (1 + \frac{3\lambda_2}{\lambda_1 \lambda_4} t)^{-1/3}$$
$$C = \lambda_3 \qquad D = \lambda_4 (1 + \frac{3\lambda_2}{\lambda_1 \lambda_4} t)^{1/3}$$

Hence, for the family in Proposition 3.3, the long time behavior of the solution $g(t)$ as $t \to +\infty$ is

$$A(t) \to +\infty, \qquad B(t) \to 0^+, \qquad C(t) = \lambda_3, \qquad D(t) \to +\infty.$$

Next, we compute the curvature decay for $g(t)$. From (2), we find $U(X_1, X_2) = -\frac{B}{2D} X_4$, $U(X_2, X_4) = \frac{B}{2A} X_1$ and all other $U(X_i, X_j) = 0$. From (3) with $h = g$, we find the sectional curvatures

$$K(X_1, X_2) = K(X_2, X_4) = \frac{B}{4AD}, \qquad K(X_1, X_4) = -\frac{3B}{4AD},$$

and all other $K(X_i, X_j) = 0$. Hence for the family in Proposition 3.3, the curvatures of the solution $g(t)$ decay at the rate $1/t$. We now pick a point $p \in M_q$. It is clear that the volume-normalized solution $(M_q, g_N(t), p)$ collapses to a plane in the pointed Gromov–Hausdorff topology. Note that since the Lie algebra is the direct sum $n_3 \oplus \mathbb{R}$, we can get the same conclusions from the analysis in [8](p. 734).

3.A5. $U1[2,1]$.

For $U1[2,1]$, we use $Y_i = \Lambda^k{}_i X_k$ with

$$\Lambda = \begin{bmatrix} 1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & 0 \\ 0 & 0 & 1 & 0 \\ a_4 & a_5 & a_6 & 1 \end{bmatrix}$$

to diagonalize the initial metric g_0.

Proposition 3.4. *For the class $U1[2,1]$, suppose the initial metric g_0 is diagonal in the basis Y_i. Then the Ricci flow solution $g(t)$ remains diagonal in the basis Y_i if and only if $a_1 = a_3 = 0$.*

Proof. We compute

$$[Y_2, Y_3] = 0, \qquad [Y_3, Y_1] = 0, \qquad [Y_1, Y_2] = 0,$$
$$[Y_1, Y_4] = -\frac{1}{2}Y_1 + Y_2 + \beta Y_2, \quad [Y_2, Y_4] = -\frac{1}{2}Y_2 + \alpha Y_3, \quad [Y_3, Y_4] = Y_3.$$

where

$$\alpha = \frac{3}{2}a_1 \qquad \beta = \frac{3}{2}(a_3 - a_1).$$

We compute the off-diagonal components of the Ricci tensor in the basis $\{\bar{Y}_i\}$ using (1) as in Section 2.A2 and get

$$\mathrm{Ric}(\bar{Y}_1, \bar{Y}_2) = -\frac{\alpha\beta\lambda_3}{2\sqrt{\lambda_1\lambda_2}\lambda_4} \qquad \mathrm{Ric}(\bar{Y}_1, \bar{Y}_3) = -\frac{3\beta\sqrt{\lambda_3}}{4\sqrt{\lambda_1}\lambda_4}$$
$$\mathrm{Ric}(\bar{Y}_1, \bar{Y}_4) = 0 \qquad \mathrm{Ric}(\bar{Y}_2, \bar{Y}_3) = \frac{(-3\alpha\lambda_1 + 2\beta\lambda_2)\sqrt{\lambda_3}}{4\lambda_1\sqrt{\lambda_2}\lambda_4}$$
$$\mathrm{Ric}(\bar{Y}_2, \bar{Y}_4) = 0 \qquad \mathrm{Ric}(\bar{Y}_3, \bar{Y}_4) = 0.$$

In order for these off-diagonal components to be zero, we must have $\alpha = \beta = 0$ and the proposition follows. □

If $\alpha = \beta = 0$, the Y_i and X_i both satisfy the same Lie algebra bracket relations. As in Section 2.A2, we use X_i and carry out the analysis of the long time behavior of the Ricci flow solution for the family in Proposition 3.4. Proceeding as in Section 2.A2, we find

$$[W, \bar{X}_1] = \frac{1}{2\sqrt{D}} w_4 \bar{X}_1 - \sqrt{\frac{B}{AD}} w_4 \bar{X}_2,$$
$$[W, \bar{X}_2] = \frac{1}{2\sqrt{D}} w_4 \bar{X}_2, \qquad [W, \bar{X}_3] = -\frac{1}{\sqrt{D}} w_4 \bar{X}_3,$$
$$[W, \bar{X}_4] = -\frac{1}{2\sqrt{D}} w_1 \bar{X}_1 + \left(\sqrt{\frac{B}{AD}} w_1 - \frac{1}{2\sqrt{D}} w_2 \right) \bar{X}_2 + \frac{1}{\sqrt{D}} w_3 \bar{X}_3.$$

We have from (1)

$$\mathrm{Ric}(W, W) = -\frac{B}{2AD} w_1^2 + \frac{B}{2AD} w_2^2 + 0 \cdot w_3^2 - \frac{1}{2}(\frac{3}{D} + \frac{B}{AD}) w_4^2,$$

so the Ricci flow is

$$\frac{dA}{dt} = \frac{B}{D} = \frac{B}{AD} A, \qquad \frac{dB}{dt} = -\frac{B^2}{AD} = -\frac{B}{AD} B,$$
$$\frac{dC}{dt} = 0, \qquad \frac{dD}{dt} = 3 + \frac{B}{A}.$$

It is clear that

$$C(t) = \lambda_3. \tag{4}$$

From $\frac{1}{A}\frac{dA}{dt} + \frac{1}{B}\frac{dB}{dt} = 0$, we get $AB = \lambda_1 \lambda_2$. Since $\frac{d}{dt}\left[\frac{A}{B}\right] = \frac{2}{D}$, A/B is increasing, so

$$3 \leq \frac{dD}{dt} = 3 + \frac{B}{A} \leq 3 + \frac{\lambda_2}{\lambda_1},$$

from which we get

$$3t + \lambda_4 \leq D(t) \leq (3 + \frac{\lambda_2}{\lambda_1}) t + \lambda_4. \tag{5}$$

To bound $A(t)$ from below and above, we compute

$$\frac{dA^2}{dt} = 2A \cdot \frac{B}{D} = \frac{2\lambda_1 \lambda_2}{D},$$

from which it follows that

$$\frac{2\lambda_1^2 \lambda_2}{(3\lambda_1 + \lambda_2)t + \lambda_1 \lambda_4} \leq \frac{dA^2}{dt} \leq \frac{2\lambda_1 \lambda_2}{3t + \lambda_4}.$$

Then, by, integrating we obtain

$$\lambda_1\sqrt{\frac{2\lambda_2}{3\lambda_1+\lambda_2}\ln\left(1+\frac{3\lambda_1+\lambda_2}{\lambda_1\lambda_4}t\right)+1} \le A(t) \le \lambda_1\sqrt{\frac{2\lambda_2}{3\lambda_1}\ln\left(1+\frac{3}{\lambda_4}t\right)+1}. \tag{6}$$

From $AB = \lambda_1\lambda_2$, we then get

$$\frac{\lambda_2}{\sqrt{\frac{2\lambda_2}{3\lambda_1}\ln\left(1+\frac{3}{\lambda_4}t\right)+1}} \le B(t) \le \frac{\lambda_2}{\sqrt{\frac{2\lambda_2}{3\lambda_1+\lambda_2}\ln\left(1+\frac{3\lambda_1+\lambda_2}{\lambda_1\lambda_4}t\right)+1}}. \tag{7}$$

Hence, for the family in Proposition 3.4, the long time behavior of the solution $g(t)$ as $t \to +\infty$ is

$$A(t) \to +\infty, \qquad B(t) \to 0^+, \qquad C(t) = \lambda_3, \qquad D(t) \to +\infty.$$

Next, we compute the curvature decay of $g(t)$. From (2) we find

$$\begin{aligned}
&U(X_1,X_1) = \frac{A}{2D}X_4 && U(X_2,X_2) = \frac{B}{2D}X_4 && U(X_3,X_3) = -\frac{C}{D}X_4\\
&U(X_4,X_4) = 0 && U(X_1,X_2) = -\frac{B}{2D}X_4 && U(X_1,X_3) = 0\\
&U(X_2,X_3) = 0 && U(X_1,X_4) = -\frac{1}{4}X_1 && U(X_2,X_4) = \frac{B}{2A}X_1 - \frac{1}{4}X_2\\
&U(X_3,X_4) = \frac{1}{2}X_3.
\end{aligned}$$

From (3) with $h = g$, we find the sectional curvatures

$$\begin{aligned}
&K(X_1,X_2) = \frac{-1+\frac{B}{A}}{4D} && K(X_1,X_3) = \frac{1}{2D} && K(X_2,X_3) = \frac{1}{2D}\\
&K(X_1,X_4) = -\frac{1+3\frac{B}{A}}{4D} && K(X_2,X_4) = \frac{-1+\frac{B}{A}}{4D} && K(X_3,X_4) = -\frac{1}{D}.
\end{aligned}$$

Hence, for the family in Proposition 3.4, the curvatures of the solution $g(t)$ decay at the rate $1/t$. We now pick a point $p \in M_q$. It is clear that $(M_q, g_N(t), p)$ collapses to a line in the pointed Gromov–Hausdorff topology.

3.A6. $U1[3]$.

For $U1[3]$, we use $Y_i = \Lambda^k{}_i X_k$ with

$$\Lambda = \begin{bmatrix} 1 & a_2 & a_3 & 0\\ 0 & 1 & a_1 & 0\\ 0 & 0 & 1 & 0\\ a_4 & a_5 & a_6 & 1 \end{bmatrix}$$

to diagonalize the initial metric g_0.

Proposition 3.5. *For the class $U1[3]$, suppose the initial metric g_0 is diagonal in the basis Y_i. Then the Ricci flow solution $g(t)$ remains diagonal in the basis Y_i if and only if $a_1 = a_2$.*

Proof. We compute

$$[Y_2, Y_3] = 0, \qquad [Y_3, Y_1] = 0, \qquad [Y_1, Y_2] = 0,$$
$$[Y_1, Y_4] = Y_2 + \alpha Y_3, \qquad [Y_2, Y_4] = Y_3, \qquad [Y_3, Y_4] = 0.$$

where $\alpha = a_2 - a_1$. We compute the off-diagonal components of the Ricci tensor in the basis $\{\bar{Y}_i\}$ as in Section 2.A2 and get

$$\mathrm{Ric}(\bar{Y}_1, \bar{Y}_2) = -\frac{\alpha\lambda_3}{2\sqrt{\lambda_1\lambda_2}\lambda_4}, \qquad \mathrm{Ric}(\bar{Y}_2, \bar{Y}_3) = \frac{\alpha\sqrt{\lambda_2\lambda_3}}{2\lambda_1\lambda_4},$$

and $\mathrm{Ric}(\bar{Y}_1, \bar{Y}_3) = \mathrm{Ric}(\bar{Y}_1, \bar{Y}_4) = \mathrm{Ric}(\bar{Y}_2, \bar{Y}_4) = \mathrm{Ric}(\bar{Y}_3, \bar{Y}_4) = 0$. In order for these off-diagonal components to be zero, we must have $\alpha = 0$ and the proposition is proved. □

If $\alpha = 0$, the Y_i and X_i both satisfy the same Lie bracket relations. As in Section 2.A2, we use X_i and carry out the analysis of the long time behavior of the Ricci flow solutions for the family in Proposition 3.5. Proceeding as in Section 2.A2, we compute

$$[W, \bar{X}_1] = -\sqrt{\frac{B}{AD}} w_4 \bar{X}_2 \qquad [W, \bar{X}_2] = -\sqrt{\frac{C}{BD}} w_4 \bar{X}_3,$$
$$[W, \bar{X}_3] = 0 \qquad [W, \bar{X}_4] = \sqrt{\frac{B}{AD}} w_1 \bar{X}_2 + \sqrt{\frac{C}{BD}} w_2 \bar{X}_3.$$

We have from (1)

$$\mathrm{Ric}(W, W) = -\frac{B}{2AD} w_1^2 + \frac{1}{2}\left(\frac{B}{AD} - \frac{C}{BD}\right) w_2^2 + \frac{C}{2BD} w_3^2 - \frac{1}{2}\left(\frac{B}{AD} + \frac{C}{BD}\right) w_4^2,$$

so the Ricci flow is

$$\frac{dA}{dt} = \frac{B}{D} \qquad \frac{dB}{dt} = \frac{AC - B^2}{AD}$$
$$\frac{dC}{dt} = -\frac{C^2}{BD} \qquad \frac{dD}{dt} = \frac{B}{A} + \frac{C}{B}.$$

Note that

$$\frac{1}{A}\frac{dA}{dt} = \frac{B}{AD} \qquad \frac{1}{B}\frac{dB}{dt} = \frac{C}{BD} - \frac{B}{AD}$$
$$\frac{1}{C}\frac{dC}{dt} = -\frac{C}{BD} \qquad \frac{1}{D}\frac{dD}{dt} = \frac{B}{AD} + \frac{C}{BD}.$$

Hence, $ABC = \lambda_1\lambda_2\lambda_3$ and $\frac{A}{CD} = \frac{\lambda_1}{\lambda_3\lambda_4}$. Define $E \doteq \frac{B}{AD}$ and $F \doteq \frac{C}{BD}$, we compute

$$\frac{dE}{dt} = -3E^2 \qquad \frac{dF}{dt} = -3F^2.$$

Solving these gives

$$E(t) = \frac{E_0}{3E_0t+1} \qquad F(t) = \frac{F_0}{3F_0t+1}$$

where $E_0 \doteq \frac{\lambda_2}{\lambda_1\lambda_4}$ and $F_0 \doteq \frac{\lambda_3}{\lambda_2\lambda_4}$. Using these in the equations for $(1/A)(dA/dt)$ and $(1/C)(dC/dt)$, we can integrate to get

$$A(t) = \lambda_1\left(3E_0t+1\right)^{1/3} \qquad C(t) = \lambda_3\left(3F_0t+1\right)^{-1/3}. \tag{8}$$

Using these with the conserved quantities ABC and $\frac{A}{CD}$, we get

$$\begin{aligned} B(t) &= \lambda_2\left(3E_0t+1\right)^{-1/3}\left(3F_0t+1\right)^{1/3} \\ D(t) &= \lambda_4\left(3E_0t+1\right)^{1/3}\left(3F_0t+1\right)^{1/3}. \end{aligned} \tag{9}$$

Hence, for the family in Proposition 3.5, the long time behavior of the solution $g(t)$ is

$$A(t) \to +\infty \qquad B(t) \to (\lambda_1\lambda_2\lambda_3)^{1/3} \qquad C(t) \to 0^+ \qquad D(t) \to +\infty.$$

Next, we compute the curvature decay of $g(t)$. From (2), we find

$$\begin{array}{lll} U(X_1,X_1) = 0 & U(X_2,X_2) = 0 & U(X_3,X_3) = 0 \\ U(X_4,X_4) = 0 & U(X_1,X_2) = -\dfrac{B}{2D}X_4 & U(X_1,X_3) = 0 \\ U(X_2,X_3) = -\dfrac{C}{2D}X_4 & U(X_1,X_4) = 0 & U(X_2,X_4) = \dfrac{B}{2A}X_1 \\ U(X_3,X_4) = \dfrac{C}{2B}X_2. & & \end{array}$$

From (3) with $h = g$, we find the sectional curvatures

$$\begin{array}{lll} K(X_1,X_2) = \dfrac{\frac{B}{A}}{4D} & K(X_1,X_3) = 0 & K(X_2,X_3) = \dfrac{\frac{C}{B}}{4D} \\ K(X_1,X_4) = -\dfrac{3\frac{B}{A}}{4D} & K(X_2,X_4) = \dfrac{\frac{B}{A} - 3\frac{C}{B}}{4D} & K(X_3,X_4) = \dfrac{\frac{C}{B}}{4D}. \end{array}$$

Hence, for the family in Proposition 3.5, the curvatures of the solution $g(t)$ decay at the rate $1/t$. Now pick a point $p \in M_q$, it is easy to see that the volume-normalized solution $(M_q, g_N(t), p)$ collapses to a plane in the pointed Gromov–Hausdorff topology.

3.A7. $U3I0$.

For $U1[2,1]$, we use $Y_i = \Lambda^k{}_i X_k$ with

$$\Lambda = \begin{bmatrix} 1 & a_4 & a_5 & a_6 \\ 0 & 1 & a_2 & a_3 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to diagonalize the initial metric g_0.

Proposition 3.6. *For the class* $U3I0$*, suppose the initial metric* g_0 *is diagonal in the basis* Y_i*. Let*

$$\alpha \doteq a_2 \qquad \beta \doteq a_1 a_2 - a_3 - a_4 \qquad \gamma \doteq a_1 - a_1 a_2^2 + a_2 a_3 + a_2 a_4 - a_5.$$

Then the Ricci flow solution $g(t)$ *remains diagonal in the basis* Y_i *if and only if either*
(i) $\alpha = \beta = \gamma = 0$*; or*
(ii) $\beta = \gamma = 0$ *and* $\lambda_2 = (1 - \alpha^2)\lambda_3$.

Proof. We compute

$$[Y_1, Y_4] = 0 \qquad [Y_2, Y_4] = 0 \qquad [Y_3, Y_4] = 0 \qquad [Y_2, Y_3] = Y_4$$
$$[Y_3, Y_1] = Y_2 - \alpha Y_3 + \beta Y_4 \qquad [Y_1, Y_2] = -\alpha Y_2 + (\alpha^2 - 1)Y_3 + \gamma Y_4.$$

We compute the off-diagonal components of the Ricci tensor in the basis $\{\bar{Y}_i\}$ using (1) as in Section 2.A2 and get

$$\mathrm{Ric}(\bar{Y}_1, \bar{Y}_2) = \frac{\beta\lambda_4}{2\sqrt{\lambda_1\lambda_2}\lambda_3}$$
$$\mathrm{Ric}(\bar{Y}_1, \bar{Y}_3) = \frac{\gamma\lambda_4}{2\sqrt{\lambda_1\lambda_3}\lambda_2}$$
$$\mathrm{Ric}(\bar{Y}_1, \bar{Y}_4) = 0$$
$$\mathrm{Ric}(\bar{Y}_2, \bar{Y}_3) = \frac{-2\alpha\lambda_2 + 2\alpha(1-\alpha^2)\lambda_3 + \beta\gamma\lambda_4}{2\lambda_1\sqrt{\lambda_2\lambda_3}}$$
$$\mathrm{Ric}(\bar{Y}_2, \bar{Y}_4) = \frac{(\beta\lambda_2 - \alpha\gamma\lambda_3)\sqrt{\lambda_4}}{2\lambda_1\sqrt{\lambda_2}\lambda_3}$$
$$\mathrm{Ric}(\bar{Y}_3, \bar{Y}_4) = \frac{(-\alpha\beta\lambda_2 + (\alpha^2 - 1)\gamma\lambda_3)\sqrt{\lambda_4}}{2\lambda_1\lambda_2\sqrt{\lambda_3}}$$

In order for these off-diagonal components to be zero, we must have either (i) or (ii) in the proposition. To finish the proof of (ii), we need to ensure that the condition $B(t) = (1 - \alpha^2)C(t)$ holds for all $t > 0$. We prove this near the end of this subsection. □

A7i First, we study the family (i) in Proposition 3.6. If $\alpha = \beta = \gamma = 0$, the bases Y_i and X_i both satisfy the same Lie bracket relations. As in Section 2.A2, we use X_i and carry out the analysis of the long time behavior of the Ricci flow solutions for the family (i) in Proposition 3.6. Proceeding as in Section 2.A2, we find

$$\begin{aligned}
[W, \bar{X}_1] &= \sqrt{\frac{B}{AC}} w_3 \bar{X}_2 + \sqrt{\frac{C}{AB}} w_2 \bar{X}_3 \\
[W, \bar{X}_2] &= -\sqrt{\frac{C}{AB}} w_1 \bar{X}_3 - \sqrt{\frac{D}{BC}} w_3 \bar{X}_4 \\
[W, \bar{X}_3] &= -\sqrt{\frac{B}{AC}} w_1 \bar{X}_2 + \sqrt{\frac{D}{BC}} w_2 \bar{X}_4 \\
[W, \bar{X}_4] &= 0.
\end{aligned}$$

We have from (1)

$$\begin{aligned}
\mathrm{Ric}(W, W) = -\frac{1}{2}\left(\frac{B}{AC} + \frac{C}{AB} + \frac{2}{A}\right) w_1^2 - \frac{1}{2}\left(\frac{C}{AB} + \frac{D}{BC} - \frac{B}{AC}\right) w_2^2 \\
-\frac{1}{2}\left(\frac{B}{AC} + \frac{D}{BC} - \frac{C}{AB}\right) w_3^2 + \frac{1}{2}\frac{D}{BC} w_4^2,
\end{aligned}$$

so the Ricci flow equation is

$$\begin{aligned}
\frac{dA}{dt} &= \frac{B}{C} + \frac{C}{B} + 2 & \qquad \frac{dB}{dt} &= \frac{C}{A} + \frac{D}{C} - \frac{B^2}{AC} \\
\frac{dC}{dt} &= \frac{B}{A} + \frac{D}{B} - \frac{C^2}{AB} & \qquad \frac{dD}{dt} &= -\frac{D^2}{BC}.
\end{aligned}$$

Straightforward calculations give us

$$\frac{1}{BC}\frac{d}{dt}[BC] = 2\frac{D}{BC} \qquad \frac{1}{D}\frac{dD}{dt} = -\frac{D}{BC},$$

from which we conclude that $BCD^2 = \lambda_2 \lambda_3 \lambda_4^2$. Now, we can write $\frac{dD}{dt} = -\frac{D^4}{\lambda_2 \lambda_3 \lambda_4^2}$ and solve to get

$$D(t) = \lambda_4 \left(1 + 3\frac{\lambda_4}{\lambda_2 \lambda_3} t\right)^{-1/3}. \tag{10}$$

Another set of simple calculations give us

$$\frac{1}{B-C}\frac{d}{dt}[B-C] = \frac{AD-(B+C)^2}{ABC} \qquad \frac{1}{AD}\frac{d}{dt}[AD] = -\frac{AD-(B+C)^2}{ABC}$$

from which we conclude that $AD(B-C) = \lambda_1\lambda_4(\lambda_2-\lambda_3)$. We get

$$\frac{(B-C)^2}{BC}A^2 = \frac{A^2D^2(B-C)^2}{BCD^2} = \frac{\lambda_1^2(\lambda_2-\lambda_3)^2}{\lambda_2\lambda_3} \doteq 4k_3^2.$$

for some $k_3 \geq 0$. With this and the identity

$$\frac{(B+C)^2}{BC} = \frac{(B-C)^2}{BC} + 4,$$

we can write

$$\frac{dA}{dt} = \frac{4k_3^2}{A^2} + 4.$$

Integrating gives us

$$A - k_3 \tan^{-1}\left(\frac{A}{k_3}\right) = 4t + \lambda_1 - k_3 \tan^{-1}\left(\frac{\lambda_1}{k_3}\right). \tag{11}$$

For large t, we have $A(t) \sim 4t$.

Using the conserved quantities $AD(B-C)$ and BCD^2, we could solve to get $B(t)$ and $C(t)$ explicitly. More importantly, we see that for large t,

$$B(t) \sim C(t) \sim \frac{1}{D(t)} \sim t^{1/3}. \tag{12}$$

Hence, for the family (i) in Proposition 3.6, the long time behavior of the solution $g(t)$ is

$$A(t) \to +\infty \qquad B(t) \to +\infty \qquad C(t) \to +\infty \qquad D(t) \to 0^+.$$

Next, we compute the curvature decay of $g(t)$. From (2), we find

$$\begin{array}{lll}
U(X_1,X_1) = 0 & U(X_2,X_2) = 0 & U(X_3,X_3) = 0 \\
U(X_4,X_4) = 0 & U(X_1,X_2) = \frac{B}{2C}X_3 & U(X_1,X_3) = \frac{C}{2B}X_2 \\
U(X_2,X_3) = -\frac{B+C}{2A}X_1 & U(X_1,X_4) = 0 & U(X_2,X_4) = -\frac{D}{2C}X_3 \\
U(X_3,X_4) = \frac{D}{2B}X_2. & &
\end{array}$$

From (3) with $h = g$, we find the sectional curvatures

$$K(X_1, X_2) = \frac{\frac{B}{C} - 3\frac{C}{B} - 2}{4A} \qquad K(X_1, X_3) = \frac{\frac{C}{B} - 3\frac{B}{C} - 2}{4A}$$
$$K(X_1, X_4) = 0 \qquad K(X_2, X_3) = -\frac{3D}{4BC} + \frac{\frac{B}{C} + \frac{C}{B} + 2}{4A}$$
$$K(X_2, X_4) = \frac{D}{4BC} \qquad K(X_3, X_4) = \frac{D}{4BC}.$$

Hence, for the family (i) in Proposition 3.6, the curvatures of the solution $g(t)$ decay at the rate $1/t$. For the volume-normalized solution $g_N(t)$, the metric components have the following long time behavior: $A_N(t) \to +\infty$, $B_N(t)$ and $C_N(t)$ approach some positive constant, and $D_N(t) \to 0^+$. It follows that the volume-normalized solution $(M_q, g_N(t), p)$ collapses to a strip in the pointed Gromov–Hausdorff topology for $p \in M_q$.

A7ii For the rest of this subsection, we address the family (ii) in Proposition 3.6. Suppose $\beta = \gamma = 0$. The Lie brackets from the proof of Proposition 3.6 take the form

$$[Y_1, Y_4] = 0 \qquad [Y_2, Y_4] = 0 \qquad [Y_3, Y_4] = 0 \qquad [Y_2, Y_3] = Y_4$$
$$[Y_3, Y_1] = Y_2 - \alpha Y_3 \qquad [Y_1, Y_2] = -\alpha Y_2 + (\alpha^2 - 1)Y_3.$$

Recall we must show the condition $B(t) = (1 - \alpha^2)C(t)$ is preserved under Ricci flow. Let ω_i be the frame dual to Y_i. Assume the Ricci flow solution g takes the form

$$g(t) = A(t)(\omega_1)^2 + B(t)(\omega_2)^2 + C(t)(\omega_3)^2 + D(t)(\omega_4)^2$$

with

$$g_0 = \lambda_1(\omega_1)^2 + \lambda_2(\omega_2)^2 + \lambda_3(\omega_3)^2 + \lambda_4(\omega_4)^2.$$

Let $\bar{Y}_1 \doteq \frac{1}{\sqrt{A}} Y_1, \cdots, \bar{Y}_4 \doteq \frac{1}{\sqrt{D}} Y_4$ and let $W = w_1\bar{Y}_1 + w_2\bar{Y}_2 + w_3\bar{Y}_3 + w_4\bar{Y}_4$. We first compute $[W, \bar{Y}_i]$ and then compute the Ricci curvature of $g(t)$ using (1). We find that $\mathrm{Ric}(W, W)$ is given by

$$\begin{aligned}\mathrm{Ric}(W, W) = &-\frac{B^2 + 2(1+\alpha^2)BC + (1-\alpha^2)^2C^2}{2ABC} w_1^2 \\ &+ \frac{-AD + B^2 - (1-\alpha^2)^2C^2}{2ABC} w_2^2 + \frac{-AD - B^2 + (1-\alpha^2)^2C^2}{2ABC} w_3^2 \\ &+ \frac{D}{2BC} w_4^2 + \frac{\alpha(-B + (1-\alpha^2)C)}{A\sqrt{BC}} w_2 w_3.\end{aligned}$$

The Ricci flow equation is

$$\frac{dA}{dt} = \frac{B^2 + 2(1+\alpha^2)BC + (1-\alpha^2)^2C^2}{BC} \qquad \frac{dB}{dt} = \frac{AD - B^2 + (1-\alpha^2)^2C^2}{AC}$$
$$\frac{dC}{dt} = \frac{AD + B^2 - (1-\alpha^2)^2C^2}{AB} \qquad \frac{dD}{dt} = -\frac{D^2}{BC}.$$

Hence,

$$\frac{d}{dt}(-B + (1-\alpha^2)C) = \frac{AD - (B + (1-\alpha^2)C)^2}{ABC}\left(-B + (1-\alpha^2)C\right).$$

Since $-B + (1-\alpha^2)C = 0$ at time $t = 0$, it remains 0 for all time which implies that $g(t)$ is diagonal in the basis Y_i.

With $-B + (1-\alpha^2)C = 0$, the Ricci flow equations reduce to

$$\frac{dA}{dt} = 4 \qquad \frac{dB}{dt} = (1-\alpha^2)\frac{D}{B} \qquad \frac{dD}{dt} = -(1-\alpha^2)\frac{D^2}{B^2}.$$

A simple calculation gives $\frac{d}{dt}(B^2D^2) = 0$ which implies $BD = \lambda_2\lambda_4$. Now, we can solve the Ricci flow equations to obtain

$$A = \lambda_1 + 4t \qquad B = \left(\lambda_2^3 + 3(1-\alpha^2)\lambda_2\lambda_4 t\right)^{1/3}$$
$$C = \frac{1}{1-\alpha^2}\left(\lambda_2^3 + 3(1-\alpha^2)\lambda_2\lambda_4 t\right)^{1/3} \quad D = \lambda_2\lambda_4\left(\lambda_2^3 + 3(1-\alpha^2)\lambda_2\lambda_4 t\right)^{-1/3}.$$

Hence for the family (ii) in Proposition 3.6, the long time behavior of the Ricci flow $g(t)$ as $t \to \infty$ is

$$A(t) \to +\infty \qquad B(t) \to +\infty \qquad C(t) \to +\infty \qquad D(t) \to 0^+$$

Finally, we compute the curvature decay of $g(t)$. From (2), we find

$$U(Y_1, Y_1) = 0 \qquad U(Y_2, Y_2) = -\frac{\alpha B}{A}Y_1$$
$$U(Y_3, Y_3) = \frac{\alpha C}{A}Y_1 \qquad U(Y_4, Y_4) = 0$$
$$U(Y_1, Y_2) = \frac{\alpha}{2}Y_2 + \frac{B}{2C}Y_3 \qquad U(Y_1, Y_3) = \frac{(1-\alpha^2)C}{2B}Y_2 - \frac{\alpha}{2}Y_3$$
$$U(Y_1, Y_4) = 0 \qquad U(Y_2, Y_3) = -\frac{B + (1-\alpha^2)C}{2A}Y_1$$
$$U(Y_2, Y_4) = -\frac{D}{2C}Y_3 \qquad U(Y_3, Y_4) = \frac{D}{2B}Y_2.$$

From (3) with $h = g$, we find the sectional curvatures

$$\begin{aligned} &K(Y_1, Y_2) = -\frac{1}{A} && K(Y_1, Y_3) = -\frac{1}{A} && K(Y_2, Y_3) = -\frac{3D}{4BC} + \frac{1}{A} \\ &K(Y_1, Y_4) = 0 && K(Y_2, Y_4) = \frac{D}{4BC} && K(Y_3, Y_4) = \frac{D}{4BC}. \end{aligned}$$

Hence for the family (ii) in Proposition 3.6, the curvatures of the solution $g(t)$ decay at the rate $1/t$. For the volume-normalized solution $g_N(t)$, the metric components have the following long time behavior: $A_N(t) \to +\infty$, $B_N(t)$ and $C_N(t)$ approach some positive constants, and $D_N(t) \to 0^+$. It follows that the volume-normalized solution $(M_q, g_N(t), p)$ collapses to a strip in the pointed Gromov–Hausdorff topology for $p \in M_q$.

3.A8. $U3I2$.

For $U3I2$, we use $Y_i = \Lambda^k{}_i X_k$ with

$$\Lambda = \begin{bmatrix} 1 & a_4 & a_5 & a_6 \\ 0 & 1 & a_2 & a_3 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to diagonalize the initial metric g_0.

Proposition 3.7. *For the class $U3I2$, suppose the initial metric g_0 is diagonal in the basis Y_i. Then, the Ricci flow solution $g(t)$ remains diagonal if and only if $a_2 = 0$, $a_1 = a_5$ and $a_3 = a_4$.*

Proof. We compute

$$\begin{aligned} &[Y_1, Y_4] = 0 \qquad [Y_2, Y_4] = 0 && [Y_3, Y_4] = 0 \qquad [Y_2, Y_3] = -Y_4 \\ &[Y_3, Y_1] = Y_2 + \alpha Y_3 + \beta Y_4 && [Y_1, Y_2] = \alpha Y_2 + (1 + \alpha^2) Y_3 + \gamma Y_4. \end{aligned}$$

where

$$\alpha = -a_2 \qquad \beta = a_1 a_2 - a_3 + a_4 \qquad \gamma = -a_1 - a_1 a_2^2 + a_2 a_3 - a_2 a_4 + a_5.$$

We compute the off-diagonal components of the Ricci tensor in the basis $\{\bar{Y}_i\}$ using (1) as in Section 2.A2 and get

$$\mathrm{Ric}(\bar{Y}_1,\bar{Y}_2) = -\frac{\beta\lambda_4}{2\sqrt{\lambda_1\lambda_2}\lambda_3} \qquad \mathrm{Ric}(\bar{Y}_1,\bar{Y}_3) = -\frac{\gamma\lambda_4}{2\sqrt{\lambda_1\lambda_3}\lambda_2}$$

$$\mathrm{Ric}(\bar{Y}_1,\bar{Y}_4) = 0 \qquad \mathrm{Ric}(\bar{Y}_2,\bar{Y}_3) = \frac{2\alpha\lambda_2 + 2\alpha(1+\alpha^2)\lambda_3 + \beta\gamma\lambda_4}{2\lambda_1\sqrt{\lambda_2\lambda_3}}$$

$$\mathrm{Ric}(\bar{Y}_2,\bar{Y}_4) = \frac{(\beta\lambda_2 + \alpha\gamma\lambda_3)\sqrt{\lambda_4}}{2\lambda_1\sqrt{\lambda_2}\lambda_3} \quad \mathrm{Ric}(\bar{Y}_3,\bar{Y}_4) = \frac{(\alpha\beta\lambda_2 + (1+\alpha^2)\gamma\lambda_3)\sqrt{\lambda_4}}{2\lambda_1\lambda_2\sqrt{\lambda_3}}.$$

In order for these off-diagonal components to be zero, we must have $\alpha = \beta = \gamma = 0$ and the proposition follows. □

If $\alpha = \beta = \gamma = 0$, the bases Y_i and X_i both satisfy the same Lie bracket relations. As in Section 2.A2, we use X_i and carry out the analysis of the long time behavior of the Ricci flow solutions for the family in Proposition 3.7. Proceeding as in Section 2.A2, we find

$$\begin{aligned}
[W,\bar{X}_1] &= \sqrt{\frac{B}{AC}}w_3\bar{X}_2 - \sqrt{\frac{C}{AB}}w_2\bar{X}_3\\
[W,\bar{X}_2] &= \sqrt{\frac{C}{AB}}w_1\bar{X}_3 + \sqrt{\frac{D}{BC}}w_3\bar{X}_4\\
[W,\bar{X}_3] &= -\sqrt{\frac{B}{AC}}w_1\bar{X}_2 - \sqrt{\frac{D}{BC}}w_2\bar{X}_4\\
[W,\bar{X}_4] &= 0.
\end{aligned}$$

We have from (1)

$$\begin{aligned}
\mathrm{Ric}(W,W) = \frac{1}{2}\left(\frac{2}{A} - \frac{C}{AB} - \frac{B}{AC}\right)w_1^2 + \frac{1}{2}\left(\frac{B}{AC} - \frac{C}{AB} - \frac{D}{BC}\right)w_2^2\\
+ \frac{1}{2}\left(\frac{C}{AB} - \frac{B}{AC} - \frac{D}{BC}\right)w_3^2 + \frac{D}{2BC}w_4^2.
\end{aligned}$$

The Ricci flow is

$$\frac{dA}{dt} = \frac{C}{B} + \frac{B}{C} - 2 \qquad \frac{dB}{dt} = -\frac{B^2}{AC} + \frac{C}{A} + \frac{D}{C}$$

$$\frac{dC}{dt} = -\frac{C^2}{AB} + \frac{B}{A} + \frac{D}{B} \qquad \frac{dD}{dt} = -\frac{D^2}{BC}.$$

The equations here are similar to those of the case A7(i), with the only difference being in the equation for A. Because the equations for B, C and D

are the same, we know that $BCD^2 = \lambda_2\lambda_3\lambda_4^2$. It follows that $\frac{dD}{dt} = -\frac{D^4}{\lambda_2\lambda_3\lambda_4^2}$, and hence,

$$D(t) = \lambda_4 \left(1 + \frac{3\lambda_4}{\lambda_2\lambda_3} t\right)^{-1/3}. \tag{13}$$

Calculations similar to those in the case A7(i) show that $AD(B+C) = \lambda_1\lambda_4(\lambda_2+\lambda_3)$. So $A^2(\frac{C}{B} + \frac{B}{C} + 2) = \frac{(AD(B+C))^2}{BCD^2} \doteq k_4^2$ is a constant where $k_4 \geq 0$, and

$$\frac{dA}{dt} = \frac{k_4^2}{A^2} - 4.$$

Integrating the equation gives us

$$\frac{k_4}{2} \tanh^{-1}\left(\frac{2A}{k_4}\right) - A = 4t + k_5 \tag{14}$$

where k_5 is a constant. Since A increases for all t and $\tanh x$ asymptotes to 1, we see that $A(t) \to k_4/2$ as $t \to +\infty$.

Using the conserved quantity BCD^2 and $AD(B+C)$, we conclude that for the family in Proposition 3.7 both B and C grow at the rate $t^{1/3}$, and the long time behavior of the Ricci flow $g(t)$ as $t \to +\infty$ is

$$A(t) \to k_4/2 \qquad B(t) \to +\infty \qquad C(t) \to +\infty \qquad D(t) \to 0^+.$$

Next, we compute the curvature decay of $g(t)$. From (2), we find

$$\begin{array}{lll}
U(X_1,X_1) = 0 & U(X_2,X_2) = 0 & U(X_3,X_3) = 0 \\
U(X_4,X_4) = 0 & U(X_1,X_2) = \dfrac{B}{2C}X_3 & U(X_1,X_3) = -\dfrac{C}{2B}X_2 \\
U(X_2,X_3) = \dfrac{-B+C}{2A}X_1 & U(X_1,X_4) = 0 & U(X_2,X_4) = \dfrac{D}{2C}X_3 \\
U(X_3,X_4) = -\dfrac{D}{2B}X_2. & &
\end{array}$$

From (3) with $h = g$, we find the sectional curvatures

$$\begin{array}{ll}
K(X_1,X_2) = \dfrac{\frac{B}{C} - 3\frac{C}{B} + 2}{4A} & K(X_1,X_3) = \dfrac{-3\frac{B}{C} + \frac{C}{B} + 2}{4A} \\
K(X_2,X_3) = -\dfrac{3D}{4BC} + \dfrac{\frac{B}{C} + \frac{C}{B} - 2}{4A} & K(X_1,X_4) = 0 \\
K(X_2,X_4) = \dfrac{D}{4BC} & K(X_3,X_4) = \dfrac{D}{4BC}.
\end{array}$$

Here, the decay rate is not obvious for all sectional curvatures. Note that $\frac{B}{C} \to 1$. The decay rate of $\frac{B}{C} - 1$ follows from the equation of $\frac{dA}{dt}$. It suffices to show that $\frac{dA}{dt}$ decays at the rate e^{-ct} for some $c > 0$. From (14), we get

$$A = \frac{k_4}{2} \tanh\left[\frac{8}{k_4}t + \frac{2}{k_4}A + \frac{2k_5}{k_4}\right]$$

and the decay rate of $\frac{dA}{dt}$ follows from taking the time derivative of this equation. Thus $\frac{B}{C} - 1$ decays exponentially. Hence, for the family in Proposition 3.7, the curvatures of the solution $g(t)$ decay at the rate $1/t$.

For the volume-normalized solution $g_N(t)$, the metric components have the following long time behavior: $A_N(t) \to 0^+$, $B_N(t)$ and $C_N(t)$ approach some positive constants, and $D_N(t) \to 0^+$. The volume-normalized flow $(M_q, g_N(t), p)$ collapses to a plane in the pointed Gromov–Hausdorff topology for $p \in M_q$.

3.A9. $U3S1$.

The Lie bracket relations for cases 3.A9 and 3.A10 differ only in $[X_1, X_2]$. To unify some of the calculations for these two cases, we introduce a constant δ and write $[X_1, X_2] = \delta X_3$ with $\delta = -1$ corresponding to 3.A9 and $\delta = 1$ corresponding to 3.A10.

For $U3S1$ and $U3S3$, we use $Y_i = \Lambda^k{}_i X_k$ with

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & 1 \end{bmatrix}$$

to diagonalize the initial metric g_0.

Proposition 3.8. *For the class $U3S1$, suppose the initial metric g_0 is diagonal in the basis Y_i. Then*

(i) if $\lambda_1 \neq \lambda_2$, the Ricci flow solution $g(t)$ remains diagonal if and only if $a_1 = a_2 = a_3 = 0$; and

(ii) if $\lambda_1 = \lambda_2$, the Ricci flow solution $g(t)$ remains diagonal if and only if $a_1 = a_2 = 0$.

Proof. We compute

$$[Y_1, Y_4] = -a_3Y_2 + \delta a_2 Y_3 \quad [Y_2, Y_4] = a_3Y_1 - \delta a_1 Y_3 \quad [Y_3, Y_4] = -a_2Y_1 + a_1Y_2$$
$$[Y_2, Y_3] = Y_1, \qquad [Y_3, Y_1] = Y_2, \qquad [Y_1, Y_2] = \delta Y_3.$$

We compute the off-diagonal components of the Ricci tensor in the basis $\bar{Y}_i$ using (1) as in Section 2.A2 and get

$$\begin{aligned}
\mathrm{Ric}(\bar{Y}_1,\bar{Y}_2) &= \frac{(\lambda_3^2-\lambda_1\lambda_2)a_1a_2}{2\lambda_3\lambda_4\sqrt{\lambda_1\lambda_2}} &\quad \mathrm{Ric}(\bar{Y}_1,\bar{Y}_3) &= \frac{(\lambda_2^2-\delta\lambda_1\lambda_3)a_1a_3}{2\lambda_2\lambda_4\sqrt{\lambda_1\lambda_3}}\\
\mathrm{Ric}(\bar{Y}_2,\bar{Y}_3) &= \frac{(\lambda_1^2-\delta\lambda_2\lambda_3)a_2a_3}{2\lambda_1\lambda_4\sqrt{\lambda_2\lambda_3}} &\quad \mathrm{Ric}(\bar{Y}_1,\bar{Y}_4) &= -\frac{(\lambda_2-\delta\lambda_3)^2a_1}{2\lambda_2\lambda_3\sqrt{\lambda_1\lambda_4}}\\
\mathrm{Ric}(\bar{Y}_2,\bar{Y}_4) &= -\frac{(\lambda_1-\delta\lambda_3)^2a_2}{2\lambda_1\lambda_3\sqrt{\lambda_2\lambda_4}} &\quad \mathrm{Ric}(\bar{Y}_3,\bar{Y}_4) &= -\frac{(\lambda_1-\lambda_2)^2a_3}{2\lambda_1\lambda_2\sqrt{\lambda_3\lambda_4}}.
\end{aligned}$$

The diagonal components are given by

$$\begin{aligned}
\mathrm{Ric}(\bar{Y}_1,\bar{Y}_1) &= \frac{(\lambda_1^2-\lambda_3^2)\lambda_2a_2^2+(\lambda_1^2-\lambda_2^2)\lambda_3a_3^2+\big(\lambda_1^2-(\lambda_2-\delta\lambda_3)^2\big)\lambda_4}{2\lambda_1\lambda_2\lambda_3\lambda_4}\\
\mathrm{Ric}(\bar{Y}_2,\bar{Y}_2) &= \frac{(\lambda_2^2-\lambda_3^2)\lambda_1a_1^2+(\lambda_2^2-\lambda_1^2)\lambda_3a_3^2+\big(\lambda_2^2-(\lambda_1-\delta\lambda_3)^2\big)\lambda_4}{2\lambda_1\lambda_2\lambda_3\lambda_4}\\
\mathrm{Ric}(\bar{Y}_3,\bar{Y}_3) &= \frac{(\lambda_3^2-\lambda_2^2)\lambda_1a_1^2+(\lambda_3^2-\lambda_1^2)\lambda_2a_2^2+\big(\lambda_3^2-(\lambda_1-\lambda_2)^2\big)\lambda_4}{2\lambda_1\lambda_2\lambda_3\lambda_4}\\
\mathrm{Ric}(\bar{Y}_4,\bar{Y}_4) &= -\frac{(\lambda_2-\delta\lambda_3)^2\lambda_1a_1^2+(\lambda_1-\delta\lambda_3)^2\lambda_2a_2^2+(\lambda_1-\lambda_2)^2\lambda_3a_3^2}{2\lambda_1\lambda_2\lambda_3\lambda_4}.
\end{aligned} \tag{15}$$

For $\delta=-1$, in order for these off-diagonal components to be zero, we must have either (i) or (ii) in Proposition 3.8. As in case A7ii, to finish the proof of (ii), in Proposition 3.8, we need to ensure that the condition $A(t)=B(t)$ holds for all $t>0$. We prove this at the end of this subsection. □

Remark. Note that there are many initial metrics g_0 that cannot be diagonalized by the choice of Λ we use here. For 3.A9 and 3.A10, the Lie group G is a product $G_1\times\mathbb{R}$ with $\dim(G_1)=3$. After transforming with Λ as given above, one can use a Milnor frame on G_1 (with respect to a chosen initial metric on G_1) to further diagonalize, in which case the Lie algebra takes the form

$$\begin{aligned}
&[Y_1,Y_4]=a_1Y_1+a_2Y_2+a_3Y_3\\
&[Y_2,Y_4]=b_1Y_1+b_2Y_2+b_3Y_3\\
&[Y_3,Y_4]=c_1Y_1+c_2Y_2+c_3Y_3\\
&[Y_2,Y_3]=Y_1 \qquad [Y_3,Y_1]=Y_2 \qquad [Y_1,Y_2]=\delta Y_3
\end{aligned}$$

with $a_1 + b_2 + c_3 = 0$ from the unimodular condition. With these, the off-diagonal components of the Ricci curvature are given by

$$\begin{aligned}
\mathrm{Ric}(\bar{Y}_1, \bar{Y}_2) &= \frac{c_1\, c_2\lambda_1\lambda_2 + b_1(b_2 - a_1)\lambda_1\lambda_3 + a_2(a_2 - b_2)\lambda_2\lambda_3 - a_3 b_3\lambda_3^2}{2\sqrt{\lambda_1}\,\sqrt{\lambda_2}\,\lambda_3\,\lambda_4}\\
\mathrm{Ric}(\bar{Y}_1, \bar{Y}_3) &= \frac{-c_1(2a_1 + b_2)\lambda_1\lambda_2 + b_1 b_3\lambda_1\lambda_3 - a_2 c_2\lambda_2^2 + a_3(2a_1 + b_2)\lambda_2\lambda_3}{2\sqrt{\lambda_1}\,\lambda_2\,\sqrt{\lambda_3}\,\lambda_4}\\
\mathrm{Ric}(\bar{Y}_1, \bar{Y}_4) &= -\frac{(\lambda_2 - \delta\,\lambda_3)\ (c_2\,\lambda_2 + b_3\,\lambda_3)}{2\sqrt{\lambda_1}\,\lambda_2\,\lambda_3\,\sqrt{\lambda_4}}\\
\mathrm{Ric}(\bar{Y}_2, \bar{Y}_3) &= \frac{-b_1 c_1\lambda_1^2 - c_2(a_1 + 2b_2)\lambda_1\lambda_2 + b_3(a_1 + 2b_2)\lambda_1\lambda_3 + a_2 a_3\lambda_2\lambda_3}{2\,\lambda_1\sqrt{\lambda_2}\,\sqrt{\lambda_3}\,\lambda_4}\\
\mathrm{Ric}(\bar{Y}_2, \bar{Y}_4) &= \frac{(\lambda_1 - \delta\,\lambda_3)\ (c_1\,\lambda_1 + a_3\,\lambda_3)}{2\,\lambda_1\sqrt{\lambda_2}\,\lambda_3\,\sqrt{\lambda_4}}\\
\mathrm{Ric}(\bar{Y}_3, \bar{Y}_4) &= -\frac{(\lambda_1 - \lambda_2)\ (b_1\,\lambda_1 + a_2\,\lambda_2)}{2\,\lambda_1\,\lambda_2\sqrt{\lambda_3}\,\sqrt{\lambda_4}}
\end{aligned}$$

One can analyze these expressions to determine conditions under which Ricci flow preserves the diagonalization of an initial metric. The complexity of these expressions leads to many cases that must be analyzed so we have limited our attention to the transformation matrix Λ given above with the results given in Proposition 3.8 for A9 and Proposition 3.9 for A10.

3.A9i First, we study family (i) in Proposition 3.8. For $a_1 = a_2 = a_3 = 0$, we have $Y_i = X_i$. The metric $g(t)$ is a product metric on $\widehat{SL}(2,\mathbb{R}) \times \mathbb{R}$

$$g(t) = g_{SL}(t) + \lambda_4 du^2$$

where $g_{SL}(t) = A(t)(\theta_1)^2 + B(t)(\theta_2)^2 + C(t)(\theta_3)^2$ is a Ricci flow solution on $\widehat{SL}(2,\mathbb{R})$. From (15), we get the Ricci flow equations

$$\frac{dA}{dt} = \frac{(B+C)^2 - A^2}{BC} \quad \frac{dB}{dt} = \frac{(A+C)^2 - B^2}{AC} \quad \frac{dC}{dt} = \frac{(A-B)^2 - C^2}{AB}.$$

The long time behavior of $g_{SL}(t)$ has been analyzed in [8]: the curvatures of $g_{SL}(t)$ decay at the rate $1/t$ and the volume-normalized Ricci flow $\tilde{g}_{SL}(\tilde{t})$ collapses to a plane. Hence, the volume-normalized solution $(M_q, g_N(t), p)$ collapses to a plane in the pointed Gromov–Hausdorff topology for $p \in M_q$.

3.A9ii For the rest of this subsection we address family (ii) in Proposition 3.8 where $\lambda_1 = \lambda_2$ and $a_1 = a_2 = 0$. From (15), we conclude that the Ricci flow equation of $g(t)$ is

$$\frac{dA}{dt} = \frac{(B+C)^2 - A^2}{BC} + \frac{-A^2+B^2}{BD}a_3^2 \quad \frac{dB}{dt} = \frac{(A+C)^2 - B^2}{AC} + \frac{A^2-B^2}{AD}a_3^2$$
$$\frac{dC}{dt} = \frac{(A-B)^2 - C^2}{AB} \qquad \frac{dD}{dt} = \frac{(A+B)^2}{AB}a_3^2.$$

Recall that we must show that the condition $A(t) = B(t)$ is preserved under Ricci flow. To this end, we compute

$$\frac{d}{dt}(A-B) = \left[\frac{C^2 - (A+B)^2}{ABC} - \frac{(A+B)^2}{ABD}a_3^2\right](A-B).$$

Since $A - B = 0$ at time $t = 0$, this implies that $A(t) = B(t)$ and $g(t)$ remains diagonal in the basis Y_i.

With $A = B$, the Ricci flow equations reduce to

$$\frac{dA}{dt} = \frac{C}{A} + 2 \qquad \frac{dC}{dt} = -\frac{C^2}{A^2} \qquad \frac{dD}{dt} = 4a_3^2.$$

A simple computation shows $\frac{d}{dt}\left(\frac{C}{A}\right) = -2A^{-1}\left(\frac{C}{A} + (\frac{C}{A})^2\right) \leq 0$; hence $2 \leq \frac{dA}{dt} \leq 2 + \frac{\lambda_3}{\lambda_1}$ and

$$2t + \lambda_1 \leq A(t) = B(t) \leq (2 + \frac{\lambda_3}{\lambda_1})t + \lambda_1. \tag{16}$$

From the equation for $\frac{dC}{dt}$ and (16), we get $-\frac{C^2}{(2t+\lambda_1)^2} \leq \frac{dC}{dt} \leq 0$. Integrating these inequalities we find

$$\frac{2\lambda_1\lambda_3}{2\lambda_1 + \lambda_3} \leq C(t) \leq \lambda_3. \tag{17}$$

Finally,

$$D(t) = 4a_3^2 t + \lambda_4. \tag{18}$$

Hence, for family (ii) in Proposition 3.8, with $a_3 \neq 0$, the long time behavior of of the Ricci flow $g(t)$ as $t \to +\infty$ is

$$A(t) \to +\infty \qquad B(t) \to +\infty \qquad C(t) \to \text{ constant} > 0 \qquad D(t) \to +\infty.$$

Next, we compute the curvature decay of $g(t)$ as in A7ii. From (2), we find (using $A(t) = B(t)$)

$$U(Y_1, Y_3) = \frac{A+C}{2A}Y_2 \qquad U(Y_2, Y_3) = -\frac{A+C}{2A}Y_1$$
$$U(Y_1, Y_4) = \frac{1}{2}a_3 Y_2 \qquad U(Y_2, Y_4) = -\frac{1}{2}a_3 Y_1,$$

and all other $U(Y_i, Y_j) = 0$. From (3) with $h = g$, we find the sectional curvatures

$$K(Y_1, Y_2) = -\frac{4 + 3\frac{C}{A}}{4A} \qquad K(Y_1, Y_3) = K(Y_2, Y_3) = \frac{C}{4A^2}$$

and all other $K(Y_i, Y_j) = 0$. Hence for family (ii) in Proposition 3.8(ii), the curvatures of the solution $g(t)$ decay at the rate $1/t$. It follows that the volume-normalized solution $(M_q, g_N(t), p)$ collapses to Euclidean space $\mathbb{R}^3$ in the pointed Gromov–Hausdorff topology.

3.A10. U3S3.

Using the setup given in A9, we prove the following.

Proposition 3.9. *For the class $U3S1$, suppose the initial metric g_0 is diagonal in the basis Y_i. Then*

(i) if $\lambda_1, \lambda_2, \lambda_3$ are all different, the Ricci flow solution $g(t)$ remains diagonal if and only if $a_1 = a_2 = a_3 = 0$;

(ii) if $\lambda_j = \lambda_k \neq \lambda_i$ for some permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, the Ricci flow solution $g(t)$ remains diagonal if and only if $a_j = a_k = 0$; and

(iii) if the initial metric satisfies $\lambda_1 = \lambda_2 = \lambda_3$, the Ricci flow solution $g(t)$ remains diagonal for any a_1, a_2, and a_3.

Proof. Set $\delta = 1$ in the proof of Proposition 3.8. In order for the off-diagonal Ricci components to be zero, we must have either (i) or (ii) or (iii) in Proposition 3.9. As in previous cases, to finish the proof of (ii) in Proposition 3.9, we need to ensure that the condition $B(t) = C(t)$ holds for all $t > 0$ (using $j = 2, k = 3$ without loss of generality). Also, to finish the proof of (iii) in Proposition 3.9, we need to ensure that the condition $A(t) = B(t) = C(t)$ holds for all $t > 0$. These are verified below. □

3.A10i If $a_1 = a_2 = a_3 = 0$, we have $Y_i = X_i$. The metric is a product metric on $S^3 \times \mathbb{R}$

$$g(t) = g_{S^3}(t) + \lambda_4 du^2$$

where $g_{S^3}(t) = A(t)(\theta_1)^2 + B(t)(\theta_1)^2 + C(t)(\theta_1)^2$ is a Ricci flow solution on S^3. From (15), we get the Ricci flow equations

$$\frac{dA}{dt} = \frac{(B-C)^2 - A^2}{BC} \qquad \frac{dB}{dt} = \frac{(A-C)^2 - B^2}{AC} \qquad \frac{dC}{dt} = \frac{(A-B)^2 - C^2}{AB}. \tag{19}$$

The volume-normalized flow associated with $g_{S^3}(t)$ has been analyzed in [8] and is found to converge to a round sphere. Hence, the behavior of $g(t)$ is clear.

3.A10ii. For family (ii) in Proposition 3.9, without loss of generality, we may assume that $i = 1, j = 2$, and $k = 3$ so $\lambda_2 = \lambda_3$ and $a_2 = a_3 = 0$. From (15), we conclude that the Ricci flow equation of $g(t)$ is

$$\frac{dA}{dt} = \frac{(B-C)^2 - A^2}{BC} \qquad \frac{dB}{dt} = \frac{(A-C)^2 - B^2}{AC} - \frac{B^2 - C^2}{CD} a_1^2$$
$$\frac{dC}{dt} = \frac{(A-B)^2 - C^2}{AB} + \frac{B^2 - C^2}{BD} a_1^2 \quad \frac{dD}{dt} = -\frac{(B-C)^2}{BC} a_1^2.$$

Recall that we must show that the condition $B(t) = C(t)$ is preserved under Ricci flow. This follows from

$$\frac{d}{dt}(B - C) = \left[\frac{A^2 - (B+C)^2}{ABC} - \frac{(B+C)^2}{BCD} a_1^2\right](B - C).$$

With $B = C$, the Ricci flow equations reduce to

$$\frac{dA}{dt} = -\frac{A^2}{B^2} \qquad \frac{dB}{dt} = \frac{A}{B} - 2 \qquad \frac{dD}{dt} = 0.$$

This is a special case of equation (19) with $B = C$, so the conclusions from 3.A10i hold here.

3.A10iii For family (iii) in Proposition 3.9, $\lambda_1 = \lambda_2 = \lambda_3$. From (15), we conclude that the Ricci flow equation of $g(t)$ is

$$\frac{dA}{dt} = -\frac{(A^2 - C^2)Ba_2^2 + (A^2 - B^2)Ca_3^2 + \big(A^2 - (B-C)^2\big)D}{BCD}$$
$$\frac{dB}{dt} = -\frac{(B^2 - C^2)Aa_1^2 + (B^2 - A^2)Ca_3^2 + \big(B^2 - (A-C)^2\big)D}{ACD}$$
$$\frac{dC}{dt} = -\frac{(C^2 - B^2)Aa_1^2 + (C^2 - A^2)Ba_2^2 + \big(C^2 - (A-B)^2\big)D}{ABD}$$
$$\frac{dD}{dt} = \frac{(B-C)^2 Aa_1^2 + (A-C)^2 Ba_2^2 + (A-B)^2 Ca_3^2}{ABC}.$$

Recall we need to show $A(t) = B(t) = C(t)$ is preserved under Ricci flow. This follows from

$$\frac{d}{dt}(A - B) = M_{11}(A - B) + M_{12}(A - C)$$
$$\frac{d}{dt}(A - C) = M_{21}(A - B) + M_{22}(A - C)$$

where M_{ij} are continuous functions of t.

With $A = B = C$, the Ricci flow equations reduce to

$$\frac{dA}{dt} = \frac{dB}{dt} = \frac{dC}{dt} = -1 \qquad \frac{dD}{dt} = 0,$$

so

$$g(t) = (\lambda_1 - t)(\omega_1)^2 + (\lambda_1 - t)(\omega_2)^2 + (\lambda_1 - t)(\omega_3)^2 + \lambda_4(\omega_4)^2$$

where ω_i is the dual frame of Y_i. It follows from this explicit solution that the conclusions from A10i hold here.

4. The Ricci flow of locally homogeneous closed 4-manifolds modelled on Non-Lie Groups.

In this section, all of the metrics are on direct products of spheres, hyperbolic spaces, and euclidean spaces of various dimensions. Under Ricci flow, the product structure is preserved, and the pieces evolve in characteristic ways: the spheres each shrink to a point singularity in finite time (type 1 singularity); the hyperbolic spaces expand for all time, with no singularity developing; and the euclidean spaces are flat and static.

Let g_{S^n} be the metric on n-dimensional sphere S^n with sectional curvature 1 and let g_{H^n} be the metric on hyperbolic space H^n with sectional curvature -1. In this section, we again use the notations stated at the beginning of Section 3.

4.B1. $H^3 \times \mathbb{R}$.

In this case, any initial metric can be written as

$$g_0 = R^2 g_{H^3} + du^2$$

for some $R > 0$. The Ricci flow solution g is given by

$$g(t) = (R^2 + 4t)g_{H^3} + du^2 \qquad -\frac{R^2}{4} < t < +\infty.$$

4.B2. $S^2 \times \mathbb{R}^2$.

In this case, any initial metric can be written as

$$g_0 = R^2 g_{S^2} + du_1^2 + du_2^2$$

for some $R > 0$. The Ricci flow solution g is given by

$$g(t) = (R^2 - 2t)g_{S^2} + du_1^2 + du_2^2 \qquad -\infty < t < \frac{R^2}{2}.$$

4.B3. $H^2 \times \mathbb{R}^2$.

In this case, any initial metric can be written as

$$g_0 = R^2 g_{H^2} + du_1^2 + du_2^2$$

for some $R > 0$. The Ricci flow solution g is given by

$$g(t) = (R^2 + 2t)g_{H^2} + du_1^2 + du_2^2 \qquad -\frac{R^2}{2} < t < +\infty.$$

4.B4. $S^2 \times S^2$.

In this case, any initial metric can be written as

$$g_0 = R_1^2 g_{S^2}(x) + R_2^2 g_{S^2}(y)$$

for some $R_1 > 0$ and $R_2 > 0$. The Ricci flow solution g is given by

$$g(t) = (R_1^2 - 2t)g_{S^2}(x) + (R_2^2 - 2t)g_{S^2}(y) - \infty < t < \min\{\frac{R_1^2}{2}, \frac{R_2^2}{2}\}.$$

4.B5. $S^2 \times H^2$.

In this case, any initial metric can be written as

$$g_0 = R_1^2 g_{S^2} + R_2^2 g_{H^2}$$

for some $R_1 > 0$ and $R_2 > 0$. The Ricci flow solution g is given by

$$g(t) = (R_1^2 - 2t)g_{S^2} + (R_2^2 + 2t)g_{H^2} \qquad -\frac{R_2^2}{2} < t < \frac{R_1^2}{2}.$$

4.B6. $H^2 \times H^2$.

In this case, any initial metric can be written as

$$g_0 = R_1^2 g_{H^2}(x) + R_2^2 g_{H^2}(y)$$

for some $R_1 > 0$ and $R_2 > 0$. The Ricci flow solution g is given by

$$g(t) = (R_1^2 + 2t)g_{H^2}(x) + (R_2^2 + 2t)g_{H^2}(y) \quad \max\{-\frac{R_1^2}{2}, -\frac{R_2^2}{2}\} < t < +\infty.$$

4.B7. $\mathbb{C}P^2$.

Let g_{FS} be the Fubini–Study metric on $\mathbb{C}P^2$ with constant holomorphic bisectional curvature $+1$. Then, the Ricci curvature $R_{i\bar{j}}(g_{FS}) = 3(g_{FS})_{i\bar{j}}$. In this case, any initial metric can be written as (see [12], p. 277)

$$g_0 = R^2 g_{FS}$$

for some $R > 0$. The Ricci flow solution g (not the volume-normalized Kähler Ricci flow) is given by

$$g(t) = (R^2 - 6t) g_{FS} \qquad -\infty < t < \frac{R^2}{6}.$$

Note that Kähler Ricci flow with positive holomorphic bisectional curvature on $\mathbb{C}P^2$ has been studied by Chen and Tian ([2]); they prove that the (volume-normalized) Kähler Ricci flow converges exponentially fast to a Kähler metric of constant holomorphic bisectional curvature.

4.B8. $\mathbb{C}H^2$.

Let $g_{\mathbb{C}H^2}$ be the Kähler metric on $\mathbb{C}H^2$ with constant holomorphic bisectional curvature -1. Then, the Ricci curvature $R_{i\bar{j}}(g_{\mathbb{C}H^2}) = -3(g_{\mathbb{C}H^2})_{i\bar{j}}$. In this case, any initial metric can be written as (see [12], p. 277)

$$g_0 = R^2 g_{\mathbb{C}H^2}$$

for some $R > 0$. The Ricci flow solution g (not the volume normalized Kähler Ricci flow) is given by

$$g(t) = (R^2 + 6t) g_{\mathbb{C}H^2} \qquad -\frac{R^2}{6} < t < +\infty.$$

4.B9. S^4.

In this case, any initial metric can be written as

$$g_0 = R^2 g_{S^4}$$

for some $R > 0$. The Ricci flow solution g is given by

$$g(t) = (R^2 - 6t) g_{S^4} \qquad -\infty < t < \frac{R^2}{6}.$$

4.B10. H^4.

In this case, any initial metric can be written as

$$g_0 = R^2 g_{H^4}$$

for some $R > 0$. The Ricci flow solution g is given by

$$g(t) = (R^2 + 6t) g_{H^4} \qquad -\frac{R^2}{6} < t < +\infty.$$

5. Conclusion.

We have analyzed the Ricci flow for compact four dimensional homogeneous geometries for which an initial diagonal metric remains diagonal under the flow. We obtain explicit solutions in most cases. We find that if the solution has long-time existence, then it is a Type III singularity solution. For volume-normalized flow, there are examples of collapse to dimensions 1, 2, and 3.

For the non-diagonal cases, the relevent ordinary differential equation systems are of a similar nature, but considerably more complicated. Numerical techniques should be useful for verifying if the behavior is similar to that of the diagonal cases.

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