

Convergence of the Ricci flow toward a soliton

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We will consider a τ -flow, given by the equation $\frac{d}{dt}g_{ij} = -2R_{ij} + \frac{1}{\tau}g_{ij}$ on a closed manifold M , for all times $t \in [0, \infty)$. We will prove that if the curvature operator and the diameter of $(M, g(t))$ are uniformly bounded along the flow, then we have a sequential convergence of the flow toward the solitons. If we also assume that one of the limit solitons is integrable, then we have a convergence toward a unique soliton, up to a diffeomorphism.

1. Introduction.

The Ricci flow equation

$$\frac{d}{dt}g_{ij} = -2R_{ij},$$

has been introduced by Hamilton in his seminal paper [6]. We will refer to this equation as to an unnormalized Ricci flow. A normalized Ricci flow is given by the equation

$$\frac{d}{dt}\tilde{g}_{ij} = -2R(\tilde{g})_{ij} + \frac{2}{n}r\tilde{g}_{ij},$$

where $r = \frac{1}{\text{Vol}(M)} \int_M R(\tilde{g}) dV_{\tilde{g}}$. This equation is sometimes more convenient to consider, since a volume of a manifold is being fixed along the normalized Ricci flow and a volume collapsing case cannot happen in a limit, if the limit exists.

A natural question that arises in studying the evolution equations, in particular, the Ricci flow equation, is under which conditions a solution will exist for all times, that is under which conditions it will avoid the singularities at finite times. The other question one can ask is if there exists a limit to the solutions when we approach infinity and how we can describe the metrics obtained in the limit. In the case of dimension three with positive Ricci curvature and dimension four with positive curvature operator, we

know (due to Hamilton) that the solutions of the Ricci flow equation, in both cases exist for all times, converging to Einstein metrics. In general, we cannot expect to get an Einstein metric in the limit. We can expect to get in the limit a solution to the Ricci flow equation which moves under one-parameter subgroup of the symmetry group of the equation. These kinds of solutions are called solitons. Since the Ricci flow equation is a gradient flow of Perelman's functional \mathcal{W} , it is natural to expect that a soliton in the limit is unique up to diffeomorphisms.

Our goal in this paper is to prove the following theorem.

Theorem 1.1. *Let $(g_{ij})_t = -2R_{ij} + \frac{1}{\tau}g_{ij}$ be a Ricci flow on a closed manifold M with uniformly bounded curvature operators and diameters for all $t \in [0, \infty)$. Assume also that some limit soliton is integrable. Then there is an 1-parameter family of diffeomorphisms $\phi(t)$, a unique soliton $h(t)$ and constants C, δ, t_0 such that $|\phi(t)^*g(t) - h(0)|_{k,\alpha} < Ce^{-\delta t}$, for all $t \in [t_0, \infty)$. Moreover, if $\psi(t)$ is a diffeomorphism such that $h(t) = \psi^*h(0)$, then $|\phi\psi^*g(t) - h(t)|_{C^0} < Ce^{-ct}$.*

The ideas for the proof of Theorem 1.1 have been inspired by those of Cheeger and Tian in [3].

Outline of the proof of Theorem 1.1

In order to deal with this problem, we will first construct a gauge on time intervals of an arbitrary length, so that in the chosen gauge, the τ -flow equation becomes strongly parabolic. We will look at the solutions of a strictly parabolic equation. It will turn out that our metrics (in the right gauge) will satisfy a strictly parabolic equation that is almost linear and therefore their behavior is modeled on the behavior of the solutions of the linear equation. There are 3 types of the solutions of our strictly parabolic equation,

- the solutions that have an exponential growth,
- the solutions that have an exponential decay,
- the solutions that change very slowly.

Roughly speaking, the integrability condition means that the solutions of a linearized deformation equation for solitons arise from a curve of metrics satisfying the same soliton equation. To deal with those slowly changing

solutions, we will use the integrability condition to change the reference soliton metric so that at the end, we deal only with the cases of either a growth or a decay. We will rule out the possibility of the exponential growth, by using the fact that our flow sequentially converges toward solitons and by using the similar arguments established by Simon in [15] and also later used by Cheeger and Tian in [3]. We will be left with the exponential decay which will allow us to continue our gauge up to infinity.

The organization of the paper is as follows. In Section 2, we will give a necessary background and notation. In Section 4, we will prove a sequential convergence of a τ -flow with uniformly bounded curvature operators and diameters toward the solitons. In Section 5, using the sequential convergence of the τ -flow, we will construct a gauge on time intervals of an arbitrary length, so that in the chosen gauge the τ -flow equation becomes strongly parabolic. In Section 6, we will use the integrability assumption to prove that a soliton that we get in the limit is unique up to a diffeomorphism.

2. Background.

Perelman's functional \mathcal{W} and its properties will play an important role in the paper. M will always denote a closed manifold. \mathcal{W} has been introduced in [11].

$$\mathcal{W}(g, f, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f} [\tau(|\nabla f|^2 + R) + f - n] dV_g.$$

We will consider this functional restricted to f satisfying

$$\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1. \quad (2.1)$$

\mathcal{W} is invariant under simultaneous scalings of τ and g and under a diffeomorphism change, i.e. $\mathcal{W}(g, f, \tau) = \mathcal{W}(c\phi^*g, \phi^*f, c\tau)$ for a constant $c > 0$ and a diffeomorphism ϕ . Perelman showed that the Ricci flow can be viewed as a gradient flow of a functional \mathcal{W} , which is one of the reasons why this functional plays an important role throughout [11]. Let $\mu(g, \tau) = \inf \mathcal{W}(g, f, \tau)$ over smooth f satisfying (2.1). It has been showed by Perelman that $\mu(g, \tau)$ is achieved by some smooth function f on a closed manifold M , that $\mu(g, \tau)$ is negative for small $\tau > 0$ and that it tends to zero as $\tau \rightarrow 0$.

We will explain the motivation why we have decided to study this flow instead of a normalized one in which a volume of a manifold has been fixed along the flow. First of all, there is a simple reparametrization that allows us to go from a τ -flow to an unnormalized flow and many smoothing

regularity properties that have been proved for the unnormalized flow continue to hold for a τ -flow as well. For example, Hamilton's compactness theorem also holds for the τ -flow. This is because Shi's estimates hold for τ -flow as well, and therefore, since we have a uniform curvature bound on the solutions to a τ -flow, we may assume uniform bounds on all covariant derivatives of the curvature, $|D^p \text{Rm}| \leq C(p)$. The reparametrization that we use to go from a τ -flow to an unnormalized flow is as follows. Let $c(s) = 1 - \frac{s}{\tau}$ and $t(s) = -\tau \ln(1 - \frac{s}{\tau})$. Let $\tilde{g}(s) = c(s)g(t(s))$. $\tilde{g}(s)$ is a solution to an unnormalized Ricci flow. On the other hand, we have that $\mathcal{W}(g(t(s)), f(t(s)), \tau) = \mathcal{W}(\tilde{g}(s), \tilde{f}(s), \tau - s)$. By the monotonicity formula for \mathcal{W} , we have that the latter quantity is increasing along an unnormalized Ricci flow and therefore, the former quantity is increasing along the τ flow as well. The monotonicity formula for a τ -flow gets the simpler form; $\mathcal{W}(g(t), f(t), \tau)$ is increasing along the τ -flow, while $f(t)$ changes by the evolution equation $\frac{d}{dt}f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}$ and τ is just a constant. The fact that τ is now a constant will be very useful in taking the limits of the minimizers for \mathcal{W} .

One of the most important properties of \mathcal{W} is the monotonicity formula.

Theorem 2.1 (Perelman). $\frac{d}{dt}\mathcal{W} = \int_M 2\tau|R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau}g_{ij}|^2(4\pi\tau)^{-\frac{n}{2}} e^{-f}dV \geq 0$ and therefore, \mathcal{W} is increasing along the flow described by the following equations

$$\begin{aligned} \frac{d}{dt}g_{ij} &= -2R_{ij}, \\ \frac{d}{dt}f &= -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \\ \dot{\tau} &= -1. \end{aligned}$$

One of the very important applications of the monotonicity formula is non-collapsing theorem for the Ricci flow that has been proved by Perelman in [11].

Definition 2.2. Let $g_{ij}(t)$ be a smooth solution to the Ricci flow $(g_{ij})_t = -2R_{ij}(t)$ on $[0, T)$. We say that $g_{ij}(t)$ is locally collapsing at T , if there is a sequence of times $t_k \rightarrow T$ and a sequence of metric balls $B_k = B(p_k, r_k)$ at times t_k , such that $\frac{r_k^2}{t_k}$ is bounded, $|\text{Rm}|(g_{ij}(t_k)) \leq r_k^{-2}$ in B_k and $r_k^{-n}\text{Vol}(B_k) \rightarrow 0$.

Theorem 2.3 (Perelman). *If M is closed and $T < \infty$, then $g_{ij}(t)$ is not locally collapsing at T .*

The corollary of Theorem 2.3 is

Corollary 2.4. *Let $g_{ij}(t)$, $t \in [0, T)$ be a solution to the Ricci flow on a closed manifold M , where $T < \infty$. Assume that for some sequences $t_k \rightarrow T$, $p_k \in M$ and some constant C we have $Q_k = |\text{Rm}|(x, t) \leq C$, whenever $t < t_k$. Then a subsequence of scalings of $g_{ij}(t_k)$ at p_k with factors Q_k converges to a complete ancient solution to the Ricci flow, which is κ -non-collapsed on all scales for some $\kappa > 0$.*

We would like to recall a definition of a soliton that will appear in later sections.

Definition 2.5. A Ricci soliton $g(t)$ is a solution to a Ricci flow equation that moves by 1-parameter group of diffeomorphisms $\phi(t)$, i.e. $g(t) = \phi(t)^*g(0)$.

The equation for a metric to move by a diffeomorphism in the direction of a vector field V is $2\text{Ric}(g) = \mathcal{L}_V(g)$, or $R_{ij} = g_{ik}D_jV^k + g_{jk}D_iV^k$. If the vector field V is the gradient of a function f , we say that the soliton is the gradient Ricci soliton. Moreover, we can consider the solutions to the Ricci flow that move by diffeomorphisms and also shrink or expand by a factor at the same time. The stationary solutions of the unnormalized Ricci flow are the Ricci flat metrics. The Ricci solitons are the generalizations of those, namely they are the stationary solutions to the Ricci flow equations, up to diffeomorphisms.

3. Sequential convergence of a τ -flow.

Definition 3.1. τ -flow is given by the equation

$$\frac{d}{dt}g_{ij} = -2R_{ij} + \frac{1}{\tau}g_{ij}, \quad (3.1)$$

for $\tau > 0$.

We want to prove the following theorem in this section

Theorem 3.2. *Consider the flow*

$$\frac{dg_{ij}}{dt} = -2R_{ij} + \frac{1}{\tau}g_{ij}, \quad (3.2)$$

on a compact manifold M , where $\tau > 0$ is fixed, $|Rm| \leq C$ and $\text{diam}(M, g(t)) \leq C$, $\forall t \in [0, \infty)$. Then for every sequence of times $t_i \rightarrow \infty$ there exists a subsequence, so that $g(t_i+t) \rightarrow h(t)$ and $h(t)$ is a Ricci soliton.

3.1. Convergence toward the solutions of the Ricci flow.

In order to prove Theorem 3.2, we will first show that it is reasonable to expect a convergence toward a smooth manifold, i.e. that a limit manifold will not collapse.

Claim 3.3. *Consider the flow as above. For every fixed $\tau > 0$, there exists a constant C such that $\text{Vol}_{g(t)}(M) \geq C$ for every t , i.e. we have a uniform lower bound on the volumes.*

Proof. Assume that the claim is not true, i.e. that there exists a sequence t_i s.t. $\text{Vol}_{g(t_i)}(M) \rightarrow 0$ as $i \rightarrow \infty$. Let $\bar{g}(s) = c(s)g(t(s))$ be unnormalized flow, for $s \in [0, \tau)$, where:

$$\begin{aligned} t(s) &= -\tau \ln\left(1 - \frac{s}{\tau}\right), \\ c(s) &= 1 - \frac{s}{\tau}, \\ R(\bar{g}) &= \frac{R(g)}{c(s)}. \end{aligned}$$

Find s_i , such that $t(s_i) = t_i$. We get that $s_i = \tau(1 - e^{-\frac{t_i}{\tau}})$. $s_i \rightarrow \tau$ as $i \rightarrow \infty$.

Let

$$\max_{M \times [0, s_i]} |Rm|(\bar{g}(s)) = Q_i, \quad (3.3)$$

and assume that the maximum is achieved at p_i . By the corollary of Perelman's non-collapsing theorem, we have that:

$$\frac{\text{Vol}_{\bar{g}(t)} B(p_i, r)}{r^n} \geq C_1,$$

for $r \leq C\sqrt{\frac{\tau}{Q_i}}$ and $t \in [0, s_i]$. Choose $r = C\sqrt{\frac{\tau}{Q_i}}$ and $t = s_i$.

$$(\sqrt{Q_i})^n \text{Vol}_{\tilde{g}(s_i)} B(p_i, C\sqrt{\frac{\tau}{Q_i}}) \geq (C\sqrt{\tau})^n C_1 = \tilde{C}.$$

Since $\text{Vol}_{\tilde{g}(s_i)} B(p_i, r) = c(s_i)^{\frac{n}{2}} \text{Vol}_{g(t_i)} B(p_i, \tilde{r})$, where \tilde{r} might be a different radius as a matter of scaling and since $Q_i \leq \frac{C}{c(s_i)}$ (because the curvature of $g(t)$ is uniformly bounded), we get that:

$$\text{Vol}_{g(t_i)}(M) \geq \tilde{C}/C,$$

where \tilde{C} and C do not depend on i . Let $i \rightarrow \infty$ in the previous inequality to get a contradiction. Therefore, we have a uniform lower bound on volumes. □

Remark 3.4. The assumptions of the Theorem 3.2 and the result of Claim 3.3 imply the uniform bounds on the curvature tensors, uniform upper bound on the diameters and uniform lower bounds on the volumes. Similarly, like in the case of unnormalized flow, uniform bounds on the curvatures gives us uniform bounds on all covariant derivatives, so by Hamilton’s compactness theorem, for every sequence $t_i \nearrow \infty$ as $i \rightarrow \infty$, there exists a subsequence (call it again t_i), such that $(M, g(t_i + t))$ converges to $(M, h(t))$, in the sense that there exist diffeomorphisms $\phi_i : M \rightarrow M$, so that $\phi_i^* g(t_i + t)$ converge uniformly together with their covariant derivatives to metrics $h(t)$ on compact subsets of $M \times [0, \infty)$. Moreover, $h(t)$ is a solution of a τ -flow as well.

3.2. Continuity of the minimizers for \mathcal{W} .

We will recall a definition of Perelman’s functional $\mathcal{W} = (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f} [\tau(R + |\nabla f|^2) + f - n] dV$. The constraint on f for this functional is $(4\pi\tau)^{-\frac{n}{2}} \int e^{-f} dV = 1$. Let $\mu(g, \tau) = \inf \mathcal{W}(g, f, \tau)$ under the constraint. This infimum has been achieved by some smooth minimizer f . Perelman has also proved that for a fixed metric g , $\lim_{\tau \rightarrow 0} \mu(g, \tau) = 0$ and $\mu(g, \tau) < 0$ for a small value of $\tau > 0$.

In the case of a τ -flow $g(t)$, $\tau > 0$ is being fixed in time, and by the monotonicity formula for \mathcal{W} , we have that $\mu(g(t), \tau)$ is increasing along the flow. Therefore, there exists $\lim_{t \rightarrow \infty} \mu(g(t), \tau)$.

Claim 3.5. $\lim_{t \rightarrow \infty} \mu(g(t), \tau)$ is finite.

Proof. Assume that $\lim_{t \rightarrow \infty} \mu(g(t), \tau) = \infty$. Then, $\forall i, \exists t_i$ s.t. $\mu(g(t_i), \tau) \geq i$. There exists a subsequence (call it t_i) such that (M, g_i) converges to (M, h) , for some metric h . By the proof Lemma 3.6 up to estimate, we get that $\mu(g(t_i), \tau) < \mu(h, \tau) + \epsilon$, for i big enough. Letting $i \rightarrow \infty$, we get a contradiction. \square

Lemma 3.6. If (M, g_i) tend to (M, h) when $i \rightarrow \infty$, where $g_i = g(t_i)$ and $t_i \nearrow \infty$, then $\lim_{i \rightarrow \infty} \mu(g_i, \tau) = \mu(h, \tau)$.

Proof.

$$\mu(h, \tau) = \int_M (\tau(|\nabla f|^2 + R(h)) + f - n)(4\pi\tau)^{-\frac{n}{2}} dV_h.$$

Since, $\phi_i^* g_i \rightarrow h$ uniformly with their covariant derivatives, if $\epsilon > 0$ is fixed, there exists some big i_0 , so that for $i \geq i_0$

$$\mu(h, \tau) \geq \int_M (\tau(|\nabla f|^2 + R(\tilde{g}_i)) + f - n)(4\pi\tau)^{-\frac{n}{2}} dV_{\tilde{g}_i} - \frac{\epsilon}{2},$$

where $\tilde{g}_i = \phi_i^* g_i$. Change the variables in the above integral by diffeomorphism ϕ_i .

$$\mu(h, \tau) \geq \int_M (\tau(|\nabla_i f_i|^2 + R(g_i)) + f_i - n)(4\pi\tau)^{-\frac{n}{2}} dV_{g_i} - \frac{\epsilon}{2},$$

where $f_i = \phi_i^* f$. Perturb a little bit f_i to get \tilde{f}_i , by a quantity that tends to zero, so that $\int_M e^{-\tilde{f}_i} (4\pi\tau)^{-\frac{n}{2}} dV_{g_i} = 1$. Since our geometries are uniformly bounded, for big enough i_0 , we will have

$$\mu(h, \tau) \geq \mathcal{W}(g_i, \tilde{f}_i, \tau) - \epsilon \geq \mu(g_i, \tau) - \epsilon. \quad (3.4)$$

Let $u_i = e^{-\frac{f_i}{2}}$. We have seen that minimizing $\mu(g_i, \tau)$ by f_i is equivalent to minimizing the following expression in u_i :

$$\int_M \tau(4|\nabla_i u_i|^2 + R_i u_i^2) - 2u_i^2 \ln u_i - nu_i^2(4\pi\tau)^{-\frac{n}{2}} dV_{g_i}.$$

The minimizer u_i has to satisfy the following elliptic differential equation

$$\tau(-4\Delta_i u_i + R_i u_i) - 2u_i \ln u_i - nu_i = \mu_{i,\tau} u_i. \tag{3.5}$$

$\mu_{i,\tau}$ is uniformly bounded, since there is a finite $\lim_{t \rightarrow \infty} \mu(g(t), \tau)$. Now, we can easily get:

$$\int_M u_i^2 (4\pi\tau)^{-\frac{n}{2}} dV_i \leq C, \tag{3.6}$$

$$\tau \int_M |\nabla_i u_i|^2 (4\pi\tau)^{-\frac{n}{2}} dV_i \leq C, \tag{3.7}$$

i.e. $u_i \in W^{1,2}$ with

$$\|u_i\|_{W^{1,2}} \leq C \quad \forall i.$$

From (3.5), by standard regularity theory of partial differential equations and Sobolev embedding theorems, we get that $u_i \in W^{k,p}$ with uniformly bounded $W^{k,p}$ norms, where $p < \frac{2n}{n-2}$, and therefore, with uniformly bounded $C^{2,\alpha}$ norms, i.e. $\|u_i\|_{C^{2,\alpha}} \leq C$. Furthermore,

$$\begin{aligned} \mu(g_i, \tau) &= \int_M (\tau(4|\nabla_i u_i|^2 + R_i u_i^2) - 2u_i^2 \ln u_i - nu_i^2)(4\pi\tau)^{-\frac{n}{2}} dV_i \\ &= \int_M \tau(|\tilde{\nabla}_i \tilde{u}_i|^2 + 4\tilde{R}_i \tilde{u}_i^2) - 2\tilde{u}_i \ln \tilde{u}_i - n\tilde{u}_i^2(4\pi\tau)^{-\frac{n}{2}} dV_{\tilde{g}_i} \end{aligned} \tag{3.8}$$

where $\tilde{u}_i = \phi_i^* u_i$. $\phi_i^* g_i$ is close to h and therefore, for i big enough, ϕ_i is almost an isometry, so $D_j \phi_i^{-1}$ can be uniformly bounded in terms of bounds on g_i and h , g_i can be bounded in terms of h . We cover M with finitely many geodesic balls of fixed radius ρ (we can do it since, we have a uniform bound on the injectivity radii from below). We use local coordinates in each of the balls to get:

$$|\tilde{\nabla}_i \tilde{u}_i|^2 = \tilde{g}_i^{jk} D_j(u_i \circ \phi_i^{-1}) D_k(u_i \circ \phi_i^{-1}).$$

$$|\tilde{\nabla} \tilde{u}_i|^2 = \tilde{g}_i^{jk} (D_j u_i)(D_k u_i)(\phi_i^{-1}) D_j \phi_i^{-1} D_k \phi_i^{-1}.$$

Now, we can easily conclude that we have a uniform bound on $|\tilde{\nabla} u_i|^2$. Since the integrand in (3.8) is uniformly bounded in i , and since \tilde{g}_i uniformly converge with their covariant derivatives to h , we have that for i large enough

$$\mu(g_i, \tau) \geq \int_M (\tau(4|\nabla_h \tilde{u}_i|^2 + R_h \tilde{u}_i^2) - 2\tilde{u}_i \ln \tilde{u}_i - n\tilde{u}_i^2)(4\pi\tau)^{-\frac{n}{2}} dV_h - \epsilon.$$

Since $l_i = \int_M \tilde{u}_i^2 (4\pi\tau)^{-\frac{n}{2}} dV_h$ is close to 1 when $i \rightarrow \infty$, taking $\bar{u}_i = \frac{\tilde{u}_i}{l_i}$ and using all the uniform bounds that we have got by now

$$\mu(g_i, \tau) \geq \mathcal{W}(h, \bar{u}_i, \tau) - \epsilon \geq \mu(h, \tau) - \epsilon.$$

By the previous inequality (for i big enough) and by (3.4), we get

$$\lim_{i \rightarrow \infty} \mu(g_i, \tau) = \mu(h, \tau).$$

□

Following the notation from the previous lemma, by Arzela–Ascoli theorem, there exists a subsequence, u_i , so that it converges in $C^{2,\alpha}$ norm to some function u . We can also get the higher order uniform estimates on u_i in a similar manner as in Lemma 3.6. Therefore, to show that a sequence of minimizers for $\mu(g_i, \tau)$ converges to a minimizer of $\mu(h, \tau)$ it is enough to show the following lemma.

Lemma 3.7. $\exists C > 0$ so that $u_i \geq C > 0 \quad \forall i$ and $\forall x \in M$

Proof. Assume that there exists a sequence u_i and $p_i \in M$, such that $0 < u_i(p_i) < \frac{1}{2i}$. M is compact and therefore, there is a subsequence, $\{p_i\}$ converging to $p \in M$ when $i \rightarrow \infty$. $C^{2,\alpha}$ norms of u_i are uniformly bounded in i and therefore, $u_i(p) < u_i(p_i) + C \text{dist}_{g_i}(p, p_i) \rightarrow 0$ as $i \rightarrow \infty$. Let u be a limit of $\{u_i\}$ in $C^{2,\alpha}$ norm. Then $u(p) = 0$. Take a geodesic ball $B(p, r)$. Let $f \in C_0^\infty(M)$ be a C^∞ function of r alone, compactly supported in $B(p, r) \setminus \{p\}$.

$$\int_M (\tau(\nabla u_i \nabla f + R_i u_i f) - 2u_i f \ln u_i - n u_i f - \mu(g_i, \tau) u_i f) dV_i = 0.$$

For this f , letting $i \rightarrow \infty$, using the result of the previous lemma and the fact that the integrand in the previous integral is uniformly bounded in i , we get

$$\int_M (\tau(\nabla u \nabla f + fuR(h)) - 2uf \ln u - nuf - \mu(h, \tau)fu) dV_h = 0.$$

Proceeding in the same manner as in [12], we can get that $u \equiv 0$ in some small ball around p . Using the connectedness argument, $u \equiv 0$ in M . On the other hand, $\int_M u_i^2 (4\pi\tau)^{-\frac{n}{2}} dV_i = 1$ and letting $i \rightarrow \infty$, we get a contradiction. □

If we write down the equations (3.5) for all $\{u_i\}$, letting $i \rightarrow \infty$, keeping in mind the previous lemma, we get

$$\tau(-4\Delta u + R(h))u - 2u \ln u - nu = \mu(h, \tau)u,$$

i.e. u is the minimizer for $\mu(h, \tau)$.

So far, we have proved the following theorem

Theorem 3.8. *If $(M, g_i) \rightarrow (M, h)$ as $i \rightarrow \infty$, then for a given $\tau > 0$, if $\mu(g_i, \tau) = \mathcal{W}(g_i, f_i, \tau)$, then $f_i \rightarrow f$ in $C^{2,\alpha}$ norm, where $\mu(h, \tau) = \mathcal{W}(h, f, \tau)$.*

3.3. Further estimates on the minimizers.

In this subsection, we want to use the minimizers f_t for \mathcal{W} at different times to construct the functions $f_t(s)$ for $s \in [0, t]$. By using the parabolic regularity, we will be able to get the uniform estimates on $C^{k,\alpha}$ norms of $f_t(s)$. This will enable us to take a limit of these functions along the sequences. These limits are the functions that will turn out to be the potential functions that come into the equations describing the soliton type solutions arising in a limit.

For any t , we can find f_t such that $\mathcal{W}(g(t), f_t, \tau) = \mu(g(t), \tau)$. If we flow f_t backward, we will get functions $f_t(s)$ that satisfy

$$\begin{aligned} \frac{df_t(s)}{ds} &= -R(s) - \Delta f_t(s) + |\nabla f_t(s)|^2 + \frac{n}{2\tau}, \\ f_t(t) &= f_t. \end{aligned}$$

We know that minimizing \mathcal{W} in f is equivalent to minimizing the corresponding functional in \tilde{u} , where $\tilde{u}_t = e^{-\frac{f_t}{2}}$. Let $u_t(s) = \tilde{u}_t^2(s)$. The equation for $u_t(s)$ is

$$\begin{aligned} \frac{du_t}{ds} &= -\Delta u_t + \left(-\frac{n}{2\tau} + R(s)\right)u_t(s), \\ u_t(t) &= u_t. \end{aligned}$$

By the monotonicity of \mathcal{W} along the flow (3.1), we have that

$$\mu(g(s), \tau) \leq \mathcal{W}(g(s), f_t(s), \tau) \leq \mathcal{W}(g(t), f_t, \tau) = \mu(g(t), \tau).$$

First of all, there exists $\lim_{t \rightarrow \infty} \mu(g(t), \tau)$. It is finite, since for every sequence $t_i \rightarrow \infty$, there exists a subsequence such that $g(t_i) \rightarrow h(0)$ and by Lemma 3.6 from the previous section, we have that $\mu(g(t_i), \tau) \rightarrow \mu(h(0), \tau)$.

Instead of functional $\mathcal{W}(g(s), f_t(s), \tau)$, we can consider the equivalent functional which depends on $\tilde{u}_t(s) = e^{-f_t(s)/2}$.

$$\begin{aligned} \mathcal{W}(u_t(s)) &= \int_M [\tau(4|\nabla \tilde{u}_t(s)|^2 + R\tilde{u}_t(s)^2) - \tilde{u}_t(s)^2 \log \tilde{u}_t(s)^2 \\ &\quad - n\tilde{u}_t(s)^2](4\pi\tau)^{-n/2} dV, \end{aligned} \quad (3.9)$$

where \tilde{u}_t satisfy

$$\tau(-4\Delta \tilde{u}_t + R\tilde{u}_t) - 2\tilde{u}_t \ln \tilde{u}_t - n\tilde{u}_t = \mu(g(t), \tau)\tilde{u}_t,$$

since f_t is a minimizer for \mathcal{W} . Since $\mu(g(t), \tau)$ is uniformly bounded, as in the previous section, we can get that $C^{2,\alpha}$ norms of \tilde{u}_t are uniformly bounded. This implies that $C^{2,\alpha}$ norms of u_t are uniformly bounded. Before we proceed with further discussion, notice the following.

Remark 3.9. $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f_t(s)} dV_{g(s)} = 1$. This is a simple consequence of the fact that $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f_t} dV_{g(t)} = 1$, since f_t is a minimizer for \mathcal{W} with respect to $g(t)$, and the following backward parabolic equation

$$\frac{d}{ds} f_t(s) = -\Delta f_t(s) + |\nabla f_t(s)|^2 - R + \frac{n}{2\tau}.$$

Namely,

$$\begin{aligned} \frac{d}{ds} \left(\int_M e^{-f_t(s)} dV_{g(s)} \right) &= \int_M e^{-f_t(s)} \left(\Delta f_t(s) - |\nabla f_t(s)|^2 + R - \frac{n}{2\tau} - R + \frac{n}{2\tau} \right) dV_{g(s)} \\ &= \int_M \Delta(e^{-f_t(s)}) dV_{g(s)} = 0 \end{aligned}$$

Since \log is a concave function and $\tilde{u}_t(s)^2(4\pi\tau)^{-n/2}dV$ is a probability measure, we have by Jensen and Sobolev inequalities

$$\begin{aligned} \int_M \tilde{u}_t(s)^2 \log \tilde{u}_t(s)^2 (4\pi\tau)^{-n/2} dV &= \frac{n-2}{2} \int_M \tilde{u}_t(s)^2 \log \tilde{u}_t(s)^{4/(n-2)} (4\pi\tau)^{-n/2} dV \\ &\leq \frac{n-2}{2} \log \int_M \tilde{u}_t(s)^{2n/(n-2)} (4\pi\tau)^{-n/2} dV \\ &\leq \frac{n-2}{2} \log [C \int_M (|\nabla \tilde{u}_t(s)|^2 + \tilde{u}_t(s)^2) dV]^{(n-2)/n} \\ &\quad + \frac{n-2}{2} \log (4\pi\tau)^{-n/2} \\ &= \frac{n}{2} \log C \int_M \tau (|\nabla \tilde{u}_t(s)|^2 + \tilde{u}_t(s)^2) (4\pi\tau)^{-n/2} dV. \end{aligned}$$

This inequality shows that

$$\tau \int_M |\nabla \tilde{u}_t(s)|^2 (4\pi\tau)^{-n/2} dV \leq C. \tag{3.10}$$

The constant C does not depend either on t or $s \in [0, t]$. To conclude, we have the following estimates

$$\begin{aligned} \int_M |\tilde{u}_t(s)|^2 (4\pi\tau)^{-\frac{n}{2}} dV_s &\leq C_1 \\ \tau (4\pi\tau)^{-\frac{n}{2}} \int_M |\nabla_s \tilde{u}_t(s)|^2 dV_s &\leq C_2, \end{aligned}$$

that is, we have that $|\tilde{u}_t|_{W_{1,2}} \leq C$ for a uniform constant C .

Take a sequence $t_i \rightarrow \infty$. There exists a subsequence such that $g(t_i+t) \rightarrow h(t)$ when $i \rightarrow \infty$, where $h(t)$ is a Ricci flow on M . This follows from Hamilton's compactness theorem ([8]). Fix $A > 0$. f_t will be a minimizer for \mathcal{W} with respect to $g(t)$, which we flow backward, for every t . Let $s \in [0, A]$.

Lemma 3.10. *For every $A > 0$, there exists $\delta = \delta(A) > 0$ such that $u_{t+A}(t+s) \geq \delta > 0$ for all t and all $s \in [0, A]$.*

Proof. Assume that the statement of the lemma is not true. In that case, there would exist a sequence s_i such that $\min_M u_{s_i+A}(s_i+a_i) \rightarrow 0$ as $i \rightarrow \infty$, for some $a_i \in [0, A]$. Consider the equation

$$\begin{aligned} \frac{d}{dt} u_{s_i+A}(s_i+t) &= -\Delta u_{s_i+A}(s_i+t) + \left(R - \frac{n}{2\tau}\right) u_{s_i+A}(s_i+t), \\ u_{s_i+A}(s_i+A) &= u_{s_i+A}, \end{aligned}$$

for $t \in [0, A]$. Let $\hat{u}_i(s_i+t) = \min_M u_{s_i+A}(s_i+t)$. Then $\Delta \hat{u}_{s_i+A}(s_i+t) \geq 0$ and

$$\frac{d}{dt} \hat{u}_i(s_i+t) \leq C \hat{u}_i(s_i+t),$$

where C is a uniform constant. If we integrate it with respect to t , we get

$$\hat{u}_i(s_i+A) \leq e^{CA} \hat{u}_i(s_i+t).$$

Since $\hat{u}_i(s_i+A) = \min_M u_{s_i+A}$ and since by Lemma 3.7, we know that there exists a constant δ such that $u_{s_i+A} \geq \delta > 0$, we have that $u_{s_i+A}(s_i+t) \geq \delta(A) > 0$ for all i and all $t \in [0, A]$. This contradicts our assumption that $\hat{u}_i(s_i+a_i) \rightarrow 0$ as $i \rightarrow \infty$. \square

Lemma 3.11. *For every $A > 0$, there exists $C(A)$ such that*

1. $\int_M u_t(s)^2 dV_{g(s)} \leq C(A)$.
2. $\int_M |\nabla u_t(s)|^2 dV_{g(s)} \leq C(A)$,

for all $t \geq A$, $s \in [t-A, t]$.

Proof. We will consider the equation

$$\begin{aligned} \frac{d}{ds} u_t(s) &= -\Delta u_t(s) + \left(R - \frac{n}{2\tau}\right) u_t(s) \\ u_t(t) &= u_t, \end{aligned}$$

where $u_t = e^{-f_t}$ and f_t is a minimizer for \mathcal{W} with respect to metric $g(t)$. Let $\hat{u}_t(s) = \max_M u_t(s)$. Then

$$\frac{d}{ds}\hat{u}_t(s) \geq -C\hat{u}_t(s),$$

where $C > 0$ is a uniform constant that does not depend either on s or t , but on the uniform bounds on geometries $g(t)$. If we integrate it with respect to s , we get

$$\hat{u}_t = \hat{u}_t(t) \geq e^{-CA}\hat{u}_t(s),$$

for any $s \in [t - A, t]$. On the other hand, we have already proved in the previous section that $C^{2,\alpha}$ norms of u_t are uniformly bounded in $t \in [0, \infty)$. Therefore, we get that $0 \leq u_t(s) \leq C(A)$ on M for all $t \in [A, \infty)$ and all $s \in [t - A, t]$. Now we immediately get part 1 of our claim. For part 2, notice that

$$\int_M |\nabla u_t(s)|^2 dV_{g(s)} = 4 \int_M u_t(s) |\nabla \tilde{u}_t(s)|^2 dV_{g(s)} \leq \tilde{C}(A),$$

since $\int_M |\nabla \tilde{u}_t(s)|^2$ is uniformly bounded for all $t \geq A$ and $s \in [t - A, t]$. \square

The previous two lemmas tell us that in order to find the uniform estimates on $f_{t_i+A}(t_i + s)$ for $s \in [0, A]$, it is enough to find the uniform $C^{k,\alpha}$ estimates on $u_{t_i+A}(t_i + s)$. Our main goal in this section is to prove the following theorem.

Theorem 3.12. *Under the assumptions of the main theorem, with the notations as above, for every $A > 0$, there exists a uniform constant C , depending on A such that $|u_t(s)|_{C^{2,\alpha}} \leq C$ for all $t \geq A$, $\forall s \in [t - A, t]$.*

Proof. Consider the equation

$$\frac{d}{ds}u_t(s) = -\Delta u_t(s) + (R(s) - \frac{n}{2\tau})u_t(s),$$

for $t \in [A, \infty)$ and $s \in [t - A, t]$. All our further estimates will depend on A . We will use C to denote different absolute constants that depend on A and the uniform bounds on our geometries $g(t)$. Denote by $h_1 = (h_1)_t(s) = (-\frac{n}{2\tau} + R(s))u_t(s)$. Omit the subscript t .

$$\frac{d}{ds}u + \Delta u = h_1.$$

$$\int_M h_1^2 = \int_M \left(\frac{d}{ds}u\right)^2 + 2 \int_M \frac{d}{ds}u \Delta u + \int_M (\Delta u)^2, \quad (3.11)$$

where we should keep in mind that the metric depends on s .

$$\begin{aligned} \int_M \frac{d}{ds}u \Delta u &= - \int_M g^{ij} \nabla_i \left(\frac{d}{ds}u\right) \nabla_j u dV_s \\ &= - \frac{1}{2} \frac{d}{ds} \int_M |\nabla u|^2 dV_s - \int_M |\nabla u|^2 \left(\frac{n}{2\tau} - R\right) dV_s \\ &\quad + \int_M g^{pi} g^{qj} D_i u D_j u (2R_{pq} - \frac{1}{2\tau} g_{pq}) dV_s, \end{aligned} \quad (3.12)$$

where the second term on the right-hand side of (3.12) comes from taking the derivative of the volume element and the third term appears from taking the derivative of g^{ij} . Denote the former one by J_1 and the latter one by J_2 .

$$\begin{aligned} \int_M (\Delta u)^2 &= \int_M g^{ij} D_i D_j u g^{kl} D_k D_l u \\ &= - \int_M g^{ij} g^{kl} D_j u D_i D_k D_l u \\ &= - \int_M g^{ij} g^{kl} D_j u D_k D_i D_l u + \int_M g^{ij} g^{kl} D_j u R_{iksl} D_s u \\ &= I + \int_M g^{ij} g^{kl} D_k D_j u D_i D_l u \\ &= I + \int_M |\nabla^2 u|^2, \end{aligned}$$

where $I = \int_M g^{ij} g^{kl} D_j u R_{iksl} D_s u$. Let $l \in (t - A, t)$ where $A > 0$. Integrating, the equation (3.11) in s , from l to t gives

$$\begin{aligned} \int_l^t \left(\int_M \left(\frac{d}{ds}u\right)^2 dV_s \right) ds + \int_M |\nabla u|^2 dV_s|_{s=l} + \int_l^t \int_M |\nabla^2 u|^2 dV_s ds \\ = \int_l^t \int_M h_1^2 + \int_M |\nabla u|^2 dV_s|_{s=t} + \int_l^t (2J_1 + 2J_2 + I). \end{aligned}$$

$$\int_l^t J_1 \leq AC \sup_{s \in (t-A, t)} \int_M |\nabla u|^2 dV_s \leq \tilde{C},$$

for every t . Similarly, we get estimates for J_2 and I . From all these estimates, we can conclude the following

$$\int_{t-A}^t \int_M \left(\frac{d}{ds} u_t(s)\right)^2 dV_s ds \leq C. \tag{3.13}$$

$$\int_{t-A}^t \int_M |\nabla^2 u_t(s)|^2 dV_s ds \leq C. \tag{3.14}$$

$$\sup_{s \in (t-A, t)} \int_M |\nabla u|^2 dV_s \leq C, \tag{3.15}$$

where $C = C(A)$. Let $\tilde{u}_t = \frac{d}{ds} u_t(s)$ (we will not confuse this \tilde{u}_t with one defined at the beginning of this section). Omit the subscript t .

$$\frac{d}{ds} \tilde{u} = -D_s \Delta_s u + \frac{d}{ds} \left[\left(R - \frac{n}{2\tau} \right) u \right].$$

Multiply the equation by \tilde{u} and integrate it along M .

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \int_M \left| \frac{d}{ds} u \right|^2 dV_s &= - \int_M \frac{d}{ds} (g(s)^{ij} D_i D_j u) \tilde{u} + \int_M \left(\frac{d}{ds} \left(R - \frac{n}{2\tau} \right) \right) u \tilde{u} \\ &\quad + \frac{1}{2} \int_M \left(R - \frac{n}{2\tau} \right) \left| \frac{d}{ds} u \right|^2 dV_s \\ &= 2 \int_M \left(-R_{pq} + \frac{1}{2\tau} g_{pq} \right) g^{pi}(s) g^{qj}(s) D_i D_j u \tilde{u} \\ &\quad - \int_M g(s)^{ij} D_i D_j \left(\frac{d}{ds} u \right) \tilde{u} \\ &\quad + \int_M \left(\frac{d}{ds} \left(R - \frac{n}{2\tau} \right) \right) u \tilde{u} \int_M g^{jk} \left(\frac{d}{dt} \Gamma_{ij}^k \right) \frac{\partial u}{\partial x_k} \tilde{u} \\ &\quad + \frac{1}{2} \int_M \left(R - \frac{n}{2\tau} \right) \left| \frac{d}{ds} u \right|^2 dV_s. \end{aligned}$$

Since $\int_M g(s)^{ij} D_i D_j \frac{d}{ds} u \tilde{u} = - \int_M |\nabla_s (\frac{d}{ds} u)|^2$ and since we are on the Ricci flow, metrics $g(s)$ are uniformly bounded, after applying Cauchy-Schwartz

inequality and using the uniform boundedness of the curvature operator, we get

$$\begin{aligned} \int_{t-A}^t \int_M |\nabla(\frac{d}{ds}u)|^2 dV_s ds + \sup_{s \in (t-A, t)} \int_M |\frac{d}{ds}u|^2 \\ \leq C \int_{t-A}^t \int_M |\frac{d}{ds}u|^2 dV_s ds \\ + C \int_{t-A}^t \int_M |\nabla^2 u|^2 dV_s ds \\ + \int_M |\frac{d}{ds}u|^2 dV_s|_{s=t} + C \int_M |\nabla u|^2. \end{aligned}$$

$\int_M |\frac{d}{ds}u(s)|^2 dV_s|_{s=t} \leq C(\int_M |\Delta u_t|^2 + \int_M h_1(t)^2)$ where $h_1(s) = (\frac{n}{2\tau} - R(s))u(s)$. Since $u_t = e^{-f_t}$, where f_t are the minimizers for \mathcal{W} , like in the previous section, we can conclude that $u_t \in W^{k,p}$, with uniform bounds on $W^{k,p}$ norms (these bounds depend on k) and therefore, $\int_M |\frac{d}{ds}u(s) dV_s|_{s=t}$ are uniformly bounded in t . This estimate together with estimates (3.13) and (3.14) gives that

$$\int_{t-A}^t \int_M |\nabla(\frac{d}{ds}u)|^2 dV_s ds \leq C. \quad (3.16)$$

$$\sup_{s \in (t-A, t)} \int_M |\frac{d}{ds}u|^2 \leq C. \quad (3.17)$$

If $\tilde{u} = \frac{d}{ds}u$ and $\tilde{h} = \frac{d}{ds}h_1$ then:

$$\frac{d}{ds}\tilde{u} = -D_s\Delta u + \tilde{h}.$$

$$\begin{aligned} D_s\Delta u = \frac{d}{ds}(g(s)^{ij}D_iD_ju) &= g(s)^{ip}g(s)^{jq}(\frac{1}{\tau}g_{pq} - 2R_{pq})D_iD_ju \\ &+ g(s)^{ij}D_iD_j\tilde{u} \\ &+ g(s)^{ij}\frac{d}{ds}(\Gamma_{ij}^k)D_ku. \end{aligned}$$

$$\begin{aligned} H &= \tilde{h} - g^{ip}g^{jq}(\frac{1}{\tau}g_{pq} - 2R_{pq})D_iD_ju - g(s)^{ij}\frac{d}{ds}(\Gamma_{ij}^k)D_ku \quad (3.18) \\ &= \frac{d}{ds}\tilde{u} + \Delta\tilde{u}. \end{aligned}$$

All the estimates that we have got so far tell that $\int_{t-A}^t \int_M H^2$ is uniformly bounded in t . The analogous estimates to the estimates (3.13), (3.14) and (3.15) for u , we can get for $\frac{d}{ds}u$ (by using the evolution equation for $\frac{d}{ds}u$ and all the estimates that we have got so far by analyzing the evolution equation for u).

$$\int_{t-A}^t \int_M \left| \nabla^2 \left(\frac{d}{ds}u \right) \right|^2 dV_s ds \leq C. \tag{3.19}$$

$$\int_{t-A}^t \int_M \left(\frac{d^2}{ds^2}u \right)^2 dV_s ds \leq C. \tag{3.20}$$

$$\sup_{s \in (t-A, t)} \int_M \left| \nabla \left(\frac{d}{ds}u \right) \right|^2 dV_s \leq C. \tag{3.21}$$

To obtain these estimates, we have used the fact that

$$\int_M \left| \nabla \frac{d}{ds}u \right|^2 dV_{g(s)}|_{s=t} \leq C \left(\int_M |\nabla \Delta u_t|^2 + \int_M \left| \nabla \left(R - \frac{n}{2\tau} \right) u_t \right|^2 \right),$$

where the right-hand side is uniformly bounded in t , since $u_t = e^{-f_t}$ and f_t are the minimizers for \mathcal{W} .

By standard regularity theory, considering $\Delta u_t(s) = -\frac{d}{ds}u_t(s) + (h_1)_t(s)$ as an elliptic equation whose right-hand side has uniformly bounded $W^{1,2}$ norms for $s \in (t - A, t)$ and all $t \geq A$, we have that $|u_t(s)|_{W^{3,2}} \leq C$, for a uniform constant C that depends on A . Take a derivative in s of the equation $\frac{d}{ds}\tilde{u} = -\Delta\tilde{u} + H$, with $\tilde{u} = \frac{d}{ds}u$. Denote by $\bar{u} = \frac{d}{ds}\tilde{u}$. By using the estimates that we have got for \tilde{u} , it is easy to conclude that \bar{u} satisfies the equation

$$\frac{d}{ds}\bar{u} = -\Delta\bar{u} + H_1,$$

where

$$H_1 = \frac{d}{ds}H + g^{ip}g^{jq}(-2R_{pq} + \frac{1}{\tau})D_iD_j\tilde{u} + g(s)^{ij}\frac{d}{ds}(\Gamma_{ij}^k)D_k\tilde{u}$$

and $\int_{t-A}^t \int_M H_1^2 dV_{g(s)} ds$ is uniformly bounded in t . As in the case of the previous estimates, we can conclude that

$$\sup_{s \in (t-A, t)} \int_M \left| \frac{d}{ds} \tilde{u} \right|^2 dV_s \leq C,$$

$$\sup_{s \in (t-A, t)} \int_M \left| \nabla \left(\frac{d}{ds} \tilde{u} \right) \right|^2 dV_s \leq C.$$

By regularity theory applied to the equation $\Delta \tilde{u} = -\frac{d}{ds} \tilde{u} + H$, we can get that $\frac{d}{ds} u_t(s)$ has uniformly bounded $W^{3,2}$ norms. If we go back to the parabolic equation for $u_t(s)$, we can get that $|u_t(s)|_{W^{5,2}} \leq C$ for all $t \geq A$ and all $s \in (t-A, t)$. Continuing this process by taking more and more derivatives in t of our original parabolic equation, we can conclude that $W^{p,2}$ norms of $u_t(s)$ are uniformly bounded for every p , by the constants that depend on A and p . Sobolev embedding theorem now gives that all $C^{k,\alpha}$ norms of $u_t(s)$ are uniformly bounded for all $t > A$ and all $s \in [t-A, t]$, by constants that depend on A and k . \square

Combining Theorem 3.12 and Lemma 3.10, we get that for every A there, exist constants $C_k = C(k, A)$ such that $|f_t(s)|_{C^{k,\alpha}} \leq C_k$, for all $t \geq A$ and all $s \in [t-A, t]$.

3.4. Ricci soliton in the limit.

In this subsection, we want to finish the proof of Theorem 3.2.

We have uniform curvature and diameter bounds for our flow $g(t)$. We have already proved that we also have a volume non-collapsing condition along the flow, for all times $t \geq 0$. This gives a uniform lower bound on the injectivity radii. Hamilton's compactness theorem (modified to the case of our flow) gives that for every sequence $t_i \rightarrow \infty$, there exists a subsequence so that $g(t_i + t) \rightarrow h(t)$ uniformly on compact subsets of $M \times [0, \infty)$ and that $h(t)$ is a solution to the Ricci flow (3.1). We will show below that for each t , $h(t)$ satisfies actually a Ricci soliton equation with the Hessian of function $f_h(t)$ involved, where $f_h(t)$ is a smooth one parameter family of functions. We will now see how we get the functions $f_h(t)$, using the estimates on $f_t(s)$ from the previous subsection and Perelman's monotonicity formula.

Take any t and let f_t be a function so that $\mu(g(t), \tau) = \mathcal{W}(g(t), f_t, \tau)$. Flow f_t backward. Fix $A > 0$. Then:

$$\begin{aligned} I(t) &= \mathcal{W}(g(t+A), f_{t+A}, \tau) - \mathcal{W}(g(t), f_{t+A}(t), \tau) \\ &\leq \mu(g(t+A), \tau) - \mu(g(t), \tau) \rightarrow 0 (t \rightarrow \infty). \\ 0 \leq I(t) &= \int_0^A \frac{d}{du} W(g(t+s), f_{t+A}(t+s), \tau) ds \rightarrow 0, \end{aligned}$$

as $t \rightarrow \infty$. We will consider $u_{t_i+A}(t_i+s)$ where $s \in [0, A]$. We will divide the proof of the theorem in a few steps.

Step 3.13. $\forall A > 0, \lim_{i \rightarrow \infty} \frac{d}{du} W(g(s+t_i), f_{t_i+A}(s+t_i), \tau) = 0$ for almost all $s \in [0, A]$.

Proof. $I(t_i) \rightarrow 0$ by Claim 3.5. On the other hand

$$\begin{aligned} I(t_i) &= \mathcal{W}(g(t_i+A), f_{t_i+A}, \tau) - \mathcal{W}(g(t_i), f_{t_i+A}(t_i), \tau) \\ &= \int_0^A \frac{d}{du} W(g(t_i+s), f_{t_i+A}(t_i+s), \tau) ds, \end{aligned}$$

and

$$\begin{aligned} \lim_{i \rightarrow \infty} I(t_i) &= \lim_{i \rightarrow \infty} \int_0^A \frac{d}{du} W(g(t_i+s), f_{t_i+A}(t_i+s), \tau) ds \\ &\geq \int_0^A \liminf_{i \rightarrow \infty} \frac{d}{du} W(g(t_i+s), f_{t_i+A}(t_i+s), \tau) ds, \end{aligned}$$

by Fatou's Lemma. Since by Perelman's monotonicity formula

$\frac{d}{du} W(g(t_i+s), f_{t_i+A}(t_i+s), \tau) \geq 0$, we have that

$\lim_{i \rightarrow \infty} \frac{d}{du} W(g(t_i+s), f_{t_i+A}(t_i+s), \tau) = 0$ for almost all $s \in [0, A]$, for

$$\int_0^A \lim_{i \rightarrow \infty} \frac{d}{du} W(g(t_i+s), f_{t_i+A}(t_i+s), \tau) ds \leq \lim_{i \rightarrow \infty} I(t_i),$$

by Fatuous lemma. □

Step 3.14. $|\tilde{u}_t(s)|_{C^{2,\alpha}} \leq C, \forall t$, where $\tilde{u}_t(s) = \frac{d}{ds} u_t(s)$.

Proof. Following the notation of the previous subsection, we get that:

$$\frac{d}{ds} \tilde{u}_t(s) = -\Delta \tilde{u}_t(s) + H_t(s),$$

where $H_t(s) = \frac{d}{ds}h_t(s) + g^{ip}g^{jq}(\frac{1}{\tau}g_{pq} - 2R_{pq})D_iD_ju + g^{ij}\frac{d}{ds}(\Gamma_{ij}^k)D_ku$.

$$\tilde{u}_t(t) = \frac{d}{ds}u_t(s) = -\Delta u_t + (-\frac{n}{2\tau} + R)u_t.$$

In the previous subsection, we have proved that there exists a uniform lower and an upper bound on $u_t(s)$ and that $|u_t(s)|_{W^{3,p}} \leq C(p, A)$ for all $t \geq A$ and all $s \in [t - A, t]$. Similarly, we can get that $|u_t(s)|_{W^{k,p}} \leq C(k, p, A)$ and therefore $|\tilde{u}_t(s)|_{W^{k-2,p}} \leq C(k, p, A)$, $\forall t \geq A$ and all $s \in [t - A, t]$. We can get that $|\tilde{u}_t(s)|_{C^{2,\alpha}} \leq C$, for all $t \geq A$ and $\forall s \in [t - A, t]$. We can extend this to all higher order time derivatives of $u_t(s)$. \square

Step 3.15. *For every $A > 0$, there exists a subsequence t_i , so that the limit metric $h(s)$ of a sequence $g(t_i + s)$ is a Ricci soliton for $s \in [0, A]$.*

Proof. By step 3.13, we have that

$$\lim_{i \rightarrow \infty} R_{jk}(t_i + s) + \nabla_j \nabla_k f_{t_i+A}(t_i + s) - \frac{1}{2\tau} g_{jk}(t_i + s) = 0,$$

for almost all $s \in [0, A]$ and almost all $x \in M$, since

$$\begin{aligned} & \frac{d}{ds} \mathcal{W}(g(t_i + s), f_{t_i+A}(t_i + s), \tau) \\ &= (4\pi\tau)^{-\frac{n}{2}} \int_M 2\tau |R_{jk} + \nabla_j f_{t_i+A} \nabla_k f_{t_i+A} - \frac{1}{2\tau} g_{jk}|^2 dV_{g(t_i+s)}. \end{aligned}$$

By Lemma 3.10 and Theorem 3.12, we have that $0 < C_1 \leq |u_{t_i+A}(s + t_i)| \leq C_2$ for all $i \geq i_0$ and all $s \in [0, A]$, for some constants C_1 and C_2 that depend on A . By step 3.14 and Theorem 3.12, we can find a subsequence, say $\{t_i\}$ such that $f_{t_i+A}(t_i + s)$ converges in $C^{2,\alpha}$ norm to $\tilde{f}_A(s)$ for all $s \in [0, A]$ and all $x \in M$. More precisely, for a countable dense subset $\{s_j\}$ of $[0, A]$ there exists a subsequence so that $f_{t_i+A}(t_i + s_j)$ converges in $C^{2,\alpha}$ norm to $\tilde{f}_A(s_j)$ on M . For any $s \in [0, A]$ there exists a subsequence t_{i_k} so that $f_{t_{i_k}+A}(t_{i_k} + s)$ converges to $\tilde{f}_A(s)$ in $C^{2,\alpha}$ norm. We want to show that actually $f_{t_i+A}(t_i + s) \xrightarrow{C^{2,\alpha}} \tilde{f}_A(s)$. For that we use the fact that $\frac{d}{ds}f_{t_i+A}(t_i + s)$ is uniformly bounded in $C^{2,\alpha}$ norm, and therefore

$$|\tilde{f}_A(s) - \tilde{f}_A(s_0)|_{C^{2,\alpha}} < \epsilon,$$

for some small $\epsilon > 0$ and some $s_0 \in \{s_j\}$ that is sufficiently close to s . We also have

$$|\tilde{f}_A(s_0) - f_{t_i+A}(t_i + s_0)|_{C^{2,\alpha}} < \epsilon,$$

for $i \geq i_0$ and

$$|f_{t_i+A}(t_i + s) - f_{t_i+A}(t_i + s_0)|_{C^{2,\alpha}} < \epsilon,$$

since $|\frac{d}{ds}f_{t_i+A}(t_i + s)|_{C^{2,\alpha}} \leq C(A)$, for all $i \geq i_0$ and all $s \in [0, A]$. By triangle inequality, we now get that for every $\epsilon > 0$ there exists i_0 so that

$$|\tilde{f}_A(s) - f_{t_i+A}(t_i + s)|_{C^{2,\alpha}} < 3\epsilon,$$

for all $i \geq i_0$ and all $s \in [0, A]$.

$f_{t_i+A}(t_i + s)$ converges in $C^{2,\alpha}$ norm on M to $\tilde{f}_A(s)$, for all $s \in [0, A]$.

Finally, we get that

$$R_{jk} + \nabla_j \nabla_k \tilde{f}_A(s) - \frac{1}{2\tau} h_{jk}(s) = 0, \tag{3.22}$$

for all $s \in [0, A]$, and for almost all $x \in M$. Because of the continuity, it will hold for all $x \in M$. Since $h(s)$ is a Ricci flow, all covariant derivatives of h and the covariant derivatives of a curvature operator are uniformly bounded, and therefore $|\nabla^p \tilde{f}_A(s)| \leq C(p)$, $\forall s \in [0, A]$ and all $p \geq 2$. Also we have that $|\frac{d^p}{ds^k} \nabla^p \tilde{f}_A(s)| \leq C(p, k)$ where $C(p, k)$ does not depend on A , for $p \geq 2$. \square

Step 3.16. *We can glue all the functions \tilde{f}_A that we get for different values of A , to get a function $f_h(s)$ defined on $M \times [0, \infty)$, which defines our metric $h(s)$ as a soliton type solution for all times $s \geq 0$.*

Proof. Take any increasing sequence $A_j \rightarrow \infty$. For every A_j , by the previous step, we can extract a subsequence t_i so that $f_{t_i+A_j}(t_i + s) \xrightarrow{C^{2,\alpha}} \tilde{f}_{A_j}(s)$ for all $s \in [0, A_j]$. Diagonalization procedure gives a subsequence so that

$f_{t_i+A_j}(s) \xrightarrow{C^{2,\alpha}} \tilde{f}_{A_j}(s)$ for all j and all $s \in [0, A_j]$. For this subsequence t_i we have that $g(t_i + t) \rightarrow h(t)$, uniformly on compact subsets of $M \times [0, \infty)$. Compare the functions \tilde{f}_{A_j} and \tilde{f}_{A_k} for $j < k$, on the interval $[0, A_j]$. We know that they both satisfy

$$\Delta_{h(s)} \tilde{f}_{A_r} + R(h(s)) - \frac{n}{2\tau} = 0,$$

and therefore, $\Delta_{h(s)}(\tilde{f}_{A_j} - \tilde{f}_{A_k}) = 0$. Since M is compact, this implies that $\tilde{f}_{A_k}(s) = \tilde{f}_{A_j}(s) + c_{A_j}^{A_k}(s)$, for $s \in [0, A_j]$, where $c_{A_j}^{A_k}(s)$ is a constant function for every $s \in [0, A_j]$. On the other hand, because of the integral normalization condition, we have

$$(4\pi\tau)^{-\frac{n}{2}} \int_M e^{-\tilde{f}_{A_j}(s)} dV_{h(s)} = 1,$$

$$(4\pi\tau)^{-\frac{n}{2}} \int_M e^{-\tilde{f}_{A_k}(s)} dV_{h(s)} = 1 = e^{-c_{A_j}^{A_k}(s)} (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-\tilde{f}_{A_j}(s)} dV_{h(s)},$$

which implies that $c_{A_j}^{A_k}(s) = 0$ for all $s \in [0, A_j]$ and all $k \geq j$. Therefore, $\tilde{f}_{A_j}(s) = \tilde{f}_{A_k}(s)$ for all $s \in [0, A_j]$. Define a function $f_h(s)$ in the following way. Let $f_h(s) = \tilde{f}_{A_j}(s)$, for all $s \in [0, A_j]$ and all $A_j \rightarrow \infty$. $f_h(s)$ is a well defined function because of the previous discussion. We also have that

$$R(h(s))_{pq} + \nabla_p \nabla_q f_h(s) - \frac{1}{2\tau} h(s)_{pq} = 0, \quad (3.23)$$

holds for all $s \in [0, \infty)$. The definition of $f_h(s)$ does not depend on a choice of an increasing sequence A_j . Namely, if B_j were another increasing sequence and if $f_{h'}(s)$ were functions defined using the sequences B_j and t_i (t_i is the same sequence as above), then at each time, both functions $f_h(s)$ and $f_{h'}(s)$ would satisfy the same equation (3.23) and the same integral normalization condition. Therefore, $f_h(s) = f_{h'}(s)$ for all $s \in [0, \infty)$. \square

3.5. Some properties of the limit solitons.

Let t_i be any sequence converging to infinity. Then as we have seen earlier, there exists a subsequence such that $g(t_i + s) \rightarrow h(s)$, where $h(s)$ is a Ricci soliton. Let $\hat{R}(h(t)) = \min R(h(t))$. We will first state a theorem that Hamilton proved in his paper [10].

Theorem 3.17 (Hamilton). *Under the normalized Ricci flow, whenever $\hat{R} \leq 0$, it is increasing, whereas if ever $\hat{R} \geq 0$ it remains so forever.*

We will use the proof of Theorem 3.17 to prove the following lemma.

Lemma 3.18. *Under the assumptions of Theorem 3.2, $\hat{R}(h(t)) \geq 0, \forall t$, for the limit metric $h(t)$ of any sequence of metrics $g(t_i)$, where $g(t)$ is a solution of*

$$\frac{d}{dt}g_{jk} = -2R_{jk}(g(t)) + \frac{1}{\tau}g_{jk}(t).$$

Proof. Assume that there exists t_0 such that $\hat{R}(h(t_0)) < 0$. Without loss of generality assume that $t_0 = 0$. Since $g(t_i) \rightarrow h(0)$ as $i \rightarrow \infty$, there exists i_0 , so that for all $i \geq i_0$ $\hat{R}(g(t_i)) < 0$. The evolution equation for R is

$$\frac{d}{dt}R = \Delta R + 2|\text{Ric}|^2 + \frac{2}{n}R\left(R - \frac{n}{2\tau}\right).$$

This implies

$$\frac{d}{dt}\hat{R} \geq \frac{2}{n}\hat{R}\left(\hat{R} - \frac{n}{2\tau}\right).$$

If $\hat{R} \leq 0$, then \hat{R} is increasing (since $\frac{d}{dt}\hat{R} \geq 0$). If $\hat{R} \geq 0$ at some time it can not go negative at later times. If there existed $t > t_{i_0}$ such that $\hat{R}(g(t)) \geq 0$, then $\hat{R} \geq 0$ would remain so forever, for all $s \geq t$ and therefore, we could not have $\hat{R}(g(t_i)) < 0$ for $t_i > t$. That contradicts the fact that $\hat{R}(g(t_i)) < 0$ for all $i \geq i_0$. Therefore, $\forall t \geq t_{i_0}$ we have that $\hat{R}(g(t)) < 0$.

$$\frac{d\hat{R}}{dt} \geq \frac{2}{n}\hat{R}\left(\hat{R} - \frac{n}{2\tau}\right) \geq 0,$$

for all t big enough. That implies \hat{R} is increasing and therefore, there exists $\lim_{t \rightarrow \infty} \hat{R}(g(t)) = -C \leq 0$. Moreover, $\hat{R}(h(s)) = -C$ for all s . Since $\lim_{i \rightarrow \infty} \hat{R}(g(t_i)) = \hat{R}(h(0)) < 0$, $C > 0$. We also have that

$$\frac{d\hat{R}(h(s))}{ds} \geq -\frac{2}{n}\hat{R}(h(s))\left(\frac{n}{2\tau} - \hat{R}(h(s))\right) = \frac{2}{n}C\left(\frac{n}{2\tau} + C\right) \geq 0.$$

The left-hand side of the above inequality is zero and therefore, we get that $C = -\frac{n}{2\tau}$ or $C = 0$. Since $C > 0$, we get a contradiction. Therefore, $R(h(t)) \geq 0$ for all t , what we wanted to prove. \square

Remark 3.19. Let (M, g) be a compact manifold and $g(t)$ be a Ricci flow on M . Since

$$\frac{d}{dt}\mathcal{W} = \int_M 2\tau|R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau}g_{ij}|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV,$$

$\mathcal{W}(g, f, \tau) = \text{const}$ along the flow, if g is a Ricci soliton satisfying the equation

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau}g_{ij} = 0.$$

Let $t_i \rightarrow \infty$ and $s_i \rightarrow \infty$ be two sequences such that $g(t_i + t) \rightarrow h(t)$ and $g(s_i + t) \rightarrow h'(t)$ where $h(t)$ and $h'(t)$ are 2 Ricci solitons on M that have been constructed earlier. We have proved that

$$R_{jk}(h) + \nabla_j \nabla_k f_h(t) - \frac{1}{2\tau}h_{jk} = 0,$$

$$R_{jk}(h') + \nabla_j \nabla_k f_{h'}(t) - \frac{1}{2\tau}h'_{jk} = 0,$$

where

$$f_h(t) = \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} f_{A_j + t_i}(t_i + t),$$

$$f_{h'}(t) = \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} f_{B_j + s_i}(s_i + t),$$

for some increasing sequences $A_j \rightarrow \infty$ and $B_j \rightarrow \infty$. By Remark 3.19, we know that $\mathcal{W}(h(t), f_h(t), \tau) = C_1$ and $\mathcal{W}(h'(t), f_{h'}(t), \tau) = C_2$ are constant along the flows $h(t)$ and $h'(t)$ respectively.

Lemma 3.20. $C_1 = C_2$, i.e. $\mathcal{W}(h(t), f_h(t), \tau)$ is a same constant for all solitons $h(t)$ that arise as limits of sequences of metrics of our original flow $g(t)$ (3.1) on a compact manifold M .

Proof.

$$\begin{aligned} & \mathcal{W}(g(t_i + t), f_{t_i+A_j}(t_i + t), \tau) - \mathcal{W}(g(s_i), f_{s_i+B_j}(s_i), \tau) \\ & \leq \mathcal{W}(g(t_i + A_j), f_{t_i+A_j}(t_i + A_j), \tau) \\ & \quad - \mathcal{W}(g(s_i), f_{s_i}(s_i), \tau) \\ & = \mu(g(t_i + A_j), \tau) - \mu(g(s_i), \tau) \rightarrow 0, \end{aligned} \tag{3.24}$$

where we have used the fact that $\mathcal{W}(g(t), f(t), \tau)$ increases in t along the flow (3.1) and the fact that $f_{s_i}(s_i) = f_{s_i}$ is a minimizer for $\mathcal{W}(g(s_i), f, \tau)$ over all f belonging to a set $\{f \mid \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_{g(s_i)}\}$. Similarly,

$$\begin{aligned} & \mathcal{W}(g(t_i + t), f_{t_i+A_j}(t_i + t), \tau) - \mathcal{W}(g(s_i), f_{s_i+B_j}(s_i), \tau) \\ & \geq \mathcal{W}(g(t_i + t), f_{t_i+t}(t_i + t), \tau) \\ & \quad - \mathcal{W}(g(s_i + B_j), f_{s_i+B_j}(s_i + B_j), \tau) \\ & = \mu(g(t_i + t), \tau) \\ & \quad - \mu(g(s_i + B_j), \tau) \rightarrow 0, \end{aligned} \tag{3.25}$$

when $i \rightarrow \infty$. From equations (3.24) and (3.25), letting $i \rightarrow \infty$, we get

$$\mathcal{W}(h(t), \tilde{f}_{A_j}(t), \tau) - \mathcal{W}(h'(0), \tilde{f}'_{B_j}(0), \tau) \leq 0.$$

$$\mathcal{W}(h(t), \tilde{f}_{A_j}(t), \tau) - \mathcal{W}(h'(0), \tilde{f}'_{B_j}(0), \tau) \geq 0.$$

Let $j \rightarrow \infty$ to get

$$C_1 = \mathcal{W}(h(t), f_h(t), \tau) = \mathcal{W}(h'(0), f_{h'}(0), \tau) = C_2.$$

□

Lemma 3.21. *For every Ricci soliton $h(t)$ that arises as a limit of some sequence of metrics of our original flow $g(t)$, the corresponding function $f_h(t)$, that we have constructed before, is a minimizer for Perelman's functional \mathcal{W} with respect to a metric $h(t)$.*

Proof. We will first prove the following claim.

Claim 3.22. *There exists a sequence $t_i \rightarrow \infty$ such that $g(t_i + t) \rightarrow h(t)$ as $i \rightarrow \infty$, where $h(t)$ is a Ricci soliton satisfying $R_{jk}(h) + \nabla_j \nabla_k f_h - \frac{1}{2\tau} h_{jk} = 0$ and $f_h(t)$ is a minimizer for $\mathcal{W}(h(t), f, \tau)$.*

Proof of the Claim. Let $H(t) = (4\pi\tau)^{-n/2} \int_M 2\tau |R_{ij} + \nabla_i \nabla_j f_t - \frac{1}{2\tau} g_{ij}|^2 dt$, where f_t is a function such that $\mu(g(t), \tau) = W(g(t), f_t, \tau)$. If we flow f_t backward by the equation

$$\frac{d}{dt} f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau},$$

starting at time t , for every $t > 0$, we get solutions $f_t(s)$. Look at $F_t(s) = \mathcal{W}(g(s), f_t(s), \tau)$. We know that

$$\frac{d}{ds} F_t(s) = (4\pi\tau)^{-\frac{n}{2}} \int_M 2\tau |R_{jk} + \nabla_j \nabla_k f_t(s) - \frac{1}{2\tau} g(s)_{jk}|^2 dV_{g(s)}.$$

$F_t(s)$ is a continuous function in $s \in [0, t]$ and $\lim_{s \rightarrow t} \frac{d}{ds} F_t(s) = H(t)$. Therefore, there exists a left derivative of $F_t(s)$ at point t and $(F_t)'_-(t) = H(t)$ for every $t > 0$. Moreover, $g(t)$ and all the derivatives of f_t up to the second order are Lipschitz functions in t (this follows from the estimates in the previous subsections) and therefore

$$\mu(t) := \mu(g(t), \tau) = \inf_{\{f \mid \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} = 1\}} \mathcal{W}(g(t), f, \tau)$$

is a Lipschitz function in t as well, i.e. $k(t) = F_t(t) = \mathcal{W}(g(t), f_t, \tau)$ is a Lipschitz function in t . This tells that $k(t)$ is differentiable in t , almost

everywhere. Our discussion then implies that $k'(t) = H(t)$ in a sense of distributions.

$$\begin{aligned} \int_{\delta}^{\infty} H(t)dt &= \lim_{K \rightarrow \infty} \int_{\delta}^K k'(t)dt \\ &= \lim_{K \rightarrow \infty} W(g(K), f_K, \tau) - W(g(\delta), f_{\delta}, \tau) \\ &= \lim_{K \rightarrow \infty} (\mu(g(K), \tau) - \mu(g(\delta), \tau)) \leq C, \end{aligned} \tag{3.26}$$

where $\delta > 0$ and C is some uniform constant. We have that $\int_{\delta}^{\infty} H(t) \leq C$. This implies that there exists a sequence $t_i \rightarrow \infty$ such that $H(t_i) \rightarrow 0$ as $i \rightarrow \infty$, i.e.

$$\lim_{i \rightarrow \infty} (R_{jk} + \nabla_j \nabla_k f_{t_i} - \frac{1}{2\tau} g_{jk})(t_i) = 0.$$

By what we have proved before, after extracting a subsequence, we can assume that $g(t_i) \rightarrow h(0)$ smoothly and $f_{t_i} \rightarrow \tilde{f}$ in $C^{2,\alpha}$ norm, where by Theorem 3.8 \tilde{f} is a minimizer for \mathcal{W} with respect to metric $h(0)$. Therefore,

$$R_{jk}(h(0)) + \nabla_j \nabla_k \tilde{f} - \frac{1}{2\tau} h_{jk}(0) = 0. \tag{3.27}$$

On the other hand, $g(t_i + t) \rightarrow h(t)$ as $i \rightarrow \infty$ where $h(t)$ is a Ricci soliton and

$$R_{jk}(h(t)) + \nabla_j \nabla_k f_h(t) - \frac{1}{2\tau} h_{jk}(t) = 0, \tag{3.28}$$

where $f_h(t) = \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} f_{t_i + A_j}(t_i + t)$, for some sequence $A_j \rightarrow \infty$. From equations (3.27) and (3.28), we have that $\Delta(f_h(0) - \tilde{f}) = 0$, i.e. $f_h(0) = \tilde{f} + C$ for some constant C . We know that $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-\tilde{f}} dV_{h(0)} = 1$, since \tilde{f} is a minimizer. From the construction of $f_h(t)$ it follows that $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f_h(0)} dV_{h(0)} = 1$ and therefore, $\tilde{f} = f_h(0)$. Since there exists a finite limit, $\lim_{t \rightarrow \infty} \mu(g(t), \tau)$, we have that $\mu(h(0), \tau) = \mu(h(t), \tau)$ for all t . This implies that

$$\begin{aligned} \mu(h(t), \tau) &= \mu(h(0), \tau) = \mathcal{W}(h(0), \tilde{f}, \tau) \\ &= \mathcal{W}(h(0), f_h(0), \tau) = \mathcal{W}(h(t), f_h(t), \tau), \end{aligned}$$

where we have used the fact that \mathcal{W} is constant along a soliton. This means that $f_h(t)$ is a minimizer for \mathcal{W} with respect to a metric $h(t)$, for every $t \geq 0$. \square

To continue the proof of Lemma 3.21 take any sequence $s_i \rightarrow \infty$. By a sequential convergence of our original flow $g(t)$ to Ricci solitons, after extracting a subsequence, we may assume that $g(s_i + t) \rightarrow h'(t)$ as $i \rightarrow \infty$ where $h'(t)$ is a Ricci soliton. Take a soliton $h(t)$ with the properties as in Claim 3.22. From the convergence of $\mu(g(t), \tau)$, we know that $\mu(h'(t), \tau) = \mu(h(s), \tau)$ for all t and all s .

$$\mu(h'(t), \tau) = \mu(h(s), \tau) = \mathcal{W}(h(s), f_h(s), \tau). \quad (3.29)$$

By Lemma 3.20, we have that $\mathcal{W}(h(s), f_h(s), \tau) = \mathcal{W}(h'(t), f_{h'}(t), \tau)$ for all s and t . Combining this with (3.29) gives that $\mu(h'(t), \tau) = \mathcal{W}(h'(t), f_{h'}(t), \tau)$, i.e. $f_{h'}(t)$ is a minimizer for $h'(t)$ for every t . \square

One useful property of the sequential soliton limits of our flow (3.1) is that all limit solitons are the solutions of the normalized flow equation

$$\frac{d}{dt}h_{ij} = -2R_{ij} + \frac{2}{n}r(h(t))h_{ij},$$

where $r(h(t)) = \frac{1}{\text{Vol}_{h(t)}M} \int_M R(h(t))dV_{h(t)}$. In the case of any of our soliton limits, we have that $R(h(t)) + \Delta f_h(t) - \frac{n}{2\tau} = 0$ and therefore, $r = r(h(t)) = \frac{n}{2\tau}$ for all $t \geq 0$.

Remark 3.23. Let $t_i \rightarrow \infty$ and $g(t_i + t) \rightarrow h(t)$, where $h(t)$ is an Einstein metric with an Einstein constant $\frac{1}{2\tau}$. If $\text{Vol}_{h'}(M) = \text{Vol}_h(M)$, for any other limit soliton h' , then h' is an Einstein metric with the same Einstein constant $\frac{1}{2\tau}$.

Proof. The fact that h is Einstein metric implies that $\nabla_i \nabla_j f_h = -2R_{ij} + \frac{1}{\tau}h_{ij} = 0$, that is $\Delta f_h = 0$. Since M is compact, $f_h = C$ such that $(4\pi\tau)^{-n/2}e^{-C}\text{Vol}_h(M) = 1$. An easy computation shows that $\mu(h, \tau) =$

$\mathcal{W}(h, C, \tau) = C - \frac{n}{2}$, and therefore, $\mu(h', \tau) = \mu(h, \tau) = C - \frac{n}{2}$. Then, $(4\pi\tau)^{-n/2}e^{-C}\text{Vol}_{h'}(M) = 1$, implies that $f = C$ is a minimizer for \mathcal{W} with respect to h' as well. This yields

$$\tau(2\Delta f - |\nabla f|^2 + R(h')) + f - n = C - \frac{n}{2},$$

that is

$$R(h') = \frac{n}{2\tau}.$$

From

$$\Delta f_{h'} = \frac{n}{2\tau} - R(h') = 0,$$

we get that $f_{h'} = C$ and therefore

$$R_{ij}(h') + \nabla_i \nabla_j f_{h'} - \frac{1}{2\tau} h'_{ij} = 0,$$

yields $R_{ij}(h') = \frac{1}{2\tau} h'_{ij}$. □

In the discussion that follows, we will use Moser’s weak maximum principle. We will state it below, for a reader’s convenience.

Lemma 3.24 (Moser’s weak maximum principle). *Let $g = g(t)$, $0 \leq t < T$, be a smooth family of metrics, b a non-negative constant and f a non-negative function on $M \times [0, T)$ which satisfies the partial differential inequality*

$$\frac{df}{dt} \leq \Delta f + bf,$$

on $M \times [0, T]$, where Δ refers to a Laplacian at time t . Then for any $x \in M$, $t \in [0, T)$,

$$|f(x, t)| \leq c \frac{1}{\sqrt{V}} e^{cHd} \max(1, d^{\frac{n}{2}}) (b + l + \frac{1}{t})^{\frac{1+n/2}{2}} e^{cbt} \|f_0\|_{L^2},$$

where c is a positive constant depending only on n and $d = \max_{0 \leq t \leq T} \text{diam}(M, g(t))$, $H = \max_{0 \leq t \leq T} \sqrt{\|\text{Ric}\|_{C^0}}$, $f_0 = f(\cdot, 0)$, $V = \min_{0 \leq t \leq T} \text{Vol}_{g(t)}(M)$.

The following remark will give us a condition that will imply obtaining the Einstein metrics in the limit.

Remark 3.25. If $g(t)$ is a solution to $(g_{ij})_t = -2R_{ij} + \frac{1}{\tau}g_{ij}$, for $t \in [0, \infty)$ such that

1. A curvature operator and a diameter are uniformly bounded along the flow.
2. $0 \leq R(x, t) \leq \frac{n}{2\tau}$ for all $x \in M$ and all $t \in [0, \infty)$.

Then all the solitons that arise as limits of the subsequences of our flow $g(t)$ are Einstein metrics with scalar curvatures $R = \frac{n}{2\tau}$ and $T_{ij}(t)$ converge to zero, uniformly on M as $t \rightarrow \infty$. $T_{ij} = R_{ij} - \frac{R}{n}g_{ij}$ is a traceless part of the Ricci curvature.

Proof of the Remark. Notice that now we do not make an assumption that one of the metrics that we get in a limit is an Einstein metric. Look at the evolution equation for $r(t) = \frac{1}{\text{Vol}_t(M)} \int_M R dV_t$,

$$\frac{d}{dt}r(t) = \frac{1}{\text{Vol}_t(M)} \left(2 \int_M |T|^2 + \left(1 - \frac{2}{n}\right) \int_M R \left(\frac{n}{2\tau} - R\right) + r \left(r - \frac{n}{2\tau}\right) \right).$$

$R \leq \frac{n}{2\tau}$ implies $r(t) \leq \frac{n}{2\tau}$ and therefore

$$\frac{d}{dt}r(t) \geq \frac{2}{\text{Vol}_t(M)} \int_M |T|^2 + r \left(r - \frac{n}{2\tau}\right). \quad (3.30)$$

We have proved that in the case of flow $g(t)$, a volume non-collapsing condition holds for all times $t \geq 0$. $\frac{d}{dt} \ln(\text{Vol}_t(M)) = \frac{n}{2\tau} - r$ and $C_1 \leq \text{Vol}_t(M) \leq C_2$ give that $\int_0^\infty (\frac{n}{2\tau} - r(t)) dt < \infty$. We can integrate the inequality (3.30) in $t \in [0, \infty)$. This, together with the uniform estimates on $\text{Vol}_t(M)$ and $r(t)$ give that

$$\int_0^\infty \int_M |T|^2 dV_t \leq C. \quad (3.31)$$

Following the calculations in Hamilton's paper [6], Rugang computed the evolution equation for T under a normalized Ricci flow ([18]). In the case of flow (3.1), we have

$$\frac{d}{dt}|T|^2 = \Delta|T|^2 - 2|\nabla T|^2 + 4\text{Rm}(T) \cdot T + \frac{4}{n} \left(R - \frac{n}{2\tau}\right) |T|^2. \quad (3.32)$$

Since the curvature operators of $g(t)$ are uniformly bounded, we derive from equation (3.32) that

$$\frac{d}{dt}|T| \leq \Delta|T| + C|T|.$$

Applying Lemma 3.24 to this differential inequality and intervals $[t - 1, t + 1]$ where $t > 1$, we derive

$$|T|^2(x, t) \leq \|T\|^2(t)_{C^0(M)} \leq C \left(\int_{M_{t-1}} |T|^2 \right),$$

where $M_t = (M, g(t))$. Integrate this inequality in $t \in [k, k + 1]$, for all $k \geq k_0$ and sum up all the inequalities that we get this way. We get

$$\begin{aligned} \int_{k_0}^{\infty} \|T\|^2 dt &\leq C \sum_{k \geq k_0} \int_k^{k+1} \left(\int_{M_{t-1}} |T|^2 \right) dt \\ &= C \int_{k_0}^{\infty} \int_M |T|^2 dV_{t-1} dt, \end{aligned} \tag{3.33}$$

where dV_{t-1} is a volume form for metric $g(t - 1)$. $\int_M |T|^2 dV_{t-1} \leq C \int_M |T|^2 dV_t$, because $\frac{d}{dt} \ln \text{Vol}_t = \frac{n}{2\tau} - R$ and the curvatures of $g(t)$ are uniformly bounded. The right-hand side of inequality (3.33) is bounded by a uniform constant, because of the estimate (3.31). Therefore, $\int_{k_0}^{\infty} \|T\|^2 dV_t \leq C$.

If there exists (p, t_0) such that $|T|^2(p, t_0) > \epsilon$, then there is a small neighbourhood of (p, t_0) in $M \times [0, \infty)$, say $U_\delta(p, t_0) = B_p(\delta, t_0) \times [t_0 - \delta, t_0 + \delta]$ such that $|T|^2(x, t) \geq \frac{\epsilon}{2}$ for all $(x, t) \in U_\delta(p, t_0)$. This follows from the fact that in the case of a Ricci flow, a bound $|\text{Rm}| \leq C$ implies $|D^k D_t^l \text{Rm}| \leq C(k, l)$. Costant δ does not depend on a point $(p, t_0) \in M \times [0, \infty)$, since all our bounds and estimates are uniform.

If there existed $\epsilon > 0$ and a sequence of points $(p_i, t_i) \in M \times [0, \infty)$, with $t_i \rightarrow \infty$ such that $|T(p_i, t_i)| \geq \epsilon$, then we would have that $\|T\|_{C^0} \geq \frac{\epsilon}{2}$ for all $t \in [t_i - \delta, t_i + \delta]$ and for all i . This would imply $C \geq \int_0^\infty \|T\|^2 dV_t \geq \sum_{i=0}^\infty \epsilon \delta = \infty$. This is impossible. Therefore, $\|T\|_{C^0(M_t)} \rightarrow 0$ as $t \rightarrow \infty$.

$\frac{d}{dt} \ln(\text{Vol}_t) = \frac{n}{2\tau} - R \geq 0$ for all t imply that there exists a finite $\lim_{t \rightarrow \infty} \text{Vol}_t$ for every $x \in M$ (otherwise, we can argue as in the previous paragraph). If we integrate this equation in $t \in [0, \infty)$, we will get that $\int_0^\infty (\frac{n}{2\tau} - R) dt < \infty$. As in the case for a traceless part of the Ricci curvature T , we can conclude that $\lim_{t \rightarrow \infty} R = \frac{n}{2\tau}$ uniformly on M .

We can conclude that under the assumptions given at the beginning of this remark, for every sequence $t_i \rightarrow \infty$, we can find a subsequence such that $g(t_i + t) \rightarrow h(t)$, where $h(t)$ is an Einstein soliton with scalar curvature $\frac{n}{2\tau}$. We also know that $R_{ij} - \frac{1}{2\tau} g_{ij} \rightarrow 0$ as $t \rightarrow \infty$, uniformly on M and that there exists $\lim_{t \rightarrow \infty} \text{Vol}_t$. \square

To conclude, we have proved a sequential convergence of a solution of a τ -flow towards solitons (generalizations of Einstein metrics), under uniform curvature and diameter assumptions. We still do not know whether we get a unique soliton (up to diffeomorphisms) in the limit or not. All observations in this subsection are in favor of the uniqueness of a soliton in the limit.

4. Uniqueness of a limit soliton.

In this section, we will assume that one of the limit solitons is integrable, in order to prove the uniqueness of a soliton in the limit, up to a diffeomorphism. We will first construct a gauge in which a τ -flow becomes a strictly parabolic flow. Similar ideas to those in [3] will help us finish the proof of Theorem 1.1.

4.1. The construction of a gauge.

To construct the right gauge, assume for simplicity that we are in a situation when $g(t) \rightarrow h$ as $t \rightarrow \infty$, where h is an Einstein metric, with the Einstein constant $\frac{1}{2\tau}$. We will see how we construct a gauge so that our modified Ricci flow equation becomes strictly parabolic on time intervals of an arbitrary length, if we go sufficiently far in time direction. This construction applies to our more general case, just with minor modifications and only for simplicity reasons, we have decided to consider a case of an Einstein metric in a limit. The main purpose of this section is to prove the following Proposition that will be reformulated in the next section for our more general setting.

Proposition 4.1. *Let $A > 0$ be an arbitrary real number, k an integer and $0 < \alpha < 1$. There exists $\epsilon_0(A, k)$ such that for every $\epsilon < \epsilon_0$, there exists $s_0 = s_0(A, k, h, \epsilon)$, such that for all $t_0 \geq s_0$ the equation*

$$\begin{aligned} \frac{d}{dt}\phi &= \Delta_{g(t),h}\phi, \\ \phi(t_0) &= \phi_{t_0}, \end{aligned} \tag{4.1}$$

has a solution $\phi(t)$, so that it is a diffeomorphism, $|\phi(t) - \text{Id}|_{k,\alpha,h} < \epsilon$ and $|\phi^*g(t) - h|_{k,\alpha} < \epsilon$, for every $t \in [t_0, t_0 + A]$. ϕ_{t_0} is chosen to be a diffeomorphism so that $\delta_{\phi^*(t_0)h}(g(t_0)) = 0$.

Definition 4.2. Let $\phi : M \rightarrow M$ be a smooth function. Define $e(\phi - \text{Id}) = g^{ij}h_{kl}(\phi_i^k - \text{Id}_i^k)(\phi_j^l - \text{Id}_j^l)$. Define $E(\phi - \text{Id}) = \int_M e(\phi - \text{Id})$ and $F_l = \phi^l - \text{Id}^l$.

Throughout the proof of Proposition 4.1, we will have a tendency to use the same symbol for different uniform constants.

Proof of Proposition 4.1. Fix $A > 0$. Let $\epsilon > 0$ be very small (we will see later how small we want to take it). We know that for s_0 sufficiently big, we can make $|g(t) - h|$ as small as we want, and therefore, we have that $\delta_{\phi(t_0)^*hg(t_0)} = 0$ implies that $|\phi(t_0) - \text{Id}|_{k+2,\alpha,h} < \epsilon/1000$ on M (see [3] for more details). Choose some $t_0 \geq s_0$. We can make $|F(t_0)|_{N,\alpha,h}$, for say $N \gg k$ as small as we want by choosing s_0 sufficiently big. Since $g(t) \rightarrow h$ as $t \rightarrow \infty$, the coefficients and the initial data of harmonic map flow (4.1) are uniformly bounded and uniformly close to each other for t_0 big enough. This implies that there exists a uniform constant $\delta_1 > 0$ so that a solution to (4.1) exists on $t \in [t_0, t_0 + \delta_1)$, for all $t_0 \geq s_0$. For the same reasons there exists some $\delta > 0$ such that $|F(t)|_{W^{2,N},g(t)} < \epsilon$, for $t \in [t_0, t_0 + \delta)$. We can assume that we have chosen N big enough so that as a consequence of Sobolev embedding theorems, we have that $|F|_{k,\alpha,g(t)} < \tilde{\epsilon}$ ($\tilde{\epsilon}$ differs from ϵ by a Sobolev embedding constant) for all $t \in [t_0, t_0 + \delta)$ and all $t_0 \geq s_0$. We want to show that the estimate $|F(t)|_{W^{2,N},g(t)} < \epsilon$ holds past time $t_0 + \delta$, until $\delta < A$. Then $|F|_{k,\alpha,g(t)} < \tilde{\epsilon}$ continues to hold past time $t_0 + \delta$, until

$\delta < A$. This actually gives a uniform upper bound on the energy densities on whole manifold M . To see this, notice that a bound $|F|_{k,\alpha} < \tilde{\epsilon}$ implies that $e(\phi - \text{Id}) \leq C\tilde{\epsilon}$. Since

$$e(\phi - \text{Id}) = e(\phi) + e(\text{Id}) - 2g^{ij}h_{kl}\text{Id}_i^k\phi_j^l,$$

by the Schwartz inequality for quadratic forms and the interpolation inequality, we get that

$$\begin{aligned} e(\phi) &\leq C\tilde{\epsilon} + C + 2(g^{ij}h^{kl}\phi_i^k\phi_j^l)^{1/2}(g^{ij}h^{kl}\text{Id}_i^k\text{Id}_j^l)^{1/2} \\ &\leq C\tilde{\epsilon} + C + \eta e(\phi), \end{aligned}$$

for some $\eta < 1$, which implies that $e(\phi) \leq \tilde{C}$. By the results proved by Eells and Sampson in [5] there exists $\bar{\delta}$, depending on (M, h) and the uniform bound on the energy densities \tilde{C} , so that for every $s \in [t_0, t_0 + \delta)$ a solution to a harmonic map flow (4.1) can be extended to $[s, s + \bar{\delta}]$. If $t_0 + \delta + \bar{\delta} < t_0 + A$, we can repeat the procedure above for a solution $\phi(t)$, on time interval $[t_0, t_0 + \delta + \bar{\delta})$ to get that the energy density estimates with the same constant \tilde{C} hold past time $t_0 + \delta + \bar{\delta}$. Since all our estimates depend only on A and the uniform bounds on geometries $g(t)$, we can iterate the argument till we reach time $t_0 + A$, for every $t_0 \geq s_0$. As a result, we will get $\phi(t)$, a solution to (4.1), such that $|\phi(t) - \text{Id}|_{k,\alpha} < \tilde{\epsilon}$ for all $t \in [t_0, t_0 + A]$.

We know that $(\Delta_{g(t),h}\text{Id})^\gamma = g^{\alpha\beta}(\Gamma(h)_{\alpha\beta}^\gamma - \Gamma(g)_{\alpha\beta}^\gamma)$ and that $\frac{d}{dt}\text{Id} = 0$. Therefore, we have

$$\frac{d}{dt}(\phi^k - \text{Id}^k) = \Delta_{g(t),h}(\phi^k - \text{Id}^k) + g^{ij}(\Gamma_{ij}^k(h) - \Gamma_{ij}^k(g)), \quad (4.2)$$

where we can choose s_0 so big, that the last term is arbitrarily small (since $g(t) \rightarrow h$). We will see later how small we want to make it, for now we can say it is less than some $\epsilon_1 > 0$.

Before we start establishing the estimates on $F = \phi - \text{Id}$, we will occupy ourselves with the problem of replacing equation (4.2) which in terms of local coordinates on M is a local system of equations, by some much more global

system. Passing to a global system of equations will make establishing the estimates on F much easier. We will follow a discussion in [5].

Since M is compact, there exists an embedding $\omega : M \rightarrow R^q$ and due to Eells and Sampson ([5]) it is always possible to construct a smooth Riemannian metric $g'' = (g''_{ab})_{1 \leq a, b \leq q}$ on a tubular neighborhood N of M in R^q , such that N is Riemannian fibered. They actually meant that if $\pi : N \rightarrow M$ is a projection map, it suffices to construct an appropriate smooth inner product in each space $R^q(p)$ for all $p \in M$, for which we can translate that tangent space to any point $m \in N$ along the straight line segment (that is contained in N) from $p = \pi(m)$ to m . Following the arguments of Section 7 in [5], we find that the evolution equation (4.2), given in local coordinates is satisfied by $\phi - \text{Id}$ if and only if $W - \tilde{W}$, where $W = \omega \circ \phi$ and $\tilde{W} = \omega \circ \text{Id}$ satisfies

$$\begin{aligned} \frac{d}{dt}(W^c - \tilde{W}^c) &= \Delta(W^c - \tilde{W}^c) + \pi_{ab}^c(W_i^a - \tilde{W}_i^a)(W_j^b - \tilde{W}_j^b)g^{ij} \\ &\quad + \frac{\partial \omega^c}{\partial y^k}g^{ij}(\Gamma_{ij}^k(h) - \Gamma_{ij}^k(g)), \end{aligned} \tag{4.3}$$

where (y_1, \dots, y_n) are the local coordinates on M . Moreover, since M is compact, the projection π satisfies (see [5])

$$|\pi_{ab}^c|_{k+1, \alpha} \leq C,$$

on M and there are constants A_1 and A_2 so that

$$A_1 ds_0^2 \leq ds^2 \leq A_2 ds_0^2,$$

where ds_0^2 denotes the line element induced on M by the usual metric on R^q . These estimates immediately imply that

$$\left| \frac{\partial^k \pi_{ab}^c}{\partial y^k} W_i^a W_j^b g^{ij} \right| \leq C(k)e(\phi),$$

where also $e(\phi) = g''_{ab} W_i^a W_j^b g^{ij}$, $e(\phi - \text{Id}) = g''_{ab}(W - \tilde{W})g^{ij}$. Moreover, if $\tilde{F}^c = W^c - \tilde{W}^c$, then $\left| \frac{\partial^k \pi_{ab}^c}{\partial y^k} \tilde{F}_i^a \tilde{F}_j^b g^{ij} \right| \leq C e(\phi - \text{Id})$.

The evolution equation for $e(\phi - \text{Id})$ (see for details [5] and [9]) is

$$\begin{aligned} \frac{d}{dt}e(\phi - \text{Id}) &= \Delta e(\phi - \text{Id}) - 2|D^2(\phi - \text{Id})|^2 \\ &\quad + 2\text{Rm}(D(\phi - \text{Id}), D(\phi - \text{Id}), D(\phi - \text{Id}), D(\phi - \text{Id})) \\ &\quad - \frac{1}{\tau}e(\phi - \text{Id}) + g^{ij}h_{kl}(\phi_j^k - \text{Id}_j^k) \\ &\quad [g^{pq}(\Gamma_{pq}^l(h) - \Gamma_{pq}^l(g))]_i \end{aligned} \quad (4.4)$$

where $\text{Rm}(D(\phi - \text{Id}), D(\phi - \text{Id}), D(\phi - \text{Id}), D(\phi - \text{Id})) = g^{ik}g^{jl}R_{pqmn}D_i(\phi^p - \text{Id}^p)D_j(\phi^q - \text{Id}_q)D_k(\phi^m - \text{Id}^m)D_l(\phi^n - \text{Id}^n)$ and $|D^2(\phi - \text{Id})|^2 = g^{ik}g^{jl}h_{pq}D_{ij}^2(\phi^p - \text{Id}^p)D_{kl}^2(\phi^q - \text{Id}^q)$. Applying the Schwarz inequality for quadratic forms and using the fact that $2\sqrt{\tau}(g^{pq}(\Gamma_{pq}^l(h) - \Gamma_{pq}^l(g)))_i$ can be made arbitrarily small by choosing s_0 sufficiently big (e.g. smaller than $\frac{2\epsilon}{1000}$), the last term in inequality (4.4) can be estimated as

$$g^{ij}h_{kl}(\phi_j^k - \text{Id}_j^k)[g^{pq}(\Gamma_{pq}^l(h) - \Gamma_{pq}^l(g))]_i \leq \frac{e(\phi - \text{Id})^{\frac{1}{2}}}{2\sqrt{\tau}}(2\epsilon)/1000.$$

Factor of 1000 (that we can increase if necessary) is chosen so that after multiplying $\frac{\epsilon}{1000}$ by at most a polynomial expression in A (which will become more apparent later in the proof of Proposition 4.1) can be made again much smaller than ϵ . Therefore, for $t \in [t_0, t_0 + \delta)$ we have that

Claim 4.3. *There exists C , small ϵ and sufficiently big s_0 such that for all $t_0 \geq s_0$*

1. $e(\phi - \text{Id}) < \epsilon_1$,
2. $E(\phi - \text{Id})(s) < \epsilon_1$,

for all s belonging to a time interval starting at t_0 at which ϕ exists, where ϵ_1 is a constant that can be made much smaller than ϵ .

Proof. By using the interpolation inequality in (4.4), we get

$$\begin{aligned} \frac{d}{dt}e(\phi - \text{Id}) &\leq \Delta e(\phi - \text{Id}) + C\epsilon^4 - \frac{1}{\tau}e(\phi - \text{Id}) + \frac{1}{2\tau}e(\phi - \text{Id}) + C\frac{\epsilon^2}{1000^2} \\ &\leq \Delta e(\phi - \text{Id}) - \frac{1}{2\tau}e(\phi - \text{Id}) + \frac{\epsilon}{1000}, \end{aligned}$$

since we can start with ϵ as small as we want, in particular, we may choose ϵ so that $C\epsilon^4 + C\frac{\epsilon^2}{1000^2} < \frac{\epsilon}{1000}$ and increase s_0 if necessary.

Let $f(t) = \max_M e(\phi - \text{Id})(t)$. Then

$$\begin{aligned} \frac{d}{dt}f &\leq -\frac{1}{2\tau}f + \frac{\epsilon}{1000}, \\ \frac{d}{dt}f &\leq -\frac{1}{2\tau}(f - \frac{\tau\epsilon}{500}). \end{aligned}$$

If we choose s_0 big enough, we may assume that $f(t_0) < \frac{\tau\epsilon}{500}\text{Vol}_h(M)$. If $f(t) \geq \frac{\tau\epsilon}{500}\text{Vol}_h(M)$ for some $t > t_0$, then $f(t)$ is non-increasing (because $\frac{d}{dt}f(t) \leq 0$ and since it starts as $f \leq \frac{\tau\epsilon}{500}\text{Vol}_h(M)$), it will remain so forever while ϕ exists. Denote by $\epsilon_1 = \frac{\tau\epsilon}{500} \max_t \text{Vol}_{g(t)}(M)$.

$$E(\phi - \text{Id})(s) = \int_U e(\phi - \text{Id})(s) dV_{g(s)} < \epsilon_1. \tag{4.5}$$

□

By Claim 4.3, $e(W - \tilde{W})$ can be made much smaller than ϵ whenever ϕ is defined (if t_0 is big enough and ϵ is small enough). The conditions $|F|_{W^{2,N}} < \epsilon$ and $|F|_{k,\alpha} < \tilde{\epsilon}$ actually mean that for \tilde{F} , we make an assumption that $|\tilde{F}|_{W^{2,N}} < \epsilon$ and $|\tilde{F}|_{k,\alpha} < \tilde{\epsilon}$, for $t \in [t_0, t_0 + \delta)$ (these ϵ and $\tilde{\epsilon}$ can be slightly different from those for F). In order to finish the proof of Proposition 4.1, it is enough to show that $|\tilde{F}|_{k,\alpha} < \tilde{\epsilon}$ continues to hold past time $t_0 + \delta$, for t_0 big enough. From now on, we will consider a globally defined evolution equation

$$\frac{d}{dt}(\tilde{F}^c) = \Delta \tilde{F}^c + \pi_{ab}^c \tilde{F}_i^a \tilde{F}_j^b g^{ij} + \frac{\partial \omega^c}{\partial y^k} g^{ij} (\Gamma_{ij}^k(h) - \Gamma_{ij}^k(g)), \tag{4.6}$$

Step 4.4. $\int_M |\tilde{F}^c|^2 dV_{g(t)}$ and $\int_{t_0}^{t_0+\delta} \int_M |\nabla \tilde{F}^c|^2 dV_{g(t)}$ can be made much smaller than ϵ , for all $t \in [t_0, t_0 + \delta)$ and for all t_0 big enough.

Multiply the equation (4.6) by \tilde{F}^c and integrate it over M against the metric $g(t)$.

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int (\tilde{F}^c)^2 dV_{g(t)} &< \int_M (\tilde{F}^c)^2 \left(\frac{n}{2\tau} - R \right) dV_{g(t)} - \int_M |\nabla \tilde{F}^c|^2 dV_{g(t)} \\
&+ \epsilon_1 \int_M |\tilde{F}^c| dV_{g(t)} \\
&+ C \left(\int_M e(\tilde{F})^2 dV_{g(t)} \right)^{1/2} \left(\int_M (\tilde{F}^c)^2 dV_{g(t)} \right)^{1/2} \\
&\leq \epsilon_1 \epsilon - \int_M |\nabla \tilde{F}^c|^2 + \epsilon_1 \left[\int_M |\tilde{F}^c| dV_{g(t)} \right] \\
&+ C \epsilon \epsilon_1 \int_M (\tilde{F}^c)^2 dV_{g(t)}
\end{aligned} \tag{4.7}$$

since

$$\begin{aligned}
\int_M \tilde{F}^c \frac{\partial \omega^c}{\partial y^l} g^{ij} (\Gamma(h_{ij}^l - \Gamma(g)_{ij}^l) dV_{g(t)} &\leq \epsilon_1 \int_M |\tilde{F}^c| dV_{g(t)}, \\
\int_M \tilde{F}^c g^{ij} \pi_{ab}^c \tilde{F}_i^a \tilde{F}_j^b &\leq C \left(\int_M e(\tilde{F})^2 \right)^{1/2} \left(\int_M (\tilde{F}^c)^2 \right)^{1/2} \\
&< C \epsilon \epsilon_1 \left[\int (\tilde{F}^c)^2 \right]^{1/2}.
\end{aligned}$$

In the above estimates, we have used the energy estimates (4.5), the fact that $g(t) \rightarrow h$ as $t \rightarrow \infty$ uniformly on M and that $|\tilde{F}|_{W^{2,N}} < \epsilon$ for $t \in [t_0, t_0 + \delta)$ (which implies $|\tilde{F}|_{C^{k,\alpha}} < \tilde{\epsilon}$ for sufficiently big N). For those reasons, $\epsilon_1 \ll \epsilon$ is a constant that can be made much smaller than ϵ , by taking ϵ small and s_0 big. Integrate (4.7) in t .

$$\begin{aligned}
\frac{1}{2} \sup_{t \in [t_0, t_0 + \delta)} \int (\tilde{F}^c)^2(t) dV_{g(t)} + \sup_{t \in [t_0, t_0 + \delta)} \int_{t_0}^t \int_M |\nabla \tilde{F}^c|^2 dV_{g(t)} \\
\leq \int \frac{1}{2} (\tilde{F}^c)^2(t_0) dV_h + C A \epsilon \epsilon_1.
\end{aligned}$$

Since for big t_0 the first integral on the right-hand side of the previous inequality can be made much smaller than ϵ , it follows that for big t_0 and small ϵ ,

$$\begin{aligned} \sup_{t \in [t_0, t_0 + \delta]} \int_M (\tilde{F}^c)^2(t) dV_{g(t)} &< \tilde{\epsilon}, \\ \sup_{t \in [t_0, t_0 + \delta]} \int_M |\nabla \tilde{F}^c|^2 dV_{g(t)} &< \tilde{\epsilon}, \end{aligned}$$

for some constant $\tilde{\epsilon} \ll \epsilon$ and these estimates depend on A .

Step 4.5. $\sup_{t \in [t_0, t_0 + \delta]} \int_{t_0}^t \int_M |\frac{d}{dt} \tilde{F}^c|^2 dV_{g(t)}$ and $\sup_{t \in [t_0, t_0 + \delta]} \int_{t_0}^t \int_M |\nabla^2 \tilde{F}^c|^2 dV_{g(t)}$ can be made much smaller than ϵ for big enough s_0 which depends on A and on the rate of convergence of $g(t)$ to h , for small enough ϵ .

$$\frac{d}{dt} \tilde{F}^c = \Delta \tilde{F}^c + H^c, \text{ where } H^c = \frac{\partial \omega^c}{\partial y_i} g^{ij} (\Gamma(h)_{ij}^l - \Gamma(g)_{ij}^l) + g^{ij} \pi_{ab}^c \tilde{F}_i^a \tilde{F}_j^b.$$

Then,

$$(H^c)^2 = (\Delta \tilde{F}^c)^2 + (\frac{d}{dt} \tilde{F}^c)^2 - 2\Delta \tilde{F}^c \frac{d}{dt} \tilde{F}^c. \tag{4.8}$$

$$\begin{aligned} - \int_M \frac{d}{dt} \tilde{F}^c \Delta \tilde{F}^c &= \int_M g^{ij} \nabla_i (\frac{d}{dt} \tilde{F}^c) \nabla_j \tilde{F}^c \\ &= \frac{1}{2} \frac{d}{dt} \int_M |\nabla \tilde{F}^c|^2 + \frac{1}{2} \int_M g^{ip} g^{jq} (-2R_{pq} + \frac{1}{\tau} g_{pq}) |\nabla \tilde{F}^c|^2 \\ &\quad - \frac{1}{2} \int_M |\nabla \tilde{F}^c|^2 (\frac{n}{2\tau} - R). \end{aligned} \tag{4.9}$$

$$\int_M (\Delta \tilde{F}^c)^2 = \int_M |\nabla^2 \tilde{F}^c|^2 + \int_M g^{ij} g^{ks} \nabla_j \tilde{F}^c R_{jks}^p \nabla_p \tilde{F}^c. \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we get

$$\begin{aligned} \int_M |\frac{d}{dt} \tilde{F}^c|^2 + \int_M |\nabla^2 \tilde{F}^c|^2 + \frac{d}{dt} \int_M |\nabla \tilde{F}^c|^2 \\ \leq \int_M (H^c)^2 + C \int_M |\nabla \tilde{F}^c|^2 dV_{g(t)}. \end{aligned} \tag{4.11}$$

since

$$\int_M g^{ip} g^{jq} (2R_{pq} - \frac{1}{\tau} g_{pq}) |\nabla \tilde{F}^c|^2 \leq \epsilon_1 \int_M |\nabla \tilde{F}^c|^2,$$

$$\int_M g^{ij} g^{ks} \nabla_j \tilde{F}^c R_{jkp}^s \nabla_p \tilde{F}^c \leq C \int_M |\nabla \tilde{F}^c|^2,$$

Notice also that

$$\begin{aligned} \int_M (H^c)^2 dV_{g(t)} &\leq \int_M C(g^{ij}(\Gamma_{ij}^l(h) - \Gamma_{ij}^l(g)))^2 dV_{g(t)} + C \int_M e(\tilde{F})^2 dV_{g(t)} \\ &< \epsilon_1 + C\epsilon\epsilon_1, \end{aligned} \quad (4.12)$$

since $e(\tilde{F}) < \epsilon$ for $t \in [t_0, t_0 + \delta)$ and $\int_M e(\tilde{F}) dV_{g(t)} < \epsilon_1$ by energy estimates (4.5). We will sometimes use the same constant ϵ_1 to denote any constant that can be made much smaller than ϵ , (the estimates above are possible if we start with ϵ small enough and increase s_0 if necessary, depending on how big A is). If we integrate (4.11) in t and use the above estimates, we get

$$\begin{aligned} &\int_M |\nabla \tilde{F}^c(t)|^2 dV_{g(t)} + \int_{t_0}^t \int_M \left| \frac{d}{dt} \tilde{F}^c \right|^2 + \int_{t_0}^t \int_M |\nabla^2 \tilde{F}^c|^2 \\ &\leq \int_M |\nabla \tilde{F}^c(t_0)|^2 dV_{g(t_0)} + \int_{t_0}^t \int_M (H^c)^2 + C \int_{t_0}^t \int_M |\nabla \tilde{F}^c|^2 dV_{g(s)} ds \\ &\leq \epsilon_1, \end{aligned} \quad (4.13)$$

because of Step 4.4, the fact that $\int_M |H^c|^2 < \epsilon_1$ for big t_0 and the fact that for big t_0 $\int_M |\nabla^k \tilde{F}^c|^2(t_0) dV_{g(t_0)}$ can be made very small.

Step 4.6. $\sup_{t \in [t_0, t_0 + \delta)} \int_M \left| \frac{d}{ds} \tilde{F}^c \right|^2, \int_{t_0}^t \int_M \left| \nabla \frac{d}{dt} \tilde{F}^c \right|^2$ can be made much smaller than ϵ for big t_0 , for all $t \in [t_0, t_0 + \delta)$.

Let $\hat{F}^c = \frac{d}{dt} \tilde{F}^c$. Then

$$\frac{d}{dt} \hat{F}^c = \Delta \hat{F}^c + g^{ip} g^{jq} (2R_{pq} - \frac{1}{\tau} g_{pq}) \nabla_i \nabla_j \tilde{F}^c + \frac{d}{dt} H^c - g^{ij} \frac{d}{dt} \Gamma_{ij}^k \nabla_k \tilde{F}^c.$$

Multiply this equation by \hat{F}^c and integrate it over M .

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_M (\hat{F}^c)^2 &- \frac{1}{2} \int_M (\hat{F}^c)^2 \left(\frac{n}{2\tau} - R \right) = - \int_M |\nabla \hat{F}^c|^2 \\ &+ \int_M g^{ip} g^{jq} (2R_{pq} - \frac{1}{\tau} g_{pq}) \nabla_i \nabla_j \tilde{F}^c \hat{F}^c \\ &+ \int_M \hat{F}^c \frac{d}{dt} H^c - \int_M g^{ij} \frac{d}{dt} \Gamma_{ij}^k \nabla_k \tilde{F}^c \hat{F}^c. \end{aligned}$$

Integrate it in t to get

$$\begin{aligned} \frac{1}{2} \int_M (\hat{F}^c(t))^2 + \int_{t_0}^t \int_M |\nabla \hat{F}^c|^2 &\leq \frac{1}{2} \int_M (\hat{F}^c(t_0))^2 + \frac{1}{2} \int_{t_0}^t \int_M (\hat{F}^c)^2 \left(\frac{n}{2\tau} - R \right) \\ &+ \epsilon_1 \left(\int_{t_0}^t \int_M |\nabla^2 \tilde{F}^c|^2 \right)^{1/2} \left(\int_{t_0}^t \int_M |\hat{F}^c|^2 \right)^{1/2} \\ &+ \left(\int_{t_0}^t \int_M (\hat{F}^c)^2 \right)^{1/2} \left(\int_{t_0}^t \int_M \frac{d}{dt} H^c \right)^{1/2} \\ &+ C \left(\int_{t_0}^t \int_M |\nabla \tilde{F}^c|^2 \right)^{1/2} \left(\int_{t_0}^t \int_M (\hat{F}^c)^2 \right)^{1/2}. \end{aligned} \tag{4.14}$$

Notice that

$$\begin{aligned} \int_{t_0}^t \int_M \left(\frac{d}{dt} H^c \right)^2 &\leq C \left(\int_{t_0}^t \int_M \left(\frac{d}{dt} \left(\frac{\partial \omega^c}{\partial y^t} g^{ij} \right) \left(\Gamma(h)_{ij}^l - \Gamma(g)_{ij}^l \right) \right) \right)^2 \\ &+ \int_{t_0}^t \int_M \left(\left(\frac{d}{dt} g^{ij} \right) \pi_{ab}^c \tilde{F}_i^a \tilde{F}_j^b \right)^2 \end{aligned} \tag{4.15}$$

$$+ \int_{t_0}^t \int_M \left(g^{ij} \pi_{ab}^c \left(\frac{d}{dt} \tilde{F}_i^a \right) \tilde{F}_j^b \right)^2, \tag{4.16}$$

where

$$\begin{aligned} \int_{t_0}^t \int_M \left(\frac{d}{dt} \left(g^{ij} \frac{\partial \omega^c}{\partial y^t} \left(\Gamma(h)_{ij}^l - \Gamma(g)_{ij}^l \right) \right) \right)^2 &< \epsilon_1, \\ \int_{t_0}^t \int_M \left(\left(\frac{d}{dt} g^{ij} \right) \pi_{ab}^c \tilde{F}_i^a \tilde{F}_j^b \right)^2 &\leq \epsilon_1, \end{aligned}$$

if t_0 is big enough, since $g(t) \rightarrow h$ uniformly on M and $\frac{d}{dt} g^{ij} = g^{pi} g^{qj} (2R_{pq} - \frac{1}{\tau} g_{pq})$, and

$$\int_{t_0}^t \int_M \left(g^{ij} \pi_{ab}^c \left(\frac{d}{dt} \tilde{F}_i^a \right) \tilde{F}_j^b \right)^2 \leq C\epsilon \int_{t_0}^t \int_M |\nabla \hat{F}^c|^2 < \frac{1}{2} \int_{t_0}^t \int_M |\nabla \hat{F}^c|^2,$$

if we choose ϵ small enough, such that $C\epsilon < \frac{1}{2}$, since $|\nabla_j \tilde{F}| < \epsilon$ for $t \in [t_0, t_0 + \delta)$.

$$\int_M |\hat{F}^c|^2 dV_{g(t)} \leq C \left(\int_{t_0}^t \int_M |\nabla^2 \tilde{F}^c|^2 + C \int_{t_0}^{t_0+\delta} \int_M |\nabla \tilde{F}^c|^2 + \epsilon_1 \right) \leq C(\epsilon_1 + \epsilon_1),$$

by Step 4.5. The assertion of Step 4.6 follows now immediately from (4.14).

From the estimate (4.15), we can now get (using the estimates of Steps 4.4, 4.5 and 4.6) that $\int_{t_0}^t \int_U (\frac{d}{dt} H^c)^2$ can be much smaller than ϵ . Consider the equation

$$\frac{d}{dt} \hat{F}^c = \Delta \hat{F}^c + \hat{H}^c, \quad (4.17)$$

where $\hat{F}^c = \frac{d}{dt} \tilde{F}^c$ and

$$\hat{H}^c = g^{ip} g^{jq} \left(2R_{pq} - \frac{1}{\tau} g_{pq} \right) \nabla_i \nabla_j \tilde{F}^c + \frac{d}{dt} H^c - g^{ij} \nabla_k \tilde{F}^c \frac{d}{dt} (\Gamma_{ij}^k)$$

. Since $g(t) \rightarrow h$, where $\text{Ric}(h) = \frac{1}{2\tau} h$, by using the previous estimates, we can easily see that $\int_{t_0}^t \int_U (\hat{H}^c)^2 dV_{g(s)} ds$ can be made much smaller than ϵ . In the same manner as we have obtained the estimates in step 4.5 for \tilde{F}^c , we can get the following estimates for $\hat{F}^c = \frac{d}{dt} \tilde{F}^c$ by considering the evolution equation (4.17).

$$\begin{aligned} & \sup_{t \in [t_0, t_0 + \delta)} \int_U \left| \nabla \frac{d}{dt} \tilde{F}^c \right|^2, \\ & \int_{t_0}^{t_0 + \delta} \int_U \left| \nabla^2 \frac{d}{dt} \tilde{F}^c \right|^2, \\ & \int_{t_0}^{t_0 + \delta} \int_U \left(\frac{d^2}{dt^2} \tilde{F}^c \right)^2, \end{aligned}$$

can be made much smaller than ϵ for big t_0 .

We have that $\Delta \tilde{F}^c = \frac{d}{dt} \tilde{F}^c - H^c$ where $W^{1,2}$ norm of $\frac{d}{dt} \tilde{F}^c$ and L^2 norm of H^c can be made much smaller than ϵ . By elliptic regularity theory, we can get that $W^{2,2}$ norm of \tilde{F}^c can be made much smaller than ϵ (since it can be estimated in terms of $W^{1,2}$ norm of $\frac{d}{dt} \tilde{F}^c$ and L^2 norm of H^c). Using that and the fact that $|\tilde{F}|_{1,\alpha,g} < \epsilon$ notice that

$$\begin{aligned}
\int_M |\nabla H^c|^2 &< C(\epsilon_1 + \int_M g^{rs}(\nabla_r(g^{ij}\pi_{ab}^c))\tilde{F}_i^a\tilde{F}_j^b)(\nabla_s g^{i'j'}\pi_{a'b'}^c\tilde{F}_{i'}^a\tilde{F}_{j'}^b) \\
&\quad + \int_M g^{rs}(g^{ij}\pi_{ab}^c\nabla_r\tilde{F}_i^a\tilde{F}_j^b)(g^{i'j'}\pi_{a'b'}^c\nabla_s\tilde{F}_{i'}^a\tilde{F}_{j'}^b) \\
&< C(\epsilon_1 + \epsilon \sum_a \int_M |\nabla\tilde{F}^a|^2 + \epsilon \sum_b \int_M |\nabla^2\tilde{F}^b|^2) \\
&< \tilde{\epsilon},
\end{aligned}$$

for some small constant $\tilde{\epsilon}$, that can be assumed to be much smaller than ϵ , since $W^{2,2}$ norm of \tilde{F}^c can be made much smaller than ϵ . By elliptic regularity theory this implies that $W^{3,2}$ norm of \tilde{F}^c can be made much smaller than ϵ for t_0 very big.

We can continue our proof by studying the equation $\frac{d}{dt}\hat{F}^c = \Delta\hat{F}^c + \hat{H}^c$. $|\tilde{F}^c|_{2,\alpha} < \epsilon$ for $t \in [t_0, t_0 + \delta)$. By a standard parabolic regularity we can get the higher order estimates of \tilde{F}^c , by constants that are comparable to ϵ . Therefore, by the similar analysis as above, we can get that $W^{3,2}$ norms of \hat{F}^c can be made much smaller than ϵ , since from the estimates that we have got till this point, we can again easily get that $W^{1,2}$ norm of \hat{H}^c can be made much smaller than ϵ . Consider again the equation

$$\Delta\tilde{F}^c = \frac{d}{dt}\tilde{F}^c - H^c. \quad (4.18)$$

We know that $W^{3,2}$ norm of $\frac{d}{dt}\tilde{F}^c$ and $W^{3,2}$ norms of \tilde{F}^c can be made much smaller than ϵ . Let's check that $W^{3,2}$ norm of H^c can be made much smaller than ϵ as well. In order for it to be true, it is enough to check that $\int_M |\nabla^3(g^{ij}\pi_{ab}^c\tilde{F}_i^a\tilde{F}_j^b)|^2$ can be made much smaller than ϵ .

$$\begin{aligned}
\int |\nabla^3(g^{ij}\pi_{ab}^c\tilde{F}_i^a\tilde{F}_j^b)|^2 &\leq C(\epsilon \int_M |\nabla\tilde{F}|^2 \\
&\quad + C \sum_{a,b} \int |\nabla^4\tilde{F}^a|^2|\nabla\tilde{F}^b|^2 \\
&\quad + C \sum_{a,b} \int_M |\nabla^3\tilde{F}^a|^2|\nabla^2\tilde{F}^b|^2) \\
&< C(\epsilon_1 + C\epsilon\epsilon_1 + \epsilon_1^2) < \tilde{\epsilon},
\end{aligned}$$

since $|\tilde{F}|_{W^{2,N},g(t)} < \epsilon$ for all $t \in [t_0, t_0 + \delta)$ and all $t_0 \geq s_0$. From here, again by elliptic regularity theory applied to equation (4.18), it follows that $W^{5,2}$ norm of \tilde{F}^c can be made much smaller than ϵ .

We can continue the proof in a similar manner as above, by taking the higher order derivatives of our original equation $\Delta \tilde{F}^c = \frac{d}{dt} \tilde{F}^c - H^c$ in t , using the estimates that we get on the way and then go backward to our original equation to improve a regularity of \tilde{F}^c . As a result, we can get (performing the previously described procedure sufficiently many times) that $|\tilde{F}|_{W^{N,2},g(t)} < \epsilon$ continues to hold past time $t_0 + \delta$. \square

So far, we have proved that for every $A > 0$ and an integer k there exists $\epsilon_0 = \epsilon_0(A, k)$ such that for every $\epsilon < \epsilon_0$ we can find $s_0 = s_0(A, \epsilon, k)$, so that $\forall t_0 \geq s_0$ there exists a solution of

$$\begin{aligned} \frac{d}{dt} \phi(t) &= \Delta_{g(t),h} \phi(t) \\ \phi(t_0) &= \phi_{t_0}, \end{aligned} \tag{4.19}$$

for all $t \in [t_0, t_0 + A]$ and $|\phi - \text{Id}|_{k,\alpha} < \epsilon$.

We want to show that these maps $\phi(t) : M \rightarrow M$ are actually diffeomorphisms which will imply that we have constructed an 1- parameter family of gauges such that for $\bar{g}(t) = (\phi(t)^*)^{-1}g(t)$ the linearization of the Ricci-DeTurck flow

$$\frac{d}{dt} \bar{g} = -2\bar{R}_{ij} + \frac{1}{\tau} \bar{g}_{ij} + \nabla_i W_j + \nabla_j W_i,$$

with $\bar{g}(t_0) = (\phi_{t_0}^{-1})^*g(t_0)$ is strictly parabolic ($W_j = \bar{g}_{jk} \bar{g}^{pq} (\Gamma_{pq}^k(\bar{g}) - \Gamma(h)_{pq}^k)$).

Corollary 4.7. *Adopt the notation from Proposition 4.1. $\phi(t)$ are diffeomorphisms for all $t \in [t_0, t_0 + A]$ and all $t_0 \geq s_0$.*

Proof. Fix any $t_0 \geq s_0$. Consider the equation

$$\begin{aligned} \frac{d}{dt} \tilde{g}_{ij} &= -2\tilde{R}_{ij} + \frac{1}{\tau} \tilde{g}_{ij} + \nabla_i V^j + \nabla_j V^i, \\ \tilde{g}(t_0) &= (\phi_{t_0}^{-1})^*g(t_0), \end{aligned} \tag{4.20}$$

where $V^k = \tilde{g}^{pq}(\Gamma_{pq}^k(\tilde{g}) - \Gamma_{pq}^k(h))$. This is a strictly parabolic system of equations and therefore, there exists some $\delta > 0$ so that a solution \tilde{g} exists for all times $t \in [t_0, t_0 + \delta)$. On the other hand, look at the system

$$\begin{aligned} \frac{d}{dt}\psi(t) &= -V \circ \psi(t), \\ \psi(t_0) &= \phi_{t_0}. \end{aligned} \tag{4.21}$$

Vector fields $V(t)$ are defined for $t \in [t_0, t_0 + \delta)$ and therefore, the system (4.21) has a solution $\psi(t)$ for all those times. It is easy to show (a classical result) that all $\psi(t)$ are diffeomorphisms for $t \in [t_0, t_0 + \delta)$. The simple computation (due to the fact that $g(t)$ is a solution of the Ricci flow equation) shows that

$$\frac{d}{dt}\psi(t) = \Delta_{g(t),h}\psi(t),$$

with $\psi(t_0) = \phi_{t_0}$. Because of the uniqueness of a harmonic map flow with the same initial data (we know that our solutions are smooth and uniformly bounded, so the uniqueness follows by the arguments of Eells and Sampson in [5]), we have that $\psi(t) = \phi(t)$ for all $t \in [t_0, t_0 + \delta)$. This means $\phi(t)$ is a diffeomorphism for $t \in [t_0, t_0 + \delta)$ and $\tilde{g}(t) = (\phi(t)^{-1})^*g(t)$. We know that for all $t \in [t_0, t_0 + A]$, for t_0 sufficiently big, we have that $|\phi(t) - \text{Id}|_{k,\alpha,h} < \epsilon$. Therefore, $|\phi^{-1} - \text{Id}|_{k,\alpha}$ can be made small which implies that $|\tilde{g}(t) - g(t)|$ can be made very small, comparable to ϵ , for all $t \in [t_0, t_0 + \delta)$. We want to extend a solution $\tilde{g}(t)$ of (4.20) all the way up to $t_0 + A$. Since $|\tilde{g}(t) - h| < \tilde{\epsilon}$ and since our flow (4.20) is strictly parabolic, there exists $t_1 = t_1(h, \tilde{\epsilon})$ so that for every $t \in [t_0, t_0 + \delta)$, a solution to (4.20) exists for all times $s \in [t, t + t_1)$. That means, we can extend our solution past time $t_0 + \delta$. Since our estimates on $|\tilde{g}(t) - h|$ for those times for which a solution $\tilde{g}(t)$ exists are independent of $\delta \leq A$, we can easily extend our solution all the way up to $t_0 + A$, with $|\tilde{g}(t) - h|$ staying very small (comparable to ϵ) for all $t \in [t_0, t_0 + A]$. Existence of $\tilde{g}(t)$ for $t \in [t_0, t_0 + A]$ gives that $\phi(t)$ stays a diffeomorphism for all times up to $t_0 + A$, because it solves the equation (4.21). \square

4.2. The integrable case.

The proofs in this subsection are motivated by those in [3], where Cheeger and Tian have considered the uniqueness problem of tangent cones under the assumption of integrability of one of the tangent cones and under some curvature and volume bounds.

Remark 4.8. So far, we have proved that if a limit h is an Einstein metric (this assumption is not essential and will be removed later, it only stands for technical simplicity) then for every $A > 0$ and an integer k there exists $\epsilon_0 = \epsilon_0(A, k)$ such that for every $\epsilon \leq \epsilon_0$ there exists $s_0 = s_0(\epsilon, A, k)$ with the property that for every $t_0 \geq s_0$ there is an 1-parameter family of diffeomorphisms $\phi(t)$ so that

1. ϕ^{-1} solves a harmonic map flow equation

$$\begin{aligned} \frac{d}{dt}\phi^{-1} &= \Delta_{g,h}\phi^{-1}, \\ \phi^{-1}(t_0) &= \phi_{t_0}, \end{aligned}$$

where $\delta_{\phi_{t_0}^* h}(g(t_0)) = 0$, for $t \in [t_0, t_0 + A]$,

2. $\tilde{g} = \phi^* g$ solves strictly parabolic equation on $[t_0, t_0 + A]$

$$\frac{d}{dt}\tilde{g} = -2\text{Ric}(\tilde{g}) + \frac{1}{\tau}\tilde{g} + \nabla_i V_j + \nabla_j V_i,$$

where $V^i = \tilde{g}^{pq}(\Gamma_{pq}^i(\tilde{g}) - \Gamma_{pq}^i(h))$. We will say that \tilde{g} is in a **standard form** around h . We will denote by $P_{h_0}(\tilde{g}) = \nabla_i V_j + \nabla_j V_i$.

3. $|\phi - \text{Id}|_{k,\alpha} < \epsilon$.
4. $|\tilde{g} - h|_{k,\alpha} < \epsilon$.

From now on, we will simply write ϕg instead of $\phi^* g$. By the assumptions of Theorem 1.1 there exists a limit soliton, say $h(t)$ which is integrable. There is a sequence t_i such that $g(t_i + t) \rightarrow h(t)$ as $i \rightarrow \infty$ and

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} h_{ij}(t) = 0,$$

for some function f . From before, we know that $f(t)$ is a minimizer for \mathcal{W} with respect to a metric $h(t)$, for every t . Let $\psi(t)$ be 1-parameter family of diffeomorphisms induced by a vector field $-\nabla f$. Then $h(t) = \psi^*(t)h_0$, where $h_0 = h(0)$. Since $h_0 = (\psi^{-1})^*h(t)$, it satisfies the equation

$$0 = \frac{d}{dt}h_0 = -2\text{Ric}(h_0) + \frac{1}{\tau}h_0 - \mathcal{L}_{\psi_* \frac{d}{dt}\psi^{-1}}h_0, \tag{4.22}$$

From $\psi \circ \psi^{-1} = \text{Id}$, by taking a time derivative, we see that $\psi^* \frac{d}{dt}\psi^{-1} + \psi^* \mathcal{L}_{\frac{d}{dt}\psi}\psi^{-1} = 0$ and since ψ is a diffeomorphism, we get that

$$\frac{d}{dt}\psi^{-1} = -\mathcal{L}_{\frac{d}{dt}\psi}\psi^{-1} = \mathcal{L}_{\nabla f(\psi)}\psi^{-1}. \tag{4.23}$$

Since $\{f(t)\}_{0 \leq t < \infty}$ are the minimizers for \mathcal{W} , there are uniform $C^{k+2,\alpha}$ estimates on $f(t)$. Since $\frac{d}{dt}\psi = -\nabla f(\psi)$, there are uniform $C^{k+1,\alpha}$ bounds on ψ , for $t \in [0, B]$. This together with (4.23) yields $|\psi^{-1}|_{k,\alpha} \leq C(B)$, for $t \in [0, B]$. Let $\tilde{g}(t) = \psi^{-1}g(t)$. Then $\tilde{g}(t)$ satisfies the equation

$$\frac{d}{dt}\tilde{g} = -2\text{Ric}(\tilde{g}) + \frac{1}{\tau}\tilde{g} - \mathcal{L}_{\psi_* \frac{d}{dt}\psi^{-1}}\tilde{g},$$

and

$$|\tilde{g}(t_i + t) - h_0|_{k,\alpha} \leq |\psi^{-1}| |g(t_i + t) - h(t)| \leq C(B)|g(t_i + t) - h(t)| \rightarrow 0,$$

when $i \rightarrow \infty$, uniformly on $M \times [0, B]$ (that implies $\tilde{g}(t_i + t) \rightarrow h_0$ uniformly on compact subsets of $M \times [0, \infty)$). The proof of Proposition 4.1, after minor modifications can be used to get the following result that tell us how to find an appropriate gauge in the case of convergence toward the solitons instead of Einstein metrics.

Theorem 4.9. *For every $L > 0$ and an integer k , there exists $\epsilon_0 = \epsilon_0(L, k)$ such that for every $\epsilon < \epsilon_0$, we can find $i_0 = i_0(L, \epsilon, k)$, so that whenever $i \geq i_0$ there is a gauge $\phi(t)$ on $M \times [t_i, t_i + L]$ such that ϕg is in a standard form around h_0 (see Remark 4.11 below), $|\phi \tilde{g} - h_0|_{k,\alpha} < \epsilon$ and $|\phi - \text{Id}|_{k,\alpha} < \epsilon$.*

Definition 4.10. A limit soliton $h(0)$ is said to be **integrable** if for every solution \mathbf{a} of a linearized deformation equation

$$\frac{d}{du}(\text{Ric}_{g_u} + \mathcal{L}_{\psi_* \frac{d}{dt}\psi^{-1}}g_u - \frac{1}{\tau}(g_u)_{ij})|_{u=0} = 0,$$

with $g_0 = h_0$ there exists a path of solitons h_u , satisfying the soliton equation

$$\text{Ric}_{h_u} + \mathcal{L}_{\psi_* \frac{d}{dt} \psi^{-1}} h_u - \frac{1}{\tau} (h_u)_{ij} = 0, \tag{4.24}$$

with $u \in (-\epsilon, \epsilon)$ and $h_0 = h(0)$ such that

$$\frac{d}{du} \Big|_{u=0} h_u = \mathbf{a}.$$

Remark 4.11. In the context of Theorem 1.1, to say that $\bar{g}(t)$ is in a **standard form around** h_0 means that \bar{g} satisfies the following equation

$$\frac{d}{dt} \bar{g} = -2\text{Ric}(\bar{g}) + \frac{1}{\tau} \bar{g} + P_{h_0}(\bar{g}) - \mathcal{L}_{\psi_* \frac{d}{dt} \psi^{-1}} \bar{g}, \tag{4.25}$$

where $P_{h_0}(\tilde{g}) = \nabla_i V_j + \nabla_j V_i$ and $V^k = \tilde{g}^{pq}(\Gamma_{pq}^k(\tilde{g}) - \Gamma_{pq}^k(h_0))$. We will write h_0 for $h(0)$ in a further discussion.

Choose i_0, ϕ as in Theorem 4.9 with $3L$ instead of L . Denote by $\|\cdot\|_{a,b} = \int_a^b |\cdot|$, where $|\cdot|$ is just the L^2 norm. Let π denote an orthogonal projection on the subspace $\ker(-\frac{d}{dt} + \Delta + \frac{1}{\tau} + U)_{M \times [t_{i_0}, t_{i_0} + L]}$, with respect to norm $\|\cdot\|_{t_{i_0}, t_{i_0} + L}$, where U is a linear first-order expression that comes out after linearizing the equation (4.25). Let g_1 be a suitable chosen soliton. Denote by $k = \phi g - g_1$ and put $\pi k = (\pi k)_\uparrow + (\pi k)_\downarrow + (\pi k)_0$. The integrability assumption on h_0 enters when we choose g_1 so that $(\pi k)_0 = 0$. Look at the explanation for $(\cdot)_\uparrow, (\cdot)_\downarrow$ and $(\cdot)_0$, just after the equation (4.27) below.

Lemma 4.12. *Let h_0 be an integrable limit soliton. Then if $\tau < \tau(n, L)$, for any cylinder $M \times [t_{i_0}, t_{i_0} + L]$ there is a soliton g_1 satisfying $P_{h_0}(g_1) = 0$ and equation (4.24), and such that $(\pi k)_0 = 0$. Moreover, if*

$$\sup_{[t_{i_0}, t_{i_0} + L]} |\phi g(t) - h_0| < \tau,$$

then

$$\|g_1 - h_0\|_{t_{i_0}, t_{i_0} + L} \leq 2 \|\pi(\phi g(t) - h_0)\|_{t_{i_0}, t_{i_0} + L}, \tag{4.26}$$

where the left-hand side of estimate (4.26) is time independent.

Proof. The proof of this lemma follows the proof of Lemma 5.56 in [3]. The integrability assumption implies that the set of metrics \tilde{g} satisfying

$$\begin{aligned} \text{Ric}(\tilde{g}) - \frac{1}{\tau}\tilde{g} + \mathcal{L}_{\psi_* \frac{d}{dt}\psi^{-1}\tilde{g}} &= 0, \\ P_{h_0}(\tilde{g}) &= 0, \end{aligned}$$

has a natural smooth manifold structure near h_0 . Let \mathcal{V} be a sufficiently small Euclidean neighborhood of h_0 . The tangent space to \mathcal{V} at h_0 is naturally identified with

$$\mathcal{K} = \{a \in \ker(\Delta + \frac{1}{\tau} + U) \mid DP_{h_0}a = 0\}.$$

Define $\psi : \mathcal{V} \rightarrow \mathcal{K}$ by

$$\psi(\tilde{g}) = \sum_i \langle \tilde{g}, B_i \rangle B_i,$$

where B_i is an orthonormal basis for \mathcal{K} with respect to a natural inner product. ψ is a smooth map and the differential of ψ is the identity map. We can use now the implicit function theorem and Lemma 4.13 to finish the proof of the Lemma 4.12. \square

The inequality (4.26) implies that $|g_1 - h_0| \leq 2 \sup_{[t_{i_0}, t_{i_0} + L]} |\pi(\phi g(t) - h_0|$, where $|\cdot|$ is just the usual L^2 norm. The linearization of the right-hand side of the equation $\frac{d}{dt}\phi g = Q(\phi g)$, satisfied by ϕg , where ϕ is a gauge chosen as in Theorem 4.9 is $DQ(k) = \Delta k + \frac{1}{\tau}k + U$, where U is a linear first-order expression in k and a Laplacian and U are with respect to metric ϕg . Let F be a solution of

$$\frac{d}{dt}F = \mathcal{L}F, \tag{4.27}$$

where $\mathcal{L} = \Delta + \frac{1}{\tau} + U$ and the Laplacian and U are this time given with a respect to a fixed metric (in our case, we will take metric h_0). Let $\{\lambda_k\}$ be the set of eigenvalues of \mathcal{L} . We can write $F = F_{\uparrow} + F_{\downarrow} + F_0$, where $F_{\uparrow}(t) = \sum_{\lambda_k < 0} a_k e^{-\lambda_k t}$, $F_{\downarrow}(t) = \sum_{\lambda_k > 0} a_k e^{-\lambda_k t}$, and F_0 is a projection of F to a kernel of \mathcal{L} .

The basic parabolic estimates (for example similarly as in [15] and [3]) yield the following.

Lemma 4.13. *There exists $\tau > 0$ such that for any solution η of (4.27) with $|g_1 - h_0|_{k+2,\alpha} \leq \tau$, we have that*

$$\sup_{(t_0, t_0+L)} |\eta|_{k,\alpha} \leq C \sup_{(t_0, t_0+L)} |\eta|,$$

where the first norm is $C^{k,\alpha}$ norm and the last norm is L^2 norm.

Lemma 4.14. *There exists $\alpha > 1$ such that*

$$\sup_{[L,2L]} |F_\uparrow| \geq \alpha \sup_{[0,L]} |F_\uparrow|, \quad (4.28)$$

$$\sup_{[L,2L]} |F_\downarrow| \leq \alpha^{-1} \sup_{[0,L]} |F_\downarrow|. \quad (4.29)$$

The norms considered above are standard L^2 norms.

Proof. We will prove only (4.28), since the proof of (4.29) is similar. Let $\delta = \min\{|\lambda_k| \neq 0\} > 0$.

$$\begin{aligned} \sup_{[L,2L]} |F_\uparrow| - \alpha \sup_{[0,L]} |F_\uparrow| &= \sup_{[0,L]} \sum_{\lambda_k < 0} a_k^2 e^{-2\lambda_k t} e^{-2\lambda_k L} - \alpha \sup_{[0,L]} \sum_{\lambda_k < 0} a_k^2 e^{-2\lambda_k t} \\ &\geq \sup_{[0,L]} \sum_{\lambda_k < 0} a_k^2 e^{-2\lambda_k t} (e^{2\delta L} - \alpha), \end{aligned}$$

which is positive, if $e^{2\delta L} > \alpha$. We can choose $\alpha = e^{\delta L} > 1$. \square

Lemma 4.15. *There exists $\beta < \alpha$ such that if*

$$\sup_{[L,2L]} |F| \geq \beta \sup_{[0,L]} |F|, \quad (4.30)$$

then

$$\sup_{[2L,3L]} |F| \geq \beta \sup_{[L,2L]} |F|, \quad (4.31)$$

and if

$$\sup_{[2L,3L]} |F| \leq \beta^{-1} \sup_{[L,2L]} |F|, \quad (4.32)$$

then

$$\sup_{[L,2L]} |F| \leq \beta^{-1} \sup_{[0,L]} |F|. \tag{4.33}$$

Moreover, if $F_0 = 0$ at least one of (4.31), (4.33) holds.

The proof of Lemma 4.15 is almost the same to the proof of analogous lemma (5.31) in [3]. We can choose β to be of order $e^{\frac{L\delta}{4}}$.

Let $\eta = \phi g - g_1$, where ϕ is chosen as in Theorem 4.9 and g_1 is a soliton as in Lemma 4.12 which does not depend on t for a considered time interval of length L .

Lemma 4.16.

$$\frac{d}{dt}(\phi g - g_1) = \Delta_{h_0}(\phi g - g_1) + \frac{1}{\tau}(\phi g - g_1) + F(\phi g, h_0, g_1) + U(\phi g - g_1) \tag{4.34}$$

where $|F(\phi g, h, g_1)|_{k,\alpha} \leq C(|g_1 - h_0| + |\eta|_{k,\alpha})|\nabla^2\eta|_{k-2,\alpha} + C(|\nabla(g_1 - h_0)|_{k-1,\alpha} + |\nabla\eta|_{k-1,\alpha})|\nabla\eta|_{k-1,\alpha}$ and U is a first order linear expression in $\phi g - g_1$.

Proof. Since both ϕg and g_1 are in a standard form around h_0 (recall that $P_{h_0}(g_1) = 0$), by using a formula for linearization of a second order operator $-2\text{Ric}(\phi g) + P_{h_0}(\phi g)$, we get

$$\begin{aligned} \frac{d}{dt}(\phi g - g_1) &= (-2\text{Ric}(\phi g) + P_{h_0}(\phi g) - \mathcal{L}_{\phi_*\psi_*\frac{d}{dt}\psi^{-1}\phi g} - \tag{4.35} \\ &\quad -(-2\text{Ric}(g_1) + P_{h_0}(g_1) - \mathcal{L}_{\psi_*\frac{d}{dt}\psi}g_1) + \frac{1}{\tau}(\phi g - g_1) \\ &= \Delta_{\phi g}(\phi g - g_1) + \frac{1}{\tau}(\phi g - g_1) + U(\phi g - g_1) + \tilde{F}(\phi g, g_1), \end{aligned}$$

where $|\tilde{F}(\phi g, g_1)|_{k,\alpha} \leq C(|\eta|_{k,\alpha}|\nabla^2\eta|_{k-2,\alpha} + |\nabla\eta|_{k,\alpha}^2)$, by a similar computation to a computation in [3]. Furthermore, $\Delta_{\phi g}\eta = \Delta_{h_0}\eta + (\Delta_{\phi g} - \Delta_{h_0})\eta$ and since $|\phi g - h_0|_{k,\alpha} \leq C(|\eta|_{k,\alpha} + |g_1 - h_0|_{k,\alpha})$, we have that $|(\Delta_{\phi g} - \Delta_{h_0})\eta| \leq C(|\eta|_{k,\alpha} + |g_1 - h_0|_{k,\alpha})|\nabla^2\eta|_{k,\alpha}$. The Lemma 4.16 now follows. \square

We assume that $|g_1 - h_0|_{k,\alpha} < \epsilon$. Let k be a solution to (4.34). Then, we have the following Proposition.

Proposition 4.17. *There exists $\epsilon_0 > 0$, depending on the uniform bounds on the geometries $g(t)$, such that if $\epsilon < \epsilon_0$, then if*

$$\sup_{[L,2L]} |k| \geq \beta \sup_{[0,L]} |k|, \quad (4.36)$$

then

$$\sup_{[2L,3L]} |k| \geq \beta \sup_{[L,2L]} |k|, \quad (4.37)$$

and if

$$\sup_{[2L,3L]} |k| \leq \beta^{-1} \sup_{[L,2L]} |k|, \quad (4.38)$$

then

$$\sup_{[L,2L]} |k| \leq \beta^{-1} \sup_{[0,L]} |k|, \quad (4.39)$$

Moreover, if $(\pi k)_0 = 0$, at least one of (4.37), (4.39) holds.

Proof. Assume there exist a sequence of gauges ϕ_i and constants $\tau_i \rightarrow 0$, such that $|\eta_i|_{k,\alpha} = |\phi_i g - h|_{k,\alpha} \leq \tau_i \rightarrow 0$, but for which none of the assertions in Proposition 4.17 holds. Let $\psi_i = \frac{\eta_i}{\sup_{[L,2L]} |\eta_i|}$. Then in view of Lemma 4.13, from standard compactness results (as in [3]), we get that for a subsequence $\psi_i \xrightarrow{C^{k,\alpha}} \psi$ and

$$\frac{d}{dt} \psi = \Delta_h \psi + U(\psi) + \frac{1}{\tau} \psi,$$

where ψ has a property that contradicts Lemma 4.15. Recall that β is of order $e^{\frac{\epsilon L}{4}}$. \square

Proof of Theorem 1.1. We will adopt the notation from above. Take some $L > 0$ big enough (we will see later how big we want to make it) and choose $\epsilon_0 > 0$ as in Theorem 4.9 so that the Theorem holds for ϵ_0 , and $3L$. For every $\epsilon < \epsilon_0$ there exists i_0 such that for every $i \geq i_0$ there exists a gauge ϕ so that ϕ satisfies all the conditions in Theorem 4.9, that is ϕg is in a standard

form around h_0 , $|\phi g - h_0|_{k,\alpha} < \epsilon$ and $|\frac{d}{dt}\phi g| < \tilde{\epsilon}$ on $M \times [t_i, t_i + 3L]$, where $\tilde{\epsilon}$ is comparable to ϵ . For each t_i pick up the largest possible L' (we will omit emphasizing a dependence of L' on i and we will call it just L' , since it is irrelevant for further discussion) such that ϕ is defined on $M \times [t_i, t_i + L']$, ϕg is in a standard form around h and $|\phi g - h|_{k,\alpha} < \epsilon$ and $\min_{[t_i, t_i + 3L]} |\phi g - h|_{k,\alpha} < \frac{\epsilon}{1000}$. Divide $[t_i, t_i + L']$ into the subintervals of length L and assume that N is the largest number such that $[t_i + (N - 1)L, t_i + NL] \subset [t_i, t_i + L']$.

Notice that for L chosen above, from the proof of Theorem 4.9, all the estimates that we have got on $|\phi - \text{Id}|_{k,\alpha}$ in the previous subsection depend on a polynomial in L (call it $q(L)$), whose coefficients depend only on a dimension, an integer k and the uniform bounds on geometries $g(t)$. By the estimates established in Proposition 4.1, we can increase i_0 if necessary, so that

1. For every $i \geq i_0$, we can find a gauge on $M \times [t_i, t_i + 3L]$, such that $\sup_{[t_i, t_i + 3L]} |\phi g(t) - h_0|_{k,\alpha} < \frac{\epsilon}{1000e^{L\delta/4}}$.
2. If the initial data $\phi(s)$ is such that $|\phi(s) - \text{Id}|_{k,\alpha} < \frac{\epsilon}{e^{L\delta/8}}$ and $|\phi(s)^*g(s) - h_0|_{k,\alpha} < \frac{\epsilon}{e^{L\delta/8}}$, where $s \in [t_i, t_i + L']$, for $i \geq i_0$, then ϕ can be extended to interval $[s, s + 3L]$ such that $\sup_{[s, s + 3L]} |\phi g - h_0|_{k,\alpha} < \frac{\epsilon}{100p(L)}$ (we might need increase i_0 for this to hold). Polynomial $p(L)$ can be any polynomial with leading coefficient 1 and with a degree that is e.g. one more than a degree of $q(L)$.
3. If the initial data is such that $|\phi(s) - \text{Id}|_{k,\alpha} < \frac{\epsilon}{p(L)}$ and $|\phi(s)^*g(s) - h_0|_{k,\alpha} < \frac{\epsilon}{p(L)}$, where $s \in [t_i, t_i + L']$, for $i \geq i_0$, then ϕ can be extended on interval $[s, s + 3L]$ such that $\sup_{[s, s + 3L]} |\phi g - h_0|_{k,\alpha} < \epsilon$.

We want to show that there exists i (for sufficiently big L , so that, above holds) such that a corresponding $L' = \infty$. Assume that for all $i \geq i_0$ and all $\epsilon > 0$, $L' < \infty$. Denote by $I_j = [t_i + jL, t_i + jL + L]$. Assume that ϵ is small enough so that we can apply Lemma 4.12, that is for every j there exists a soliton g_j such that $(\pi(\phi g - g_j))_0 = 0$ on I_j and therefore by Proposition 4.17, $\phi g - g_j$ either satisfies a growth condition ((4.36) \Rightarrow (4.37)) or a decay

condition ((4.38) \Rightarrow (4.39)). Moreover, $|g_j - h_0| \leq 2 \sup_{I_j} |\pi(\phi g - h_0)| \leq C \sup_{I_j} |\phi g - h_0|_{k,\alpha}$. We need to consider two cases.

Case 4.18. *Assume that for all i_0 and all $i \geq i_0$, where $i_0 = i_0(L)$ is chosen as in Theorem 4.9 for L big enough (so that (1), (2) and (3) hold), and for all the intervals I_j (that are defined with respect to t_i ; we want to omit double indices) for which we have $\sup_{I_j} |\phi g - h_0|_{k,\alpha} \leq \frac{\epsilon}{100p(L)}$, $\phi g - g_j$ satisfies a decay condition on I_j (recall that L^2 norms are considered in a growth and a decay condition).*

By using Proposition 4.17 inductively, we get that

$$\sup_{I_l} |\phi g - g_j| \leq \frac{1}{\beta^l} \sup_{I_1} |\phi g - g_j|,$$

for all $l \leq j$. Moreover, $\sup_{I_1} |\phi g - g_j| \leq \sup_{I_1} |\phi g - h_0| + |g_j - h_0| \leq \sup_{I_1} |\phi g - h_0| + 2 \sup_{I_j} |\phi g - h_0| < \frac{3\epsilon}{100p(L)}$, which yields

$$\sup_{I_l} |\phi g - g_j| \leq \frac{1}{\beta^l} \frac{3\epsilon}{100p(L)},$$

By Lemma 4.13, we may assume that $\sup_{I_l} |\phi g - g_j|_{k+2,\alpha} \leq \frac{\epsilon}{\beta^l}$. Whenever, we increase L (the necessity for L being increased will depend only on the uniform estimates), we can choose an appropriate ϵ_0 as in Theorem 4.9 and take any $\epsilon < \epsilon_0$. Each time, we do that we might have to increase i_0 (depending on $\epsilon < \epsilon_0$). Therefore, on $M \times I_l$, for $l \leq j$, we have

$$\begin{aligned} \left| \frac{d}{dt} \phi g \right|_{k,\alpha} &= \left| \frac{d}{dt} (\phi g - g_j) \right|_{k,\alpha} \\ &= (-2\text{Ric}(\phi g) + 2\text{Ric}(g_j)) + \frac{1}{\tau} (\phi g - g_j) + (P_{h_0}(\phi g) - P_{h_0}(g_j)) \\ &\quad + \mathcal{L}_{\psi_* \frac{d}{dt} \psi^{-1}} (g_j - \phi g) \\ &\leq C \sup_{I_l} |\phi g - g_j|_{k+2,\alpha} < \frac{C\epsilon}{\beta^l}. \end{aligned}$$

For every $l \leq j$, since $\frac{d}{dt}\phi g = \frac{d}{dt}(\phi g - h_0)$, we have that

$$\begin{aligned} \sup_{I_l} |\phi g - h_0|_{k,\alpha} &\leq 2L \sup_{I_l \cup I_{l-1}} \left| \frac{d}{dt} \phi g \right|_{k,\alpha} + \sup_{I_{l-1}} |\phi g - h_0|_{k,\alpha} \\ &\leq 2LC \frac{\epsilon}{\beta^{l-1}} + 2LC \frac{\epsilon}{\beta^{l-2}} + \cdots + 2LC \frac{\epsilon}{\beta} + \sup_{I_2} |\phi g - h_0|_{k,\alpha} \\ &\leq \sup_{I_2} |\phi g - h_0|_{k,\alpha} + \frac{2LC\epsilon}{\beta - 1}, \end{aligned}$$

which can be made smaller than $\frac{\epsilon}{e^{L\delta/8}}$ for L chosen big enough at the beginning. By Condition 2, for big values of i , we can extend ϕ on I_{j+1} so that $\sup_{I_{j+1}} |\phi g - h_0| < \frac{\epsilon}{100p(L)}$ and it has to coincide with our previously constructed ϕ on I_{j+1} . We can continue a described procedure by looking now at intervals I_j and I_{j+1} replaced by intervals I_{j+1} and I_{j+2} respectively. If we repeat this sufficiently many times, we will reach the interval I_{N-1} with

$$\sup_{I_{N-1}} |\phi g - h_0|_{k,\alpha} < \frac{\epsilon}{100p(L)}.$$

By Condition 3, we will now be able to extend ϕ (for sufficiently big values of i) to interval $[t_i + (N - 1), t_i + (N + 1)L]$, with $\sup_{[t_i + (N-1), t_i + (N+1)L]} |\phi g - h_0| < \epsilon$ holding. Since $(N + 1)L > L'$, this estimate contradicts a maximality of L' with properties. Therefore, either there exists i such that a corresponding $L' = \infty$, or we have a following case holding.

Case 4.19. *There are some L, i and j for which $\sup_{I_j} |\phi g - h_0|_{k,\alpha} < \frac{\epsilon}{100p(L)}$, and $\phi g - h_0$ satisfies a growth condition on I_j (I_j is defined with respect to t_i).*

By using Proposition 4.17 inductively, we would have that

$$\begin{aligned}
\sup_{I_{N-1}} |\phi g - g_j| &< \frac{1}{\beta} \sup_{I_N} |\phi g - g_j| \\
&\leq \frac{1}{\beta} (\sup_{I_N} |\phi g - h_0| + |g_j - h_0|) \\
&\leq \frac{1}{\beta} (\sup_{I_N} |\phi g - h_0| + 2 \sup_{I_j} |\phi g - h_0|) \\
&< \frac{3\epsilon}{\beta}.
\end{aligned}$$

Moreover, if we use Lemma 4.13, together with the estimate

$$\begin{aligned}
\sup_{I_{N-1}} |\phi g - h_0| &\leq \sup_{I_{N-1}} |\phi g - g_j| + |g_j - h_0| \\
&\leq \sup_{I_{N-1}} |\phi g - g_j| + 2 \sup_{I_j} |\phi g - h_0| \\
&< \frac{3\epsilon}{\beta} + \frac{\epsilon}{100p(L)},
\end{aligned}$$

which can be made smaller than $\frac{\epsilon}{p(L)}$, by Condition 3 we can extend ϕ to an interval $[t_i + (N-1)L, t_i + (N+1)L]$ (if i is big enough), with $|\phi g - h_0|_{k,\alpha} < \epsilon$ holding. We again get a contradiction as in the previous case if we assume $L' < \infty$ for all i .

Therefore, there exists i_0 such that a gauge ϕ can be constructed on $M \times [t_{i_0}, t_{i_0} + L']$, satisfying properties and such that a corresponding $L' = \infty$. Consider again $I_j = [t_{i_0} + jL, t_{i_0} + jL + L]$ and the corresponding g_j that are found by Lemma 4.12, such that for $k_j = \phi g - g_j$, we have that $(\pi k_j)_0 = 0$ on $M \times I_j$. Notice that a decay condition ((4.38) \Rightarrow (4.39)) holds for all j . If there existed some j for which it were not true, by using Proposition 4.17 inductively and standard parabolic estimates (Lemma 4.13), we would find that

$$\epsilon > \sup_{[t_{i_0} + (N-1)L, t_{i_0} + NL]} |\phi g - g_j| \geq \beta^{N-j} \sup_{I_j} |\phi g - g_j|,$$

for all N and we would get a contradiction by letting N tend to infinity (if $\sup_{I_j} |\phi g - g_j| = 0$, our metric $g(t_{i_0} + jL)$ would be a soliton satisfying

(4.22) and it would stay so for all later times which is not an interesting case). This means, we have a decay for all times if we do not start with a soliton.

After passing to a subsequence, we may assume that for some metric g_∞ that satisfies a soliton type equation $\lim_{j \rightarrow \infty} |g_{j_p} - g_\infty|_{k, \alpha'} = 0$, where $\alpha' < \alpha$.

Claim 4.20. $\lim_{p \rightarrow \infty} \sup_{I_{j_p}} |k_{j_p}|_{k, \alpha} = 0$.

Proof. If it were not the case, there would exist a subsequence of j_p (denote it by the same symbol) such that $\phi g - g_{j_p}$ would satisfy a growth condition, that is

$$\sup_{[t_{i_0} + (N-1)L, t_{i_0} + NL]} |\phi g - g_{j_p}| \geq \beta^{(N-j_p)} \sup_{I_{j_p}} |\phi g - g_{j_p}|,$$

for all N , where β can be taken to be $e^{\frac{L\delta}{4}}$ and by taking $N \rightarrow \infty$ we immediately get a contradiction, since $\sup_{[t_{i_0} + (N-1)L, t_{i_0} + NL]} |\phi g - g_{j_p}| < C\epsilon$. \square

As in the proof of the claim above, we get that $\phi g - g_{j_p}$ has to satisfy a decay condition for all p . By Claim 4.20, by using Proposition 4.17 inductively and by standard parabolic estimates (Lemma 4.13), we find that for some $c > 0$,

$$|\phi g - g_\infty|_{k, \alpha} \leq ce^{-\frac{\delta(t-t_{i_0})}{4}},$$

for $t \in [t_{i_0} + (N-1)L, t_{i_0} + NL]$ and for all $N > 0$, that is

$$|\phi g - g_\infty|_{k, \alpha} \leq Ce^{-ct}, \tag{4.40}$$

for all $t \geq t_{i_0}$. (4.40) implies that $|g(t) - \phi^{-1}g_\infty|_{C^0} < Ce^{-ct}$. $\phi^{-1}g_\infty$ is a soliton that moves by diffeomorphisms $\phi(t)^{-1}$ and therefore, is determined by metric $\phi^{-1}(t_{i_0})g_\infty$. Since h_0 is a limit soliton of metrics $g(t_i)$, h_0 and $\phi^{-1}(t_{i_0})g_\infty$ differ only by a diffeomorphism, that is $\eta\phi^{-1}(t_{i_0})g_\infty = h_0$ for some diffeomorphism η . Let finally $\phi' = \eta\phi^{-1}(t_{i_0})\phi(t)$. Then,

$$|\phi'g(t) - h_0|_{k, \alpha} < Ce^{-ct},$$

that is $\phi'g(t)$ converges to a soliton h_0 exponentially as $t \rightarrow \infty$. We know that $h(t) = \psi(t)h_0$ and therefore,

$$|\psi\phi'g(t) - h(t)|_{C^0} \leq Ce^{-ct}.$$

This finishes the proof of Theorem 1.1. □

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References.

- [1] B. Kleiner, J. Lott, *Notes on Perelman's paper* (available at <http://www.math.lsa.umich.edu/research/ricciflow/perelman.html>).
- [2] A. Besse, *Einstein manifolds*; Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge Band 10; Springer-Verlag; ISBN 3-540-15279-2.
- [3] J. Cheeger, G. Tian, *On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay*, Inv. Math. **118**, (1994), 493–571.
- [4] X. X. Chen, G. Tian, *Ricci flow on Kähler-Einstein surfaces*, Inv. Math. **147** (2002), 487–544.
- [5] J. Eells, J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Ann. of Math. **86** (2) (1964) 109–160.
- [6] R. Hamilton: *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982) 225–306.
- [7] R. Hamilton, *Four-manifolds with positive curvature operator*, J. Diff. Geom. **24** (1986), 153–179.
- [8] R. Hamilton, *A compactness property for solutions of the Ricci flow*, Amer. J. Math. **117** (1995), 545–572.

- [9] R. Hamilton, The formation of singularities in the Ricci flow, *Surveys in Differential Geometry*, vol. 2, International Press, Cambridge, MA (1995) 7–136.
- [10] R. Hamilton, *Non-singular solutions of the Ricci flow on 3 manifolds*, *Communications in Analysis and Geometry*, **7** (1999), 695–729.
- [11] G. Perelman, The entropy formula for the Ricci flow and its geometric applications; arXiv:math.DG/0211159
- [12] Rothaus, *Logarithmic Sobolev inequality and spectrum*, *J. Dif. Anal.* **42** (1981), 109–120.
- [13] W. X. Shi, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, *J. Diff. Geom.* **30** (1989), 303–394.
- [14] N. Sesum, G. Tian, X. Wang, *Notes on Perelman's paper*.
- [15] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, *Ann. Math.* **118** (1983), 525–571.
- [16] M. Struwe, *On the evolution of harmonic maps in higher dimensions*, *J. Diff. Geom.* **28** (1988), 485–502.
- [17] P. Topping, *Diameter control under Ricci flow*, <http://www.maths.warwick.ac.uk/topping/diamrf.pdf>.
- [18] Rugang Ye, *Ricci flow, Einstein metrics and space forms*, *Transactions of the American Mathematical Society*, **338** (2) (1993) 871–895.