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Relative Differential Characters

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There are two natural candidates for the group of relative Cheeger–Simons differential characters. The first directly extends the work of Cheeger and Simons and the second extends the description given by Hopkins and Singer of the Cheeger–Simons group as the homology of a certain cochain complex. We discuss both approaches and relate the two relative groups.

1. Introduction.

Given a principal G-bundle with connection on a smooth manifold M, an invariant polynomial gives rise to a Chern–Weil form in the base. Lifting this to the total space gives an exact form and the well known form of Chern and Simons [4] has the property that its differential is this exact form. If one happens to have a global section, one can pull this back to the base space, but in general Chern–Simons forms live in the total space. Cheeger and Simons [3] introduced the notion of a differential character as a way of understanding Chern–Simons forms in terms of the base space. These differential characters can be assembled into a group $\widehat{H}^k(M)$ which contains both homotopical and geometrical information. One may also think of differential characters as encoding how integral cycles and differential forms with integral periods interact when viewed in real cohomology. When $k = 2$, the Cheeger–Simons group classifies $U(1)$-bundles with connection and when $k = 3$, it classifies $U(1)$-gerbes with connection (see for example [2]). The latter makes Cheeger–Simons groups relevant to "stringy" topology since a gerbe with connection can be interpreted as a line bundle over a free loop space with parallel transport over surfaces. In order to incorporate open strings into such a framework it is necessary to develop a relative theory.

In this paper, we study the notion of *relative* differential character associated to a pair of manifolds A and M together with a smooth map $\rho\colon A \to M$. There are two competing definitions for the relative group: the first a natural extension of Cheeger and Simons' work giving what we will call the relative Cheeger–Simons group, the second following the more recent work of Hopkins and Singer defining the Cheeger–Simons group as the cohomology of a certain cochain complex, giving what we will call the relative Hopkins–Singer group. The purpose of this paper is to clarify the relationship between these two groups.

For compact smooth manifolds A and M and a smooth map $\rho : A \to M$, we give in Section 2 a natural definition of the group of relative differential characters $\widehat{H}^k(\rho)$, which closely mirrors the original definition of Cheeger and Simons. We show that this group can be described as the homology of a cochain complex analogous to the description of (non-relative) Cheeger–Simons characters by Hopkins and Singer [6]. The relative group fits into certain short exact sequences as in the non-relative case. We end Section 2 with some examples. There is, on the other hand, another natural candidate for a relative group if one adopts the Hopkins–Singer point of view from the outset. In Section 3, we define the relative Hopkins–Singer group $\check{H}^k(\rho)$. Unlike the relative Cheeger–Simons group this group fits into a long exact sequence. Finally, in Section 4, we relate the relative Cheeger–Simons group to the relative Hopkins–Singer group showing that the latter is a quotient of a certain subgroup of the former.

2. Relative Cheeger–Simons groups.

We begin by briefly reviewing the definition of differential characters given in [3] and provide a variant of the Hopkins–Singer description as the homology of a certain cochain complex [6]. We then introduce the group of relative differential characters, provide a description as the homology of a complex and present three short exact sequences. Finally, in this section we give some examples of relative differential characters. We now fix for the rest of the paper a proper subring Λ of $\mathbb{R}$.

Cheeger–Simons differential characters.

Let M be a compact closed smooth manifold. We denote by C_kM and Z_kM the groups of smooth k-chains and k-cycles in M. We let Ω^kM denote the group of k-forms on M which can be viewed as a subgroup of $C^k(M;\mathbb{R})$ via integration. We let $\Omega^k_\Lambda M$ denote the subgroup of k-forms with Λ-periods, noting also that such forms are closed. One important property that we shall use many times is that a non-zero differential form can never take values only in a proper subring. It follows that we can view Ω^kM as a subgroup of $C^k(M;\mathbb{R}/\Lambda)$ and we will make this identification implicitly throughout.

The group of Cheeger–Simons differential characters of degree $k \geq 1$ is defined as

$$\widehat{H}^k(M) = \{f \in \operatorname{Hom}(Z_{k-1}M, \mathbb{R}/\Lambda) \text{ such that } f \circ \partial \in \Omega^k M\}.$$

Cheeger and Simons index this group by $k-1$, but it is convenient to shift up the dimension by 1 to be consistent with Hopkins and Singer. The example one should have in mind is when $k=2$, where a line bundle with connection gives a differential character via its holonomy.

These groups sit in three exact sequences as follows.

$$\begin{aligned}
&0 \to H^{k-1}(M;\mathbb{R}/\Lambda) \to \widehat{H}^k(M) \to \Omega^k_\Lambda M \to 0\\
&0 \to \Omega^{k-1}M/\Omega^{k-1}_\Lambda M \to \widehat{H}^k(M) \to H^k(M;\Lambda) \to 0\\
&0 \to H^{k-1}(M;\mathbb{R})/r(H^{k-1}(M;\Lambda)) \to \widehat{H}^k(M) \to R^k(M;\Lambda) \to 0,
\end{aligned}$$

where r is the natural map $H^{k-1}(M;\Lambda) \to H^{k-1}(M;\mathbb{R})$ and

$$R^k(M;\Lambda) = \{(\omega, u) \in \Omega^k_\Lambda M \times H^k(M;\Lambda) \mid [\omega] = r(u)\}.$$

The closed differential form associated to the differential character is to be thought of as a curvature form and the class in $H^k(M;\Lambda)$ as a characteristic class analogous to the first Chern-Class for $U(1)$-bundles.

We now turn to the description of Cheeger–Simons groups as the homology of a cochain complex following Hopkins and Singer. Consider the cochain complex $\check{C}^*(M)$ defined by

$$\check{C}^k(M) = C^k(M,\Lambda) \times C^{k-1}(M;\mathbb{R}) \times \Omega^k_\Lambda M$$

with differential $\Delta\colon \check{C}^k(M) \to \check{C}^{k+1}(M)$ given by

$$\Delta(c, h, \omega) = (\delta c, \omega - c - \delta h, 0).$$

This differs from the Hopkins–Singer approach in that we include only forms with Λ-periods in the complex above. This allows us to obtain all Cheeger–Simons groups as the homology of a single chain complex. Indeed, denoting the subgroup of cocycles by $\check{Z}^k(M)$, we can define a map $\varphi\colon \check{Z}^k(M) \to \widehat{H}^k(M)$ by $\varphi(c,h,\omega) = \widetilde{h}|_{Z_{k-1}M}$ (here the tilde means mod Λ reduction) and Hopkins and Singer show that ϕ induces an isomorphism

$$H_k(\check{C}^*(M)) \simeq \widehat{H}^k(M).$$

Relative Cheeger–Simons groups.

Let A and M be compact smooth manifolds, possibly with boundary, and $\rho : A \to M$ a smooth map. In order to define a relative version of Cheeger–Simons, we find it convenient to consider the following chain complex:

$$C_k(\rho) = C_k M \times C_{k-1} A$$

with differential $\partial : C_k(\rho) \to C_{k-1}(\rho)$ given by

$$\partial(\sigma, \tau) = (\partial\sigma + \rho_*\tau, -\partial\tau).$$

We will denote the cycles and boundaries of this complex by $Z_k(\rho)$ and $B_k(\rho)$ respectively. For a general map ρ the homology of this complex is the homology $H_k(C_\rho)$ of the mapping cone C_ρ, while in the special case of submanifold $A \subset M$ with ρ the inclusion map we recover the usual relative homology groups $H_k(M, A)$.

The associated dual complex $C^*(\rho; G) = \mathrm{Hom}(\mathrm{C}_*(\rho), \mathrm{G})$ with coefficient group G has differential $\delta(h, e) = (\delta h, \rho^* h - \delta e)$. In general, taking homology gives $H^k(C_\rho; G)$ and when ρ is an inclusion this is isomorphic to $H^k(M, A; G)$. Notice the symbols ∂ and δ are used both to denote the relative differentials and as the usual operators in the non-relative complexes.

We will also use the complex of pairs of differential forms

$$\Omega^*(\rho) = \Omega^* M \times \Omega^{*-1} A$$

with differential $\delta(\omega, \theta) = (d\omega, \rho^*\omega - d\theta)$. The homology of this complex gives the relative de Rham cohomology (see for instance [1]). As above in the non-relative version, we can view $\Omega^k(\rho)$ as a subgroup of $C^k(\rho; \mathbb{R}/\Lambda)$ via integration: for $(\omega, \theta) \in \Omega^k(\rho)$ set

$$(\omega, \theta)(\sigma, \tau) = \int_\sigma \omega + \int_\tau \theta$$

where $(\sigma, \tau) \in C_k(\rho)$. For ρ the inclusion map, this induces the relative de Rham isomorphism.

There is now an obvious definition to be made for the notion of *relative* differential character.

Definition 2.1. Let $\rho : A \to M$ be a smooth map. The group of *relative Cheeger–Simons differential characters* of degree k associated to the triple (A, ρ, M) is defined as

$$\widehat{H}^k(\rho) = \{f \in \mathrm{Hom}(Z_{k-1}(\rho), \mathbb{R}/\Lambda) \text{ such that } f \circ \partial \in \Omega^{\mathrm{k}}(\rho)\}.$$

This definition has already appeared in [7] in the case ρ is an inclusion and a close variant has also appeared in [5] for the restricted case of manifolds with boundary. It is immediately seen that if we take $A = \emptyset$, we recover the non-relative group $\widehat{H}^k(M)$ and we present two further examples at the end of this section.

Associated to a differential character is a pair of differential forms and a characteristic class in the cohomology of the mapping cone. We shall denote by $\Omega^{k+1}_\Lambda(\rho) < \Omega^{k+1}(\rho)$ the subgroup of pairs (ω, θ) taking Λ values on relative cycles and say that such pairs have *relative* Λ*-periods.* In fact, if ρ is the inclusion, $\Omega^*_\Lambda(\rho)$ consists of precisely those forms representing the image of Λ-classes in relative real cohomology (via the de Rham isomorphism). In particular, a pair with relative Λ-periods satisfies the following properties.

Lemma 2.2. *If* (ω, θ) *has relative* Λ*-periods, then*

1. $\delta(\omega, \theta) = 0$ *in the complex* $\Omega^*(\rho)$ *i.e.* $d\omega = 0$ *and* $d\theta = \rho^*\omega$.
2. ω *has* Λ*-periods.*

Proof. For (1) let $\alpha \in C_k A$. Then

$$\int_\alpha (\rho^*\omega - d\theta) = \int_{\rho_*\alpha} \omega - \int_{\partial\alpha} \theta = (\omega, \theta)(\rho_*\alpha, -\partial\alpha).$$

Since (ω, θ) has relative Λ-periods and $(\rho_*\alpha, -\partial\alpha)$ is a relative cycle, the right-hand side is in Λ. Recalling that a non-zero form never takes values only in a proper subring, we conclude that $d\theta - \rho^*\omega = 0$. Showing $d\omega = 0$ is similar.

For (2), let $\sigma \in Z_k M$, then

$$\int_\sigma \omega = \int_\sigma \omega - \int_0 \theta = (\omega, \theta)(\sigma, 0)$$

and the right-hand side in an element of Λ since (ω, θ) has relative Λ-periods. □

We now define two homomorphisms

$$\delta_1 : \widehat{H}^k(\rho) \to \Omega^k_\Lambda(\rho),$$
$$\delta_2 : \widehat{H}^k(\rho) \to H^k(C_\rho; \Lambda).$$

Let $f \in \widehat{H}^k(\rho)$. Since $\mathbb{R}/\Lambda$ is divisible and $Z_{k-1}(\rho)$ free, f can be extended to a map $C_{k-1}(\rho) = C_{k-1}M \times C_{k-2}A \to \mathbb{R}$, which we denote by the pair (h, e).

From the definition of differential character, there exists a pair $(\omega,\theta)\in\Omega^k(\rho)$ and pair $(c,b)\in C^k(\rho;\Lambda)$ such that

$$\delta(h,e)=(\omega,\theta)-(c,b).$$

The above homomorphisms are defined by

$$\delta_1(f)=(\omega,\theta)\qquad\qquad \delta_2(f)=[c,b].$$

To see that δ_1 is well defined, observe that for a relative cycle $(\sigma,\tau)\in Z_k(\rho)$, we have,

$$(\omega,\theta)(\sigma,\tau)=(\delta(h,e)+(c,b))(\sigma,\tau)=(c,b)(\sigma,\tau)\in\Lambda$$

and hence, (ω,θ) has relative Λ-periods. For δ_2 observe that

$$\delta(c,b)=\delta((\omega,\theta)-\delta(h,e))=\delta(\omega,\theta)-0=0$$

since (ω,θ) is closed by Lemma 2.2. Moreover, δ_1 and δ_2 are independent of the choice of the extension (h,e) of f.

It would now be possible to determine the kernels of δ_1 and δ_2 directly and fit them into short exact sequences. We prefer to first describe the relative Cheeger–Simons groups as the homology of a cochain complex along the lines of the non-relative case described above. We can then easily deduce the short exact sequences using this description.

Consider the complex

$$\widehat{C}^*(\rho)=C^*(\rho;\Lambda)\times C^{*-1}(\rho;\mathbb{R})\times\Omega^*_\Lambda(\rho)$$

with differential

$$\Delta((c,b),(h,e),(\omega,\theta))=(\delta(c,b),(\omega,\theta)-(c,b)-\delta(h,e),0).$$

The last entry is in fact $\delta(\omega,\theta)$, but this is zero since $(\omega,\theta)\in\Omega^*_\Lambda(\rho)$ and hence is closed. Let $\widehat{Z}^k(\rho)$ denote the group of k-cocycles in $\widehat{C}^*(\rho)$ and define a map

$$\varphi\colon\widehat{Z}^k(\rho)\to\widehat{H}^k(\rho)$$

by

$$\varphi((c,b),(h,e),(\omega,\theta))=\widetilde{(h,e)}|_{Z_{k-1}(\rho)}$$

where, as above, the tilde denotes mod Λ reduction. Since we start with a cocycle, we have $\delta(h,e)=(\omega,\theta)-(c,b)$ and so $\widetilde{(h,e)}\circ\partial=\widetilde{(\omega,\theta)}$ which shows $\widetilde{(h,e)}|_{Z_{k-1}(\rho)}$ is indeed a differential character.

Theorem 2.3. *The homomorphism φ defined above induces an isomorphism*

$$H_k(\widehat{C}^*(\rho)) \cong \widehat{H}^k(\rho).$$

Proof. To show φ is surjective, let $f : Z_{k-1}(\rho) \to \mathbb{R}/\Lambda$ be a relative differential character. From the discussion of the maps δ_1 and δ_2 above, we can find pairs $(h, e) \in C^{k-1}(\rho; \mathbb{R})$, $(c, b) \in C^k(\rho; \Lambda)$ and $(\omega, \theta) \in \Omega^k_\Lambda(\rho)$ such that $\delta(c, b) = 0$ and $\delta(h, e) = (\omega, \theta) - (c, b)$. In other words, $((c, b), (h, e), (\omega, \theta))$ is a cocycle which maps to f under φ.

The proof will be completed by showing that $\mathrm{Ker}\varphi = \mathrm{Im}\Delta$.

Firstly, an element in the image of Δ has the form $(\delta(c', b'), (\omega', \theta') - (c', b') - \delta(h', e'), 0)$ where $((c', b'), (h', e'), (\omega', \theta')) \in \widehat{C}^{k-1}(\rho)$. Since, (ω', θ') has relative Λ-periods and (c', b') is Λ-valued, this is clearly mapped to 0 under φ. Hence, $\mathrm{Im}\Delta \subset \mathrm{Ker}\varphi$.

Now, suppose $((c, b), (h, e), (\omega, \theta)) \in \mathrm{Ker}\varphi$. Then, $\widetilde{(h, e)}|_{Z_{k-1}(M,A)} = 0$, so $\delta\widetilde{(h, e)} = 0$. Since, $(\omega, \theta) = (c, b) + \delta(h, e)$, we have

$$\widetilde{(\omega, \theta)} = \widetilde{(c, b)} + \delta\widetilde{(h, e)} = 0$$

as so $(\omega, \theta) = 0$. Using the splitting of the standard short exact sequence relating cycles, chains and boundaries, we can extend the Λ-valued map $-(h, e)|_{Z_{k-1}(\rho)} \colon Z_{k-1}(\rho) \to \Lambda$ to a map $C_{k-1}(\rho) \to \Lambda$. Denoting this extension by (c', b'), we have

$$\delta(c', b') = -\delta(h, e) = (c, b) - (\omega, \theta) = (c, b).$$

Finally, since $-(c', b') - (h, e)$ is zero on cycles, it is a real coboundary, namely $-(c', b') - (h, e) = \delta(h', e')$ for some $(h', e') \in C^{k-2}(\rho; \mathbb{R})$. Hence ,

$$\Delta((c', b'), (h', e'), 0) = (\delta(c', b'), -(c', b') - \delta(h', e'), 0) = ((c, b), (h, e), (\omega, \theta)),$$

and so $((c, b), (h, e), (\omega, \theta)) \in \mathrm{Im}\Delta$. Thus, $\mathrm{Ker}\varphi \subset \mathrm{Im}\Delta$ which finishes the proof. □

Using this description is now easy to fit the relative Cheeger–Simons groups into three short exact sequences. Let $r : H^k(C_\rho; \Lambda) \to H^k(C_\rho; \mathbb{R})$ be the natural map induced by the inclusion $\Lambda \to \mathbb{R}$ and set

$$R^k(\rho, \Lambda) = \{((\omega, \theta), u) \in \Omega^k_\Lambda(\rho) \times H^k(C_\rho; \Lambda) \mid [\omega, \theta] = r(u)\}$$

where $[\omega, \theta]$ denotes the de Rham cohomology class.

Theorem 2.4. *The following sequences are exact.*

$$0 \longrightarrow H^{k-1}(C_\rho;\mathbb{R}/\Lambda) \longrightarrow \widehat{H}^k(\rho) \xrightarrow{\delta_1} \Omega^k_\Lambda(\rho) \longrightarrow 0$$

$$0 \longrightarrow \Omega^{k-1}(\rho)/\Omega^{k-1}_\Lambda(\rho) \longrightarrow \widehat{H}^k(\rho) \xrightarrow{\delta_2} H^k(C_\rho;\Lambda) \longrightarrow 0$$

$$0 \longrightarrow H^{k-1}(C_\rho;\mathbb{R})/r(H^{k-1}(C_\rho;\Lambda)) \longrightarrow \widehat{H}^k(\rho) \xrightarrow{(\delta_1,\delta_2)} R^k(\rho;\Lambda) \longrightarrow 0$$

Proof. Consider the following short exact sequences of chain complexes.

$$\begin{aligned}
&0 \to C^*(\rho;\Lambda)\times C^{*-1}(\rho;\mathbb{R}) \to \widehat{C}^*(\rho) \to \Omega^*_\Lambda(\rho) \to 0\\
&0 \to C^{*-1}(\rho;\mathbb{R})\times \Omega^*_\Lambda(\rho) \to \widehat{C}^*(\rho) \to C^*(\rho;\Lambda) \to 0\\
&0 \to C^{*-1}(\rho;\mathbb{R}) \to \widehat{C}^*(\rho) \to C^*(\rho;\Lambda)\times\Omega^*_\Lambda(\rho) \to 0
\end{aligned}$$

Each of the differentials in the complexes on the left is obtained by restriction of the differential of $\widehat{C}^*(\rho)$. Then as in [6], the three exact sequences of the theorem can be deduced from the long exact sequences obtained from the short exact sequences above. □

Notice that taking $A = \emptyset$ recovers the usual non-relative short exact sequences. Unfortunately, there is no long exact sequence for the relative group of the triple (A, ρ, M). This is easily seen just at the level of differential forms: a differential character with associated pair of forms (ω, θ) induces a differential character on M with form ω, however unless θ is closed, there is no guarantee that $\rho^*\omega = 0$ which would be required by any long exact sequence.

Examples. We end this section by presenting a couple of examples. In both, ρ is the inclusion map and we adopt the standard relative notation $Z_k(M, A)$ and $\widehat{H}^k(M, A)$ for $Z_k(\rho)$ and $\widehat{H}^k(\rho)$.

Example 2.5. Let M be a smooth manifold with submanifold A. Let $P \to M$ be a principal S^1-bundle with connection together with a given trivialisation of the bundle over A. The relative holonomy $H : Z_1(M, A) \to S^1$ of P is defined as follows. Given $\gamma \in Z_1(M, A)$, split γ into a (finite) collection of piecewise smooth paths γ_i with endpoints in A. For each γ_i, define $H(\gamma_i)$ to be the holonomy along γ_i where the endpoint fibres are identified via the given trivialisation over A. Then, $H(\gamma)$ is defined to be the product (in the circle group) of the $H(\gamma_i)$.

We now define the relative differential character $\widehat{\chi} : Z_1(M, A) \to \mathbb{R}/\mathbb{Z}$ associated to P by the equation

$$e^{2\pi i \widehat{\chi}(\gamma)} = H(\gamma).$$

To see that $\widehat{\chi}$ is indeed a differential character consider the curvature 2-form ω of P. Since P is trivial over A and hence ω exact on A, there exists a 1-form θ such that on A, we have $\omega = d\theta$. This form satisfies $H(\tau) = e^{2\pi i \int_\tau \theta}$ for $\tau \in C_1 A$, which in turn immediately implies that $H(\partial(\sigma, \tau)) = e^{2\pi i (\int_\sigma \omega + \int_\tau \theta)}$ for a relative boundary $\partial(\sigma, \tau)$. Using the defining equation for $\widehat{\chi}$, we see that

$$\widehat{\chi}(\partial(\sigma, \tau)) = \int_\sigma \omega + \int_\tau \theta = (\omega, \theta)(\sigma, \tau) \mod \mathbb{Z}$$

and hence, $\widehat{\chi} \in \widehat{H}^2(M, A)$.

Example 2.6. In this example, we consider $A = S^{n-1}$ as the boundary of $M = D^n$. Using the exact sequences in Theorem 2.4 and some elementary topology, one obtains the following.

$$\widehat{H}^k(D^n, S^{n-1}) = \begin{cases} C^\infty(D^n, \mathbb{R}/\Lambda) & k = 1 \\ \Omega^k_\Lambda(D^n, S^{n-1}) & 2 \le k \le n \\ \mathbb{R}/\Lambda & k = n+1 \\ 0 & k \ge n+2 \end{cases}$$

3. Relative Hopkins–Singer groups.

Following the description of Hopkins and Singer, there is another candidate for the relative group. This group, which we refer to as the relative Hopkins–Singer group can be identified with a quotient of a subgroup of the relative Cheeger–Simons group and does fit into a long exact sequence.

Recall from the previous section the chain complex

$$\check{C}^k(M) = C^k(M, \Lambda) \times C^{k-1}(M; \mathbb{R}) \times \Omega^k_\Lambda M$$

with differential $\delta(c, h, \omega) = (\delta c, \omega - c - \delta h, 0)$ which, as we have seen, has homology $\widehat{H}^*(M)$. Following the approach used to define relative real and de Rham cohomology, consider the chain complex

$$\check{C}^*(M) \times \check{C}^{*-1}(A)$$

with differential $\delta(S,T) = (\Delta S, \rho^* S - \Delta T)$, where S, T are triples in $\check{C}^*(M)$ and $\check{C}^{*-1}(A)$ respectively.

Definition 3.1. The *relative Hopkins–Singer groups* are defined by

$$\check{H}^k(\rho) = H_k(\check{C}^*(M) \times \check{C}^{*-1}(A)).$$

These groups can be fitted into short exact sequences analogous to those above for the relative Cheeger–Simons groups. However, the groups of forms appearing are not particularly enlightening and it is better to understand this group in terms of its relation to the relative Cheeger–Simons group which is the subject of the next section.

A main feature of the relative Hopkins–Singer groups is that they do fit into a long exact sequence. Let q be the projection $\check{C}^*(M) \times \check{C}^{*-1}(A) \to \check{C}^*(M)$ and let l be the inclusion $\check{C}^{*-1}(A) \to \check{C}^*(M) \times \check{C}^{*-1}(A)$. Using the same notation for induced maps, we have the following theorem.

Theorem 3.2. *The following sequence is exact.*

$$\cdots \longrightarrow \widehat{H}^{k-1}(A) \xrightarrow{l} \check{H}^k(\rho) \xrightarrow{q} \widehat{H}^k(M) \xrightarrow{\rho} \widehat{H}^k(A) \longrightarrow \cdots$$

where ρ is the natural pullback map from differential characters on M to those on A.

Proof. This is the long exact sequence induced by the following exact sequence of chain complexes

$$0 \to \check{C}^{*-1}(A) \to \check{C}^*(M) \times \check{C}^{*-1}(A) \to \check{C}^*(M) \to 0$$

with the boundary map of this sequence coinciding with the map ρ above.□

4. Relating the Relative Hopkins–Singer Groups and the Relative Cheeger–Simons Groups.

We now relate the relative Hopkins–Singer group to the relative Cheeger–Simons group by identifying $\check{H}^k(\rho)$ as a quotient of a subgroup of $\widehat{H}^k(\rho)$.

In order to do this, it will be helpful to view the relative Hopkins–Singer groups as the homology of a slightly different chain complex. Consider the chain complex

$$\check{C}^*(\rho) = C^*(\rho; \Lambda) \times C^{*-1}(\rho; \mathbb{R}) \times \Omega^*_\Lambda M \times \Omega^{*-1}_\Lambda A$$

with differential $\Delta : \check{C}^*(\rho) \to \check{C}^{*+1}(\rho)$ given by

$$\Delta((c,b),(h,e),(\omega,\theta)) = (\delta(c,b),(\omega,\theta)-(c,b)-\delta(h,e),(0,\rho^*\omega))$$

Notice that the last term is in fact $\delta(\omega,\theta) = (d\omega, \rho^*\omega - d\theta) = (0,\rho^*\omega)$.

This chain complex $\check{C}^*(\rho)$ is isomorphic (by changing the sign of e) to $\check{C}^*(M) \times \check{C}^{*-1}(A)$ and so, we have

$$\check{H}^k(\rho) \simeq H_k(\check{C}^*(\rho)).$$

Let $\widehat{H}^k_0(\rho) < \widehat{H}^k(\rho)$ be the subgroup of relative differential characters defined by

$$\widehat{H}^k_0(\rho) = \{f \in \widehat{H}^k(\rho) \text{ such that } f|_{Z_{k-1}A} = 0\}.$$

Here, we are regarding $Z_{k-1}A \subset Z_{k-1}M \subset Z_{k-1}(M,A)$ i.e. regard $\tau \in Z_{k-1}A$ as $(\tau,0) \in Z_{k-1}(M,A)$. Notice that

$$f(\tau,0) = f \circ \partial(0,\tau) = (\omega,\theta)(0,\tau) = \widetilde{\int_\tau \theta},$$

so the condition $f|_{Z_{k-1}A} = 0$ is equivalent to θ having Λ-periods.

These groups can also be identified with the cohomology of a cochain complex. Let

$$\widetilde{\Omega}^k M = \{\omega \in \Omega^k M \mid (\omega,0) \in \Omega^k_\Lambda(\rho)\},$$

noticing that $\widetilde{\Omega}^k M \subset \Omega^k_\Lambda M$. Then $\widehat{H}^k_0(\rho)$ is the homology of the cochain complex

$$\widehat{C}^*_0(\rho) = C^*(\rho;\Lambda) \times C^{*-1}(\rho;\mathbb{R}) \times \widetilde{\Omega}^* M \times \Omega^{*-1}_\Lambda A$$

with differential

$$\Delta((c,b),(h,e),(\omega,\theta)) = (\delta(c,b),(\omega,\theta)-(c,b)-\delta(h,e),0).$$

We can easily relate $\widehat{H}^k_0(\rho)$ to $\widehat{H}^k(\rho)$. Let

$$\Omega^{k-1}_{Im} A = \mathrm{Im}(\Omega^k_\Lambda(\rho) \to \Omega^{k-1}A).$$

and note that given $\theta \in \Omega^{k-1}_\Lambda A$, then $(0,\theta) \in \Omega^k_\Lambda(\rho)$ and so $\Omega^{k-1}_\Lambda A < \Omega^{k-1}_{Im} A$.

Proposition 4.1. *The following sequence is exact.*

$$0 \to \widehat{H}^k_0(\rho) \to \widehat{H}^k(\rho) \to \Omega^{k-1}_{Im} A / \Omega^{k-1}_\Lambda A \to 0$$

where the left map is the obvious inclusion and the right map the projection.

Proof. Consider the short exact sequence of complexes

$$0 \longrightarrow \widehat{C}^*_0(\rho) \xrightarrow{j} \widehat{C}^*(\rho) \xrightarrow{p} \Omega^{*-1}_{Im} A/\Omega^{*-1}_{\Lambda} A \longrightarrow 0$$

where complex on the right has trivial differential and j and p are the inclusion and projection maps. To show this is exact the only point worth noting is that given $\theta \in \Omega^{k-1}_{Im} A$, then by definition, there exists ω such that $(\omega, \theta) \in \Omega^k_\Lambda(\rho)$ and so $((0,0),(0,0),(\omega,\theta))$ is a cocycle mapped to θ under p i.e. p is onto.

The differential of the associated long exact sequence is trivial giving the result. □

To relate $\widehat{H}^k_0(\rho)$ to the relative Hopkins–Singer group we need to define two maps ϕ and J. We define the map

$$\phi : \Omega^{k-1}_\Lambda M/\widetilde{\Omega}^{k-1} M \to \widehat{H}^k_0(\rho)$$

as follows. For $\nu \in \Omega^{k-1}_\Lambda M$, define $\phi(\nu) : Z_{k-1}(\rho) \to \mathbb{R}/\Lambda$ by

$$\phi(\nu)(\gamma, -\partial\gamma) = \int_\gamma \nu \qquad \mathrm{mod}\ \Lambda\ .$$

If $(\sigma, \tau) \in C_k M \times C_{k-1} A$, we have $\phi(\nu) \circ \partial(\sigma,\tau) = (0, \rho^*\nu)(\sigma,\tau)$ and so $\nu(\rho) \circ \partial \in \Omega^k(\rho)$. Thus, $\phi(\nu)$ is a differential character as required and moreover, if $\nu \in \widetilde{\Omega}^{k-1} M$, then $\rho^*\nu = 0$ and so ϕ is well defined.

To define J, recall from Section 2 that given a relative differential character f, we can find pairs $(c,b),(h,e)$ and (ω,θ) such that $((c,b),(h,e),(\omega,\theta))$ is a cocycle in $\widehat{C}^*(\rho)$. If $f \in \widehat{H}^k_0(\rho)$, then in addition, we have θ has Λ-periods and $\rho^*\omega = 0$. Since, ω also has Λ-periods (by Lemma 2.2) it follows that $((c,b),(h,e),(\omega,\theta))$ is in fact a cocycle in $\check{C}^*(\rho)$. Thus, we can define a map

$$J \colon \widehat{H}^k_0(\rho) \to \check{H}^k(\rho)$$

by

$$J(f) = [((c,b),(h,e),(\omega,\theta))].$$

The relationship between the relative Cheeger–Simons groups and the relative Hopkins–Singer groups is given by Proposition 4.1 and the following result.

Theorem 4.2. *The following sequence is exact.*

$$0 \longrightarrow \Omega^{k-1}_\Lambda M/\widetilde{\Omega}^{k-1} M \xrightarrow{\phi} \widehat{H}^k_0(\rho) \xrightarrow{J} \check{H}^k(\rho) \longrightarrow 0$$

Proof. Consider the short exact sequence of complexes

$$0 \longrightarrow \widehat{C}_0^*(\rho) \xrightarrow{j} \check{C}^*(\rho) \xrightarrow{p} \Omega_\Lambda^k M/\widetilde{\Omega}^k M \longrightarrow 0$$

where the complex on the right has trivial differential, p is the projection and j is the inclusion. Note that j is well defined since by Lemma 2.2 $\omega \in \Omega_\Lambda^k M$ and by definition $\theta \in \Omega_\Lambda^{k-1} A$. It is also a map of cochain complexes since for $((c,b),(h,e),(\omega,\theta)) \in \widehat{C}_0^k(\rho)$, we have $\rho^*\omega = d\theta = 0$.

The only non-trivial part in showing that the above sequence is indeed exact is the middle term. If $((c,b),(h,e),(\omega,\theta)) \in \mathrm{Ker}(p)$, then $\omega \in \widetilde{\Omega}^k M$ i.e. $(\omega, 0) \in \Omega_\Lambda^k(\rho)$. Note that since θ has Λ-periods, we have $(0,\theta) \in \Omega_\Lambda^k(\rho)$. Hence, $(\omega,\theta) = (\omega,0) + (0,\theta) \in \Omega_\Lambda^k(\rho)$ showing that $((c,b),(h,e),(\omega,\theta)) \in \mathrm{Im}(j)$ i.e. $\mathrm{Ker}(p) \subset \mathrm{Im}(r)$. Conversely, if $((c,b),(h,e),(\omega,\theta)) \in \mathrm{Im}(j)$, then $\omega \in \widetilde{\Omega}^k M$. Thus, $\mathrm{Im}(r) \subset \mathrm{Ker}(p)$.

Using the same notation for induced maps, consider the long exact homology sequence of the above sequence.

$$\cdots \longrightarrow \Omega_\Lambda^{k-1} M/\widetilde{\Omega}^{k-1} M \xrightarrow{\delta} \widehat{H}_0^k(\rho) \xrightarrow{j} \check{H}^k(\rho) \xrightarrow{p} \Omega_\Lambda^k M/\widetilde{\Omega}^k M \longrightarrow \cdots$$

We now claim Im(p) = 0. If $((c,b),(h,e),(\omega,\theta))$ is a cocycle in $\check{C}^k(\rho)$, then $(\omega,\theta) = (c,b) + \delta(h,e)$ and so (ω,θ) has relative integral periods. Since $(0,\theta)$ also has relative integral periods, we obtain $\omega \in \widetilde{\Omega}^k M$. Hence, $p(((c,b),(h,e),(\omega,\theta))) = 0$.

Finally, $\mathrm{Ker}(\delta) = \mathrm{Im(p)} = 0$ and we obtain the desired sequence.

The induced map j clearly agrees with J and moreover the boundary map $\delta\colon \Omega_\Lambda^{k-1} M/\widetilde{\Omega}^{k-1} M \to \widehat{H}_0^k(\rho)$ is given by $\delta([\eta]) = ((0,0),(\eta,0),(0,\rho^*\eta))$ and hence agrees with ϕ. □

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