Branched Covers of Hyperbolic Manifolds and Harmonic Maps

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1. Introduction and statements of the results.

Let $f\colon M \to N$ be a homotopy equivalence between closed negatively curved manifolds. The fundamental existence results of Eells and Sampson [5] and uniqueness of Hartmann [15] and Al'ber [1] grant the existence of a unique harmonic map h homotopic to f. Based on the enormous success of the harmonic map technique, Lawson and Yau conjectured that the harmonic map h should be a diffeomorphism. This conjecture was proved to be false by Farrell and Jones [6] in every dimension in which exotic spheres exist. They constructed examples of homeomorphisms $f\colon M \to N$ between closed negatively curved manifolds for which f is not homotopic to a diffeomorphism. These counterexamples were later obtained also in dimension six by Ontaneda [21] and later generalized by Farrell, Jones and Ontaneda to all dimensions > 5 [8]. In fact, in [21] and [8] examples are given for which f is not even homotopic to a PL homeomorphism. The fact that f is not homotopic to a PL homeomorphism has several interesting strong consequences that imply certain limitations of well known powerful analytic ethods in geometry [9], [10], [11], [12] (see [13] for a survey).

In all the examples mentioned above, one of the manifolds is always a closed hyperbolic manifold. Hence, both manifolds M and N have the homotopy type of a closed hyperbolic manifold (hence the homotopy type of a closed locally symmetric space). We call these *examples of hyperbolic homotopy type*.

In [2], Ardanza also gave counterexamples to the Lawson–Yau conjecture. In his examples, the manifolds M and N are not homotopy equivalent to a closed locally symmetric space; in particular, they are not homotopy

equivalent to a closed hyperbolic manifold. We call these *examples of non-hyperbolic homotopy type*. In these examples, the map f is not homotopic to a diffeomorphism and exists in dimensions $4n - 1$, $n \geq 2$. Hence Ardanza's result is an analogue of Farrell-Jones result [6] for examples of non-hyperbolic homotopy type. His constructions use branched covers of hyperbolic manifolds. Recall that Gromov and Thurston [14] proved that large branched covers of hyperbolic manifolds do not have the homotopy type of a closed locally symmetric space.

In this paper, we show that most of the results obtained for examples of hyperbolic homotopy type in [21], [8], [9], [10], [11], [12], [13] are also true for examples of non-hyperbolic homotopy type. We now state our main results.

First, we extend Ardanza's result to every dimension > 5 and also with the stronger property that f is not even homotopic to a PL homeomorphism. This is an analogue of the result in [8] for examples of non-hyperbolic homotopy type.

Theorem 1.1. *Given $\epsilon > 0$ and $n > 5$, there are closed n-dimensional Riemannian manifolds M_1, M_2 and a homeomorphism $f \colon M_1 \to M_2$ such that:*

1. *f is not homotopic to a PL homeomorphism. In particular, the unique harmonic map homotopic to f is not a diffeomorphism.*

2. *M_1 and M_2 have ϵ-pinched to -1 sectional curvatures.*

3. *M_1 and M_2 do not have the homotopy type of a closed locally symmetric space. In particular, they are not homotopy equivalent to a closed hyperbolic manifold.*

Remark 1.2. This result is a little weaker than the result in [2] (for examples of non-hyperbolic homotopy type) and than the one in [8] (for examples of hyperbolic homotopy type): in [2], M_1 is not diffeomorphic to M_2 and in [8] M_1 is not PL homeomorphic to M_2. The Theorem above states the existence of a particular map f that is not homotopic to a PL homeomorphism, but we do not know whether M_1 and M_2 are PL homeomorphic. The missing ingredient is "differentiable rigidity" which, for examples of hyperbolic homotopy type, is provided by Mostow's Rigidity Theorem: every homotopy equivalence between hyperbolic manifolds is homotopic to a diffeomorphism (in fact an isometry). We do not know whether every homotopy equivalence between branched covers of hyperbolic manifolds is homotopic to a diffeomorphism.

As for the case of examples of hyperbolic homotopy type, the fact that the map f is not homotopic to a PL homeomorphism has the following interesting consequence:

Corollary 1.3. *Given $\epsilon > 0$ and $n > 5$, there are closed n-dimensional Riemannian manifolds M_1, M_2 and a harmonic homotopy equivalence $h\colon M_1 \to M_2$ such that:*

1. *h is not one-to-one.*

2. *M_1 and M_2 have ϵ-pinched to -1 sectional curvatures.*

3. *M_1 and M_2 do not have the homotopy type of a closed locally symmetric space. In particular, they are not homotopy equivalent to a closed hyperbolic manifold.*

This Corollary can be directly deduced from Theorem 1.1 and the $C^\infty-$ Hauptvermutung of Scharlemann and Siebenmann [24] (the proof of Corollary 1.3 appears after the proof of Theorem 1.1). Also, if Poincare's conjecture for three dimensional manifolds is true, then the map h above is not even a cellular map, hence it cannot be approximated by homeomorphisms. The proof of this fact is similar to the proof for examples of hyperbolic homotopy type. For more details see [9] or [13].

In the examples provided by the Corollary above, the main obstruction to h being a diffeomorphism or a homeomorphism is that h is not homotopic to a PL homeomorphism. We may ask then what happens if this obstruction vanishes, that is, if h is homotopic to a PL homeomorphism or even homotopic to a diffeomorphism. This was considered in Problem 111 of the list compiled by Yau in [28]. Here is a restatement of this problem.

Problem 111 of [28] Let $f\colon M_1 \to M_2$ be a diffeomorphism between two compact manifolds with negative curvature. If $h : M_1 \to M_2$ is the unique harmonic map which is homotopic to f, is h a homeomorphism?, or equivalently, is h one-to-one?

(This problem had been reposed in [27] as Grand Challenge Problem 3.6.) The answer to the problem was proved to be yes when $\dim M_1 = 2$ by Schoen–Yau [25] and Sampson [23]. But it was proved by Farrell, Ontaneda and Raghunathan [10] that the answer to this question is in general negative for dimensions > 5. The counterexamples constructed in [10] are examples of hyperbolic homotopy type. Here, we construct counterexamples of non-hyperbolic homotopy type:

Theorem 1.4. *For every integer $n > 5$ and $\epsilon > 0$, there is a diffeomorphism $f : M_1 \to M_2$ between a pair of closed n-dimensional Riemannian manifolds such that:*

1. *The unique harmonic map homotopic to f is not one-to-one.*

2. *M and N have ϵ-pinched to -1 sectional curvatures.*

3. *M and N do not have the homotopy type of a closed locally symmetric space. In particular, they are not homotopy equivalent to a closed hyperbolic manifold.*

Also, in [11], we constructed examples of harmonic maps $h\colon M \to N$ between ϵ-pinched to -1 closed Riemannian manifolds such that h can be approximated by diffeomorphisms, but h is not a diffeomorphism (in particular h is a cellular map). The examples in [11] are examples of hyperbolic homotopy type. Here we also construct counterexamples of non-hyperbolic homotopy type:

Theorem 1.5. *For every integer $n > 10$ and $\epsilon > 0$, there is a harmonic map $h : M_1 \to M_2$ between a pair of closed n-dimensional Riemannian manifolds such that:*

1. *The harmonic map h is not a diffeomorphism.*

2. *The harmonic map h can be approximated by diffeomorphisms (in the C^∞ topology).*

3. *M_1 and M_2 have ϵ-pinched to -1 sectional curvatures.*

4. *M_1 and M_2 do not have the homotopy type of a closed locally symmetric space. In particular, they are not homotopy equivalent to a closed hyperbolic manifold.*

As for examples of hyperbolic homotopy type, if Poincare's Conjecture for three dimensional manifolds is true, we can say a little more in this case (see [11] or [13]). Also, it seems that it is straightforward to verify (but we do not do this here) that in Theorems 1.1, 1.4 and Corollary 1.3 above (and may be also in Theorem 1.5) we can replace the word "harmonic" by "natural". The concept of natural map was defined by Besson, Courtois and Gallot (e.g. see [3]). These maps have very powerful dynamic and geometric properties and are also used to prove rigidity results. (For more details, see [13]).

In [12], the results of [11] were used to construct examples of ϵ-pinched to -1 closed Riemannian manifolds for which the Ricci flow does not converge smoothly to a negatively curved metric. These examples are examples of hyperbolic homotopy type. Here, we show also that the constructions used to prove Theorem 1.5 (which are analogous to the constructions in [11]) can also be used to produce examples of non-hyperbolic homotopy type of ϵ-pinched to -1 closed Riemannian manifolds for which the Ricci flow does not converge smoothly to a negatively curved metric.

Remark 1.6. We say that the Ricci flow for a negatively curved Riemannian metric h *converges smoothly to a negatively curved metric* if the Ricci flow, starting at h, is defined for all t and converges (in the C^∞ topology) to a well defined negatively curved (Einstein) metric. The next Theorem shows the existence of pinched negatively curved metrics for which the Ricci flow does not converge smoothly to a negatively curved metric.

Theorem 1.7. *For every integer $n > 10$ and $\epsilon > 0$, there is a closed smooth n-dimensional Riemannian manifold N such that:*

1. *N admits a Riemannian metric with sectional curvatures in $[-1 - \epsilon, -1 + \epsilon]$ for which the Ricci flow does not converge smoothly to a negatively curved metric.*

2. *N does not have the homotopy type of a closed locally symmetric space. In particular, N is not homotopy equivalent to a closed hyperbolic manifold.*

The proofs of all Theorems above use the following Proposition:

Proposition 1.8. *For every integer $n > 5$ and $r, s > 0$, there are closed connected orientable Riemannian manifolds M, N, T, P such that:*

1. *M is a n-dimensional hyperbolic manifold, N and P are totally geodesic $(n-1)$-dimensional submanifolds of M and T is a totally geodesic $(n-2)$-dimensional submanifold of M. N intersects T and P transversally.*

2. *The isometry class of N does not depend on r (only on n and s).*

3. *$0 \neq [N \cap T] \in H_{n-3}(M, \mathbb{Z}_2)$, where $[N \cap T]$ is the $\mathbb{Z}_2$-homology class represented by the $(n-3)$-dimensional submanifold $N \cap T$.*

4. *The normal geodesic tubular neighborhood of N has width $> r$ and the normal geodesic tubular neighborhood of $N \cap P$ has width $> s$.*

Remark 1.9. The Proposition remains valid if we replace N in item 2 by T. (We cannot choose *both* N and T to be independent of the widths of their normal geodesic tubular neighborhoods.) Also, we can have $T \subset P$, if we choose so.

In the section 2, we prove Proposition 1.8 and show how it implies the Theorems. The proofs of the Theorems resemble the proofs of the corresponding Theorems for the case of examples of hyperbolic homotopy type presented in [21], [8], [9], [10], [11], [12], [13]. We will refer to these papers.

2. Proofs of the results.

First we prove Proposition 1.8.

Proof. of the Proposition 1.8 We use all notation from [10]. Let $\mathbf{G}$, $\mathbf{G}_1$, $\mathbf{G}_2$ be as in the proof of lemma of [10], p. 233. We have that $\mathbf{G}_i = R_{k/\mathbb{Q}}\mathbf{G}'_i$ where $\mathbf{G}'_i$ denotes the k-algebraic group $SO(f_i)$ and f_i is the restriction of the form f to the subspace of E generated by $\mathcal{B}_i$, see[10] p.245. To be specific, choose $\mathcal{B}_1 = \mathcal{B} \setminus \{e_1\}$ and $\mathcal{B}_2 = \mathcal{B} \setminus \{e_2, e_3\}$.

For ideals $\underline{a}$, $\underline{b}$, $\underline{c}$, let $\Phi(\underline{a}, \underline{b}, \underline{c})$ be the arithmetic subgroup of $\mathbf{G}$ constructed in p. 234 of [10]. We will need the following two properties of Φ. These properties can easily be checked directly from the definition.

a. $\Gamma(\underline{c}) \subset \Phi(\underline{a}, \underline{b}, \underline{c}) \subset \Gamma(\underline{a})$

b. If $\underline{a}' \subset \underline{a}$, $\underline{b}' \subset \underline{b}$, $\underline{c}' \subset \underline{c}$ then $\Phi(\underline{a}', \underline{b}', \underline{c}')$ is a subgroup of finite index of $\Phi(\underline{a}, \underline{b}, \underline{c})$.

The following is shown in [10]:

There is an ideal $\underline{a}_0$ of $\mathbb{Z}$ such that for every ideal $\underline{a} \subset \underline{a}_0$, there is an ideal $\underline{b}(\underline{a}) \subset \underline{a}$ with the following properties. For any ideal $\underline{c} \subset \underline{b}(\underline{a})$, define $\Phi = \Phi(\underline{a}, \underline{b}(\underline{a}), \underline{c})$, and Φ_i, M, N, T, as in p. 234 of [10]. Then M, N and T are closed connected orientable manifolds that satisfy:

1. M is a n-dimensional hyperbolic manifold, N is a totally geodesic $(n-1)$-dimensional submanifold of M and T is a totally geodesic $(n-2)$-dimensional submanifold of M. N and T intersect transversally.

2. The isometry class of N does not depend on $\underline{c}$.

3. $0 \neq [N \cap T] \in H_{n-3}(M, \mathbb{Z}_2)$, where $[N \cap T]$ is the $\mathbb{Z}_2$-homology class represented by the $(n-3)$-dimensional submanifold $N \cap T$.

Moreover, given r we can choose $\underline{c}$ such that

4. The normal geodesic tubular neighborhood of N has width $> r$.

We have to define P. For this, let $\mathcal{B}_3 = \mathcal{B} \setminus \{e_2\}$ and $\mathbf{G}_3 = R_{k/\mathbb{Q}}\mathbf{G}'_3$ where $\mathbf{G}'_3$ denotes the k-algebraic group $SO(f_3)$ and f_3 is the restriction of the form f to the subspace of E generated by $\mathcal{B}_3$. It can be verified from the results of [10], Section 2, that there is an ideal $\underline{a}_1$ of $\mathbb{Z}$ such that $P = X_3/\Phi_3$ is a connected closed orientable totally geodesic $(n-1)$-dimensional sub-manifold of X/Φ, where Φ is any subgroup of finite index of $\Gamma(\underline{a}_1)$. Here, $\Phi_3 = \Phi \cap \mathbf{G}_3(\mathbb{Q})$ and $X_3 = (K \cap G_3)/G_3$. Note that, since $\mathcal{B}_2 \subset \mathcal{B}_3$, we have that $T \subset P$.

Now take an ideal $\underline{a}' \subset \underline{a}_0 \cap \underline{a}_1$. Define $\Phi' = \Phi(\underline{a}', \underline{b}', \underline{b}')$ where $\underline{b}' = \underline{b}(\underline{a}')$. Define Φ'_1, Φ'_2, Φ'_3, M', N', T', P' accordingly. Then M', N', T', P' satisfy 1,2,3 of the statement of the Proposition. That is, they satisfy all required properties, except the ones about the widths of the normal geodesic tubular neighborhoods. (Note that N' and P' intersect transversally because both are different connected totally geodesic hypersurfaces of M'.)

By using an argument similar to the one in pp. 235–236 of [10], we can find an ideal $\underline{a}'' \subset \underline{b}'$ such that the following holds. Define $\Phi'' = \Phi(\underline{a}'', \underline{b}'', \underline{b}'')$ where $\underline{b}'' = \underline{b}(\underline{a}'')$. Define Φ''_1, Φ''_2, Φ''_3, M'', N'', T'', P'' accordingly. Then we can choose $\underline{a}''$ such that the normal geodesic tubular neighborhood of $P'' \cap N''$ is larger that s. Note that M'', N'', T'', P'' also satisfy 1,2,3 of the statement of the Proposition. (Here we required $\underline{a}'' \subset \underline{b}'$ so that Φ'' is a subgroup of Φ'; it follows that $M'' \to M'$ is a finite cover.)

Finally, choose $\underline{c} \subset \underline{b}''$ as in [10], pp. 235–236, such that the following holds. Define $\Phi = \Phi(\underline{a}'', \underline{b}'', \underline{c})$ and define Φ_1, Φ_2, Φ_3, M, N, T, P accordingly. Then, we can choose $\underline{c}$ such that the normal geodesic tubular neighborhood of N is larger than r. Note that we also have M, N, T, P satisfy 1, 2, 3 of the statement of the Proposition and that the normal geodesic tubular neighborhood of $P \cap N$ is larger than s. (Note also that $N = N''$.) Then M, N, T, P satisfy 1, 2, 3, 4 of the statement of the Proposition. This completes the proof of the Proposition. $\square$

Remark 2.1. To prove the assertion in the Remark 1.9, just replace the roles of N and T in the proof above.

We now recall the construction of branched covers and introduce some notation (for more details, see [22]). Let M be a hyperbolic manifold of dimension n and let R and Q be compact totally geodesic submanifolds of M of dimensions $n-1$ and $n-2$ with $\partial Q = R$. Assume that the normal bundle of Q is trivial. The i-branched cover $M(i)$ of M with respect to (Q, R) is obtained in the following way. Cut M along Q to produce a manifold M'. Since the normal bundle of Q is trivial, M' contains two copies of Q that intersect in R. Label these copies Q_0 and Q_1. Take now i copies of M' and identify Q_1 of the first copy of M' with Q_0 of the second copy of M', and so on. At the end identify Q_1 of the last copy of M' with Q_0 of the first copy of M'. The resulting manifold is the i-branched cover of M with respect to (Q, R). The branched cover $M(i)$ contains i copies of Q. The union of these copies is called the i-book of $M(i)$. These copies intersect in a (unique) copy of R. Hence we consider R as a submanifold of $M(i)$. There is a projection map $\pi \colon M(i) \to M$, which restricted to each copy of M' in $M(i)$ is just the identification map (identify back the two copies of Q). Note that if $R = \emptyset$, then $\pi \colon M(i) \to M$ is an i-sheeted (ordinary) covering space. In general, $\pi \colon M(i) \setminus R \to M \setminus R$ is also an i-sheeted covering space. Finally, note that there is an action of $\mathbb{Z}_i$, the cyclic group of order i, on $M(i)$ that "rotates" the copies of M' around R

Remark 2.2. We present an alternative construction of the i-sheeted cover of M with respect to (Q, R) that has an action of the dihedral group D_i of order $2i$. We use the same notation as above. Now take $2i$ copies of M'. Identify Q_1 of the first copy with Q_1 of the second copy, then identify Q_0 of the second copy with Q_0 of the second copy of M', and so on. We denote this resulting manifold by $M[i]$. Then D_i acts on $M[i]$ in the obvious way. Note that we can also consider R contained in $M[i]$ as before.

Gromov and Thurston proved that $M(i)$ admits a Riemannian metric with sectional curvatures equal to -1 outside a normal geodesic tubular neighborhood of R of width s, and sectional curvatures in the interval $[-1-\epsilon, -1]$, where $0 < \epsilon < \frac{(10\,log\,2i)^2}{s^2}$, inside a normal geodesic tubular neighborhood of R. The $\mathbb{Z}_i$-action on $M(i)$ (with this Riemannian metric) is by isometries.

Remark 2.3. It is a simple matter to verify that $M[i]$ (see remark 2.2) also admits a Riemannian metric with sectional curvatures equal to -1 outside a normal geodesic tubular neighborhood of R of width s, and sectional curvatures in the interval $[-1-\epsilon, -1]$, where $0 < \epsilon < \frac{(10\,log\,4i)^2}{s^2}$, inside a

normal geodesic tubular neighborhood of R. Also the D_i-action on $M[i]$ is by isometries.

Gromov and Thurston also proved that, for large i (how large depending on M and R), $M(i)$ does not have the homotopy type of a closed locally symmetric space (they mention that probably all $M(i)$, $i > 1$, do not have the homotopy type of a closed locally symmetric space). For the proofs of the Theorems we will need the following stronger (and more technical) result, proved also by Gromov and Thurston in Section 3.5 of [14] (p. 6):

2.1. *Let M_s, $s = 1, 2, ...$ be a sequence of closed n-dimensional hyperbolic manifolds, $n \geq 4$, and R_s a closed codimension 2 totally geodesic submanifold of M_s with trivial normal bundle that bounds a compact codimension 1 totally geodesic submanifold Q_s of M_s. Assume that the width of the normal geodesic tubular neighborhood of R_s is larger than s.*

Then, there is a sequence of branched covers $M_s(i_s)$ of M_s, with respect to (Q_s, R_s), satisfying the following property:

Given $\epsilon > 0$ there is a s_0, such that for all $s \geq s_0$ we have:

1. *$M_s(i_s)$ does not have the homotopy type of a closed locally symmetric space.*

2. *$M_s(i_s)$ admits a metric with sectional curvatures in $[-1 - \epsilon, -1]$.*

Remark 2.4.   1. Let M be a closed negatively curved manifold of dimension $\neq 3,4$. Farrell–Jones Rigidity Theorem [7] implies that M has the homotopy type of a closed locally symmetric space if and only if M admits a negatively curved locally symmetric differentiable structure, i.e. a differentiable structure that supports a negatively curved locally symmetric Riemannian metric.

2. If a closed strictly 1/4-pinched negatively curved Riemannian manifold is homotopically equivalent to a closed locally symmetric space N, then N is a (real) hyerbolic manifold. This is a consequence of [16], [17], [20] and [29], and can also be seen by comparing Pontryagin classes and using a known case of Novikov's Conjecture due to Mishchenko [19].

3. We can give an altenative construction of pinched negatively curved closed manifolds that are not homotopy equivalent to a closed locally symmetric space, that do not use the results of 3.5 of [14]. Consider M_s, Q_s and R_s as in 2.1 above. To simplify our notation, we drop the subindex "s". We show that $M[2]$ or $M[2](2)$ satisfy 1 and 2 of 2.1.

In fact (see remark 2.2), both $M[2]$ and $M[2](2)$ satisfy 2, for s large enough. If $M[2]$ is not homotopically equivalent to a closed hyperbolic manifold, we are done. Suppose that $M[2]$ is homotopically equivalent to a closed hyperbolic manifold. Then there is a homotopy equivalence $f\colon \mathrm{M}[2]\to N$, where N is a closed hyperbolic manifold. We show then that $M[2](2)$ is not homotopy equivalent to a closed locally symmetric space. Note that $D_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ acts on $M[2]$ and the fixed point sets of the three non-zero involutions are codimension 1 submanifolds H_1, H_2 and the codimension 2 submanifold $R = H_1 \cap H_2$. Note that H_i is homeomorphic to the double of Q along its boundary R. Identify Q with one of the two copies (of Q) that form, say, H_1. Note also that H_1 and H_2 split $M[2]$ in four pieces, each homeomorphic to M', where M' is obtained from M by cutting along Q. Now, by Theorem 11 of [26], D_2 acts by isometries on N and f is homotopic to a D_2-equivariant map g. Pansu in [22] shows that N also splits in four homeomorphic pieces and g induces an isomorphism of fundamental groups from each copy of M' in $M[2]$ to each of the four pieces of N (and these isomorphisms are also isomorphisms when restricted to the boundaries). Using this, it is not difficult to construct a D_2-equivariant map h homotopic to f such that h maps each copy of M' to each of the four pieces of N. In particular, $Q' = h(Q)$, $R' = h(R)$ are totally geodesic submanifolds of N and $\partial Q' = R'$. This h can be lifted to give a homotopy equivalence between $M[2](2)$ and $N(2)$, where the 2-branched covers are taken with respect to (Q, R) and (Q', R'), respectively. But 3.6 of [14] implies that $N(2)$ (and consequently, also $M[2](2)$) is not homotopy equivalent to a closed locally symmetric space, that is, $M[2](2)$ satisfies 1 and 2 of 2.1.

Throughout the proofs of all Theorems in this paper, we can replace the use of $M(i)$ (in 2.1 above) by $M[2]$ or $M[2](2)$ (whichever satisfies 1 of 2.1.), and the proofs remain valid.

Proof. of Theorem 1.1 Fix $\epsilon > 0$. For each $s = 1, 2, 3, \dots$ let M'_s, N'_s, P'_s and T'_s be as in Proposition 1.8, with the width of the normal geodesic tubular neighborhood of $N'_s \cap P'_s$ larger than s and with large r (how large will be determined in a moment; note that, by item 2 of Proposition 1.8, N'_s does not depend on r). We assume also $r > 2s$. To alleviate the notation we will drop the subindex "s" whenever this causes no confusion. Write $U' = N' \cap T'$. By item 3 of Proposition 1.8 $c' = Dual(U') \neq 0 \in H^3(M', \mathbb{Z}_2)$. Then we have a smooth structure Σ' on M' such that its corresponding PL structure

corresponds to c'. (We choose this correspondence to assign the hyperbolic differentiable structure to $0 \in H^3(M', \mathbb{Z}_2)$, see [21]. It follows that Σ' is not PL-concordant to the hyperbolic differentiable structure, see [18].) Choose r large enough so that (M', Σ') admits a Riemannian metric with sectional curvatures ϵ-close to -1, see [21] (r depends on s and ϵ). The important point here is the following. Identify (a piece of) the tubular neighborhood of N' with $N' \times [-r, r]$. Then outside $V' = N' \times [-r, -\frac{r}{2}]$ we have that Σ' coincides with the hyperbolic differentiable structure and the ϵ-pinched metric, mentioned above, is hyperbolic, see [21]. That is, the change of the differentiable structure and the change of the metric only happen inside V'. Note that the distance between V' and N' is $\frac{r}{2} > s$. $\square$

Remark 2.5. Since the differentiable structure Σ' is not PL-concordant to the hyperbolic differentiable structure, we have that the identity map $M' \to (M', \Sigma')$ is not homotopic to a PL homeomorphism. More generally, if M is any closed non positively curved manifold of dimension $\neq 3,4$, and Σ is a differentiable structure not PL-concordant to the given non positively curved differentiable structure then the identity map $M \to (M, \Sigma)$ is not homotopic to a PL homeomorphism. To see this, suppose that $h \colon M \times [0,1] \to M$ is a homotopy from id_M to a PL homeomorphism. Then, by Farrell–Jones Rigidity Theorem [7], the map $H'(x, t) = (h(x, t), t)$ is homotopic to a homeomorphism $H \colon M \times [0,1] \to M \times [0,1]$ with $H_0 = id_M$ and H_1 equal to the PL homeomorphism above. It follows that Σ is PL-concordant to the given non positively curved differentiable structure.

From item 3 of Proposition 1.8, it follows that $0 \neq [N'] \in H_{n-1}(M', \mathbb{Z}_2)$. Let M be the double cover of M' with respect to N', that is, with respect to the kernel of the map $\pi_1 M' \to \mathbb{Z}_2$ given by $\alpha \mapsto Dual(N')[\alpha]$. Let $p \colon M \to M'$ be the covering projection. Note that M consists of two copies A, B of the manifold obtained from M' by cutting along N'. A and B intersect in two copies of N'. Denote one of these copies by N. Let $T = p^{-1}(T')$. Write $U = N \cap T$. Then $p|_U \colon U \to U'$ is a homeomorphism (in fact an isometry). Hence $0 \neq [U] \in H_{n-3}(M)$. Hence, as before, $c = Dual(U) \neq 0 \in H^3(M, \mathbb{Z}_2)$. Also, the normal geodesic tubular neighborhood of N is as large as the normal geodesic tubular neighborhood of N' and let V be the piece of $p^{-1}(V')$ that lies in the r normal geodesic tubular neighborhood of N. Also we assume that, say, $V \subset A$. Note that V is a piece, of width $\frac{r}{2}$, of the normal geodesic tubular neighborhood of N which is at a distance $\frac{r}{2} > s$ from B.

Let $P = p^{-1}(P')$, $Q = P \cap B$ and $R = \partial Q$. Note that R consists of two copies of $N' \cap P'$ and that the width of the normal geodesic tubular neighborhood of R is larger than s. Note also that the distance between V and Q is $> s$.

So far, we have obtained (we write back the subindex "s") sequences M_s, N_s, T_s, R_s, Q_s, V_s, $U_s = N_s \cap T_s$, c_s satisfying:

a. M_s is a closed n-dimensional hyperbolic manifold and N_s is a closed codimension one totally geodesic submanifold of M_s. U_s is a closed codimension three totally geodesic submanifold of M_s and $U_s \subset N_s$.

b. R_s is a closed codimension 2 totally geodesic submanifold of M_s that bounds the compact codimension 1 totally geodesic submanifold Q_s. The width of the normal geodesic tubular neighborhood of R_s is larger than s.

c. V_s is a piece of the tubular neighborhood of N_s and V_s is at a distance $> s$ from Q_s.

d. $c_s = Dual(U_s)$ is a non zero cohomology class. As for M'_s, M_s, with the smooth structure Σ_s whose corresponding PL structure corresponds to c_s, admits a Riemannian metric with sectional curvatures ϵ-close to -1. Outside V_s this ϵ-pinched to -1 metric is hyperbolic.

By 2.1, there is a s_0, such that for all $s \geq s_0$, we have:

1. $M_s(i_s)$ does not have the homotopy type of a locally symmetric space.

2. $M_s(i_s)$ admits a metric with sectional curvatures in $[-1-\epsilon, -1]$. This metric is hyperbolic outside the normal geodesic tubular neighborhood of width s of R_s.

Write $(M_1)_s = M_s(i_s)$ and we again drop the subindex "s". Let $\pi \colon M_1 \to M$ be the ramified projection. Choose a copy of $M \setminus (Q \setminus R)$ in M_1. Hence, we then can find N_1, V_1, U_1 contained in this copy such that:

$$\pi|_{N_1} \colon N_1 \to N, \ \pi|_{V_1} \colon V_1 \to V, \ \pi|_{U_1} \colon U_1 \to U \text{ are homeomorphisms.}$$

Since U does not bound, it follows that $c_1 = Dual(U_1) \neq 0 \in H^3(M_1, \mathbb{Z}_2)$. Then we have a smooth structure Σ_1 on M_1 such that its corresponding PL structure corresponds to c_1. (Again, we choose this correspondence to assign the branched differentiable structure to $0 \in H^3(M', \mathbb{Z}_2)$.) Hence Σ_1 is not PL-concordant to the branched differentiable structure.

But $M_2 = (M_1, \Sigma_1)$ admits a Riemannian metric with sectional curvatures ϵ-close to -1 such that outside V_1 this metric coincides with the metric of item 2 above. It follows then that the identity map $id \colon M_1 \to M_2$ is a homotopy equivalence between ϵ-pinched to -1 closed manifolds which is not homotopic to a PL homeomorphism (see the last remark above) and M_1 and M_2 do not have the homotopy type of a locally symmetric space. This proves Theorem 1.1.

Corollary 1.3 can be directly deduced from Theorem 1.1 and the $C^\infty-$ Hauptvermutung of Scharlemann and Siebenmann [24]:

Proof. of Corollary 1.3 By Theorem 1.1, we have
In every dimension $n \geq 6$, there is a pair of closed negatively curved n-dimensional manifolds M_1 and M_2 and a homeomorphism $f : M_1 \to M_2$ that satisfy (2) and (3) of the statement of Corollary 1.3, and f is not homotopic to a PL homeomorphism.

Let $h : M_1 \to M_2$ be the (unique) harmonic map homotopic to f. We have to prove that h is not one-to-one. But this follows by just applying the following result of M. Scharlemann and L. Siebenmann [24]
A smooth homeomorphism between closed smooth manifolds of dimension $\geq$ 6 is homotopic to a PL-homeomorphism.
(Smooth homeomorphisms are not necessarily diffeomorphisms. A simple example is given by the smooth homeomorphism $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$.)

Thus, the harmonic map h cannot be a homeomorphism (equivalently, cannot be one-to one) because f is not homotopic to a PL homeomorphism. This proves Corollary 1.3.

Before we prove Theorem 1.4, we need a remark about coverings of branched covers. Let $p \colon M \to M'$ be a cover, where M and M' are hyperbolic manifolds. Let R and R' be closed codimension two totally geodesic submanifolds of M and M', respectively, that bound closed codimension one totally geodesic submanifolds Q and Q' of M and M', respectively. Assume $Q = p^{-1}(Q')$ and $R = p^{-1}(R')$. Then, for any i, we can use p to construct a cover $q \colon M(i) \to M'(i)$, where $M(i)$ and $M'(i)$ are the i-branched covers of M and M' with respect to (Q, R) and (Q', R') respectively. The covers p and q fit in the following commutative square: □

$$
\begin{array}{ccc}
M(i) & \xrightarrow{\pi} & M \\
q \downarrow & & \downarrow p \\
M'(i) & \xrightarrow{\pi'} & M'
\end{array}
$$

If p is a ℓ-sheeted cover, then so is q. If p is a regular cover, so is q. We now prove Theorem 1.4.

Proof. of Theorem 1.4 Let M', N', T', R', Q', V', U', c', Σ' be as in the proof of Theorem 1.1 (we are dropping the subindex "s" and introducing a prime on each symbol). These objects satisfy properties a, b, c, d in the proof of Theorem 1.1. We assume also $T' \subset P'$ (see Remark 1.9). Let also M'_1, N'_1, V'_1, U'_1, c'_1, π', Σ'_1, M'_2 be as in the proof of Theorem 1.1 (we are also dropping the subindex "s" and introducing a prime on each symbol). We have that M'_1 satisfies properties a and b in the proof of Theorem 1.1 (assuming s large enough).

Let $p \colon M \to M'$ be the double cover of M' with respect to N'. Note that M consists of two copies D, and E of the manifold obtained from M' by cutting along N'. D and E intersect in two copies of N'. Let $\Sigma = p^*\Sigma'$. We know from [10] that the identity $M \to (M, \Sigma)$ is now homotopic to a diffeomorphism $f \colon M \to (M, \Sigma)$, but the unique harmonic map h homotopic to f (or to the identity) is not one-to-one. This is because the PL structure corresponding to Σ corresponds to $c = p^*(c')$ which vanishes. (The cohomology class c vanishes because it is dual to $p^{-1}(U')$, which is the boundary of $D \cap p^{-1}(T')$.)

Let $M_1 = M(i)$. Since we also have (by definition) $M'_1 = M'(i)$, the commutative square given just before this proof becomes:

$$
\begin{array}{ccc}
M_1 & \xrightarrow{\pi} & M \\
q \downarrow & & \downarrow p \\
M'_1 & \xrightarrow{\pi'} & M'
\end{array}
$$

Define $\sigma_1 = q^*\Sigma'_1$, $M_2 = (M_1, \Sigma_1)$, $c_1 = q^*c'_1$ and $U_1 = q^{-1}(U'_1)$. Note that:

(i) $q \colon M_2 \to M'_2$ is a smooth cover.

(ii) $M'_1 = M_1/F$, where $F = \{id, \phi\} \cong \mathbb{Z}_2$ and $\phi \colon M_1 \to M_1$ is the unique non trivial covering transformation.

(iii) The PL structure corresponding to Σ_1 corresponds to c_1.

(iv) c_1 is dual to U_1.

Claim. M_1 is not homotopy equivalent to a locally symmetric space.
If M_1 supports a negatively curved locally symmetric differentiable structure (see item 1 of Remark 2.4) then, by Mostow's Rigidity Theorem, ϕ can be realized by an isometry. It follows that M'_1 admits a negatively curved locally symmetric differentiable structure. This is a contradiction. This proves the claim.

Now, since $p\colon M_1 \to M_1'$ and $q\colon M_2 \to M_2'$ are smooth covers, we have that both M_1, M_2 admit ϵ-pinched to -1 Riemannian metrics.

It remains to prove that Σ_1 is $DIFF$-concordant to the differentiable structure of M_1. To be able to repeat the argument given in [10], pp. 230–233, we need to prove that the cohomology class c_1 vanishes. Equivalently, we need to prove that U_1 bounds. Recall that $U' = T' \cap N'$. We assume that $T' \subset P'$ (see remark after the statement of the Proposition). Then we can also find T_1' in one of the copies that form M_1' such that $\pi'|_{T_1'}\colon T_1' \to T'$ is a homeomorphism and $U_1' \subset T_1'$. Then $U_1' = T_1' \cap N_1'$. Let $T_1 = q^{-1}(T_1')$. A simple geometric argument then shows that $U_1 = \partial(T_1 \cap D_1)$, where D_1 lies in the same copy where T_1 lies and $\pi(D_1) = D$. This proves Theorem 1.4.$\square$

Proof. of Theorem 1.5 We use all notation from the proof of Theorem 1.4, with the following changes.

1. Now, we assume that the width of the normal geodesic tubular neighborhood of T' (not of N') is larger than r (see remark after the statement of the Proposition). It follows that the width of the normal geodesic tubular neighborhood of T is also larger than r.

2. The changes of structure and metric happen now in a piece of the normal geodesic tubular neighborhood of T' and T: since the normal bundle of T' is trivial, we have $\mathcal{N}_r(T') \setminus T'$ can be identified with $T' \times \mathbb{S}^1 \times (0, r)$, where $\mathcal{N}_r(T')$ is the normal geodesic tubular neighborhood of T' of width r. In [11] it is shown that we can take now $V' = T' \times I \times (0, \frac{r}{2})$, where $I \subset \mathbb{S}^1$ is any non-trivial interval. That is, outside V', the differentiable structure Σ' coincides with the hyperbolic differentiable structure, and the metric is hyperbolic. Note that, by choosing I properly, we have that V' does not intersect P'.

3. Now $p\colon M \to M'$ denotes the finite sheeted cover given in [11]. Again, we have that Σ is now $DIFF$-concordant to the hyperbolic differentiable structure. The new feature now is that the metric pulled back from (M', Σ') can be deformed to the hyperbolic one and all this deformation happens inside V.

Now, since V does not intersect P, we can also deform the metric of M_1 to the metric pulled back from M_2, and all this happens inside V_1. This is the main ingredient needed for the proof. The rest follows exactly as in [11]. This completes the proof of Theorem 1.5. $\square$

Proof. of Theorem 1.7. The proof follows the ideas of the proofs of Theorem and Corollary 1.3 in [12]. We use all objects in the proof of Theorem 1.5 above, but change the notation a bit to match the notation in [12]. From the proof above we have a homotopy commutative diagram:

$$
\begin{array}{ccc}
M_1 & \xrightarrow{f} & M_2 \\
q \downarrow & & \downarrow p \\
M_1' & \xrightarrow{f'} & M_2'
\end{array}
$$

where f is a diffeomorphism and f' is not homotopic to a PL homeomorphism. Recall that f' is the identity and f is homotopic to the identity (the underlying topological manifolds of $M_2' = (M_1', \Sigma')$ and $M_2 = (M_1, \Sigma)$ are M_1' and M_1 respectively). We now change the notation to match the one in [12]. First identify M_1 with M_2 via f and write N. Then write M_0 for M_1', M_1 for M_2', f for f', p_0 for q and p_1 for pf. The diagram above becomes:

$$
\begin{array}{ccc}
& N & \\
p_0 \swarrow & & \searrow p_1 \\
M_0 & \xrightarrow{f} & M_1
\end{array}
$$

As in [12], we have then that M_0, M_1, N, are smooth closed manifolds of dimension $n > 10$, that admit Riemannian metrics g_0, g_1 on M_0 and M_1, respectively, and smooth regular finite covers $p_0 \colon N \to M_0$, $p_1 \colon N \to M_1$ such that:

(1) The map f is not homotopic to a PL-homeomorphism.

(2) and g_0 and g_1 have sectional curvatures in $[-1 - \epsilon, -1 + \epsilon]$.

(3) There is a C^∞ family of C^∞ Riemannian metrics h_s on N, $0 \leq s \leq 1$, with $h_0 = p_0^* g_0$, and $h_1 = p_1^* g_1$, such that every h_s has sectional curvatures in $[-1 - \epsilon, -1 + \epsilon]$.

Let $G_i \subset Diff(N)$, be (finite) subgroups of the group $Diff(N)$, of all self-diffeomorphisms of N, such that $N/G_i = M_i$, $i = 0, 1$. It was shown in [10], [11], that G_0 and G_1 are conjugate in $Top\, N$, via a homeomorphism homotopic to id_N; hence $\gamma G_0 = \gamma G_1$, where $\gamma : Diff(N) \to Out(\pi_1 N)$ is the map described in the proof of Corollary 1.3 in [12]. Note that $G_i \subset Iso(N, h_i)$, where $Iso(N, h_i) \subset Diff(N)$ is the subgroup consisting of all isometries of the negatively curved manifold (N, h_i).

If the Ricci flow does not converge smoothly to a negatively curved metric for some h_s, we are done. So, let us assume that the Ricci flow converges smoothly to a negatively curved metric for all h_s. We will show a contradiction. Write $h_{s,t}$, for the Ricci flow starting at

$h_{s,0} = h_s$, $0 \leq t < \infty$, converging to the negatively curved Einstein metric j_s. Using the same argument as the one given in the proof of Theorem of [12], we get that all j_s are equal (modulo diffeomorphism and rescaling). Moreover, there is a diffeomorphism $\phi \colon N \to N$ homotopic to the identity such that j_0 is equal to $\phi^* j_1$ up to scaling (see also the proof of Corollary 1.3 in [12]). It follows that G_1 is conjugate in $Diff(N)$ to a subgroup of $Iso(N, j_0)$ via a diffeomorphism ϕ homotopic to id_N; i.e. $\phi^{-1} G_1 \phi \subset Iso(N, j_0)$. Note that $\gamma(\phi^{-1} G_1 \phi) = \gamma(G_1)$ since $\phi \sim id_N$; hence $\gamma(\phi^{-1} G_1 \phi) = \gamma(G_0)$. This implies that $\phi^{-1} G_1 \phi = G_0$ since both $\phi^{-1} G_1 \phi$ and G_0 are subgroups of $Iso(N, j_0)$ and Borel–Conner–Raymond showed (see [4], p. 43) that γ restricted to compact subgroups of $Diff(N)$ is monic. (Recall that N is aspherical and the center of $\pi_1(N)$ is trivial.) It follows that ϕ induces a diffeomorphism φ between $M_0 = N/G_0$ and $M_1 = N/G_1$. To find a contradiction, we have to prove that φ is homotopic to the identity. Since all manifolds here are aspherical, it is enough to prove that the induced map φ_* at the fundamental group level is the identity. Note that, since ϕ_* is the identity, we have that $\varphi_*|_{\pi_1 N}$ is the identity. Then, the fact that φ_* is also the identity follows from the next Lemma: $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Lemma 2.6. *Let Γ be the fundamental group of a closed negatively curved manifold. Let H be a subgroup of finite index of Γ and let $\alpha \colon \Gamma \to \Gamma$ be an isomorphism whose restriction to H is the identity. Then α is also the identity.*

Proof. Let $x \in \Gamma$. Since H has finite index in Γ we have that there is a positive integer n such that $x^n \in H$. Hence $\alpha(x^n) = x^n$. Therefore, $(\alpha(x))^n = x^n$. But n-roots are unique in Γ, thus $\alpha(x) = x$. This proves the Lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

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