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Heegaard splittings of the form $H + nK$

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Suppose that a three-manifold M contains infinitely many distinct strongly irreducible Heegaard splittings $H + nK$, obtained by Haken summing the surface H with n copies of the surface K. We show that K is incompressible. All known examples, of manifolds containing infinitely many irreducible Heegaard splittings, are of this form. We also give new examples of such manifolds.

1. Introduction.

F. Waldhausen, in his 1978 paper [19], asked if every closed orientable three-manifold contains only finitely many unstabilized Heegaard splittings. A. Casson and C. Gordon (see [1] or [11]), using a result of R. Parris [13], obtain a definitive "no" answer; they obtain examples of closed hyperbolic three-manifolds each of which contains *strongly irreducible* splittings of arbitrarily large genus. These examples have been studied and generalized by T. Kobayashi [5], [6], M. Lustig and Y. Moriah [10], E. Sedgwick [17], and K. Hartshorn [3].

The goal of this paper is three-fold. We first show, in Section 3, that all of the examples studied so far are of the form $H + nK$: There is a pair of surfaces H and K in the manifold so that the strongly irreducible splittings are obtained via a cut-and-paste construction, *Haken sum*, of H with n copies of K. See Section 2 for a precise definition of Haken sum.

Next, and of more interest, we show when such a sequence exists the surface K must be incompressible (in Sections 5 through 6). We claim:

Theorem 1.1. *Suppose M is a closed, orientable three-manifold and H and K are closed orientable transverse surfaces in M. Suppose that a Haken sum $H+K$ is given so that, for arbitrarily large values of n, the surfaces $H+nK$*

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are pairwise non-isotopic strongly irreducible Heegaard splittings. Then the surface K is incompressible.

Theorem 1.2. *shows that all of the counter-examples to Waldhausen's question found thus far are Haken manifolds. This was already known but required somewhat subtle techniques (see Lemmas 3.2 and 3.3 and Theorem 4.9 of Y.-Q. Wu's paper [20]).*

Theorem 1.3. *was originally conjectured by Sedgwick along with the much stronger:*

Conjecture 1.4. *Let M be a closed, orientable 3-manifold which contains infinitely many irreducible Heegaard splittings that are pairwise non-isotopic. Then M is Haken*[4].

We also produce new counter-examples, which are quite different from those previously studied. These examples are discussed in Section 7. The paper concludes in Section 8 by listing several conjectures.

2. Preliminaries.

Fix M, a closed, orientable three-manifold. If X is a submanifold of M we denote an open regular neighborhood of X by $\eta(X)$.

A surface K is *incompressible* in M if K is embedded, orientable, closed, not a two-sphere, and a simple closed curve $\gamma \subset K$ bounds an embedded disk in M if and only if γ bounds a disk in K. The three-manifold M is *irreducible* if every embedded two-sphere bounds a three-ball in M. If M is irreducible and contains an incompressible surface, then M is a *Haken* manifold.

A surface H is a *Heegaard splitting* for M if H is embedded, connected, and separates M into a pair of handlebodies, say V and W. A disk D properly embedded in a handlebody V is *essential* if $\partial D \subset \partial V$ is not null-homotopic in ∂V.

Definition 2.1. A Heegaard splitting $H \subset M$ is *reducible* if there is a pair of essential disks $D \subset V$ and $E \subset W$ with $\partial D = \partial E$. If H is not reducible, it is *irreducible*.

[4]After our paper was submitted this conjecture, and the other conjectures in Section 8, were claimed by T. Li. See [8] and [9].

Definition 2.2. A Heegaard splitting $H \subset M$ is *weakly reducible* if there is a pair of essential disks $D \subset V$ and $E \subset W$ with $\partial D \cap \partial E = \emptyset$. (See [2].) If H is not weakly reducible it is *strongly irreducible*.

One reason to study strongly irreducible Heegaard splittings is that these surfaces have many of the properties of incompressible surfaces. An important example of this is:

Lemma 2.3 (Scharlemann's No Nesting Lemma [15]). *Suppose that $H \subset M$ is a strongly irreducible Heegaard splitting. Suppose $\gamma \subset H$ bounds a disk D that is embedded in M and transverse to H. Then, γ bounds a disk in either V or W.* □

We now turn from Heegaard splittings to the concept of the *Haken sum* of a pair of surfaces. See Figure 1 for an illustration.

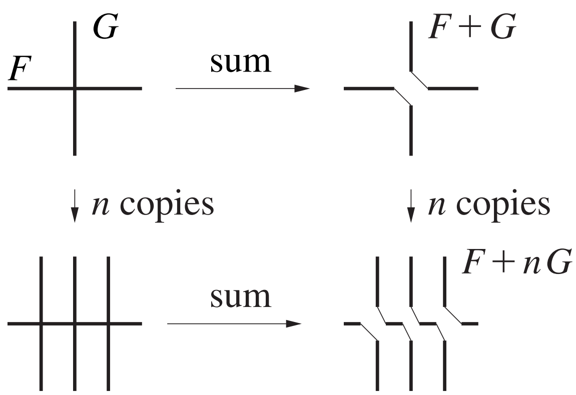

Figure 1: For every intersection of F and G, we have n intersections of F and nG. The light lines are the annuli $A_+(\gamma_i)$.

Suppose $F, G \subset M$ are a pair of closed, orientable, embedded, transverse surfaces. Assume that $\Gamma = F \cap G$ is non-empty. Note that, for every $\gamma \in \Gamma$, the open regular neighborhood $T(\gamma) = \eta(\gamma)$ is an open solid torus in M. Note that $\partial \overline{T(\gamma)} \setminus (F \cup G)$ is a union of four open annuli $A_1(\gamma) \cup A_2(\gamma) \cup A_3(\gamma) \cup A_4(\gamma)$, ordered cyclically. We collect these into two opposite pairs; $A_+(\gamma) = A_1 \cup A_3$ and $A_-(\gamma) = A_2 \cup A_4$. For every $\gamma \in \Gamma$, choose an $\epsilon(\gamma) \in \{+, -\}$ and form the *Haken sum*:

$$F + G = \left((F \cup G) \setminus \left(\bigcup_\gamma T(\gamma) \right) \right) \cup \left(\bigcup_\gamma A_{\epsilon(\gamma)}(\gamma) \right)$$

Note that the Haken sum depends heavily on our choices of $\epsilon(\gamma)$. As a bit of notation, we call the core curves of the annuli A_ϵ the *seams* of the Haken sum. Also there is an obvious generalization of Haken sum to properly embedded surfaces.

Remark 2.4. If F and G are compatible normal surfaces, carried by a single branched surface, or transversely oriented, there is a natural choice for the function $\epsilon(\gamma)$.

We now define the Haken sum $F + nG$: Take n parallel copies of G in $\eta(G)$ and number these $\{G_i\}_1^n$. For every curve $\gamma \in \Gamma$, we now have n curves $\{\gamma_i \subset F \cap G_i\}_{i=1}^n$. A Haken sum $F + G$ is determined by labelings $A_\pm(\gamma)$ and choices $\epsilon(\gamma) \in \{+, -\}$. Using the parallelism of the G_i, we take identical labelings for $A_\pm(\gamma_i)$ and make identical choices for $\epsilon(\gamma_i)$. See Figure 6 for a cross-sectional view at γ.

The surface $F+nG$ is now the usual Haken sum of F and nG with these induced choices, $A_\pm(\gamma_i)$ and $\epsilon(\gamma_i)$.

3. Existing examples.

This section shows that the Casson–Gordon examples are of the form $H + nK$. At the end of the section, we briefly discuss the examples of Kobayashi [6], and Lustig and Moriah [10].

Let $k = k(n_1, \ldots, n_m) \subset S^3$ be a *pretzel knot* [4] with *twist boxes* of order n_i. Here, we choose m and the n_i to be odd, positive, and greater than 4. See Figure 2 for an example.

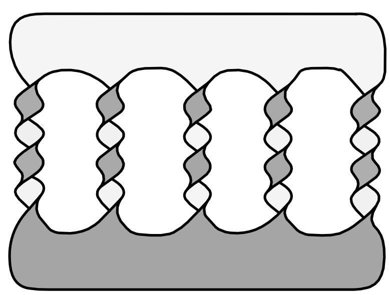

Figure 2: The $k(5, 5, 5, 5, 5)$-pretzel knot.

A pretzel knot has an associated Seifert surface, F. This is the compact checkerboard surface for the standard diagram. Again, see Figure 2. Let

B be the three-ball containing the pair of consecutive twist boxes of order n_i and n_{i+1}. Let $S = \partial B$. Note that $|k \cap S| = 4$; see Figure 3. There is a well-known twisting procedure which, twists $k = k(n_1, \ldots, n_m)$ along S giving

$$k_1 = k(n_1, \ldots, n_{i-1}, -1, n_i, n_{i+1}, 1, n_{i+2}, \ldots, n_m).$$

Again, see Figure 3.

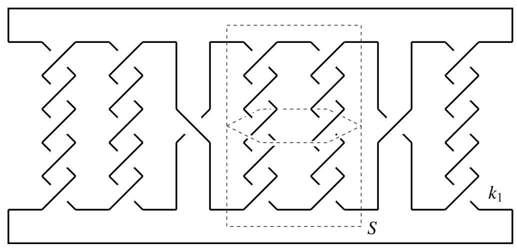

Figure 3: After twisting the $k(5,5,5,5,5)$-pretzel knot, we obtain the $k(5,5,-1,5,5,1,5)$-pretzel knot.

So, given the pretzel knot k and the sphere S, we can produce the sequence $\{k_n\}$ of *n-times twisted* pretzels:

$$k_n = k(n_1, \ldots, n_{i-1}, \overbrace{-1, \ldots, -1}^{n}, n_i, n_{i+1}, \overbrace{1, \ldots, 1}^{n}, n_{i+2}, \ldots, n_m).$$

Denote the associated Seifert surface for k_n by F_n. Note that k_n is isotopic to $k = k_0$ and that $F_0 = F$.

In his thesis, Parris proves:

Theorem 3.1 (Parris [13]). *The surfaces F_n are free incompressible Seifert surfaces for k.* □

Let $X = S^3 \setminus \eta(k_n)$. Let $\widehat{V_n}$ be a closed regular neighborhood of $F_n \cup \eta(k_n)$. So $k_n \subset \widehat{V_n}$. Let $W_n = \overline{S^3 \setminus \widehat{V_n}}$. Now, as k_n is isotopic into $H_n = \partial \widehat{V_n}$, doing $1/l$ Dehn surgery along k makes $\widehat{V_n}$ into a handlebody, which we denote by V_n. Here, l is any positive integer greater than 4. Let $M = X(1/l)$ be the $1/l$ Dehn surgery of S^3 along k. Let $H_n = \partial V_n = \partial W_n \subset M$. Note that the genus of H_n is $2n + 4$. We have:

Theorem 3.2 (Casson and Gordon [1], [11]). *The Heegaard splittings $H_n \subset M$ are strongly irreducible.* □

Now, let G be the surface $\partial(B \setminus \eta(k)) = (S \setminus \eta(k)) \cup (\partial\overline{\eta(k)} \cap B)$. We now state the main theorem of this section:

Theorem 3.3. *The Heegaard surfaces H_n are isotopic to a Haken sum $H_0 + 2nG$.*

We require several lemmas for the proof of Theorem 3.3.

Lemma 3.4. *The surface F_n is isotopic to $F_0 + nS$.*

Proof. Let α and β be the arcs of intersection between S and $F = F_0$. Let B_α be a closed regular neighborhood of α. Let S_α be the boundary of B_α. See the left side of Figure 4 for a picture of $S \cup F$ inside B_α.

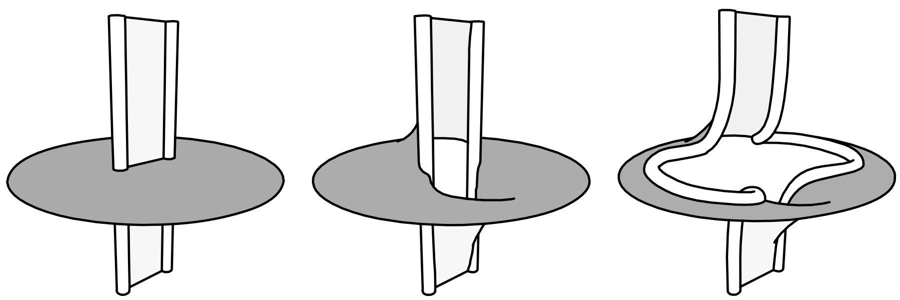

Figure 4: The knot k has been thickened a bit. On the left, F is vertical while S is horizontal. The middle is their Haken sum. The right shows the isotopy of $\alpha' \cup k' \cup \alpha'' \cup k''$ to be horizontal.

We choose the Haken sum which glues the top sheet of $(F \cap B_\alpha) \setminus \alpha$ to the back sheet of $(S \cap B_\alpha) \setminus \alpha$. Glue the bottom sheet of $(F \cap B_\alpha) \setminus \alpha$ to the front sheet of $(S \cap B_\alpha) \setminus \alpha$. See the center of Figure 4 for a picture of the Haken sum.

Let α' and α'' be the seams along which the sheets of F and S are glued. Let k' and k'' be the arcs of $k \setminus (\partial\alpha' \cup \partial\alpha'')$ inside of B_α. Do a small isotopy of the loop $\gamma = \alpha' \cup k' \cup \alpha'' \cup k''$ as shown in Figure 4. After this isotopy, the image of γ lies in a regular neighborhood of the curve $S_\alpha \cap S$.

We perform the same sequence of steps near β. Recall that $S_\alpha \cap S$ and $S_\beta \cap S$ cobound an annulus, $A \subset S$. Isotope the surface $F + S$ to move k close to the core curve of A – this isotopy is illustrated in a sequence of steps in Figure 5.

Now flatten out the right-hand side of Figure 5 by rotating the two twist boxes inside of S by 180°. Also flatten the annulus into the plane containing the standard diagram of k. See Figure 6.

Note that the result is the Seifert surface associated to the pretzel knot $k_1 = k(5, 5, -1, 5, 5, 1, 5)$. Thus, by induction, the proof of Lemma 3.4 is complete. □

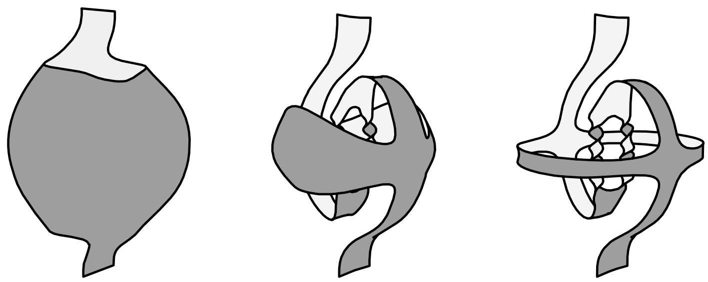

Figure 5: Isotoping $F + S$, moving k near the equator of S.

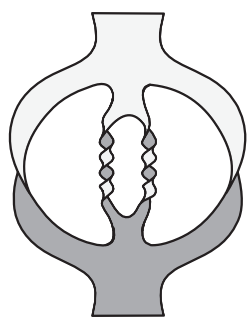

Figure 6: Flatten the resulting figure into the plane of the diagram.

Recall that k is the given pretzel knot, $F = F_0$ is the associated Seifert surface, and S is the two-sphere bounding the three-ball B, as above.

Lemma 3.5. *The surface F_n is isotopic to $F_0 + nG$.*

Proof. Consider a single component of $\eta(k) \cap \eta(B)$. This component B' is a ball. Let $k' = k \cap B'$. The disk $F' = F \cap B'$ is a boundary compression for k' in B'. The two disks $S' \cup S'' = S \cap B'$ each intersect k' in a single point. See the left-hand side of Figure 7 for a picture. (The knot k has been thickened a bit.)

The arcs $S' \cap F'$ and $S'' \cap F'$ are both part of $\alpha \subset S \cap F$. Thus, Haken summing along $S' \cap F'$ agrees with Haken summing along $S'' \cap F'$. See the right-hand side of Figure 7.

Turn now to $F + G$. Recall that $G = \partial(B \setminus \eta(k))$. Note that $G \cap \overline{\eta(k)}$ is a pair of annuli. Isotope these annuli, rel boundary, slightly into $\eta(k)$ so that $G \setminus \eta(k)$ is identical to $S \setminus \eta(k)$. Thus, obtain the picture of $F \cap B'$ and $G \cap B'$, shown on the left in Figure 8.

Finally, take the Haken sum of F' with $G' = G \cap B'$ as forced by our previous choices. See the right of Figure 8. Note that $F' + G'$ is isotopic to $F' + (S' \cup S'')$, rel boundary. The same holds inside the other component of $\eta(k) \cap B$. Finally, $F + S$ is identical to $F + G$ outside of $\eta(k)$. The lemma is proved. □

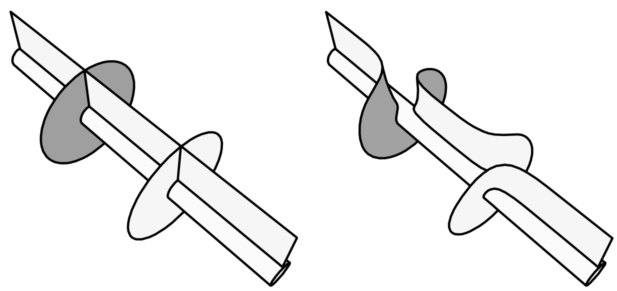

Figure 7: Forming the Haken sum of F (longitudinal) and S (meridional).

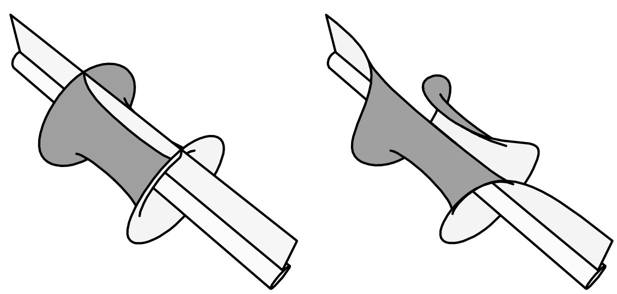

Figure 8: Forming the Haken sum of F and G.

We are now equipped to prove Theorem 3.3:

Proof. Notice now that H_n is isotopic to the boundary of a regular neighborhood of F_n. As $\partial F_n = k_n$, the splitting H_n is obtained by gluing two parallel copies of F_n with an annulus $A_n \subset \partial\overline{\eta(k_n)}$, where the core curve of A_n has longitudinal slope $\partial\overline{\eta(k_n)} \cap F_n$. Note that A_0 is taken to A_n by the twisting isotopy taking $k = k_0$ to k_n. We thus have the following:

$$\begin{aligned} H_n &= 2F_n \cup A_n && (3.1)\\ &\approx 2(F_0 + nG) \cup A_0 && (3.2)\\ &= (2F_0 \cup A_0) + 2nG && (3.3)\\ &= H_0 + 2nG. && (3.4)\end{aligned}$$

The second line follows from Lemma 3.5. The third line holds because G has no boundary. This concludes the proof of Theorem 3.3. □

Remark 3.6. The examples of [6] and [10] are very similar – they begin with a knot admitting a Conway sphere S and a natural Seifert surface F. They then isotope the knot by twisting inside S. Thus their examples of high genus Heegaard splittings may also be obtained via Haken sum.

4. Removing trivial curves.

Here we discuss a method for "cleaning" Haken sums. To be precise, we have:

Lemma 4.1. *Suppose $H + nK$ is a sequence of Haken sums. Let m be the number of curves of $H \cap K$ which are inessential on K. Then there is an isotopy of $H' = H + mK$ and a Haken sum $H' + K$ so that*

- *all curves of $H' \cap K$ are essential on K and*
- *for all $n > m$ the surface $H + nK$ is isotopic to $H' + (n - m)K$.*

Definition 4.2. We call such sequences *essential* in K.

Proof of Lemma 4.1. If $m = 0$ there is nothing to prove. If not, we claim there is a surface $\widehat{H}$ such that: $\widehat{H}$ is isotopic to $H + K$, $\widehat{H} \cap K$ has fewer inessential (on K) curves than $H \cap K$ does, and $\widehat{H} + (n-1)K$ is isotopic to $H + nK$ for all $n > 0$. Applying this m times will prove the lemma.

So suppose $\alpha \subset H \cap K$ is inessential on K. Assume that the disk $D \subset K$ bounded by α is *innermost*. That is, $D \cap H = \alpha$.

Let $N = \overline{\eta(K)} \cong K \times [0,1]$. We identify K with $K \times \{1/2\}$. Let D' be the component of $(H+K) \setminus \partial N$ containing D. Suppose that $\overline{D'}$ has boundary in $K \times \{1\}$. (The case $\overline{D'} \subset K \times \{0\}$ is similar.)

Isotope D' up, relative to $(H+K) \cap \partial N$, to lie in $\eta(K \times \{1\})$, while isotoping all other components of $K \setminus H$ down into $\eta(K \times \{0\})$. See Figure 9.

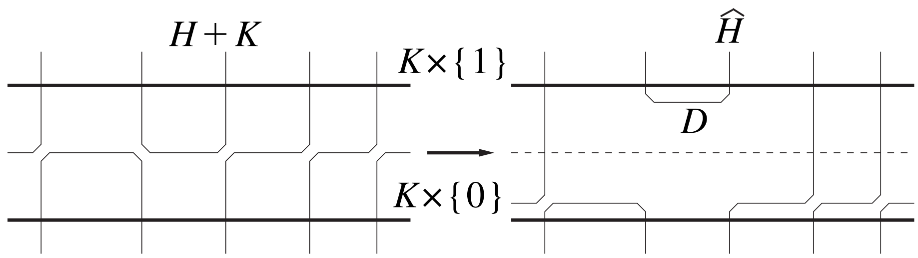

Figure 9: On the left, we see $H+K$ intersecting $\overline{\eta(K)}$. On the right, $H+K$ has been isotoped to $\widehat{H}$.

Let $\widehat{H}$ be this new position of $H+K$ and note that $\widehat{H} \cap (K \times \{1/2\})$ has at least one fewer trivial curve of intersection with K.

We now must prove that $\widehat{H} + (n-1)K$ is isotopic to $H + nK$, for all $n > 0$. Recall that α was the chosen innermost curve of $H \cap K$, bounding $D \subset K$. Form $H + nK$ and isotope all subdisks parallel to D up. Isotope the lowest copy of $K \setminus D$ down. This yields $\widehat{H} + (n-1)K$. (See Figure 10.) This completes the claim and thus the lemma. □

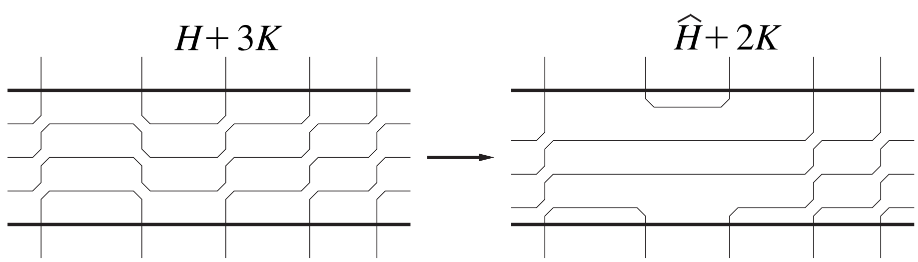

Figure 10: $H + 3K$ is isotopic to $\widehat{H} + 2K$.

5. Adding surfaces of genus greater than two.

Theorem 1.3 divides into two statements. The first addresses the case $\mathrm{genus}(K) > 1$ while the second deals with the case K a torus. We begin with:

Theorem 5.1. *Suppose M is a closed, orientable three-manifold and H and K are closed orientable transverse surfaces in M, with $\mathrm{genus}(K) \geq 2$. Suppose that a Haken sum $H + K$ is given so that the surface $H + nK$ is a strongly irreducible Heegaard splitting for arbitrarily large values of n. Then the surface K is incompressible.*

We begin by giving a brief sketch of the proof. Aiming for a contradiction, we assume that K is compressible. Using Lemma 5.2 below, we find a compressing disk D for K with ∂D separating in K.

For large n, the disk D intersects $H + nK$ in a fairly controlled way – in particular, there is a large family of parallel curves $\{\gamma_i\}$ in the intersection $(H + nK) \cap D$. We will show that many of the $\{\gamma_i\}$ are essential curves on $H + nK$. By Scharlemann's "No Nesting" Lemma 2.3, all of these γ_i's bound disks D_i in one of the two handlebodies V_n or W_n. (Here $\partial V_n = \partial W_n$ equals $H + nK$.) Finally, the two curves γ_i and γ_{i+1} cobound a subannulus $A_i \subset D$. Compressing or boundary compressing A_i, will give an essential disk E_i disjoint from D_i. This demonstrates that $H + nK$ is weakly reducible, a contradiction.

5.1. Finding a separating compressing disk.

We will need a simple lemma:

Lemma 5.2. *If $G \subset M$ is a compressible surface, which is not a torus, then there is a compressing disk $D \subset M$ so that ∂D is a separating curve, on G.*

Proof. Let E be any compressing disk for G. If ∂E is a separating curve then take $D = E$ and we are done. So suppose instead that ∂E is non-separating in G. Choose $\gamma \subset G$ to be any simple closed curve which meets ∂E exactly once. Let N be a closed regular neighborhood of $\gamma \cup E$, taken in M. Let D be the closure of the disk component of $\partial N \setminus G$. This is the desired disk. $\square$

5.2. The intersection with the compressing disk.

We now begin the proof of Theorem 5.1.

Recall that H and K are a pair of surfaces so that $H + nK$ is a strongly irreducible Heegaard splitting for arbitrarily large n. Applying Lemma 4.1, we may assume that every curve of intersection between H and K is essential in K.

In order to obtain a contradiction assume that K is compressible. Use Lemma 5.2 to obtain a compressing disk D for K, transverse to H, where ∂D is separating in K. We may choose D to minimize the size of the intersection $|(H \cap K) \cap D|$. Denote the two components of $K \smallsetminus \partial D$ by K' and K''.

For any $n > 0$ such that $H + nK$ is a strongly irreducible Heegaard splitting, proceed as follows: Label the components of nK as $K_1, \ldots, K_n$. Isotope nK so that all of the K_i lie inside of $\eta(K)$, are disjoint from K, and meet interior(D) in a single curve. Choose subscripts for the K_i consecutively so that $K_1 \cap D$ is innermost among the curves of intersection $(\cup K_i) \cap D$. See Figure 11 for a picture of how the K_i and H intersect D.

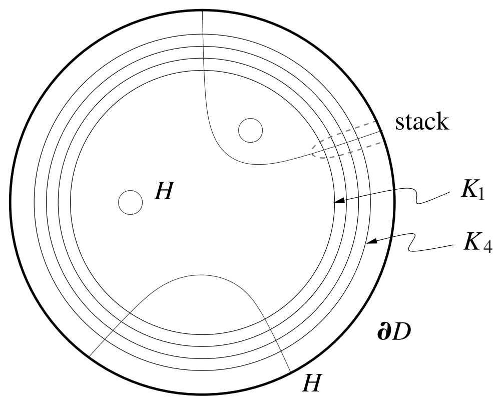

Figure 11: A picture of D. The concentric circles are the curves of $K_i \cap D$. The arcs and small circles make up $H \cap D$.

Note that $H \cap D$ is a collection of arcs and simple closed curves. The arcs' intersection with $K_i \cap D$ will give a cross-sectional view of the Haken sum of H with nK.

Fix attention on a *stack* of intersections: a collection of n consecutive points of intersection between an arc of $H \cap D$ and nK, all of which are close to a point of $H \cap \partial D$. Again, see Figure 11. Choose a transverse orientation on D. Assign a parity to the stack as follows: A stack is *positive* if, after the Haken sum, the segment of $(K_i \cap D) \smallsetminus \eta(K_i \cap H)$ on the left is attached to the segment of $(K_{i+1} \cap D) \smallsetminus \eta(K_{i+1} \cap H)$ on the right. Otherwise, the stack is *negative*. See Figure 12.

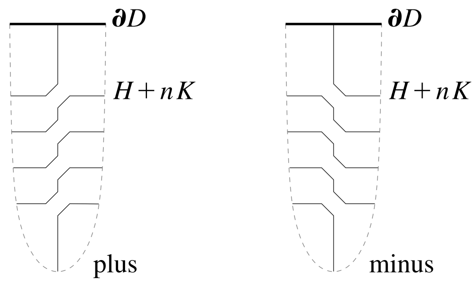

Figure 12: In both cases, we are looking at D in the direction of the transverse orientation.

Claim 5.3. *The number of positive stacks equals the number of negative stacks.*

Proof. Recall ∂D separates K into two pieces, K' and K''. So every component of $H \cap K'$ is either a simple closed curve, disjoint from ∂D, or is a properly embedded arc. Pick one of these arcs, say $\alpha \subset H \cap K'$. Note the endpoints of α lie in ∂D and give rise to stacks of opposite parity. □

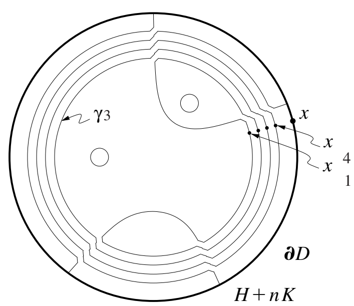

Figure 13.

Next, analyze how the intersection $(H+nK)\cap D$ lies in D: As in Figure 13, fix any point $x \in (\partial D \setminus H)$. Let x_i be the corresponding point of $K_i \cap D$.

An arc of $(K_i \cap D) \setminus \eta(H \cap nK)$ is a *horizontal arc at level* i. In particular, the arc containing x_i is at level i. Orient these arcs in a clockwise fashion. Note that horizontal arcs are also subarcs of $(H+nK)\cap D$. When a horizontal arc at level i enters a positive stack, it *ascends* and when it enters a negative stack it *descends* a single level.

Consider now an arc of $(H \cap D) \smallsetminus \eta(H \cap nK)$. These are the *vertical* arcs. If a vertical arc meets ∂D, call it an *external* arc. If a vertical arc is contained in the subdisk of D bounded by $K_1 \cap D$ call it an *internal* arc. See Figure 11.

Suppose the component of $(H + nK) \cap D$ which contains x_i does *not* contain any internal or external vertical arcs. Then, call that component γ_i. For each value of i where the property above does not hold, γ_i is left undefined.

Set

$$c_1 = |H \cap \partial D|. \tag{5.1}$$

Note that c_1 is even.

Claim 5.4. *The collection $(H + nK) \cap D$ consists of*

- *exactly $c_1/2$ arcs,*
- *the curves $\{\gamma_i\}$, and*
- *at most another $|H \cap D|$ simple closed curves.*

Furthermore, each γ_i is a simple closed curve. Also $|\{\gamma_i\}| \geq n - c_1$. Finally, γ_i and γ_{i+1} cobound an annulus component A_i of $D \smallsetminus (H + nK)$.

The claim follows from Figure 13. For completeness, a proof is included.

Proof of Claim 5.4.. The first statement in the claim is trivial: $H \cap \partial D$ and $(H + nK) \cap \partial D$ are the same set of points. Next, count the γ_i's: Choose any i with $c_1/2 < i < n - c_1/2$ and let α be the component of $(H + nK) \cap D$ containing x_i. Starting at x_i, and moving along α in a clockwise fashion, we ascend whenever we go through a positive stack and descend through the negative stacks. As there are $c_1/2$ positive stacks and the same number of negative stacks α contains no internal or external vertical arcs. Also α goes through none of the other x_j's. So α is a simple closed curve and is labeled γ_i.

It follows that there are at least $n - c_1$ of the γ_i's in $(H + nK) \cap D$. These are all parallel in D, yielding the annuli $\{A_i\}$. Again, see Figure 5.2.

To finish the claim, note that any simple closed curve of $(H + nK) \cap D$, which is not a γ_i, is either a simple closed curve component of $H \cap D$ or contains an internal vertical arc. Thus, there are at most $|H \cap D|$ such simple closed curves. □

In short, if n is sufficiently large then $(H + nK) \cap D$ cuts D into pieces and most of these pieces are the parallel annuli, A_i.

5.3. Finding a "cover" of K.

Recall that $K \setminus \partial D = K' \amalg K''$. Let $\{\alpha'_j\} = H \cap K'$. Similarly, let $\{\alpha''_j\} = H \cap K''$. Due to the minimality assumptions (see the beginning of Section 5.2) every loop of $H \cap K$ is essential in K and every arc $\alpha'_j \subset K'$ and $\alpha''_j \subset K''$ is also essential

Choose a collection of oriented arcs $\{\beta'_j\}$ with the following properties:

- Every arc β'_j is simple and is embedded in K'.
- Both endpoints of β'_j are at the point x.
- The interiors of the β'_j are disjoint.
- The union of the β'_j, together with ∂D, forms a one-vertex triangulation of K'.
- The chosen arcs $\{\beta'_j\}$ minimize the quantity $|(\bigcup_j \alpha'_j) \cap (\bigcup_j \beta'_j)|$.

Similarly, choose a collection of arcs $\{\beta''_j\}$ for K''.

Now, lift everything to a subsurface of $H + nK$ which is "almost" a cyclic cover of K: Let $\widetilde{K} = (H + nK) \cap \eta(K)$. Let $\pi\colon \widetilde{K} \to K$ be the natural projection map. So π is the composition of the homeomorphism of $\eta(K) \cong K \times (0, 1)$ with projection onto the first factor, restricted to $\widetilde{K} \subset \eta(K)$. (It is necessary to slightly tilt the vertical annuli coming from $H \setminus nK$. This makes π, a local homeomorphism.)

Thus $\{x_i\} = \pi^{-1}(x)$. As discussed above, for most values of i the curve γ_i is the component of $\pi^{-1}(\partial D)$ which contains x_i.

Now lift the set of curves $\alpha', \alpha'', \beta', \beta''$: To be precise, let $\alpha'_{j,i}$ be the component of $\pi^{-1}(\alpha'_j)$ which is contained in the annulus connecting K_i and K_{i+1}. Define $\alpha''_{j,i}$ similarly. See Figure 14.

Let $\beta'_{j,i}$ be the component of $\pi^{-1}(\beta'_j)$ which, given the orientation of β'_j, starts at the point x_i. Define $\beta''_{j,i}$ similarly. Not every $\beta'_{j,i}$ is useful. However, letting

$$c_2 = \max_k \left\{ \left| \left(\bigcup_j \alpha'_j \right) \cap \beta'_k \right|, \left| \left(\bigcup_j \alpha''_j \right) \cap \beta''_k \right| \right\} \tag{5.2}$$

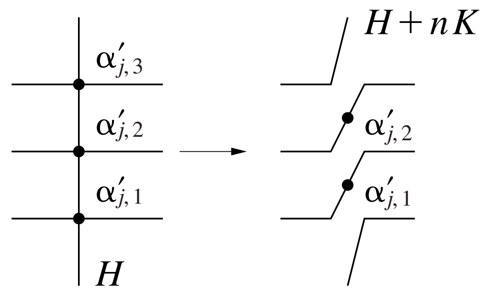

Figure 14: The left is before the Haken sum and the right is after. We have tilted the vertical annuli of H.

we have:

Claim 5.5. *For all j and for all i with $c_2 < i < n - c_2$, we have $\pi(\beta'_{j,i}) = \beta'_j$. The same holds for $\pi|\beta''_{j,i}$.*

Proof. Every time $\beta'_{j,i}$ crosses one of the $\alpha'_{j,i}$'s it goes up (or down) exactly one level. Thus, any $\beta'_{j,i}$, with i as in the hypothesis, has both endpoints on some lift of x and the claim holds. □

Definition 5.6. Suppose that $c_2 < i < n - c_2$. Suppose that the final point of $\beta'_{j,i}$ is x_k. By definition of $\beta'_{j,i}$ the starting point is x_i. Define the *shift* of $\beta'_{j,i}$ to be $\sigma(\beta'_{j,i}) = k - i$.

An important observation is:

Claim 5.7. *The shift $\sigma(\beta'_{j,i})$ does not depend on the value of i.* □

Remark 5.8. Note that c_2 is an upper bound on the absolute value of any shift $\sigma(\beta'_{j,i})$ or $\sigma(\beta''_{j,i})$.

Henceforth, we will use $\sigma(\beta'_j)$ to denote the shift of $\beta'_{j,i}$, for any i. The same notation will be used for arcs of K''.

5.4. Finding essential curves and annuli.

Now to gain some control over the parallel curves $\gamma_i \subset D$. Set

$$c_3 = \max\{c_1, 2c_2\}. \tag{5.3}$$

Claim 5.9. *If, for all j, the shifts $\sigma(\beta'_j)$ are zero then, for all i with $c_2 < i < n - c_2$, the curve γ_i separates $H + nK$ into two surfaces. One of these is homeomorphic to K' (and in fact is isotopic, relative to γ_i, to K'_i). The similar statement holds on the K'' side.* □

Remark 5.10. It follows immediately that there is at least one non-zero shift on at least one side. Otherwise, $H + nK$ would be disconnected for large n.

Claim 5.11. *For all i with $c_3 < i < n - c_3$, the curve γ_i is essential in $H + nK$.*

Proof. Consider some curve γ_i with i in the indicated range.

First, suppose that all shifts on one side, say K', are zero. Take $n > 7 \cdot c_3$ (this lower bound is used here and in Claim 5.12 below). Recall that $\chi(K) < 0$ and that Euler characteristic is additive under Haken sum. Thus $\chi(K') + 1 > \chi(H) + n\chi(K) = \chi(H + nK)$. Now, if γ_i is inessential then, by Claim 5.9, γ_i bounds a surface homeomorphic to K' on one side and bounds a disk on the other side. It would follow that $\chi(H + nK) = \chi(K') + 1$, a contradiction. So if all shifts on one side are zero, then γ_i is essential.

Now suppose that there are non-zero shifts on both sides. Reversing the orientation of some β'_j or β''_k, we may assume that the shifts $\sigma(\beta'_j) = r$ and $\sigma(\beta''_k) = s$ are both positive. We may further assume that $r \leq s$. If $r = s$, take $\delta = \beta'_{j,i} \cup \beta''_{k,i}$. If $r < s$, take $\delta = \beta'_{j,i} \cup \beta''_{k,i+r-s} \cup \beta'_{j,i-s} \cup \beta''_{k,i-s}$.

So δ is a simple closed curve embedded in $\widetilde{K}$: the important point, that δ is closed, follows from Claim 5.5 and the fact that c_3 is at least twice as large as c_2. Note that δ meets γ_i exactly once at the point x_i. So γ_i is essential. □

Similar ideas will give some control over the annuli $A_i \subset D$. Recall that $\partial A_i = \gamma_i \cup \gamma_{i+1}$.

Claim 5.12. *For all i with $3c_3 < i < n - 3c_3 - 1$, the annuli A_i and A_{i+1} are not boundary parallel into $H + nK$.*

Proof. Suppose that A_i is boundary parallel into $H + nK$. (The situation for A_{i+1} is similar.) Let $B \subset H + nK$ be the annulus with which A_i cobounds a solid torus. So $\partial B = \partial A_i = \gamma_i \cup \gamma_{i+1}$. As the other case is similar, suppose that B is adjacent to the curve γ_i from the K'-side.

As B is not homeomorphic to K', it follows from Claim 5.9 that there is a non-zero shift on the K' side. Let $r = \sigma(\beta'_j)$ be the smallest non-zero shift (in absolute value) on the K' side. Now the arc $\beta'_{j,i}$, by Claim 5.5, runs from x_i to x_{i+r}. Also the interior of $\beta'_{j,i}$ does not meet any γ_k. Since $\partial B = \gamma_i \cup \gamma_{i+1}$ it follows that $\gamma_{i+r} \subset B$. Given the assumed bounds on i it follows from Claim 5.11 that γ_{i+r} is essential in $H + nK$ and thus in B. So γ_{i+r} is parallel in B to γ_i.

Let $B' \subset B$ be the annulus cobounded by γ_i and γ_{i+r}. Now, B' is adjacent to both γ_i and γ_{i+r} on the K' side. Note that $r = \sigma(\beta'_{j,i}) = \sigma(\beta'_{j,i+r})$, by Claim 5.7. As above deduce from Claim 5.5 that the arc $\beta'_{j,i+r}$ runs from x_{i+r} to x_{i+2r}. Also the interior of $\beta'_{j,i+r}$ does not meet any γ_k. Since $\partial B' = \gamma_i \cup \gamma_{i+r}$, it follows that $\gamma_{i+2r} \subset B'$. Given the assumed bounds on i, it follows from Claim 5.11 that γ_{i+2r} is essential in $H + nK$ and thus in B'. So γ_{i+2r} separates γ_i from γ_{i+r} in B'. See Figure 15. This is a contradiction, as $\beta'_{j,i}$ connects $x_i \in \gamma_i$ to $x_{i+r} \in \gamma_{i+r}$ and does not meet γ_{i+2r}. □

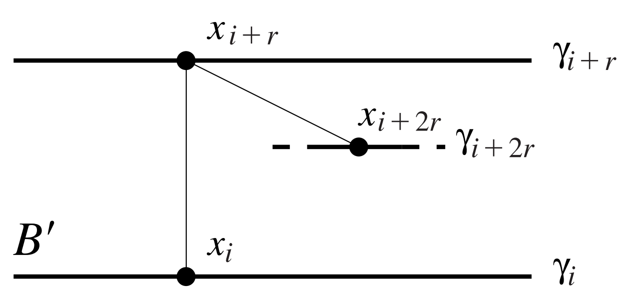

Figure 15: The curve γ_{i+2r} cannot be a core curve for B' without crossing $\beta'_{j,i}$.

5.5. Finishing the proof of the theorem.

Recall that all of the curves γ_i bound embedded disks in the manifold because they bound disks in D. Thus, by Scharlemann's "No Nesting" Lemma 2.3, all of the γ_i's bound disks in one of the two handlebodies bounded by $H + nK$, V_n or W_n. From strong irreducibility of $H + nK$ and Claim 5.11, it follows that all the γ_i's bound essential disks on the same side. As the other case is identical, suppose that γ_i bounds $D_i \subset V_n$ for all i.

Now either A_i or A_{i+1} lies in the opposite handlebody W_n. As the two possibilities are symmetric, suppose $A_i \subset W_n$. There are two final cases. If A_i is compressible in W_n, then compress to obtain two disks, say $E_i, E_{i+1} \subset W_n$. Here, $\partial E_i = \gamma_i = \partial D_i$. It follows that $H + nK$ is reducible, a contradiction.

Suppose instead that A_i is incompressible. Since A_i is not boundary parallel (Claim 5.12), there is a boundary compression of A_i yielding an essential disk E_i with ∂E_i disjoint from $\partial A_i = \gamma_i \cup \gamma_{i+1}$. So $H + nK$ is weakly reducible, another contradiction. This final contradiction completes the proof of Theorem 5.1. □

6. Adding copies of a torus.

For the remaining part of Theorem 1.3, the surface added is a torus, T. Hence, we deal with sequences of strongly irreducible Heegaard splittings of the form $H + nT$.

Theorem 6.1. *Suppose M is a closed, orientable three-manifold and H and T are closed orientable transverse surfaces in M, with T a two-torus. Suppose that a Haken sum $H + T$ is given so that the surface $H + nT$ is a strongly irreducible Heegaard splitting for arbitrarily large values of n. Assume also that no pair of these splittings are isotopic in M. Then, the surface T is incompressible.*

Assume that T is compressible to obtain a contradiction. As M is irreducible there are two cases: Either T bounds a solid torus or T bounds a *cube with a knotted hole*. Denote this submanifold which T bounds by $X \subset M$.

Before considering these cases in detail, apply Lemma 4.1 so that $H \cap T$ consists of curves essential on T. These all have the same slope. Further, assign a parity to the curves of $H \cap T$ as follows: Choose any oriented curve α in T which meets each of the components of $H \cap T$ exactly once. Then, traveling along α in the chosen direction, we cross the curves of $H \cap T$ and, according to the Haken sum, $H + nT$ either descends into the submanifold X or ascends out of X. Assign the former a negative parity and the latter a positive. As the other case is similar, we assume that there are more curves of $H \cap T$ of positive parity than negative. (There cannot be equal numbers of both as then, for large values of n, the surface $H + nT$ fails to be connected.) Recalling Definition 4.2 of an essential sequence we now have:

Lemma 6.2. *Suppose the sequence $H + nT$ is essential in T. Let m be the number of positive curves of $H \cap T$ minus the number of negative. Let $m' = (|H \cap T| - m)/2$. Then, there is an isotopy of $H' = H + m'T$ so that*

- *all curves of $H' \cap T$ are essential in T,*
- *all curves of $H' \cap T$ are positive, and*
- *for all $n > m'$, the surface $H + nT$ is isotopic to $H' + (n - m')T$.*

□

As the proof of Lemma 6.2 is essentially identical to that of Lemma 4.1, we omit it. An essential sequence $H + nT$ is *reduced* if all of the curves of $H \cap T$ have the same parity.

6.1. Bounding a solid torus.

Suppose now that T bounds a solid torus X. We have:

Claim 6.3. *If $H + nT$ is reduced and $m = |H \cap T|$ then, for any positive n, the surface $H + nT$ is isotopic in M to $H + (n + m)T$.*

Proof. Choose a homeomorphism $X \cong \mathbb{D}^2 \times S^1$, where $\overline{\eta(T) \cap X} \cong A \times S^1$ with $A \cong \{z \in \mathbb{C} \ | \ 1/2 \leq |z| \leq 1\}$. Set $D_0 = \overline{\mathbb{D}^2 \smallsetminus A}$.

If the slope of $H \cap T$ is meridional (isotopic to $\partial\mathbb{D}^2 \times \{\mathrm{pt}\}$), then the desired isotopy is $\varphi \colon M \times I \to M$ with $\varphi_t|(M \smallsetminus X) = \mathrm{Id}$, $\varphi_t(z, \theta) = (z, \theta \pm 2t\pi)$ for all $z \in D_0$, and $\varphi_t(z, \theta) = (z, \theta \pm 2t\pi \cdot (2 - 2|z|))$ for all $z \in A$. Here, the sign $\pm$ is determined by the parity of the curves $H \cap T$. Note also that we only need to do this isotopy once, not m times.

For any other slope the desired isotopy is $\varphi \colon M \times I \to M$ with $\varphi_t|(M \smallsetminus X) = \mathrm{Id}$, $\varphi_t(z, \theta) = (z \cdot \exp(\pm 2t\pi i), \theta)$ for all $z \in D_0$, and $\varphi_t(z, \theta) = (z \cdot \exp(\pm 2t\pi i(2 - 2|z|)), \theta)$ for all $z \in A$. Again, the sign $\pm$ is determined by the parity of the curves $H \cap T$. □

Thus, when T bounds X a solid torus, the sequence $H + nT$ contains only finitely many isotopy classes of Heegaard splittings. This is a contradiction.

6.2. Bounding a cube with a knotted hole.

Suppose now that the two-torus T bounds a *cube with a knotted hole*. That is, $X \subset M$ is a submanifold contained in a three-ball $Y \subset M$, and $T = \partial X$

compresses in Y but not in X. The unique slope of this compressing disk is called the *meridian*.

We require one more definition: A pair of transverse surfaces H and K in a three-manifold M are *compression-free* if all curves of $H \cap K$ are essential on both surfaces.

The main theorem of [7] is:

Theorem 6.4. *Suppose $H \subset M$ is strongly irreducible and the two-torus T bounds $X \subset M$, a cube with a knotted hole. Suppose also that H and T are compression-free with non-trivial intersection. Then:*

- *the components of $H \cap X$ are all annuli and*
- *there is at least one component of $H \setminus T$ which is a meridional annulus, boundary parallel into T.*

So, choose H and T as provided by the hypotheses of Theorem 6.1. Suppose also, as provided by Lemmas 4.1 and 6.2, that $H + nT$ is reduced – all curves of $H \cap T$ are essential and of the same parity.

Claim 6.5. *All curves of $H \cap T$ are meridional on T.*

Proof. If H and T are compression-free, then apply Theorem 6.4 and we are done. If not, then there is a curve of intersection which bounds an innermost disk in H and which is essential on T. As T is not compressible into X, we are done. □

The proof of Theorem 6.1, with X a cube with knotted hole, now splits into two subcases. Either $H \cap T$ is compression-free or not.

6.2.1. The compression-free case. Suppose that $H \cap T$ is compression-free and that $H + nT$ is a reduced sequence. We again wish to prove that infinitely many of the $H + nT$ are pairwise isotopic.

Take nT to be n parallel copies of T, all inside of X. Note that $H \cap T = (H + nT) \cap T$ and $H \setminus X = (H + nT) \setminus X$. Hence, $H + nT$ and T are compression-free.

We repeatedly isotope $H + nT$ via the following procedure: Apply Theorem 6.4 to $H + nT$ and T. Thus, there is a meridional annulus $A \subset (H + nT) \setminus T$ which is boundary parallel into T. Let $B \subset T$ be the annulus to which A is parallel. Denote by Z the solid torus which A and B cobound.

Now, if $A \subset M \setminus X$ then $Z \cap X = B$. In this case, isotope A and all components of $(H + nT) \cap Z$ into X. Begin the procedure again appliying to this new position of $H + nT$.

If $A \subset X$ then $Z \subset X$ as well. In this case all components of $(H+nT) \cap Z$ are meridional annuli which are parallel rel boundary into T. Isotope A and all of the annuli of $(H + nT) \cap Z$ out of X, but keeping them parallel to T. See Figure 16.

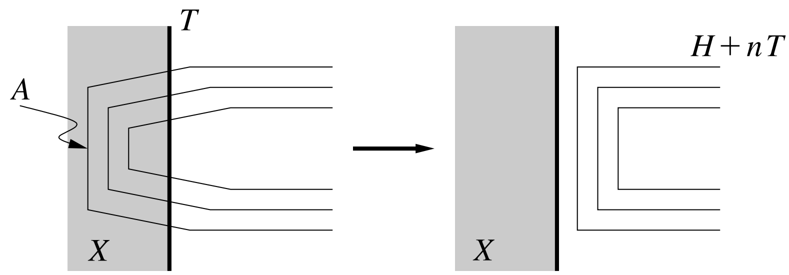

Figure 16: Isotopic pieces of $H + nT$ out of X.

At the end of the procedure, we have isotoped $H + nT$ out of X. The surface $H + nT$ is thus isotopic to the surface which is a union of components of $H \setminus X$ together with a union of annuli parallel to sub-annuli of T. There are only finitely many of the latter (as $H \cap T$ is bounded). This is a contradiction.

6.2.2. The meridional compression case. Suppose now that $H \setminus X$ contains a meridional disk $D \subset H$ for T. Let Y be the three-ball $X \cup \eta(D)$. Note that all the curves $\{\gamma_j\} = H \cap \partial Y$ are parallel in ∂Y. This is because all of the curves $(H + nT) \cap T$ are meridional for T. We think of Y as a copy of $\mathbb{D}^2 \times I$ – "a tall tuna can" – with all of the γ_j of the form $\partial \mathbb{D}^2 \times \{\mathrm{pt}\}$.

For each n, we carry out an inductive procedure: Fix n. Let $Y^0 = Y$ and let $H^0 = H_n = H + nT$. At stage i, there is a "stack of tuna cans" $Y^i \cong \mathbb{D}^2 \times I_i \subset Y^0$, where I_i is a disjoint union of finitely many closed intervals in I. See either side of Figure 17.

Each component of ∂Y^i contains at least one of the curves γ_j. Also, the surface H^0 has been isotoped to a surface H^i so that $H^i \setminus Y^i \subset H^{i-1} \setminus Y^{i-1} \subset H \setminus Y$. It follows that $\partial Y^i \cap H^i$ is a subset of $\cup \gamma_j$. Note that all the components of $\partial Y^i \setminus H^i$ are "vertical" annuli or disks.

Suppose some annulus component of $\partial Y^i \setminus H^i$ is compressible in $M \setminus (Y^i \cup H^i)$. So do the "packing tuna" isotopy: There is a disk D^i with interior in $M \setminus (Y^i \cup H^i)$ and with boundary $\partial D^i \subset \partial Y^i$ (see left side of Fig-

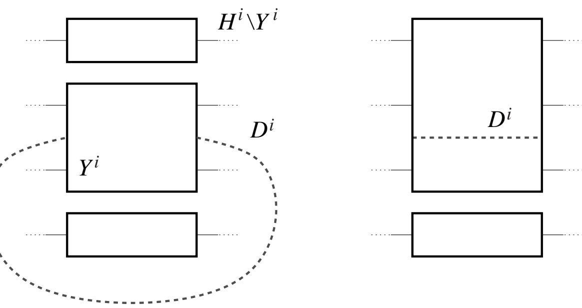

Figure 17: The packing step is illustrated on the left while the slicing step is on the right. The disk D^i is depicted by the dotted line.

ure 17). Let Z be the component of Y^i containing ∂D^i. Then ∂D^i bounds two disks in ∂Z, say E and E'. Then, either $D^i \cup E$ or $D^i \cup E'$ bound a three-ball in M which has interior disjoint from Z. (This is because M is irreducible.) So there is an isotopy of H^i which moves some components of $H^i \smallsetminus Z$ into Z. This reduces the number of curves of intersection $H^i \cap \partial Y^i$. Let H^{i+1} be the new position of $H + nT$. Let Y^{i+1} be equal to the union of all the components of Y^i which meet H^{i+1}. The induction hypotheses clearly hold.

Suppose instead some annulus component of $\partial Y^i \smallsetminus H^i$ is compressible in $Y^i \smallsetminus H^i$. Next, perform the "slice a can in half" move: Let $D^i \subset Y^i$ be such a compressing disk with $\partial D^i = \mathbb{D}^2 \times \{\text{pt}\}$ and $D^i \cap H^i = \emptyset$. See right side of Figure 17. Isotope $D^i \cup H^i$ until D^i is level ($D^i = \mathbb{D}^2 \times \{\text{pt}\}$) while maintaining $D^i \cap H^i = \emptyset$. This isotopy is supported inside of Y^i. Let H^{i+1} be the new position of $H + nT$ and let $Y^{i+1} = Y^i \smallsetminus \eta(D^i)$. Again the induction hypotheses clearly hold.

The procedure terminates after at most $|\{\gamma_j\}| = |(H+nT) \cap Y|$ steps. To see this, note that we can never have $|Y^i|$ greater than the original number of curves $\{\gamma_j\}$. So, we cannot "slice" more than that number of times. Also, the number of components of $(H + nT) \smallsetminus Y = H \smallsetminus Y$ is bounded and $H^i \smallsetminus Y^i$ is contained in $H \smallsetminus Y$. So, we cannot "pack" more than that number of times.

Let m be the largest value of i reached in the above procedure. After the procedure terminates, we have every component of $\partial Y^m \smallsetminus H^m$ being incompressible in both $M \smallsetminus (Y^m \cup H^m)$ and inside $Y^m \smallsetminus H^m$. An innermost disk argument shows that every component of $\partial Y^m \smallsetminus H^m$ is incompressible in $M \smallsetminus H^m$.

Let Z be a component of Y^m. Recall that the curves $\gamma_j \subset \partial Z$ are parallel. Now apply Scharlemann's Local Detection Theorem [15] (for three-balls) to ∂Z. It follows that $H^m \cap Z$ is either a disk or an unknotted annulus.

At the end of the procedure, the surface $H + nT$ has been isotoped to a surface which is a union of components of $H \setminus Y$ together with a union of "vertical" annuli and disks of the form $\mathbb{D}^2 \times \{\mathrm{pt}\}$. There are only finitely many of the latter (as $H \cap \partial Y$ is bounded). So for all n, the splitting $H + nT$ is isotopic to one of these finitely many surfaces, a contradiction. This completes the proof of Theorem 6.1. □

7. New examples.

The goal of the next two sections is to give new examples of $H, K, H + K \subset M$ such that for all integers n, the surface $H + nK$ is a strongly irreducible Heegaard splitting.

Note that the manifolds of Casson–Gordon have Heegaard genus four and larger. Our examples have genus as low as three. Also, our examples, unlike those of [6] and [10], do not involve twisting around a two-sphere in S^3 or require the existence of an incompressible spanning surface.

In the next two sections, we first (7.1) construct our new examples and then (7.2) prove that they have the desired properties.

7.1. Constructing the new examples.

To begin with, we sketch the construction, which has obvious generalizations. Take V a handlebody of genus three or more. Take γ to be a "sufficiently complicated" curve in $H = \partial V$. Double V across H and let W be the other copy of V. Alter the gluing of V to W by Dehn twisting along γ at least six times. This gives M, a closed orientable manifold. Now, we will have a properly embedded surface $K' \subset V$ with $K' \cap \gamma = \emptyset$. Thus K' doubles to give a surface K in M. Adding copies of K to H will give the desired sequence of Heegaard splittings.

Before giving the details, recall:

Definition 7.1. Let V be a handlebody. A simple closed curve $\gamma \subset \partial V$ is *disk-busting* if $\partial V \setminus \gamma$ is incompressible in V.

For the remainder of this section, take V' a handlebody of genus two. (Larger genus is also possible.) Let $\gamma' \subset \partial V'$ be a non-separating disk-busting curve. Set $K' = \partial V' \setminus \eta(\gamma')$. For an example of this, see Figure 18.

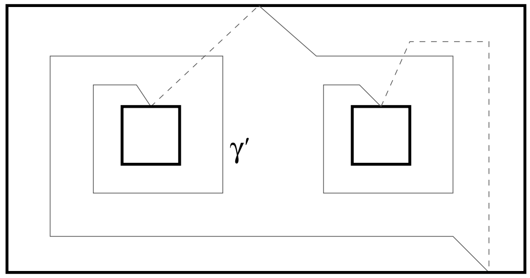

Figure 18: The curve γ' is disk-busting in V'

Take U, a solid torus, and fix a subdisk of the boundary $E \subset \partial U$. Let $V'' = (K' \times I) \cup U$ where $K' \times I$ is glued to U via some homeomorphism between a subdisk of $K' \times \{1\}$ and the disk E. Thus, E and any meridional disk of U (which is disjoint from E) are essential disks in V''. Let $\partial_+ V'' = \overline{((K' \times \{1\}) \cup \partial U) \setminus E}$. Let $\partial_- V'' = K' \times \{0\}$.

Now, choose $\gamma \subset \partial_+ V''$, a disk-busting curve for V''. See Figure 19, for example.

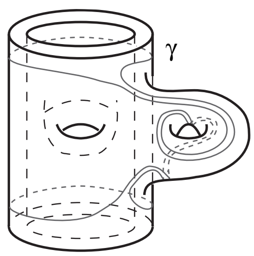

Figure 19: The curve $\gamma \subset \partial_+ V''$ is disk-busting for V''.

Form a genus three handlebody V by gluing V' to V'' via the natural map between $K' \subset \partial V'$ and $\partial_- V'' \subset \partial V''$. It is easy to check that γ is disk-busting in V. As this fact is not needed in what follows, we omit the proof. However, see Figure 20 for a picture.

Now, form a manifold $D(V)$ by *doubling* V – that is, let W be an identical copy of V and glue these two handlebodies by the identity map between their boundaries. Finally, obtain a closed three-manifold M by altering the gluing between V and W by Dehn twisting at least six times along γ. Again, we do not need the fact that H is a strongly irreducible Heegaard splitting, nor the consequence that M is irreducible.

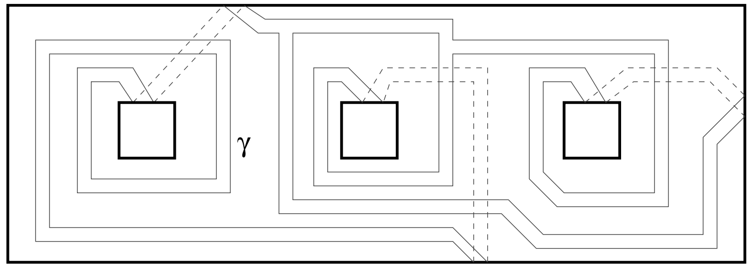

Figure 20: To obtain M, double the handlebody shown and Dehn twist at least six times along γ.

Let $K = D(K') \subset M$ be the double of K'. As K' is connected, so is K. The surface K is also incompressible in M, but as this fact is not required in the sequel, we omit any direct proof.

Next, choose the Haken sum of H and K: Label the two curves of $K \cap H$ by α and β. Recall that γ' was chosen to be disk-busting and non-separating in $\partial V'$. Note that α and β cobound an annulus $A = \eta(\gamma') = \partial V' \setminus K' \subset H$ and that α and β cut K into two halves $K' \subset V$ and $K'' \subset W$. Also, α and β cut H into two connected pieces, A and $\overline{H \setminus A} \cong \partial_+ V''$. Note that K and H are both separating surfaces in M. For a schematic picture, see the left side of Figure 21.

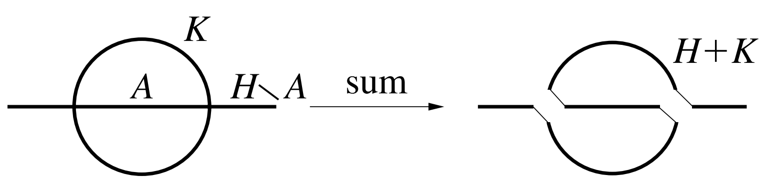

Figure 21: Picture showing (schematically) the relative positions of H, K, and $H + K$.

So choose the Haken sum of H and K as indicated by the right side of Figure 21. To be precise, let $\mathcal{H}\colon M \times I \to M$ be an ambient isotopy of M which is fixed pointwise outside of $\eta(A)$, moves α across A to β, sends the solid torus $\eta(\alpha)$ to $\eta(\beta)$, takes $K \cap \eta(\alpha)$ to $K \cap \eta(\beta)$, and takes $H \cap \eta(\alpha)$ to $H \cap \eta(\beta)$. Now choose any Haken sum of H and K along α and use $\mathcal{H}$ to transfer this choice to β. Again, see Figure 21. This defines the Haken sum $H + K$ and thus defines $H + nK$.

7.2. Demonstrating the desired properties.

We now can state:

Theorem 7.2. *Given V and γ as above, the surface $H + nK$ is a strongly irreducible Heegaard splitting of M, for any even $n > 0$.*

Remark 7.3. In fact $H + nK$ is a strongly irreducible Heegaard splitting for any integer n. We restrict n to positive and even, in order to simplify the proof.

Remark 7.4. The curve γ in Figure 20 does not give a hyperbolic manifold because the resulting M contains a pair of Klein bottles. See Figure 22 for a more complicated curve γ. This curve does yield a hyperbolic manifold with the desired sequence of Heegaard splittings.

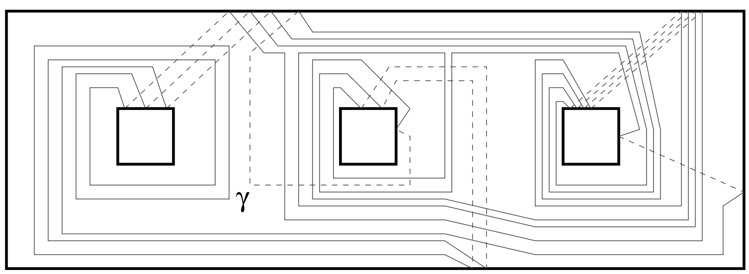

Figure 22: Doubling the handlebody and twisting along the curve shown gives a hyperbolic manifold satisfying the hypotheses of Theorem 7.2

The proof of Theorem 7.2 divides naturally into two pieces.

Claim 7.5. *For positive, even n the surface, $H+nK$ is a Heegaard splitting.*

Proof. Recall that $M \setminus \eta(H \cup K)$ is homeomorphic to the disjoint union of V', V'', W', and W''. Also, the curves $K \cap H$ are denoted by α and β.

Let nK be n evenly spaced parallel copies of K in $\eta(K)$. That $H + nK$ is connected follows from our choice of Haken sum along α and β. $H + nK$ is separating because H and K are separating. See Figure 23.

Label the closures of the two components of $M \setminus (H + nK)$ by V_n and W_n where V_n contains $V \setminus \eta(K)$ and W_n contains $W \setminus \eta(K)$. (This is where

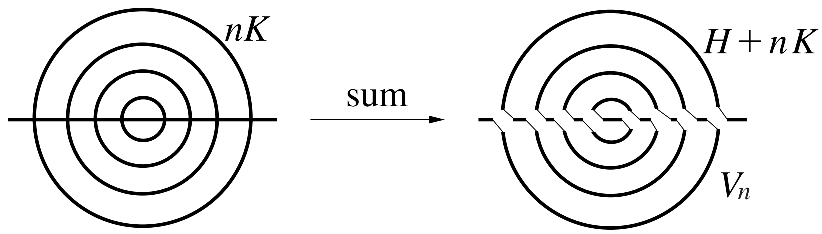

Figure 23: Adding nK to H yields a connected, separating surface.

"n positive and even" is used. Again, see the right half of Figure 23 for a picture with $n = 4$.)

Consider now the collection of closed annuli $\overline{H \cap \text{interior}(V_n)}$. Cutting V_n along all of these gives several components: The first, V'_n, contains $V' \setminus \eta(K)$ while the second, V''_n, contains $V'' \setminus \eta(K)$ and the rest are isotopic to $\eta(K')$ or $\eta(K'')$. Let V_n^P be the submanifold of V_n obtained by taking the union of all the latter (thus, not V'_n or V''_n). Here the "P" in the superscript stands for "product".

Let $A_n \cup B_n$ be the two annuli in $\overline{H \cap \text{interior}(V_n)}$ which are also in ∂V_n^P. Here we assign labels so that A_n meets the component of $H \cap \eta(K)$ which contains α. Thus, as n is even, B_n meets the component of $H \cap \eta(K)$ which contains β. We have realized V_n as the union of three pieces V'_n, V''_n, and V_n^P, glued to each other along the annuli A_n and B_n.

Recall now that $V'_n \cong V'$, $V''_n \cong V''$ and thus both are handlebodies. Also, the annulus B_n is *primitive* in V''_n: There is a disk in V''_n meeting B_n in a single co-core arc. See Figure 19 and notice that B_n is parallel to $\beta \times I \subset \partial K' \times I \subset V''$.

Since V_n^P and V''_n are handlebodies it follows that $V_n^P \cup_{B_n} V''_n$ is also a handlebody. Also, as V_n^P is a product, the annulus A_n is primitive on $V_n^P \cup_{B_n} V''_n$. So, since V'_n is a handlebody, we finally have $V_n = V'_n \cup_{A_n} V_n^P \cup_{B_n} V''_n$ is a handlebody and applying similar arguments to W_n the surface $H + nK$ is a Heegaard splitting of M. □

Claim 7.6. *For positive, even n the surface $H + nK$ is strongly irreducible.*

Proof. Recall that γ was a curve in $\partial_+ V''$ and thus also a curve in $H + nK$. Recall that M was obtained by doubling V and then twisting at least six times along γ.

We will show that γ is disk-busting for V_n and thus for W_n. The proof of the claim will then conclude with a Theorem of Casson [11] proving that $H + nK$ is strongly irreducible.

Choose D, any essential disk in V_n. Choose a hyperbolic metric on $H+nK$. Tighten $\partial D, \partial A_n, \partial B_n, \gamma$ to be geodesics. Perform a further isotopy of D relative to ∂D to minimize the intersection of D with $A_n \cup B_n$.

Now note that A_n and B_n are incompressible in V_n. If not, then some boundary component of A_n bounds a disk in V_n' or some boundary component of B_n bounds a disk in V_n''. (None of these curves bound disks in V_n^P because neither K' nor K'' is a planar surface.) The first is impossible because ∂A_n is parallel to $\gamma' \subset V_n'$ which is disk-busting. The second is impossible because $\partial_- V''$ is π_1-injective into V_n''.

So no component of $D \cap (A_n \cup B_n)$ is a simple closed curve. Let D' be an outermost disk of $D \setminus (A_n \cup B_n)$: That is, D' is the closure of a disk component of $D \setminus (A_n \cup B_n)$ and D' meets $A_n \cup B_n$ in at most one arc. It follows that D' is an essential disk in V_n', V_n^P, or V_n''. (If not we could decrease $|\partial D \cap (A_n \cup B_n)|$, an impossibility.)

There are three cases: D' lies in V_n', V_n^P, or V_n''.

Suppose first that $D' \subset V_n'$. If $D' = D$ is disjoint from A_n then, as A_n is parallel to γ' in $\partial V_n'$, we may isotope D to be disjoint from γ'. This contradicts our choice of γ' being disk-busting in V_n'. If $D' \subset D$ is a strict inclusion then $D' \cap A_n$ is a single arc. Then, D' may be isotoped either to lie disjoint from γ' ($D' \cap A_n$ is inessential in A_n) or to meet γ' in a single point ($D' \cap A_n$ is essential in A_n). Again, this is because γ' and A_n are parallel on the boundary on V_n'. The former contradicts γ' being disk-busting. For the latter take two parallel copies of D' in V_n' and band these together along $\gamma' \setminus \eta(D')$ to obtain an essential disk disjoint from γ'. This is again a contradiction.

The next possibility is that D' lies in V_n^P. However, this cannot happen as V_n^P is the trivial I-bundle over a surface.

We conclude that D' is an essential disk in V_n''. It follows that D' intersects γ because, γ was chosen to be disk-busting for $V'' \cong V_n''$. Thus, D has a non-trivial geometric intersection with γ. As our choice of D is arbitrary, we conclude that $\gamma \subset H + nK$ is disk-busting for both V_n and W_n.

Note that $D(V)$, the double of V, is reducible. To obtain M from $D(V)$, we cut open along a neighborhood of γ in $\partial_+ V''$ and Dehn twisted at least six times. It follows that $H + nK$ gives a Heegaard splitting of $D(V)$ and all of these are reducible in $D(V)$. (To see this, recall that the disk E cut the solid torus U from V''. Thus, the double $D(E)$ is a reducing sphere for all of the $H + nK$ in $D(V)$.) Thus, we are in a position to apply the following Theorem of Casson (see the appendix of [11]):

Theorem 7.7. *Suppose $\gamma \subset H \subset N$ is a curve on a weakly reducible Heegaard splitting surface of a closed orientable manifold N, and that $H \setminus \gamma$ is incompressible in N. Cutting N open along a neighborhood of γ in H and Dehn-twisting at least six times gives a strongly irreducible splitting H' of the new manifold N'.*

It follows that for all positive, even n the splittings $H + nK$ are strongly irreducible. We are done. □

Claim 7.5 and Claim 7.6 together prove Theorem 7.2. □

Remark 7.8. There is a well-known relationship, due to H. Rubinstein [14] and M. Stocking [18], between strongly irreducible splittings and almost normal surfaces. In particular, strongly irreducible surfaces should contain a single place (or "site" in Rubinstein's terminology) where the curvature is highly negative. This supposedly corresponds to the almost normal octagon or annulus of the almost normal surface. In our examples, we find that the subsurface $\partial_+ V''$ is the distinguished subsurface of $H + nK$ which presumably contains this special site.

8. Questions.

Recall that Theorem 5.1 was originally conjectured by Sedgwick along with the much stronger:

Conjecture 1.4**.** *Let M be a closed, orientable* 3*-manifold which contains infinitely many irreducible Heegaard splittings that are pairwise non-isotopic. Then, M is Haken.*

This conjecture may be split, roughly, into two parts. First we have the so-called "Generalized Waldhausen Conjecture":

Conjecture 8.1. *Let M be a closed, orientable* 3*-manifold which contains infinitely many Heegaard splittings, pairwise non-isotopic, all of the same genus. Then M is toroidal.*

Note that this has been claimed by W. Jaco and Rubinstein. However, no manuscript is available as of the writing of this paper.

The other half of Sedgwick's conjecture deals with splittings of increasing genus and is the inspiration for our current work:

Conjecture 8.2. *Let M be a closed, orientable* 3*-manifold which contains irreducible Heegaard splittings of arbitrarily large genus. Then M is Haken.*

We now turn to questions about examples. In all of the manifolds listed above, which contain splittings of arbitrarily large genus, the three-manifold has had Heegaard genus three or higher. Kobayashi asks:

Question 8.3. *Is there an example of a Heegaard genus two-manifold which admits strongly irreducible splittings of arbitrarily large genus?*

Remark 8.4. Note that there are examples of toroidal manifolds containing infinitely many strongly irreducible splittings all of the form $H+nT$. Here, H is a genus two Heegaard splitting and T is an incompressible torus; see [12].

Sedgwick, in [17], has shown that the Casson–Gordon examples satisfy the so-called "Stabilization Conjecture [16]". That is, for any two splittings H and H' obtained from the same pretzel knot, after stabilizing the higher genus splitting once, we may destabilize to find the lower genus splitting. Sedgwick's techniques apply to all of the splittings discussed in Section 3. Kobayashi suggests that the examples of $H + nK$ given in this paper, after stabilizing twice, should destabilize about $2n$ times.

Question 8.5. *Does one stabilization suffice?*

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