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From spatially periodic instantons to singular monopoles

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The main result is a computation of the Nahm transform of a SU(2)-instanton over $\mathbb{R} \times T^3$, called spatially-periodic instanton. It is a singular monopole over T^3, a solution to the Bogomolny equation, whose rank is computed and behavior at the singular points is described.

1. Introduction.

Heuristically, there is a correspondence, called the *Nahm transform*, between

1. solutions to the anti-self-dual (ASD) equation, or its appropriate dimensional reduction, on the quotient of $\mathbb{R}^4$ by a closed subgroup Λ of $\mathbb{R}^4$, and satisfying a finite energy condition, and
2. solutions to some associate equation satisfying some boundary condition on the quotient of $\mathbb{R}^{4*}$ by the dual subgroup $\Lambda^* = \{f \in \mathbb{R}^{4*} \mid f(\Lambda) \subset \mathbb{Z}\}$.

This heuristic comes from a re-engineering due to Nahm [22] of the ADHM construction of instantons on $\mathbb{R}^4$ [1]. Nahm's approach has the advantage of being transportable to quotients by non-trivial subgroup Λ as well, with some ad hoc efforts necessary in each case.

Nahm gave an outline of the correspondence for classical instantons ($\Lambda = \{0\}$) and for monopoles on $\mathbb{R}^3$ ($\Lambda = \mathbb{R}$). Corrigan and Goddard in [10] completed the details of the ADHM construction following Nahm's guideline, while Hitchin in [13] completed the story for SU(2)-monopole on $\mathbb{R}^3$. In [23], Nakajima rendered Hitchin's proof more parallel to the ADHM story.

This framework guided several other authors in the quest for an understanding of other moduli spaces of instantons (or their appropriate dimensional reduction) on various quotients of $\mathbb{R}^4$: for instantons on T^4, see [27, 5];

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for monopoles of other classical groups, see [14]; for calorons, or instantons on $S^1 \times \mathbb{R}^3$, see [25, 24]; for instantons on $T^2 \times \mathbb{R}^2$, see [19, 15, 16, 17, 4]; and for monopoles on $\mathbb{R}^2 \times S^1$, see [8, 9, 7]. Marcos Jardim wrote a survey paper [18] on the Nahm transform, and the reader is invited to consult it for some insights on an even more general framework in which to place the above referenced literature and the present paper.

Apart from some numerical approximations and remarks in [28] and a computation of the Nahm transform of charge 1 instantons in [29], the case of the *spatially periodic instantons*, instantons on $\mathbb{R} \times T^3$, has been largely ignored. The present paper starts the groundwork necessary to close that gap. We prove here that the Nahm transform of an instanton on $\mathbb{R} \times T^3$ is a singular monopole on T^3 with specific behavior at the singular points.

This paper is organized as follows. The main result on the Nahm transform of instantons on $\mathbb{R} \times T^3$ and its singular behavior is spelled out in Section 5 after the adequate language is explained. Before reaching this result, it is useful to go over a brief overview of the classical ADHM construction in Section 2, then check the bigger picture of the Nahm transform heuristic in Section 3, and then zoom in on the Fredholmness properties of the Dirac operators on $\mathbb{R} \times T^3$ in Section 4. The proof of the result splits three ways: first, the rank of the transformed bundle is computed at the end of Section 5; then, a splitting of the transformed bundle around the singularities is developed in Section 6; and finally, the asymptotic of the Higgs field is proved in Section 7.

2. The classical ADHMN.

The classical work of Atiyah, Drinfeld, Hitchin and Manin, termed ADHM construction, classifies all the solutions to the ASD equation on $\mathbb{R}^4$, up to gauge equivalence. Once viewed under the umbrella of the Nahm transform heuristic, thus adding an N to form *ADHMN*, the classification is as follows.

A connection A on a $\mathrm{SU}(n)$-bundle E over $\mathbb{R}^4$ whose curvature F_A satisfies the ASD equation $*F_A = -F_A$ and the finite energy condition $\int_{\mathbb{R}^4} |F_A|^2 < \infty$ gives rise, through an analysis of its Dirac operator $\not{\mathfrak{D}}_A$, to a set of algebraic data: two vector spaces

$$
\begin{aligned}
V &= L^2 \cap \ker(\not{\mathfrak{D}}_A^*), \text{ and} \\
W &= \text{ bounded harmonic sections of } E \text{ for } \nabla_A,
\end{aligned}
$$

and five maps

$$\Phi_1, \dots, \Phi_4 \colon V \to W,$$
$$\eta \colon V \to S^+ \otimes W.$$

Since the vector space is built using the augmented Dirac operator $\mathfrak{D}^*_A$ acting on sections of $S^- \otimes E$, the dimension of V can be computed by some index theorem, and

$$\dim V = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} |F_A|^2$$

provided the cokernel $L^2 \cap \ker(\mathfrak{D}_A)$ is $\{0\}$. It is indeed so, as the Weitzenbock formula

$$\mathfrak{D}^*_A \mathfrak{D}_A = \nabla^*_A \nabla_A + cl(F^+_A)$$

clearly establishes: for an instanton connection, the Clifford multiplication term vanishes and a L^2 solution ϕ to $\mathfrak{D}^*_A \phi = 0$ must be parallel, hence 0 since $\mathbb{R}^4$ has infinite volume.

The map $\Phi_i = P m_{x_i}$ is the composite of the multiplication by the ith coordinate, denoted m_{x_i} and the L^2-projection P on $\ker(\mathfrak{D}^*_A)$, while the map η encodes the asymptotic behavior of elements of V.

For an instanton (E, A), the associated algebraic data (V, W, Φ, η) satisfy a non-degeneracy condition and the ADHM equation, the precise formulation of which is not important here. This "ADHM transform" places in one-to-one correspondence instantons modulo gauge equivalence with non-degenerate solutions to the ADHM equation modulo some symmetry group action. A complete description of this construction can be found in [11, Chapter 3], and in the author's thesis [6].

It is a fruitful idea to interpret the set of maps $\Phi = (\Phi_1, \dots, \Phi_4)$ as a constant connection form

$$\overline{B} = \Phi_1 dx^1 + \cdots \Phi_4 dx^4$$

on the trivial bundle $\underline{V}$ over $\mathbb{R}^4$ with fiber V. The curvature $F_{\overline{B}}$ of $\overline{B}$ splits as

$$F_{\overline{B}} = (\text{ASD part}) + (\text{ SD part involving } \eta).$$

Morally, the idea is that the transformed connection $\overline{B}$ on $\mathbb{R}^{4^*}$, invariant under the action of $\Lambda^* = \mathbb{R}^{4^*}$, is almost anti-self-dual, and the self-dual part is determined by the asymptotic behavior of harmonic spinors.

3. The Nahm transform heuristic.

The work of Nahm provides a framework in which to think about the classification of all the finite energy solutions to the ASD equation on a quotient $\mathbb{R}^4/\Lambda$. Philosophically, once we find the appropriate codomain for the Nahm transform to be described in this section, it should be an isomorphism. This idea has been shown to work in many cases, as explained in the introduction.

A connection A on a $\mathrm{SU}(n)$-bundle E over $\mathbb{R}^4$, invariant under the action of a closed subgroup Λ, and whose curvature F_A satisfies the ASD equation

$$*F_A = -F_A$$

and the finite energy condition

$$\int_{\mathbb{R}^4/\Lambda} |F_A|^2 < \infty$$

gives rise, this time, to a bundle V with a connection B over $\mathbb{R}^{4^*}/\Lambda^*$. Those objects are constructed in the following way.

For an element z of $\mathbb{R}^{4^*}$, the space of $\mathbb{R}$-valued linear functions on $\mathbb{R}^4$, we define the bundle L_z over $\mathbb{R}^4$ to be a trivial $\mathbb{C}$-bundle with connection

$$\omega_z := 2\pi i z = 2\pi i \sum_{j=1}^{4} z_j dx^j.$$

For $z' \in \Lambda^*$, the flat bundles L_z and $L_{z+z'}$ over $\mathbb{R}^4/\Lambda$, both invariant under the action of Λ, are isomorphic. We write A_z for the connection $A \otimes 1 + 1 \otimes \omega_z$ on $E \otimes L_z = E$. For $z \in \mathbb{R}^{4^*}$, consider the operator

$$\not{\mathfrak{D}}^*_{A_z} \colon \Gamma(\mathbb{R}^4, S^- \otimes E \otimes L_z) \to \Gamma(\mathbb{R}^4, S^+ \otimes E \otimes L_z).$$

A section of the bundle $S^- \otimes E \otimes L_z$ is said to be in L^2_Λ if it is invariant under the action of Λ and if its L^2-norm over $\mathbb{R}^4/\Lambda$ is finite.

The first ingredient of the Nahm transform of the instanton (E, A) is the family of vector spaces

$$V_z := L^2_\Lambda \cap \ker(\not{\mathfrak{D}}^*_{A_z}).$$

Since the vector space V_z is built using the augmented Dirac operator $\not{\mathfrak{D}}^*_{A_z}$ acting on sections of $S^- \otimes E$, the dimension of V_z can often be computed by an appropriately chosen index theorem, and it is constant on connected components on which $\not{\mathfrak{D}}_{A_z}$ is Fredholm provided the cokernel $L^2_\Lambda \cap \ker(\not{\mathfrak{D}}_{A_z})$

is $\{0\}$. For a quotient $\mathbb{R}^4/\Lambda$ of infinite volume, it is indeed so, as the Weitzenbock formula

$$\not{\mathfrak{D}}_A^* \not{\mathfrak{D}}_A = \nabla_A^* \nabla_A + cl(F_A^+)$$

clearly establishes: for an instanton connection, the Clifford multiplication term vanishes and a L^2 solution ϕ to $\not{\mathfrak{D}}_A^* \phi = 0$ must be parallel, hence 0 because of the infinite volume condition. For a quotient of finite volume, we must add an extra condition to ensure the cokernel is trivial.

It turns out in many cases that $\not{\mathfrak{D}}_{A_z}^*$ is not Fredholm for every z, which is a good thing. Suppose for example that $\not{\mathfrak{D}}_{A_z}^*$ was Fredholm everywhere when $\Lambda = \mathbb{Z}^3$. As we explore in this present paper, the object created by the Nahm transform is a monopole over T^3. But as one can show (see [26, Prop. 1]), smooth monopoles over compact 3-manifolds are not very interesting.

Set $g_z(x) := e^{2\pi i z(x)}$. Notice that for any section ϕ of $S^- \otimes E$, we have $\not{\mathfrak{D}}_{A_z}^*(g_z \phi) = g_z \not{\mathfrak{D}}_A^* \phi$. Then, for all $z' \in \Lambda^*$, we have an isomorphism

$$g_{z'} \colon V_z \to V_{z+z'}, \tag{3.1}$$

hence V is a bundle over $\mathbb{R}^{4^*}/\Lambda^*$.

3.1. First viewpoint: on $\mathbb{R}^{4^*}$, a curvature computation.

In the understanding of the ADHM construction, it was beneficial to view the maps Φ_i as parts of a connection on the bundle $\underline{V}$ on $\mathbb{R}^{4^*}$, without passing to the quotient. We do similarly here and consider first the bundle V on an open subset of $\mathbb{R}^{4^*}$ on which the Dirac operator is Fredholm.

We define a connection B on V. Each fiber V_z is in fact contained in the vector space $L^2_\Lambda(S^- \otimes E)$. We can then consider the trivial connection d^z in the trivial bundle of fibers $L^2_\Lambda(S^- \otimes E)$, and its projection Pd^z to V.

The operator $\not{\mathfrak{D}}_{A_z}^* \not{\mathfrak{D}}_{A_z}$ should be invertible, and we use its inverse, the Green's operator $G_{A_z} = (\not{\mathfrak{D}}_{A_z}^* \not{\mathfrak{D}}_{A_z})^{-1}$, to define the projection P by the formula

$$P = 1 - \not{\mathfrak{D}}_{A_z} G_{A_z} \not{\mathfrak{D}}_{A_z}^*.$$

To parallel the ADHMN story, let us now compute the curvature F_B of B. To simplify the notation, we set $\Omega := 2\pi i \sum_{j=1}^4 cl(dx^j) dz^j$. Then $[d^z, \not{\mathfrak{D}}_{A_z}] = \Omega$, and similarly for $\not{\mathfrak{D}}_{A_z}^*$.

The curvature F_B can be computed as follows:

$$\begin{aligned}\left\langle (Pd^z)^2\phi, \psi \right\rangle &= \langle d^z P d^z \phi, \psi\rangle \\ &= \langle P d^z \phi, d^z\psi\rangle - \langle d^z\phi, d^z\psi\rangle \\ &= -\left\langle \not{D}_{A_z} G_{A_z} \not{D}_{A_z} d^z\phi, d^z\psi \right\rangle \\ &= \left\langle \not{D}_{A_z} G_{A_z} \Omega\phi, d^z\psi \right\rangle .\end{aligned}$$

Let ν be the normal vector field to $S^{r-1}(R) \times T^s$. The integration by parts necessary to bring $\not{D}_{A_z}$ on the right-hand side of the scalar product introduces a boundary term

$$\partial\text{-term} := \lim_{R\to\infty} \int_{S^{r-1}(R)\times T^s} \langle cl(\nu) G_{A_z} \Omega\phi, d^z\psi\rangle . \tag{3.2}$$

Performing the said integration by parts, we obtain

$$\begin{aligned}\langle F_B\phi, \psi\rangle &= \left\langle G_{A_z}\Omega\phi, \not{D}_{A_z} d^z\psi \right\rangle + \partial\text{-term} \\ &= -\langle G_{A_z}\Omega\phi, \Omega\psi\rangle + \partial\text{-term} \\ &= \langle G_{A_z}\phi, \Omega\wedge\Omega\psi\rangle + \partial\text{-term}.\end{aligned}$$

In terms of the usual basis ϵ_j and $\bar{\epsilon}_j$ of respectively $\bigwedge^+$ and $\bigwedge^-$, we have

$$\Omega\wedge\Omega = -4\pi^2 \sum_{j=1}^{3} \big(cl(\epsilon_j)\epsilon_j + cl(\bar{\epsilon}_j)\bar{\epsilon}_j \big).$$

Since $\bigwedge^+$ acts trivially on $S^- \otimes E$, the first term of the curvature is ASD.

In the case we are studying at this moment, the ∂-term is 0.

3.2. Second viewpoint: on a 3-dimensional quotient, the Bogolmolny equation.

Let us now shift our perspective and look at V and $\overline{B}$ from the viewpoint of the quotient. Suppose some $\mathbb{R}$ is in Λ^*, say as the axis z_1. In fact, suppose here that $\Lambda = \mathbb{Z}^3$, and thus that $\Lambda^* = \mathbb{R}\times\mathbb{Z}^3$. Set $g_z(x) = e^{2\pi i x_1 z_1}$. Then,

$$g(\overline{B}) = -2\pi i P m_{x_1} dz^1 + P\left(\frac{\partial}{\partial z_2} dz^2 + \cdots + \frac{\partial}{\partial z_4} dz^4 \right).$$

So using this gauge transformation, we render $\overline{B}$ independent of the z_1 coordinate. We define the Higgs field Φ by

$$\Phi = -2\pi i P m_{x_1},$$

and the connection B on $\mathbb{R}^{4^*}/\Lambda^* = T^3$ by

$$B = Pd^z,$$

where z represents here the coordinates (z_2, z_3, z_4) on T^3. As we just saw,

$$g(\overline{B}) = \Phi dz^1 + B.$$

Should we be able to prove that ∂-term $= 0$, it would be so that $g(\overline{B})$ is ASD. It is in fact so, as we see in the next section, and thus (B, Φ) satisfies the dimensional reduction of the ASD equation

$$\nabla_B \Phi = *F_B$$

called the *Bogomolny equation.*

4. Fredholmness of the Dirac operator.

It is crucial now to understand exactly for which $z \in T^3$ the Dirac operator $\not{\mathfrak{D}}^*_{A_z}$ acting on L^2 sections of $S^- \otimes E$ over $\mathbb{R} \times T^3$ is Fredholm.

Let us start with a SU(2)-instanton (E, A) on $\mathbb{R} \times T^3$ and call t the $\mathbb{R}$-coordinate. Modulo gauge transformation, we can pick a representative in temporal gauge: A has no dt term and can be seen as a path of connections on T^3, parameterized by $\mathbb{R}$. In temporal gauge, the Dirac operator splits as

$$\not{\mathfrak{D}}^*_A = -\frac{\partial}{\partial t} + D_A$$

with D_A the Dirac operator on the cross-section $\{t\} \times T^3$. Furthermore, as $t \to \infty$ and $t \to -\infty$, the connection A has flat limits Γ_+ and Γ_-; see [21, Theorem 4.3.1]. Consequently, the operator D_{A_z} limits to $D_{\Gamma_+\ z}$ and $D_{\Gamma_-\ z}$ at $+\infty$ and $-\infty$. It is a crucial observation of Atiyah–Patodi–Singer [2] that the unbounded operator $\not{\mathfrak{D}}^*_{A_z} : L^2 \to L^2$ is Fredholm if and only if 0 is not in the spectrum of either $D_{\Gamma_+\ z}$ or $D_{\Gamma_-\ z}$; see [6, Chapter 6] for a very detailed account.

As it turns out, any flat SU(2) bundle over a 3-manifold splits as a sum of flat $U(1)$-bundles. Our bundle E, restricted to $\pm\infty$, splits respectively as

$$E = L_{w_\pm} \oplus L_{-w_\pm},$$

for some $w_\pm \in \mathbb{R}^{3^*}$. The spectrum of $D_{\Gamma_+\ z}$ is thus the multiset

$$\mathrm{Spec}(D_{\Gamma_+ z}) = \pm 2\pi \left|\Lambda^*_{\mathbb{Z}} - w_+ - z\right| \cup \pm 2\pi \left|\Lambda^*_{\mathbb{Z}} + w_+ - z\right| \tag{4.1}$$

for the part $\Lambda^*_{\mathbb{Z}} \cong \mathbb{Z}^3$ of Λ^* in $\mathbb{R}^{3*}$, and similarly for $D_{\Gamma_- z}$; see [6, Chapter 3].

Thus, $\not{\mathfrak{D}}^*_{A_z}$ is Fredholm as long as z is not in the set

$$W = \{w_+, -w_+, w_-, -w_-\}.$$

Keeping a parallel with the notation for the ADHMN story, the set W is in some sense our set of "infinity data," although in a much milder way than for $\mathbb{R}^4$.

It is appropriate at this point to ask for which z is $\not{\mathfrak{D}}^*_{A_z}$ Fredholm when we change the domain to allow for more or less growth. Choosing a weight $\delta \in \mathbb{R}^2$, say $\delta = (\delta_-, \delta_+)$, and a weighing function σ_δ such that

$$\sigma_\delta = \begin{cases} e^{-\delta_- t}, & \text{for } t < -1, \\ e^{-\delta_+ t}, & \text{for } t > 1, \end{cases}$$

we define the weighted L^2-norm

$$\|f\|_{L^2_\delta} := \|\sigma_\delta f\|_{L^2},$$

and naturally

$$L^2_\delta := \left\{ f \in L^2_{loc} \mid \|f\|_{L^2_\delta} < \infty \right\}.$$

We omit the bundle from the notation, as it should always be clear which bundle is involved.

Similarly, we can define weighted Sobolev spaces. These include only those L^2_δ sections whose derivatives are also in L^2_δ. Fix a connection ∇ on E, and set

$$W^{1,2}_\delta := \{ f \in L^2_\delta \mid \nabla f \in L^2_\delta \}.$$

Keeping in mind that the first coordinate of the weight describes the growth at $-\infty$ while the second describes the growth at $+\infty$, we define the grid

$$\mathfrak{G}_A := \operatorname{Spec}(D_{\Gamma_-}) \times \mathbb{R} \;\cup\; \mathbb{R} \times \operatorname{Spec}(D_{\Gamma_+})$$

in the weight space $\mathbb{R}^2$. Naturally, the Atiyah–Patodi–Singer condition becomes

$$\not{\mathfrak{D}}^*_{A_z} \colon W^{1,2}_\delta \to L^2_\delta \ \text{ is Fredholm if and only if } \ \delta \notin \mathfrak{G}_{A_z}.$$

We define the spaces

$$\begin{aligned} \ker(\delta) &:= \ker(\not{\mathfrak{D}}_A \colon W^{1,2}_\delta \to L^2_\delta), \\ \ker^*(\delta) &:= \ker(\not{\mathfrak{D}}^*_A \colon W^{1,2}_\delta \to L^2_\delta), \end{aligned} \tag{4.2}$$

and the integers

$$
\begin{aligned}
\operatorname{ind}(\delta) &:= \operatorname{ind}(\not{\mathfrak{D}}_A \colon W^{1,2}_{\delta} \to L^2_{\delta}) \\
N(\delta) &:= \dim\ker(\delta), \text{ and} \\
N^*(\delta) &:= \dim\ker^*(\delta).
\end{aligned} \tag{4.3}
$$

Since $(L^2_\delta)^* = L^2_{-\delta}$, elliptic regularity tells us that $\dim\operatorname{coker}(\not{\mathfrak{D}}_A) = N^*(-\delta)$, hence

$$\operatorname{ind}(\delta) = N(\delta) - N^*(-\delta).$$

That the formal adjoint $\not{\mathfrak{D}}^*_A$ on $W^{1,2}_{-\delta}$ is really the adjoint of $\not{\mathfrak{D}}_A$ on $W^{1,2}_{\delta}$ is guaranteed by the following lemma.

Lemma 4.1. *The subspace* $\ker^*(-\delta)$ *of* $L^2_{-\delta} = (L^2_\delta)^*$ *kills* $\operatorname{Im}(\delta)$ *in the* L^2 *natural pairing.*

Proof. Suppose ϕ is a smooth function with compact support. Then for all $\psi \in \ker^*(-\delta)$, we have $\langle \psi, \not{\mathfrak{D}}\phi \rangle = \langle \not{\mathfrak{D}}^*\psi, \phi \rangle = 0$. Since C^∞_c is dense in $W^{1,2}_\delta$, the lemma holds. □

The operator $\not{\mathfrak{D}}^*_{A_z} \colon W^{1,2}_\delta \to L^2_\delta$ is conjugate to the operator $\not{\mathfrak{D}}^*_{A_z} + \sigma_\delta cl(\text{grad } \sigma^{-1}_\delta)$ from $W^{1,2}$ to L^2. So the family parameterized by δ in an open square delimited by $\mathfrak{G}_{A_z}$ is continuous and hence has constant index. In fact, the dimensions of the kernel and the cokernel are also constant in an open square. The proof is easy and can be found in [6, Theorem 6.3-2].

As we cross a wall in $\mathfrak{G}_A$ to change from one open square to another, the index ind of $\not{\mathfrak{D}}_A$ and the index ind* of $\not{\mathfrak{D}}^*_A$ change as follows:

$$
\begin{aligned}
\operatorname{ind}(\delta) &= \operatorname{ind}(\eta) + \dim\{D_{\Gamma_+}\phi = -\lambda\phi\}, \text{ and} \\
\operatorname{ind}^*(\delta) &= \operatorname{ind}^*(\eta) + \dim\{D_{\Gamma_+}\phi = \lambda\phi\}
\end{aligned} \tag{4.4}
$$

when $\delta_+ < \eta_+$, and δ and η are in adjacent open squares separated by the wall $\mathbb{R} \times \{\lambda\} \subset \mathfrak{G}_A$;

$$
\begin{aligned}
\operatorname{ind}(\delta) &= \operatorname{ind}(\eta) + \dim\{D_{\Gamma_-}\phi = -\lambda\phi\}, \text{ and} \\
\operatorname{ind}^*(\delta) &= \operatorname{ind}^*(\eta) + \dim\{D_{\Gamma_-}\phi = \lambda\phi\}
\end{aligned}
$$

when $\delta_- > \eta_-$, and δ and η are in adjacent open squares separated by the wall $\{\lambda\} \times \mathbb{R} \subset \mathfrak{G}_A$.

When the limit Γ_+ is such that the kernel of D_{Γ_+} is $\{0\}$, not only are $\not{\mathfrak{D}}_A$ and $\not{\mathfrak{D}}^*_A$ Fredholm, as we saw above, we also have that A decays exponentially to Γ_+. So there exist $\beta > 0$ such that $|A - \Gamma_+| \leq Ce^{-\beta t}$ for $t > 0$; this is a consequence of [21, Theorem 5.2.2] and of the embedding of $W^{1,2}$ in bounded C^0 functions, [12, Theorem 3.4]. In that case, we have the following result on harmonic spinors.

Theorem 4.2. *Suppose* $\phi \in \ker(\not{\mathfrak{D}}_A^*) \cap W_\delta^{1,2}$. *Suppose* $\lambda - \beta < \eta < \delta$ *and that* λ *is the only eigenvalue of* D_{Γ_+} *between* η *and* δ: $\operatorname{Spec}(D_{\Gamma_+}) \cap [\eta, \delta] = \{\lambda\}$. *Then, there exists an eigenvector* $\bar{\psi}$ *of* D_{Γ_+} *of eigenvalue* λ *on* T^3 *and* $\bar{\phi} \in W_\eta^{1,2}((0,\infty) \times T^3)$ *such that*

$$\phi = e^{\lambda t}\bar{\psi} + \bar{\phi} \text{ for } t > 0. \tag{4.5}$$

Furthermore, $\bar{\phi} = O(e^{\eta t})$ *as* $t \to \infty$.

Proof. The space $L^2(T^3)$ splits according to the finite dimensional eigenspaces W_λ for D_{Γ_+}. Let Π_δ^+, Π_δ^- and Π_δ be respectively the projections from $L^2(T^3)$ to

$$\bigoplus_{\lambda > \delta} W_\lambda, \quad \bigoplus_{\lambda < \delta} W_\lambda, \quad \text{and } W_\delta.$$

To simplify the notation, we omit δ when it is 0 and set $\phi^\pm := \Pi^\pm(\phi)$.

For every $\phi \in L^2(Y)$, let ϕ_λ be its W_λ component. Thus $\phi = \sum \phi_\lambda$. Using this decomposition, we can define the space $W^{\frac{1}{2},2}(T^3)$ using the norm

$$\|\phi\|^2_{W^{\frac{1}{2},2}} = \sum (1 + |\lambda|)\|\phi_\lambda\|^2_{L^2}. \tag{4.6}$$

Because T^3 is compact, the space $W^{\frac{1}{2},2}(T^3)$ defined by two different Dirac operators are equal, with commensurate norms. The + and − part of L^2, however, depend highly on D_{Γ_+}.

The operator

$$\begin{aligned} \not{\mathfrak{D}}^!_{\Gamma_+} : W^{1,2}([a,\infty) \times T^3) &\to L^2([a,\infty) \times T^3) \oplus \Pi^+ W^{\frac{1}{2},2}(\{a\} \times T^3) \\ \phi &\mapsto (\not{\mathfrak{D}}_{\Gamma_+}\phi, \Pi^+\phi(a)) \end{aligned} \tag{4.7}$$

is an isomorphism when D_{Γ_+} has no kernel.

The proof of this fact starts as one does in the full cylinder case:

$$\begin{aligned} \|\not{\mathfrak{D}}_{\Gamma_+}\phi\|^2_{L^2} &= \|\partial_t\phi\|^2_{L^2} + \|D_{\Gamma_+}\phi\|^2_{L^2} + \int_a^\infty \partial_t \left\langle \phi, D_{\Gamma_+}\phi \right\rangle_{L^2(Y)} \\ &\geq C\|\phi\|^2_{W^{1,2}} - \left\langle \phi(a), D_{\Gamma_+}\phi(a) \right\rangle_{L^2(Y)}. \end{aligned}$$

Contrary to the full cylinder case, the boundary term here cannot be made to vanish and henceforth helps control the $W^{1,2}$-norm of ϕ. Using the inequality above and the decomposition $\phi = \sum \phi_\lambda$, we find

$$\|\phi\|^2_{W^{1,2}} \leq C\left(\|\slashed{\mathfrak{D}}_{\Gamma_+}\phi\|^2_{L^2} + \|\phi^+(a)\|^2_{W^{\frac{1}{2},2}(T^3)}\right). \tag{4.8}$$

We just proved that $\|\phi\|_{W^{1,2}} \leq C\|\slashed{\mathfrak{D}}'_{\Gamma_+}\phi\|$, hence $\slashed{\mathfrak{D}}'_{\Gamma_+}$ is semi-Fredholm and injective. Suppose now that (ψ, η) is perpendicular to $\mathrm{Im}(\slashed{\mathfrak{D}}'_{\Gamma_+})$. For all $\phi \in W^{1,2}([a,\infty) \times T^3)$, we have

$$\begin{aligned}
0 &= \left\langle \slashed{\mathfrak{D}}_{\Gamma_+}\phi, \psi\right\rangle + \left\langle \eta, \phi^+(a)\right\rangle \\
&= \left\langle \phi, \slashed{\mathfrak{D}}^*_{\Gamma_+}\psi\right\rangle - \left\langle \phi(a), \psi(a)\right\rangle + \left\langle \eta, \phi^+(a)\right\rangle \\
&= \left\langle \phi, \slashed{\mathfrak{D}}^*_{\Gamma_+}\psi\right\rangle - \left\langle \phi^-(a), \psi^-(a)\right\rangle + \left\langle \eta - \psi^+(a), \phi^+(a)\right\rangle.
\end{aligned}$$

Testing against all the ϕ with $\phi(a) = 0$ in a first time, $\phi^+(a) = 0$ then, and finally $\phi^-(a) = 0$, we prove

$$\begin{aligned}
\slashed{\mathfrak{D}}^*_{\Gamma_+}\psi &= 0, \\
\eta &= \psi^+(a), \\
\psi^-(a) &= 0.
\end{aligned}$$

Thus we have $-\partial_t\psi + D_{\Gamma_+}\psi = 0$, which means that ψ is a linear combination of the $e^{\lambda t}\psi_\lambda$. The condition $\psi^-(a) = 0$ forces out all the negative λ, while the positive ones are forced out by the L^2 condition. Hence, $\psi = 0$ and $\slashed{\mathfrak{D}}'_{\Gamma_+}$ is surjective. The proof that the operator in Eq. (4.7) is an isomorphism is now complete.

For a big enough, the operator $\slashed{\mathfrak{D}}'_A$, not independant of t but close enough to $\slashed{\mathfrak{D}}'_{\Gamma_+}$, is also an isomorphism.

As in the full cylinder case, we can look at weighted version of $\slashed{\mathfrak{D}}$ and $\slashed{\mathfrak{D}}'$. For computing the asymptotic expansion of harmonic spinors, we actually need to consider the dual $\slashed{\mathfrak{D}}^*$ and its counterpart

$$\begin{aligned}
\slashed{\mathfrak{D}}'^*_A \colon W^{1,2}_\delta([a,\infty) \times T^3) &\to L^2_\delta([a,\infty) \times T^3) \oplus \Pi^-_\delta W^{\frac{1}{2},2}(T^3) \\
\phi &\mapsto (\slashed{\mathfrak{D}}^*_A\phi, \Pi^-\phi(a)),
\end{aligned}$$

which is Fredholm if and only if $\delta \notin \mathrm{Spec}(D)$, and is an isomorphism when Fredholm.

We close the proof of Theorem 4.2 with a diagram chase. We first introduce maps to compose our diagram.

Recall $\eta < \delta$ and $\operatorname{Spec}(D) \cap [\eta, \delta] = \{\lambda\}$. Then obviously, the map

$$\begin{aligned} I\colon \Pi_\eta^- W^{\frac{1}{2},2}(\{a\} \times T^3) \oplus W_\lambda &\to \Pi_\delta^- W^{\frac{1}{2},2}(\{a\} \times T^3) \\ (\phi, \psi) &\mapsto \phi + e^{a\lambda}\psi \end{aligned}$$

is an isomorphism, while the map

$$\begin{aligned} J\colon W_\eta^{1,2}([a,\infty) \times T^3) \oplus W_\lambda &\to W_\delta^{1,2}([a,\infty) \times T^3) \\ (\phi, \psi) &\mapsto \phi + e^{\lambda t}\psi \end{aligned}$$

is an injection.

Consider now the map

$$\begin{aligned} K\colon W_\eta^{1,2}([a,\infty) \times T^3) \oplus W_\lambda &\to L_\eta^2([a,\infty) \times T^3) \oplus \Pi_\eta^- W^{\frac{1}{2},2}(\{a\} \times T^3) \oplus W_\lambda \\ (\phi, \psi) &\mapsto \big(\not{\mathfrak{D}}_A(\phi + e^{\lambda t}\psi), \Pi_\eta^-\phi, \psi + e^{-a\lambda}\Pi_\lambda \phi(a)\big). \end{aligned}$$

As $\left|\not{\mathfrak{D}}_A^*(e^{\lambda t}\psi)\right| \leq Ce^{(\lambda-\beta)t}|\psi|$, then $\not{\mathfrak{D}}_A^*(e^{\lambda t}\psi) \in L_\eta^2([a,\infty) \times T^3)$, and K is well-defined.

We put all these maps in a commutative diagram

(4.9)

$$\begin{tikzcd} W_\delta^{1,2}([a,\infty) \times T^3) \arrow[r, "\not{\mathfrak{D}}'^*_A"] & L_\delta^2([a,\infty) \times T^3) \oplus \Pi_\delta^- W^{\frac{1}{2},2}(\{a\} \times T^3) \\ W_\eta^{1,2}([a,\infty) \times T^3) \oplus W_\lambda \arrow[u, "J"] \arrow[r, "K"'] & L_\eta^2([a,\infty) \times T^3) \oplus \Pi_\eta^- W^{\frac{1}{2},2}(\{a\} \times T^3) \oplus W_\lambda \arrow[u, "\iota \oplus I"'] \end{tikzcd}$$

We know that $\not{\mathfrak{D}}'^*_A$ is an isomorphism. Using the identification

$$\not{\mathfrak{D}}'^*_A\colon W_\eta^{1,2}([a,\infty) \times T^3) \equiv L_\eta^2([a,\infty) \times T^3) \oplus \Pi_\eta^- W^{\frac{1}{2},2}(\{a\} \times T^3),$$

we see that K has the form

$$\begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix}$$

for the splitting $W_\eta^{1,2}([a,\infty) \times T^3) \oplus W_\lambda$ of the domain and codomain. Hence, $K - 1$ is a compact operator, and K is thus Fredholm of index 0. If $K(x) = K(y)$, then $\not{\mathfrak{D}}'^*_A J(x) = \not{\mathfrak{D}}'^*_A J(y)$ as the diagram is commutative, hence $x = y$ and K is injective. Being of index 0, it henceforth must be an isomorphism.

Let us now exploit this fantastic diagram. Suppose

$$\phi \in \ker(\not{\mathfrak{D}}_A^*) \cap W_\delta^{1,2}(\mathbb{R} \times T^3).$$

Then for a big enough, the diagram (4.9) has rows which are isomorphism for δ and η satisfying the hypothesis of the theorem.

We now chase around the diagram. Since I is an isomorphism, we know there exist $(\chi,\nu) \in \Pi^-_\eta W^{\frac{1}{2},2}(\{a\} \times T^3) \oplus W_\lambda$ such that

$$\iota \oplus I(0,\chi,\nu) = \not{\mathfrak{D}}^{!*}_A(\phi).$$

But as K is an isomorphism, there is $(\bar\phi,\bar\psi) \in W^{1,2}_\eta([a,\infty) \times T^3) \oplus W_\lambda$ such that

$$K(\bar\phi,\bar\psi) = (0,\chi,\nu).$$

By commutativity of the diagram, we have

$$\not{\mathfrak{D}}^{!*}_A J(\bar\phi,\bar\psi) = \not{\mathfrak{D}}^{!*}_A(\phi)$$

but $\not{\mathfrak{D}}^{!*}_A$ is an isomorphism hence $\phi = e^{\lambda t}\bar\psi + \bar\phi$ for $t > a$.

As the choice of a is artificial, we set $a = 0$. The proof is now complete. ☐

Suppose now

$$\lambda \in \mathrm{Spec}(D_{\Gamma_-}) \times \mathrm{Spec}(D_{\Gamma_+}),$$
$$\delta \text{ is in the upper left open square adjacent to } \lambda,$$
$$\eta \text{ is in the lower right open square adjacent to } \lambda.$$

When A decays exponentially to its limits, we have

$$\ker(\lambda) = \ker(\eta). \tag{4.10}$$

Indeed, suppose now $\phi \in \ker(\lambda)$. Then, $\phi \in \ker(\delta)$ hence by Theorem 4.2, we expand ϕ for $t > 0$ as $\phi = e^{-\lambda_+ t}\psi_{\lambda_+} + \bar\phi$, with $\bar\phi \in W^{1,2}_{\eta+}([0,\infty) \times T^3)$. Since ϕ and $\bar\phi$ are both in $W^{1,2}_{\lambda_+}$, so is the term $e^{-\lambda_+ t}\psi_{\lambda_+}$. This fact implies that $\psi_{\lambda_+} = 0$. Using a similar proof at $-\infty$, we find $\phi \in W^{1,2}_\eta$. Obviously, the same is true for $\ker^*$.

5. Nahm transform: Instantons to singular monopoles.

Since $\not{\mathfrak{D}}^*_{A_z}$ is Fredholm $L^2 \to L^2$ outside of W, and since $\ker(\not{\mathfrak{D}}_{A_z}) = 0$ as F_{A_z} is ASD and $\mathbb{R} \times T^3$ has infinite volume, we have a bundle V over $T^3 \backslash W$ whose fiber at z is

$$V_z := \ker(\not{\mathfrak{D}}^*_{A_z}) \cap L^2.$$

As outlined earlier, this bundle is equipped with

$$\begin{aligned}&\text{a connection } B \text{ on } T^3 \setminus W,\\ &\text{a Higgs field } \Phi \in \Gamma(T^3 \setminus W, \mathrm{End} V).\end{aligned}$$

The main result of this present paper is the following theorem.

Theorem 5.1. *Outside of a set W consisting of at most four points, the family of vector spaces V described above defines a vector bundle of rank*

$$\frac{1}{8\pi^2}\int |F_A|^2,$$

and the couple (B, Φ) satisfies the Bogomolny equation

$$\nabla_B \Phi = *F_B.$$

For $w \in W$ and z close enough to w, there are maps $\Phi^\perp$ and $\Phi^\lrcorner$ such that

$$\Phi = \frac{-i}{2|z-w|}\Phi^\perp + \Phi^\lrcorner,$$

and $\Phi^\perp$ is the L^2-orthogonal projection on the orthogonal complement of a naturally defined subbundle $V_\llcorner$ of V.

The last part of the theorem is made clearer by the introduction of $V_\llcorner$ in Section 6.

Proof. The rank of V is computed in Lemma 5.2 below.

The boundary term of Eq. (3.2) is

$$\partial\text{-term} = \langle \nu\Omega G\phi, d^z\psi\rangle_{T^3}\Big|_{-\infty}^{\infty}.$$

For $z \notin W$, both $G\phi$ and $d^z\psi$ decay exponentially by Theorem 4.2 hence

$$\partial\text{-term} = 0,$$

and the connection Pd^z on $\mathbb{R} \times (T^3 \setminus W)$ is ASD. Thus, as explained in Section 3, the pair (B, Φ) satisfies outside of W the appropriate dimensional reduction of the ASD equation, which is in this case the Bogomolny Equation

$$\nabla_B \Phi = *F_B.$$

The last part of the theorem is the content of Section 7 and rests on the splitting of Section 6. □

As announced, we compute now the rank of V, and prove an L^2-index theorem for $\mathbb{R} \times T^3$.

Lemma 5.2. *For a $SU(2)$-instanton (E, A) on $\mathbb{R} \times T^3$, the index of the Dirac operator*

$$\not{\mathfrak{D}}_A \colon W^{1,2}(\mathbb{R} \times T^3) \to L^2(\mathbb{R} \times T^3)$$

when A has non-zero limits at $\pm\infty$ is given by the formula

$$\operatorname{ind}(\not{\mathfrak{D}}_A) = -\frac{1}{8\pi^2} \int |F_A|^2.$$

Proof. The fact that A has non-zero limits guarantees that the operator $\not{\mathfrak{D}}_A$ is Fredholm on $W^{1,2}$. Moreover, A decays exponentially to its limits.

Let

$$(\chi_R^+, \chi_R^-, \chi_R^0)$$

be a partition of unity subordinate to the covering

$$\left((R, \infty) \times T^3, (-\infty, -R) \times T^3, (-R-1, R+1) \times T^3\right).$$

Suppose $\Gamma_\pm = d + \gamma_\pm$, and $A = d + a$. Then a tends to γ_+ and γ_- when t tends to $+\infty$ and $-\infty$ respectively. Set

$$a_R = \chi_R^+ \gamma_+ + \chi_R^- \gamma_- + \chi_R^0 a. \tag{5.1}$$

The sequence $\not{\mathfrak{D}}_{a_n} - \not{\mathfrak{D}}_{a_R}$ of compact operators is Cauchy, and thus has a limit, K say, which is then compact. As $\not{\mathfrak{D}}_A = \not{\mathfrak{D}}_{a_R} + K$, we have that $\operatorname{ind}(\not{\mathfrak{D}}_A) = \operatorname{ind}(\not{\mathfrak{D}}_{a_R})$ for all $R > 0$. We now compute $\operatorname{ind}(\not{\mathfrak{D}}_{a_R})$ using the relative index theorem. It could be that $\Gamma_- \neq \Gamma_+$, but this case is easily converted to a situation where $\Gamma_- = \Gamma_+$, as we now see.

Choose a path Γ_s in the space of flat connections on T^3 starting at Γ_+ and ending at Γ_-, and avoiding the trivial connection. Hence, $0 \notin \operatorname{Spec}(D_{\Gamma_s})$ for all s; recall Eq. (4.1). Suppose $\Gamma_s = d + \gamma_s$ and set

$$a_R^s = \chi_R^+ \gamma_s + \chi_R^- \gamma_- + \chi_R^0 a. \tag{5.2}$$

The family $\not{\mathfrak{D}}_{a_R^s}$ of Fredholm operator depends continuously on s. Hence

$$\operatorname{ind}(\not{\mathfrak{D}}_A) = \operatorname{ind}(\not{\mathfrak{D}}_{a_R}) = \operatorname{ind}(\not{\mathfrak{D}}_{a_R^0}) = \operatorname{ind}(\not{\mathfrak{D}}_{a_R^1}).$$

Note that the connection a^1_R equals Γ_- outside $[-R-1, R+1] \times T^3$. Hence the relative index theorem tells us

$$\operatorname{ind}(\slashed{\mathfrak{D}}_{a^1_R}) - \operatorname{ind}(\slashed{\mathfrak{D}}_{\Gamma_-}) = \operatorname{ind}(\tilde{\slashed{\mathfrak{D}}}_{a^1_R}) - \operatorname{ind}(\tilde{\slashed{\mathfrak{D}}}_{\Gamma_-}), \tag{5.3}$$

where the tilded operators are extensions to some compact manifold of the restriction of the operators $\slashed{\mathfrak{D}}_{a^1_R}$ and $\slashed{\mathfrak{D}}_{\Gamma_-}$ to $[-R-1, R+1] \times T^3$.

Because D_{Γ_-} has no kernel, $\slashed{\mathfrak{D}}_{\Gamma_-} : W^{1,2} \to L^2$ is an isomorphism, and thus $\operatorname{ind}(\slashed{\mathfrak{D}}_{\Gamma_-}) = 0$. Hence, the left-hand side of Eq. (5.3) is equal to $\operatorname{ind}(\slashed{\mathfrak{D}}_A)$.

To compute the right-hand side, we embed $[-R-1, R+1] \times T^3$ in some flat T^4. The spinor bundles S^+ and S^- on $[-R-1, R+1] \times T^3$ agree very nicely with those of T^4. We extend both a^1_R and Γ_- by the trivial bundle with connection Γ_-.

The Atiyah–Singer index theorem tells us that

$$\begin{aligned}
\operatorname{ind}(\tilde{\slashed{\mathfrak{D}}}_{\Gamma_-}) &= \{ch(\Gamma_-) \cdot \hat{\mathbf{A}}(T^4)\}[T^4] \\
\operatorname{ind}(\tilde{\slashed{\mathfrak{D}}}_{a^1_R}) &= \{ch(a^1_R) \cdot \hat{\mathbf{A}}(T^4)\}[T^4] \\
&= \left(\frac{c_1^2}{2} - c_2\right)[T^4].
\end{aligned}$$

Since a^1_R is in SU(2), we have $c_1 = 0$, while

$$c_2[T^4] = \frac{1}{8\pi^2} \int_{T^4} |(|F^-_{a^1_R}|^2 - |F^+_{a^1_R}|^2)|^2.$$

Note that on the complement of $[-R-1, R+1] \times T^3$ in T^4, the connection a^1_R equals Γ_- hence is flat there. Furthermore, on $[-R, R] \times T^3$, we have $a^1_R = A$. On $[R, R+1] \times T^3$ and $[-R-1, -R] \times T^3$, the curvature $F_{a^1_R}$ involves cut off functions, their derivatives and $(A - \Gamma_-)$ terms. Since A tends to Γ_- exponentially fast, we therefore have constant C and β such that

$$\left|\operatorname{ind}(\slashed{\mathfrak{D}}_A) + \frac{1}{8\pi^2} \int_{[-R,R] \times T^3} |F_A|^2\right| \leq Ce^{-\beta R}.$$

As $R \to \infty$, we have the wanted result. □

6. A geometric splitting and exact sequences.

In this section, we analyze a splitting of V in a neighborhood of a point $w \in W$ where the solution (B, ϕ) to Bogomolny equation is singular. This point w is associated, say, to the limit $\Gamma = \Gamma_+$ of A at $+\infty$, in the sense that Γ splits E as $L_w \oplus L_{-w}$ on T^3.

Suppose the connection A decays at most with rate β, as in $|A - \Gamma_+| \leq Ce^{-\beta t}$ for $t > 0$ and $|A - \Gamma_-| \leq Ce^{\beta t}$ for $t < 0$. Set

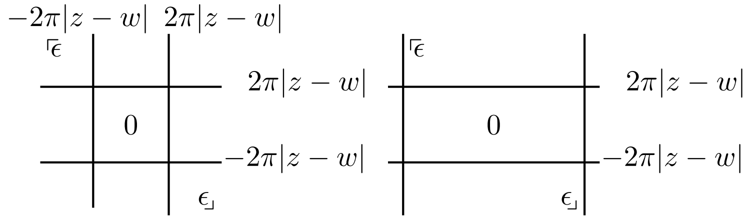
$$\epsilon := \frac{1}{4} \min\left(\beta, \operatorname{dist}\big(w, \Lambda^* + W \setminus \{w\}\big)\right),$$

and define the six weights

$$\begin{array}{lll} {}^{\ulcorner}\epsilon := (-\epsilon, \epsilon) & \overline{\epsilon} := (0, \epsilon) & \epsilon^{\urcorner} := (\epsilon, \epsilon) \\ {}_{\llcorner}\epsilon := (-\epsilon, -\epsilon) & \underline{\epsilon} := (0, -\epsilon) & \epsilon_{\lrcorner} := (\epsilon, -\epsilon) \end{array}$$

displayed here in a way which is reminiscent of their position in $\mathbb{R}^2$.

Consider the ball $B^3(w)$ of radius 2ϵ around w. As z varies in $B^3(w)$, and depending on whether $\Gamma_+ = \Gamma_-$ or not, there are two or one walls to cross to pass from 0 to ${}^{\ulcorner}\epsilon$ and from $\epsilon_{\lrcorner}$ to 0. In a picture, we have



$\Gamma_+ = \Gamma_-$ $\Gamma_+ \neq \Gamma_-$

As z varies in $B^3(w)$, those walls move around without ever touching $\epsilon_{\lrcorner}$ and ${}^{\ulcorner}\epsilon$. Hence, for $L^2_{\epsilon_{\lrcorner}}$ and $L^2_{{}^{\ulcorner}\epsilon}$, the operators $\not{\mathfrak{D}}_{A_z}, \not{\mathfrak{D}}^*_{A_z}$ and $\not{\mathfrak{D}}^*_{A_z}\not{\mathfrak{D}}_{A_z}$ are Fredholm for all $z \in B^3(w)$.

Hence, for $z \in B^3(w)$, the six vector spaces

$$\begin{array}{ll} {}^{\ulcorner}V_z := \ker(\not{\mathfrak{D}}^*_{A_z}) \cap L^2_{{}^{\ulcorner}\epsilon}, & {}^{\ulcorner}K_z := \ker(\not{\mathfrak{D}}_{A_z}) \cap L^2_{{}^{\ulcorner}\epsilon}, \\ V_{\lrcorner z} := \ker(\not{\mathfrak{D}}^*_{A_z}) \cap L^2_{\epsilon_{\lrcorner}}, & K_z := \ker(\not{\mathfrak{D}}_{A_z}) \cap L^2 \\ \mathcal{H}_z := \ker(\nabla^*_{A_z}\nabla_{A_z}) \cap L^2_{{}^{\ulcorner}\epsilon}, & K_{\lrcorner z} := \ker(\not{\mathfrak{D}}_{A_z}) \cap L^2_{\epsilon_{\lrcorner}}, \end{array}$$

are kernels of Fredholm operators. By contrast, the space V_z, already defined as $\ker(\not{\mathfrak{D}}^*_{A_z}) \cap L^2$, is not the kernel of a Fredholm operator at w.

Notice that none of those vector spaces form *a priori* a bundle over $B^3(w)$ as the dimensions could jump at random. However, for $L^2_{{}^{\ulcorner}\epsilon}$ and $L^2_{\epsilon_{\lrcorner}}$, the operators $\not{\mathfrak{D}}_{A_z}$, $\not{\mathfrak{D}}^*_{A_z}$, and $\nabla^*_{A_z}\nabla_{A_z}$ are Fredholm operators for all

$z \in B^3(w)$. The various indices are therefore constant and we have that, for example,

$$\dim V_{\lrcorner z} - \dim {}^{\ulcorner}\!K_z \text{ is constant on } B^3(w).$$

We have the following obvious results:

$$\begin{aligned} V_{\lrcorner} \subset V \subset {}^{\ulcorner}V, \qquad & \qquad K_{\lrcorner} \subset K \subset {}^{\ulcorner}K, \\ \not{\mathfrak{D}}\mathcal{H} \subset {}^{\ulcorner}V, \qquad & \qquad {}^{\ulcorner}K \subset \mathcal{H}, \\ & \qquad K_{\lrcorner} = K = \{0\}. \end{aligned}$$

Equation (4.10) signifies here that $V_{\lrcorner w} = V_w$. The following few lemmas describe in more detail the relationship between the various spaces.

The smallest eigenvalues of D_{Γ_z} are $\pm 2\pi|z - w|$. For simplicity, we set

$$\lambda := 2\pi|z - w|,$$

and define

$$W_\lambda := \lambda \text{ eigenspace of } D_{\Gamma_z} \text{ on } T^3.$$

The family W_λ defines a bundle over the sphere $|z - w| = \lambda/2\pi$ around w. Its rank is given by

$$\text{(6.1)} \qquad \operatorname{rk} W_\lambda = \begin{cases} 1, & \text{if } \lambda \neq 0 \text{ and } 2w \notin \Lambda^*; \\ 2, & \text{if } \lambda \neq 0 \text{ and } 2w \in \Lambda^*, \text{ or } \lambda = 0 \text{ and } 2w \notin \Lambda^*; \\ 4, & \text{if } \lambda = 0 \text{ and } 2w \in \Lambda^*. \end{cases}$$

This W_λ plays an important role in understanding the relations between the various spaces just introduced.

For any instanton connection A' on $\mathbb{R} \times T^3$, set

$$\begin{aligned} V(\delta) &:= \ker(\not{\mathfrak{D}}^*_{A'}) \cap L^2_\delta, \\ K(\delta) &:= \ker(\not{\mathfrak{D}}_{A'}) \cap L^2_\delta, \end{aligned}$$

and let $[\delta]$ denote the open square in $\mathbb{R}^2 \setminus \mathfrak{S}_{A'}$ containing δ.

Lemma 6.1 (one wall). *Suppose $\delta, \eta \in \mathbb{R}^2 \setminus \mathfrak{S}_{A'}$ are weights for which $[\delta]$ and $[\eta]$ are adjacent and separated by the wall $\{\mu\} \times \mathbb{R}$ or $\mathbb{R} \times \{\mu\}$. Then, the sequence*

(6.2)

$$0 \longrightarrow V(\delta) \longrightarrow V(\eta) \xrightarrow{\lim(e^{-\mu t}\cdot)} W_\mu \xrightarrow{\left(\lim(e^{\mu t}\cdot)\right)^*} K(-\delta)^* \longrightarrow K(-\eta)^* \longrightarrow 0,$$

where the limits are both evaluated at $+\infty$ when $[\eta]$ is above $[\delta]$ and at $-\infty$ when $[\eta]$ is to the left of $[\delta]$, is exact.

Proof. Theorem 4.2 ensures that the limits give functions α and β^* which are well defined, and that

$$0 \longrightarrow V(\delta) \longrightarrow V(\eta) \longrightarrow W_\mu \quad \text{and} \quad 0 \longrightarrow K(-\eta) \longrightarrow K(-\delta) \longrightarrow W_\mu$$

are exact.

It only remains to prove that Sequence (6.2) is exact at W_μ. Suppose $\phi \in V(\eta)$ and $\psi \in K(-\delta)$. Then,

$$\begin{aligned} 0 &= \left\langle \not{\mathfrak{D}}^*_{A'}\phi, \psi \right\rangle - \left\langle \phi, \not{\mathfrak{D}}_{A'}\psi \right\rangle \\ &= \lim_{t\to\infty} \left\langle \phi, cl\left(\frac{\partial}{\partial t}\right)\psi \right\rangle - \lim_{t\to-\infty} \left\langle \phi, cl\left(\frac{\partial}{\partial t}\right)\psi \right\rangle \\ &= \lim_{t\to\infty} \left\langle e^{-\mu t}\phi, cl\left(\frac{\partial}{\partial t}\right) e^{\mu t}\psi \right\rangle - \lim_{t\to-\infty} \left\langle e^{-\mu t}\phi, cl\left(\frac{\partial}{\partial t}\right) e^{\mu t}\psi \right\rangle. \end{aligned}$$

One of those limits is $\beta^*\alpha(\phi)(\psi)$ while the other one vanishes as we now see. Suppose $[\eta]$ is above $[\delta]$, and suppose $\{\mu'\} \times \mathbb{R}$ is the wall to their right. Then, $\phi = O(e^{\mu' t})$ as $t \to -\infty$ by Theorem 4.2. But for some $\mu'' < \mu'$, the wall $\{-\mu''\} \times \mathbb{R}$ is exactly to the right of $[-\eta]$; hence, $\psi = O(e^{-\mu'' t})$ as $t \to -\infty$. But then

$$\beta^*\alpha(\phi)(\psi) = \lim_{t\to-\infty} O(e^{(\mu'-\mu'')t}) = 0,$$

hence $\operatorname{Im}(\alpha) \subset \ker(\beta^*)$. A similar argument establishes the same fact when $[\eta]$ is to the left of $[\delta]$.

The sequence is then exact if $\dim \operatorname{Im}(\alpha) = \dim \ker(\beta^*)$. We have two short exact sequences:

$$\begin{gathered} 0 \longrightarrow V(\delta) \longrightarrow V(\eta) \longrightarrow \operatorname{Im}(\alpha) \longrightarrow 0, \qquad \text{and} \\ 0 \longrightarrow W_\mu/\ker(\beta^*) \longrightarrow K(-\delta)^* \longrightarrow K(-\eta)^* \longrightarrow 0. \end{gathered}$$

Using those short exact sequences and notation from Eq. (4.3), we have

$$\begin{aligned} \dim \operatorname{Im}(\alpha) - \dim\ker(\beta^*) &= N^*(\eta) - N^*(\delta) - \dim W_\mu + N(-\delta) - N(-\eta) \\ &= \operatorname{ind}^*(\eta) - \operatorname{ind}^*(\delta) - \dim W_\mu. \end{aligned}$$

The Wall Crossing Equation (4.4) forces the last line to be 0. The proof is thus complete. □

Corollary 6.2. *Suppose* $\Gamma_+ \neq \Gamma_-$*. Then the sequences*

$$0 \longrightarrow V_z \longrightarrow {}^{\ulcorner}V_z \longrightarrow W_\lambda \longrightarrow 0, \qquad \text{for } \lambda \neq 0, \tag{6.3}$$

$$0 \longrightarrow V_{\lrcorner z} \longrightarrow V_z \longrightarrow W_{-\lambda} \longrightarrow {}^{\ulcorner}K_z \longrightarrow 0, \qquad \text{for } \lambda \neq 0, \tag{6.4}$$

$$0 \longrightarrow V_w \longrightarrow {}^{\ulcorner}V_w \longrightarrow W_0 \longrightarrow {}^{\ulcorner}K_w \longrightarrow 0, \tag{6.5}$$

are exact.

Proof. Apply Lemma 6.1 to the choice of weights $\{{}^{\ulcorner}\epsilon, 0\}$ and $\{0, \epsilon_{\lrcorner}\}$ for the connection $A' = A_z$, and remember that $K_{\lrcorner} = K = \{0\}$. $\square$

Corollary 6.3. *Suppose* $\Gamma_+ = \Gamma_-$. *Then the sequences*

$$0 \longrightarrow V_z \longrightarrow {}^{\ulcorner}V_z \longrightarrow W_\lambda \oplus W_{-\lambda} \longrightarrow 0, \qquad \text{for } \lambda \neq 0, \tag{6.6}$$

$$0 \longrightarrow V_{\lrcorner z} \longrightarrow V_z \longrightarrow W_\lambda \oplus W_{-\lambda} \longrightarrow {}^{\ulcorner}K_z \longrightarrow 0, \qquad \text{for } \lambda \neq 0, \tag{6.7}$$

$$0 \longrightarrow V_w \longrightarrow {}^{\ulcorner}V_w \longrightarrow W_0 \oplus W_0 \longrightarrow {}^{\ulcorner}K_w \longrightarrow 0, \tag{6.8}$$

are exact.

Proof. Suppose we have the following choice of weights:

$$\begin{array}{c|cl} {}^{\ulcorner}\delta & \delta^{\urcorner} & \\ \hline {}_{\llcorner}\delta & \delta_{\lrcorner} & \mu \\ \multicolumn{2}{c}{-\mu} & \end{array}$$

Denote ι any inclusion map, and $L^{\pm}_{\mu}$ the maps

$$L^+_\mu(\phi) = \lim_{t\to\infty} e^{\mu t}\phi, \qquad \text{and} \qquad L^-_\mu = \lim_{t\to-\infty} e^{\mu t}\phi.$$

Then sequences akin to Sequence (6.2) fit in a diagram

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & & V(\delta^{\urcorner}) \arrow[rr, "L^+_{-\mu}"] \arrow[dr, "\iota"] & & W_\mu \arrow[rr, "{L^+_\mu}^*"] \arrow[dr, "{L^+_\mu}^*"] & & K(-{}_{\llcorner}\delta)^* \arrow[rr] \arrow[dr, "\iota"] & & 0 \\
& V(\delta_{\lrcorner}) \arrow[ur, "\iota"] \arrow[dr, "\iota"] & & V({}^{\ulcorner}\delta) \arrow[ur, "L^+_{-\mu}"] \arrow[dr, "L^-_\mu"] & & K(-\delta_{\lrcorner})^* \arrow[ur, "\iota"] \arrow[dr, "\iota"] & & K(-{}^{\ulcorner}\delta)^* \arrow[ur] \arrow[dr] & & . \\
0 \arrow[rr] \arrow[ur] & & V({}_{\llcorner}\delta) \arrow[rr, "L^-_\mu"'] \arrow[ur, "\iota"] & & W_{-\mu} \arrow[rr, "{L^-_{-\mu}}^*"'] \arrow[ur, "{L^-_{-\mu}}^*"] & & K(-\delta^{\urcorner})^* \arrow[rr] \arrow[ur, "\iota"] & & 0
\end{tikzcd} \tag{6.9}$$

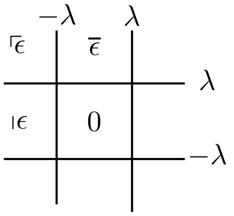
Suppose $\phi \in V({}^\ulcorner\!\delta)$, and $\psi \in K(-\delta_\lrcorner)$. Then

$$\begin{aligned}
0 &= \big\langle \not{\mathfrak{D}}_{A'}\phi, \psi \big\rangle - \big\langle \phi, \not{\mathfrak{D}}_{A'}\psi \big\rangle \\
&= \left\langle \phi, cl\left(\frac{\partial}{\partial t}\right)\psi \right\rangle \Big|_{-\infty}^{\infty} \\
&= \lim_{t\to\infty} \left\langle e^{-\mu t}\phi, cl\left(\frac{\partial}{\partial t}\right) e^{\mu t}\psi \right\rangle - \lim_{t\to-\infty} \left\langle e^{\mu t}\phi, cl\left(\frac{\partial}{\partial t}\right) e^{-\mu t}\psi \right\rangle \\
&= \big({L_{\mu}^{+}}^{*} L_{-\mu}^{+}(\phi) - {L_{-\mu}^{-}}^{*} L_{\mu}^{-}(\phi) \big)(\psi),
\end{aligned}$$

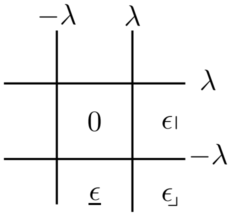
hence the middle square commutes. It is quite obvious that all the other squares and triangles commute. From Diagram (6.9), we extract, for an obvious choice of maps, the exact sequence

$$0 \longrightarrow V(\delta_\lrcorner) \longrightarrow V({}^\ulcorner\!\delta) \longrightarrow W_\mu \oplus W_{-\mu} \longrightarrow K(-\delta_\lrcorner)^* \longrightarrow K(-{}^\ulcorner\!\delta)^* \longrightarrow 0.$$

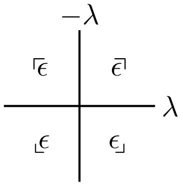
In particular, the sets of weights



at $z \neq w$
$\Gamma_+ = \Gamma_-$



at $z \neq w$
$\Gamma_+ = \Gamma_-$



at $z = w$
$\Gamma_+ = \Gamma_-$

yield for $A' = A_z$ the exact sequences (6.6), (6.7) and (6.8). □

An analysis for $\nabla^*_{A_z}\nabla_{A_z}$ brings a very similar wall crossing formula

$$\operatorname{ind}(\nabla^*_{A_z}\nabla_{A_z}, {}^\ulcorner\!\epsilon) - \operatorname{ind}(\nabla^*_{A_z}\nabla_{A_z}, \epsilon_\lrcorner) = \begin{cases} 2\dim W_0, & \text{for } \Gamma_+ \neq \Gamma_-; \\ 4\dim W_0, & \text{for } \Gamma_+ = \Gamma_-. \end{cases}$$

However, since $\nabla^*_{A_z}\nabla_{A_z}$ is self-adjoint, $\operatorname{ind}(\nabla^*_{A_z}\nabla_{A_z}, {}^\ulcorner\!\epsilon) = -\operatorname{ind}(\nabla^*_{A_z}\nabla_{A_z}, \epsilon_\lrcorner)$, whence

$$\operatorname{rk}\mathcal{H} = \begin{cases} \dim W_0, & \text{for } \Gamma_+ \neq \Gamma_-; \\ 2\dim W_0, & \text{for } \Gamma_+ = \Gamma_-. \end{cases}$$

Using Equation (6.1), we can even say

$$\operatorname{rk}\mathcal{H} = \begin{cases} 2, & \text{for } \Gamma_+ \neq \Gamma_- \text{ and } 2w \notin \Lambda^*; \\ 4, & \text{for } \Gamma_+ \neq \Gamma_- \text{ and } 2w \in \Lambda^*, \text{ or } \Gamma_+ = \Gamma_- \text{ and } 2w \notin \Lambda^*; \\ 8, & \text{for } \Gamma_+ = \Gamma_- \text{ and } 2w \in \Lambda^*. \end{cases}$$

Similarly, we have for the Laplacian the following isomorphisms:

$$0 \longrightarrow \mathcal{H}_z \longrightarrow W_\lambda \oplus W_{-\lambda} \longrightarrow 0, \qquad \text{for } z \neq w \text{ and when } \Gamma_+ \neq \Gamma_-, \tag{6.10}$$

$$0 \longrightarrow \mathcal{H}_w \longrightarrow W_0 \longrightarrow 0, \qquad \text{when } \Gamma_+ \neq \Gamma_-, \tag{6.11}$$

$$0 \longrightarrow \mathcal{H}_z \longrightarrow \big(W_\lambda \oplus W_{-\lambda}\big)^{\oplus 2} \longrightarrow 0, \qquad \text{for } z \neq w \text{ and when } \Gamma_+ = \Gamma_-, \tag{6.12}$$

$$0 \longrightarrow \mathcal{H}_w \longrightarrow W_0 \oplus W_0 \longrightarrow 0, \qquad \text{when } \Gamma_+ = \Gamma_-. \tag{6.13}$$

Bringing all of those sequences together allows us to conclude the following.

Theorem 6.4. *On $B^3(w)$, we have*

$$^{\ulcorner}V = V_{\lrcorner} \oplus \not{\mathfrak{D}}\mathcal{H}.$$

Proof. Denote W'_λ the space

$$W'_\lambda := \begin{cases} W_\lambda \oplus W_{-\lambda}, & \text{if } \Gamma_+ = \Gamma_-; \\ W_\lambda, & \text{if } \Gamma_+ \neq \Gamma_-. \end{cases}$$

Let $p\colon W'_\lambda \oplus W'_{-\lambda} \to W'_\lambda$ denote the map $p(a, b) = 2\lambda a$.

For $\lambda \neq 0$, we use the Snake Lemma on the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d, "\not{\mathfrak{D}}"'] & W'_\lambda \oplus W'_{-\lambda} \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & V_{\lrcorner} \arrow[r] & {}^{\ulcorner}V \arrow[r] & W'_\lambda \arrow[r] & 0
\end{tikzcd}$$

coming from Sequences (6.3), (6.6), (6.10), and (6.12), to produce an exact sequence

$$\begin{array}{ccccccccccc}
\ker(0) & \longrightarrow & \ker(\not{\mathfrak{D}}) & \longrightarrow & \ker(p) & \longrightarrow & \operatorname{coker}(0) & \longrightarrow & \operatorname{coker}(\not{\mathfrak{D}}) & \longrightarrow & \operatorname{coker}(p) \\
0 & \longrightarrow & {}^{\ulcorner}K_z & \longrightarrow & W'_{-\lambda} & \longrightarrow & V_{\lrcorner z} & \longrightarrow & \operatorname{coker}(\not{\mathfrak{D}}) & \longrightarrow & 0
\end{array} \tag{6.14}$$

Note that the map $V_{\lrcorner} \to \operatorname{coker}(\not{\mathfrak{D}})$ being surjective forces ${}^{\ulcorner}V$ to be spanned by $V_{\lrcorner}$ and $\not{\mathfrak{D}}\mathcal{H}$.

Sequences (6.4) and (6.7) imply

$$\dim {}^{\ulcorner}V_z = \dim V_{\lrcorner z} + \dim W'_\lambda - \dim {}^{\ulcorner}K_z$$

while Sequences (6.3) and (6.6) imply

$$\dim {}^{\ulcorner}V_z = \dim V_z + \dim {W'}_\lambda.$$

Thus,

$$\dim {}^{\ulcorner}V_z = \dim V_{\lrcorner z} + 2\dim {W'}_\lambda - \dim {}^{\ulcorner}K_z = \dim V_{\lrcorner z} + \dim \not{\mathfrak{D}}\mathcal{H}.$$

Since Lemma 4.1 guarantees that $\left\langle \not{\mathfrak{D}}\mathcal{H}, V_{\lrcorner}\right\rangle = \{0\}$, we have $V \cap \not{\mathfrak{D}}\mathcal{H}$ perpendicular to $V_{\lrcorner}$ for the L^2 inner product. Hence, $\not{\mathfrak{D}}\mathcal{H} \cap V_{\lrcorner} = \{0\}$, and ${}^{\ulcorner}V_z = V_{\lrcorner z} \oplus \not{\mathfrak{D}}\mathcal{H}$.

It remains to prove the theorem for $z = w$. We already know $V_{\lrcorner w} = V_w$ and $\not{\mathfrak{D}}\mathcal{H}_w \subset {}^{\ulcorner}V_w$. We also know from Sequences (6.5) and (6.8) that

$$\begin{aligned}\dim {}^{\ulcorner}V_w &= \dim V_w + \dim {W'}_0 - \dim {}^{\ulcorner}K_w \\ &= \dim V_{\lrcorner w} + \dim \not{\mathfrak{D}}\mathcal{H}_w.\end{aligned}$$

We therefore only have to prove that the intersection $V_w \cap \not{\mathfrak{D}}_{A_w}\mathcal{H}_w$ is $\{0\}$ to complete the proof.

The asymptotic behavior of $\phi \in \mathcal{H}_w$ is

$$\phi = \begin{cases} t\phi_0^+ + \phi_1^+ + o(1), & \text{as } t \to \infty; \\ t\phi_0^- + \phi_1^- + o(1), & \text{as } t \to -\infty; \end{cases}$$

for some $\phi_0^\pm, \phi_1^\pm \in W_0$. If $\Gamma_+ \neq \Gamma_-$, we must have $\phi_0^- = \phi_1^- = 0$, as w is associated to Γ_+.

The asymptotic behavior of $\not{\mathfrak{D}}_{A_w}\phi$ is

$$\not{\mathfrak{D}}_{A_w}\phi = \begin{cases} \phi_0^+ + o(1), & \text{as } t \to \infty; \\ \phi_0^- + o(1), & \text{as } t \to -\infty. \end{cases}$$

Suppose $\not{\mathfrak{D}}_{A_w}\phi \in L^2$. Then,

$$\begin{aligned}\|\not{\mathfrak{D}}_{A_w}\phi\|_{L^2}^2 &= \left\langle \not{\mathfrak{D}}_{A_w}^* \not{\mathfrak{D}}_{A_w}\phi, \phi\right\rangle + \lim_{t\to\infty}\left\langle \not{\mathfrak{D}}_{A_w}\phi, cl\left(\frac{\partial}{\partial t}\right)\phi\right\rangle \\ &\quad + \lim_{t\to-\infty}\left\langle \not{\mathfrak{D}}_{A_w}\phi, cl\left(\frac{\partial}{\partial t}\right)\phi\right\rangle \\ &= \left\langle \phi_0^+, \phi_1^+\right\rangle + \lim_{t\to\infty} t|\phi_0^+|^2 - \left\langle \phi_0^-, \phi_1^-\right\rangle - \lim_{t\to-\infty} t|\phi_0^-|^2.\end{aligned}$$

For $\|\not{\mathfrak{D}}_{A_w}\phi\|_{L^2}$ to be finite, we must get rid of the limits, thus forcing $\phi_0^\pm = 0$ and consequently, we have $\not{\mathfrak{D}}_{A_w}\phi = 0$. The proof is now complete. $\square$

For a continuous family of Fredholm operators, like $\not{\mathfrak{D}}_{A_z}$ on $L^2_{\ulcorner\epsilon}$ parameterized on $B^3(w)$, the dimension of the kernel can only drop in a small neighborhood of a given point, it cannot increase. However, not any random behavior is acceptable.

Lemma 6.5 (see [20, p. 241]). *Let* $T\colon X \to Y$ *be Fredholm and* $S\colon X \to Y$ *a bounded operator. Then the operator* $T+tS$ *is Fredholm and* $\dim\ker(T+tS)$ *is constant for small* $|t| > 0$.

We obviously use this lemma with $T = \not{\mathfrak{D}}_{A_w}, X = W^{1,2}_{\ulcorner\epsilon}, Y = L^2_{\ulcorner\epsilon}$, and $S = cl(e)$ for some direction $e \in \mathbb{R}^3$. Let us note that three scenarios are possible.

1. $\dim {}^{\ulcorner}K_z$ is constant on a neighborhood around w, say $B^3(w)$;
2. $\dim {}^{\ulcorner}K_z$ is constant for $z \in B^3(w) \setminus \{w\}$, but is smaller than $\dim {}^{\ulcorner}K_w$;
3. $\dim {}^{\ulcorner}K_{w+\lambda e} \neq \dim {}^{\ulcorner}K_{w+\lambda' e'}$ for small $\lambda, \lambda' > 0$ and some $e \neq e'$.

7. Asymptotic of the Higgs field.

We now study the behavior of the Higgs field Φ as z approaches of a given element w of W. We know w is associated to the limit Γ of A at ∞ or $-\infty$, in the sense that Γ splits E as $L_w \oplus L_{-w}$. Without loss of generality, we suppose

$$\Gamma_+ = \Gamma.$$

When $\Gamma_+ \neq \Gamma_-$, and for $2\pi|z-w| < \epsilon$, notice that

$$\begin{aligned} {}^{\ulcorner}V_z &= L^2_{\ulcorner\epsilon} \cap \ker(\not{\mathfrak{D}}^*_{A_z}) = L^2_{\overline{\epsilon}} \cap \ker(\not{\mathfrak{D}}^*_{A_z}) = L^2_{\epsilon\urcorner} \cap \ker(\not{\mathfrak{D}}^*_{A_z}), \text{ and} \\ V_{\lrcorner z} &= L^2_{\llcorner\epsilon} \cap \ker(\not{\mathfrak{D}}^*_{A_z}) = L^2_{\underline{\epsilon}} \cap \ker(\not{\mathfrak{D}}_{A^*_z}) = L^2_{\epsilon\lrcorner} \cap \ker(\not{\mathfrak{D}}^*_{A_z}). \end{aligned}$$

When $\Gamma_+ = \Gamma_-$, those spaces are *a priori* all different.

Theorem 7.1. *On a closed ball* $B^3(w)$ *around* w, *there exist families of operators* $\Phi^\perp$ *and* $\Phi^\lrcorner$, *bounded independently of* z , *such that*

$$\Phi = \frac{-i}{2|z-w|}\Phi^\perp + \Phi^\lrcorner. \tag{7.1}$$

Furthermore, $\Phi^\perp$ *is the* L^2*-orthogonal projection on* $\not{\mathfrak{D}}_{A_z}\mathcal{H}_z \cap V_z$.

Proof. Obviously, $\underset{\lrcorner}{V}$ supports many different norms, and amongst those are the L^2 and $L^2_{\epsilon_\lrcorner}$ norms. For $\phi \in \underset{\lrcorner}{V}_z$, observe that

$$\|t\phi\|_{L^2} \leq C_\epsilon \|\phi\|_{L^2_{\epsilon_\lrcorner}}.$$

We would really like to bound this last quantity by a multiple of $\|\phi\|_{L^2}$.

Let Q denote the projection $L^2 \to V_w$. Of course, since $L^2_{\epsilon_\lrcorner} \subset L^2$, the projection is also defined on $L^2_{\epsilon_\lrcorner}$. Let $V_w^\perp$ be the L^2-orthogonal complement, and $V_w^0 = V_w^\perp \cap L^2_{\epsilon_\lrcorner}$. In fact, we have

$$L^2_{\epsilon_\lrcorner} = V_w \oplus V_w^0$$

since at w, we have $V_w = \underset{\lrcorner}{V}_w$.

Since $\not{\mathfrak{D}}^*_{A_w}$ is injective on V_w^0, there is a constant such that

$$\|u\|_{L^2_{\epsilon_\lrcorner}} \leq C \|\not{\mathfrak{D}}^*_{A_w} u\|_{L^2_{\epsilon_\lrcorner}} \text{ for } u \in V_w^0.$$

But then for $u \in \underset{\lrcorner}{V}_{w+\lambda e}$, we have

$$\begin{aligned}
\|u\|_{L^2_{\epsilon_\lrcorner}} &\leq \|Qu\|_{L^2_{\epsilon_\lrcorner}} + \|(1-Q)u\|_{L^2_{\epsilon_\lrcorner}} \\
&\leq \|Qu\|_{L^2_{\epsilon_\lrcorner}} + C\|\not{\mathfrak{D}}^*_{A_w}(1-Q)u\|_{L^2_{\epsilon_\lrcorner}} \\
&= \|Qu\|_{L^2_{\epsilon_\lrcorner}} + C\|\not{\mathfrak{D}}^*_{A_w} u\|_{L^2_{\epsilon_\lrcorner}} \\
&= \|Qu\|_{L^2_{\epsilon_\lrcorner}} + C\lambda \|u\|_{L^2_{\epsilon_\lrcorner}}.
\end{aligned}$$

Hence, for λ small enough,

$$\|u\|_{L^2_{\epsilon_\lrcorner}} \leq 2\|Qu\|_{L^2_{\epsilon_\lrcorner}}.$$

Of course, since V_w is finite dimensional, there exists a constant C for which $\|Qu\|_{L^2_{\epsilon_\lrcorner}} \leq C\|Qu\|_{L^2}$ and thus for $u \in \underset{\lrcorner}{V}_z$ with z close to w,

$$\|tu\|_{L^2} \leq C_\epsilon \|u\|_{L^2_{\epsilon_\lrcorner}} \leq 2C_\epsilon \|Qu\|_{L^2_{\epsilon_\lrcorner}} \leq C\|Qu\|_{L^2} \leq C\|u\|_{L^2}.$$

Denote $P^\lrcorner$ the L^2-orthogonal projection of V on $\underset{\lrcorner}{V}$. We just proved that

$$\Phi \circ P^\lrcorner \text{ is bounded independently of } z \in B^3(w).$$

It is part of the map $\Phi^\lrcorner$ announced in the statement of the theorem.

One of the crucial features of this proof is our ability to find a uniform bound for m_t on $\underset{\lrcorner}{V}$.

As suggested above, let $\Phi^{\perp}$ denote the L^2-orthogonal projection on $\not{\mathfrak{D}}_{A_z}\mathcal{H}_z \cap V_z$. Then,

$$\begin{aligned}\Phi = -2\pi i P m_t &= \Phi P^{\lrcorner} - 2\pi i\big(P^{\lrcorner} + \Phi^{\perp}\big) m_t \Phi^{\perp}\\ &= \Phi P^{\lrcorner} + 2\pi i P^{\lrcorner} m_t \Phi^{\perp} - 2\pi i \Phi^{\perp} m_t \Phi^{\perp}.\end{aligned}$$

For $\phi_1 \in V_{\lrcorner}$, and $\phi_2 \in V$, we have $\langle \phi_1, t\Phi^{\perp}\phi_2\rangle = \langle t\phi_1, \Phi^{\perp}\phi_2\rangle$. Thus, $P^{\lrcorner} m_t \Phi^{\perp}$ is also bounded independently of $z \in B^3(w)$.

It remains only to analyze $\Phi^{\perp} m_t \Phi^{\perp}$. Pick a vector $e \in \mathbb{R}^3$ of length 1. Let

$$\mathcal{R} = \{w + \frac{\lambda}{2\pi} e\} \subset B^3(w)$$

be a ray inside $B^3(w)$ emerging from w. As the notation suggests, we parameterize this ray by $\lambda = 2\pi|z - w|$. Pick a family $\phi_z \in \not{\mathfrak{D}}_{A_z}\mathcal{H}_z$ for $z \in \mathcal{R}$, with

$$\begin{aligned}&\phi_z \in V_z \text{ for } \lambda > 0,\\ &\|\phi_z\|_{L^2_{\ulcorner\epsilon}} = 1.\end{aligned} \tag{7.2}$$

But then,

$$\|\phi_z\|_{L^2} \to \infty \text{ as } \lambda \to 0.$$

To prove this claim, suppose it is not true. Then, there is a subsequence $\phi_{z_j} \rightharpoonup \tilde{\phi}_w$ weakly in L^2. Hence, $\langle \phi_{z_j}, f\rangle \to \left\langle \tilde{\phi}_w, f\right\rangle$ for all $f \in L^2$, in particular for all $f \in L^2_{\epsilon\lrcorner} = (L^2_{\ulcorner\epsilon})^*$, whence $\phi_{z_j} \rightharpoonup \tilde{\phi}_w$ weakly in $L^2_{\ulcorner\epsilon}$. Since $\phi_z \to \phi_w$ in $L^2_{\ulcorner\epsilon}$, we have $\tilde{\phi}_w = \phi_w$, which is impossible as $\tilde{\phi}_w$ is in L^2 while ϕ_w is not.

Because Γ_w is independent of t, and because $-\epsilon$ is not an eigenvalue of D_{Γ_w}, the operator $\not{\mathfrak{D}}^*_{\Gamma_w}$ is an isomorphism $W^{1,2}_{\llcorner\epsilon} \to L^2_{\llcorner\epsilon}$, and $W^{1,2}_{\epsilon\urcorner} \to L^2_{\epsilon\urcorner}$, hence there exists a constant C such that

$$\|u\|_{W^{1,2}_{\llcorner\epsilon}} \le C\|\not{\mathfrak{D}}^*_{\Gamma_w} u\|_{L^2_{\llcorner\epsilon}}, \quad \text{for } u \in W^{1,2}_{\llcorner\epsilon}, \tag{7.3}$$

$$\|u\|_{W^{1,2}_{\epsilon\urcorner}} \le C\|\not{\mathfrak{D}}^*_{\Gamma_w} u\|_{L^2_{\epsilon\urcorner}}, \quad \text{for } u \in W^{1,2}_{\epsilon\urcorner}. \tag{7.4}$$

Because $\phi_z \in V_z$ for $\lambda > 0$, for $t > 0$, we can write $\phi_z = e^{-\lambda t}\psi_{-\lambda} + g_z$ for some eigenvector $\psi_{-\lambda}$ of eigenvalue $-\lambda$ of D_{Γ_z} and some $g_z \in W^{1,2}_{-\epsilon}([0,\infty) \times T^3)$. When $\Gamma_- = \Gamma_+$, and for $t < 0$, we can write $\phi_z = e^{\lambda t}\psi_{\lambda} + j_z$ for some eigenvector ψ_{λ} of eigenvalue λ of D_{Γ_z} and some $j_z \in W^{1,2}_{\epsilon}((-\infty, 0] \times T^3)$.

While g_z and j_z appear to be defined only for $t > 0$ and $t < 0$ respectively, let us define them globally on $\mathbb{R} \times T^3$ by $g_z = \phi_z - e^{-\lambda t}\psi_{-\lambda}$ and $j_z = \phi_z - e^{\lambda t}\psi_{\lambda}$.

Notice that

$$\not{\mathfrak{D}}^*_{\Gamma_z} g_z = \not{\mathfrak{D}}^*_{\Gamma_z}\phi_z = (\not{\mathfrak{D}}^*_{\Gamma_z} - \not{\mathfrak{D}}^*_{A_z})\phi_z = cl(\Gamma - A)\phi_z, \tag{7.5}$$

and similarly

$$\not{\mathfrak{D}}^*_{\Gamma_z} j_z = \not{\mathfrak{D}}^*_{\Gamma_z}\phi_z = (\not{\mathfrak{D}}^*_{\Gamma_z} - \not{\mathfrak{D}}^*_{A_z})\phi_z = cl(\Gamma - A)\phi_z, \tag{7.6}$$

Overall, there is a constant such that $|cl(A - \Gamma)| \leq C\sigma_{(0,\beta)}$, and this estimate can be improved to $|cl(A - \Gamma)| \leq C\sigma_{(-\beta,\beta)}$ when $\Gamma_- = \Gamma_+$. Hence, $cl(A - \Gamma)$ gives a bounded map $L^2_{\ulcorner\epsilon} \to L^2_{\llcorner\epsilon}$ in all cases and $L^2_{\ulcorner\epsilon} \to L^2_{\epsilon\urcorner}$ when $\Gamma_- = \Gamma_+$. Thus, Eq. (7.5) yields

$$\|\not{\mathfrak{D}}^*_{\Gamma_z} g_z\|_{L^2_{\llcorner\epsilon}} \leq C\|\phi_z\|_{L^2_{\ulcorner\epsilon}}\,, \tag{7.7}$$

and for the special case $\Gamma_- = \Gamma_+$, Eq. (7.6) yields

$$\|\not{\mathfrak{D}}^*_{\Gamma_z} j_z\|_{L^2_{\epsilon\urcorner}} \leq C\|\phi_z\|_{L^2_{\ulcorner\epsilon}}\,. \tag{7.8}$$

From Eqs. (7.3), and (7.7), we derive

$$\begin{aligned}
\|g_z\|_{W^{1,2}_{\llcorner\epsilon}} &\leq C\|\not{\mathfrak{D}}^*_{\Gamma_w} g_z\|_{L^2_{\llcorner\epsilon}} \\
&= C\|\not{\mathfrak{D}}^*_{\Gamma_z} g_z + \lambda cl(e) g_z\|_{L^2_{\llcorner\epsilon}} \\
&\leq C\|\phi_z\|_{L^2_{\ulcorner\epsilon}} + C\lambda\|g_z\|_{L^2_{\llcorner\epsilon}}\,,
\end{aligned}$$

After rearranging, we notice that $\|g_z\|_{W^{1,2}_{\llcorner\epsilon}}$ is bounded independently of small z, and similarly $\|j_z\|_{W^{1,2}_{\epsilon\urcorner}}$ is bounded independently of small z. This last fact is also true for $\Gamma_- \neq \Gamma_+$, for in that case $j_z = \phi_z$ and its $L^2_{\ulcorner\epsilon}$-norm is equivalent to the $L^2_{\epsilon\urcorner}$-norm, as both as defined on ${}^{\ulcorner}V$ over $B^3(w)$.

While it is agreeable to work with a smooth splitting, nothing prevents us from considering the functions

$$h_\lambda = \begin{cases} e^{\lambda t}\psi_\lambda, & \text{for } t < 0, \\ e^{-\lambda t}\psi_{-\lambda}, & \text{for } t > 0, \end{cases} \qquad \text{and} \qquad r_z = \begin{cases} j_z, & \text{for } t < 0, \\ g_z, & \text{for } t > 0, \end{cases}$$

and the associate splitting

$$\phi_z = h_\lambda + r_z.$$

That $\|r_z\|_{L^2_{\epsilon_\lrcorner}}$ is bounded independently of small z follows from the similar fact concerning g_z and j_z.

Consider the families

$$\begin{aligned}\bar{\phi}_z &:= \phi_z/\|\phi_z\|_{L^2},\\ \bar{h}_\lambda &:= h_\lambda/\|\phi_z\|_{L^2},\\ \bar{r}_z &:= r_z/\|\phi_z\|_{L^2}.\end{aligned}$$

Since $\|\phi_z\|_{L^2} \to \infty$ and $\|r_z\|_{L^2_{\epsilon_\lrcorner}}$ is bounded, we have $\|\bar{r}_z\|_{L^2_{\epsilon_\lrcorner}} \to 0$ as $\lambda \to 0$, and *a fortiori*, $\|\bar{r}_z\|_{L^2} \to 0$. The triangle inequality then guarantees

$$\left|\|\bar{h}_\lambda\|_{L^2} - \|\bar{r}_z\|_{L^2}\right| \leq \|\bar{\phi}_z\|_{L^2} \leq \|\bar{h}_\lambda\|_{L^2} + \|\bar{r}_z\|_{L^2}.$$

Since $\|\bar{\phi}_z\|_{L^2} = 1$, and $\|\bar{r}_z\|_{L^2} \to 0$, we must have

$$\|\bar{h}_\lambda\|_{L^2} \to 1 \text{ as } \lambda \to 0.$$

Let us now come back to our main worry. We study

$$\left\langle t\bar{\phi}_z, \bar{\phi}_z\right\rangle = \left\langle t\bar{h}_\lambda, \bar{h}_\lambda\right\rangle + 2\left\langle \bar{h}_\lambda, t\bar{r}_z\right\rangle + \left\langle t\bar{r}_z, \bar{r}_z\right\rangle.$$

The last two terms are bounded by a multiple of $\|t\bar{r}_z\|_{L^2}$. But

$$\|t\bar{r}_z\|_{L^2} \leq C\|\bar{r}_z\|_{L^2_{\epsilon_\lrcorner}},$$

hence it is going to 0.

As for the first term, we have

$$\begin{aligned}\left\langle t\bar{h}_\lambda, \bar{h}_\lambda\right\rangle &= \frac{1}{\|\phi_\lambda\|^2_{L^2}}\left(\int_0^\infty te^{-2\lambda t}|\psi_{-\lambda}|^2 + \int_{-\infty}^0 te^{2\lambda t}|\psi_\lambda|^2\right)\\ &= \frac{1}{2\lambda}\frac{1}{\|\phi_\lambda\|^2_{L^2}}\left(\int_0^\infty e^{-2\lambda t}|\psi_{-\lambda}|^2 + \int_{-\infty}^0 e^{2\lambda t}|\psi_\lambda|^2\right)\\ &= \frac{1}{2\lambda}\|\bar{h}_\lambda\|^2_{L^2},\end{aligned}$$

hence

$$\left\langle t\bar{\phi}_\lambda, \bar{\phi}_\lambda\right\rangle = \frac{1}{2\lambda} + o(1) \text{ as } \lambda \to 0.$$

Suppose now $\bar{\phi}^1_z$ and $\bar{\phi}^2_z$ are two such families, but so that

$$\left\langle \bar{\phi}^1_z, \bar{\phi}^2_z\right\rangle_{L^2} = 0.$$

Then

$$\begin{aligned}\left\langle t\bar{\phi}_z^1, \bar{\phi}_z^2 \right\rangle &= \left\langle t\bar{h}_\lambda^1, \bar{h}_\lambda^2 \right\rangle + \left\langle \bar{h}_\lambda^1, t\bar{r}_z^2 \right\rangle + \left\langle t\bar{r}_z^1, \bar{h}_\lambda^2 \right\rangle + \left\langle t\bar{r}_z^1, \bar{r}_z^2 \right\rangle \\ &= \frac{1}{2\lambda} \left\langle \bar{h}_\lambda^1, \bar{h}_\lambda^2 \right\rangle + o(1),\end{aligned}$$

and of course $\left\langle \bar{h}_\lambda^1, \bar{h}_\lambda^2 \right\rangle \to 0$, hence the result. □

Finally, let us note that in fact, Scenario 3 of page 206 cannot happen. We can take the trace of (B, Φ) to obtain an abelian monopole (b, φ) on $B^3(w) \setminus \{w\}$. The Bogomolny equation reduces to

$$d\,\varphi = *d\,b,$$

and thus $\Delta\varphi = 0$. Since φ is harmonic, not every possible behavior is acceptable as $z \to w$. For one thing, there is a unique set of homogeneous harmonic polynomials p_m and q_m of degree m which give a decomposition of φ on $B^3(w) \setminus \{w\}$ as a Laurent series

$$\varphi = \sum_{m=0}^{\infty} p_m(z-w) + \sum_{m=0}^{\infty} \frac{q_m(z-w)}{|z-w|^{2m+1}};$$

see for example [3, Theorem 10.1, p. 209].

Whether or not the rank is constant, we can find for any sequence of points approaching w a subsequence of points $z_j \to w$ for which the decomposition of Eq. 7.1 is valid. We then have

$$\lim_{j\to\infty} 2|z_j - w|\varphi_{z_j} = i \dim \not{\mathfrak{D}}_{A_{z_j}} \mathcal{H}_{z_j} = i(\mathrm{rk}\mathcal{H} - \dim \overline{K}_{z_j}).$$

By the Laurent series decomposition given above, this number must be the same in any way we approach w, hence $\dim \overline{K}_z$ must be constant on $B^3(w) \setminus \{w\}$.

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