COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 14, Number 1, 107-134, 2006

Counting Curves in Elliptic Surfaces by Symplectic Methods

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We explicitly compute family GW invariants of elliptic surfaces for primitive classes. That involves establishing a TRR formula and a symplectic sum formula for elliptic surfaces and then determining the GW invariants using an argument from [9]. In particular, as in [2], these calculations also confirm the well-known Yau–Zaslow Conjecture [22] for primitive classes in $K3$ surfaces.

In [13], we introduced "family GW invariants" for Kähler surfaces with $p_g > 0$. Since these invariants are defined by using *non-compact* family of almost Kähler structures, we can easily extend several existing techniques for calculating GW invariants to the family GW invariants. In particular, the 'TRR formula' applies to the family invariants, and at least some special cases of the symplectic sum formula [9] apply, with appropriate minor modifications to the formula. Those formulas enable us to enumerate the curves in the elliptic surfaces $E(n)$ for the class $A=$ section plus multiples of the fiber.

Theorem 0.1. *Let $E(n) \to \mathbb{P}^1$ be a standard elliptic surface with a section of self-intersection $-n$. Denote by S and F the homology class of the section and the fiber. Then the genus g family GW invariants for the classes $S+dF$ are given by the generating function*

$$\sum_{d\geq 0} GW^{\mathcal{H}}_{S+dF,g}(E(n))\big(\, pt^g\,\big)\, t^d \;=\; \big(\, tG'(t)\,\big)^g \prod_{d\geq 1}\Big(\frac{1}{1-t^d}\Big)^{12n} \tag{0.1}$$

where $G(t) = \sum\limits_{d\geq 1} \sigma(d)\, t^d$ *and* $\sigma(d) = \sum_{k|d} k$.

Bryan and Leung [2, 3] defined family invariants for K3 and Abelian surfaces by using the Twistor family. They used algebraic methods to show (0.1) for GW invariants of the rational elliptic surface $E(1)$ and for family invariants of $E(2) = K3$ surfaces. For K3, that confirms the famous

Yau–Zaslow Conjecture [22] for those cases when the homology class A is primitive. They also pointed out that one can define family invariants of $E(n)$ for $n \geq 3$ using compact family of complex structures induced from the fiber sum, and then use the algebraic methods of [2] to show that those invariants also satisfy (0.1) [5], see also Section 5 of [4]. An analogous problem for the Seiberg–Witten invariants was studied by Liu [14].

On the other hand, Ionel and Parker used analytic methods to compute the GW invariants of $E(1)$ [9]. They related TRR formula and their sum formula for the relative invariants to obtain a quasi-modular form as in (0.1). We follow the same argument – relating TRR formula and sum formula – to show Theorem 0.1. This theorem also confirm the Yau–Zaslow Conjecture for primitive classes, since our invariants of K3 surfaces are equivalent to the invariants define by Bryan and Leung (cf. Theorem 4.3 of [13]).

For $E(1)$ and $E(2)$, the invariants are known to be enumerative, that is, formula (0.1) actually counts (irreducible) holomorphic curves in the primitive classes for generic complex structures on those surfaces [2]. At the moment, it is not clear whether, or in what sense, that is true for the $E(n)$ with $n \geq 3$ (cf. Remark 5.12 of [4]).

The construction of family invariants for Kähler surfaces is briefly described in Section 1. We give an overview of the proof of Theorem 0.1 in Section 2. This argument is an extension of the elegant argument used by Ionel and Parker to compute the GW invariants of $E(1)$ [9]. It involves computing the generating function for the invariants in two ways, first using the so-called TRR formula, and second using a symplectic sum formula as in [9]. Roughly, the only modification needed is a shift in the dimension counts. The extended TRR formula is proved in Section 3 and the sum formula are established in the last 5 sections.

Section 4 gives an alternative definition of the family invariants for $E(n)$ based on the idea of perturbing the J_α-holomorphic map equations as in [20, 21]. This alternative definition is better suited to adapt the analytic arguments in [8, 9] to a family version of sum formula. The proof of the sum formula begins by studying holomorphic maps into a degeneration of $E(n)$. Because $E(n)$ is a Kähler surface we are able to degenerate within a holomorphic family, rather than the symplectic family used in [9].

The degeneration family Z is described in Section 5. It is a family $\lambda : Z \to D^2$ whose fiber Z_λ at $\lambda \neq 0$ is a copy of $E(n)$ and whose central fiber is a union of $E(n)$ with $E(0) = T^2 \times S^2$ along a fixed elliptic fiber V. As $\lambda \to 0$ maps into Z_λ converge to maps into Z_0, and by bumping α to zero along the fiber V, we can ensure that the limits satisfy a simple matching condition along V (there is a single matching condition for the classes A

that we consider). Section 6 shows this splitting argument.

Conversely, if a map into Z_0 satisfies the matching condition, then it can be smoothed to produce a map into Z_λ for small λ. That smoothing is the Gluing Theorem in [9], which relates family invariants of $E(n)$ with relative invariants of $(E(n), V)$ and $(E(0), V)$. We define a family version of relative invariants of $E(n)$ in Section 7. Using the Gluing Theorem, we prove the required sum formulas for the family invariants of $E(n)$ in Section 8.

1. Family Invariants for Kähler surfaces.

Let X be a closed complex surface with Kähler structure (ω, J, h). In this section, we briefly describe the family Gromov–Witten invariants associated to (X, J) which were defined in [13]. First, set

$$\mathcal{H} = \{ \ \alpha + \overline{\alpha} \mid \alpha \in H^{2,0}(X) \ \}.$$

This is a $2p_g$-dimensional space of harmonic forms which are J-*anti-invariant*, that is, $\alpha(Ju, Jv) = -\alpha(u, v)$. Each $\alpha \in \mathcal{H}$ defines an endormorphism K_α of TX by the equation

$$h(u, K_\alpha v) = \alpha(u, v).$$

One can check that, for each $\alpha \in \mathcal{H}$, $Id + JK_\alpha$ is invertible, so defines an almost complex structure

$$J_\alpha \ = \ (\, Id + JK_\alpha\,)^{-1} \ J \ (\, Id + JK_\alpha\,)\,.$$

Let $\overline{\mathcal{F}} = \overline{\mathcal{F}}_{g,k,A}$ the space of all (not necessarily holomorphic) stable maps $f : (C, j) \to X$ of genus g with k marked points which represent homology class A. For each such map, collapsing unstable components of the domain determines a point in the Deligne–Mumford space $\overline{\mathcal{M}}_{g,k}$ and evaluation of marked points determines a point in X^k. Thus, we have a map

$$\overline{\mathcal{F}} \ \xrightarrow{st \times ev} \ \overline{\mathcal{M}}_{g,k} \times X^k \tag{1.1}$$

where st and ev denote stabilization map and evaluation map, respectively. On the other hand, there is a *generalized orbifold bundle* E over $\overline{\mathcal{F}} \times \mathcal{H}$ whose fiber over $(\, f, j, \alpha\,)$ is $\Omega^{0,1}_{jJ_\alpha}(f^*TX)$. This bundle has a section Φ defined by

$$\Phi(f, j, \alpha) = df + J_\alpha\, df\, j\,. \tag{1.2}$$

By definition, the right-hand side of (1.2) vanishes for J_α-holomorphic maps. Thus, $\Phi^{-1}(0)$ is the moduli space of J_α-holomorphic maps which we denote by

$$\overline{\mathcal{M}}^{J,\mathcal{H}}_{g,k}(\, X, A\,).$$

It is, unfortunately, not always compact. When it is compact, it gives rise to family Gromov–Witten invariants in the usual way (cf. [13]).

Proposition 1.1 ([17]). *Suppose the moduli space $\Phi^{-1}(0)$ is compact. Then the bundle E has a rational homology "Euler class" $[\,\overline{\mathcal{M}}^{J,\mathcal{H}}_{g,k}(X,A)\,]^{\mathrm{vir}} \in H_{2r}(\overline{\mathcal{F}};\mathbb{Q})$ for*

$$r = c_1(X)[A] + g - 1 + k + p_g.$$

Definition 1.2. Whenever the moduli space $\overline{\mathcal{M}}^{J,\mathcal{H}}_{g,k}(\,X,A\,)$ is compact, we define the family GW invariants of (X,J) to be the map

$$GW^{J,\mathcal{H}}_{g,k}(X,A)\ :\ H^*(\overline{\mathcal{M}}_{g,k};\mathbb{Q}) \times [H^*(X;\mathbb{Q})]^k \to \mathbb{Q}$$

defined on $\beta \in H^*(\,\overline{\mathcal{M}}_{g,k};\mathbb{Q}\,)$ and $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_k \in [H^*(\,X;\mathbb{Q}\,)]^{\otimes k}$ by

$$GW^{J,\mathcal{H}}_{g,k}(X,A)(\beta;\alpha) = [\overline{\mathcal{M}}^{J,\mathcal{H}}_{g,k}(X,A)]^{\mathrm{vir}} \cap (st^*(\beta) \cup ev^*(\alpha))\,.$$

Remark 1.3. When the homology class A is of type $(1,1)$ and the family moduli space $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ is compact, it consists of only truly holomorphic maps (see Theorem 2.4 of [13]). Namely, for any (f,α) in $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$, $\alpha = 0$. In that case, the construction of the virtual fundamental class in [17] shows that one can replace the parameter space $\mathcal{H}$ by an open neighborhood of $0 \in \mathcal{H}$ to define the same rational homology class $[\,\overline{\mathcal{M}}^{J,\mathcal{H}}_{g,k}(X,A)\,]^{\mathrm{vir}}$.

This paper will focus on the case where X is a standard elliptic surface $E(n)$ with a section of self intersection number $-n$. Note that the elliptic surfaces $E(2)$ are K3 surfaces. Denote by S and F the homology class of the section and the fiber of $E(n)$. Since $c_1(E(n)) = (2-n)F$ and $p_g = n-1$, we have

$$\text{(formal) } \dim \overline{\mathcal{M}}^{J,\mathcal{H}}_{g,k}(\,E(n), S + dF\,) \ = \ 2(g+k) \tag{1.3}$$

where d is an integer.

Proposition 1.4. ([13]) *Let (X,J) be an elliptic surface $E(n)$ and $A = S + dF$, where d is an integer.*

(*a*) *The moduli space $\overline{\mathcal{M}}^{J,\mathcal{H}}_{g,k}(X,A)$ is compact and hence the invariants $GW^{J,\mathcal{H}}_{g,k}(X,A)$ are well-defined. Moreover, they are invariant under deformations of the complex structure which preserve the $(1,1)$-type of A.*

(*b*) *For K3 surfaces (i.e. n=2) the* $GW^{J,\mathcal{H}}_{g,k}(X,A)$ *are same as the invariants defined by Bryan and Leung in* [2]*.*

Thus, for elliptic surfaces, the family of J_α-holomorphic maps parameterized by the family $\mathcal{H}$ gives rise to well-defined invariants, which we will denote variously as

$$GW^{\mathcal{H}}_{g,k}(E(n),A), \quad GW^{\mathcal{H}}_{A,g}(E(n)), \qquad \text{or simply } \; GW^{\mathcal{H}}_{A,g}.$$

The goal of this paper is to *calculate* these family GW invariants.

The family invariants have a property analogous to the composition law of ordinary GW invariants. Consider a node of a stable curve C in the Deligne–Mumford space $\overline{\mathcal{M}}_{g,k}$. When the node is separating, the normalization of C has two components. The genus and the number of marked points decompose as $g = g_1 + g_2$ and $k = k_1 + k_2$ and there is a natural map

$$\sigma : \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \to \overline{\mathcal{M}}_{g,k}. \tag{1.4}$$

defined by identifying (k_1+1)-th marked points of the first component to the first marked point of the second component. We denote by $PD(\sigma)$ the Poincaré dual of the image of this map σ. For non-separating node, there is another natural map

$$\theta : \overline{\mathcal{M}}_{g-1,k+2} \to \overline{\mathcal{M}}_{g,k} \tag{1.5}$$

defined by identifying the last two marked points. We also write $PD(\theta)$ for the Poincaré dual of the image of this map θ.

Proposition 1.5 ([13]). *Let* $\{H_a\}$ *be any basis of* $H^*(E(n);\mathbb{Z})$ *and* $\{H^a\}$ *be its dual basis with respect to the intersection form of* $E(n)$*.*

(a) *For the gluing map* σ *in* (1.4)*, we have*

$$GW^{\mathcal{H}}_{A,g}(PD(\sigma);\alpha_1,\cdots,\alpha_k) = \sum_{A=A_1+A_2} I_{A_1+A_2}(\alpha_1,\cdots,\alpha_k)$$

where if the family invariant $GW^{\mathcal{H}}_{A_1,g_1}$ *is well-defined*

$$\begin{aligned} &I_{A_1+A_2}(\alpha_1,\cdots,\alpha_k) \\ &\quad = \sum_a GW^{\mathcal{H}}_{A_1,g_1}(\alpha_1,\cdots,\alpha_{k_1},H_a)\, GW_{A_2,g_2}(H^a,\alpha_{k_1+1},\cdots,\alpha_k), \end{aligned}$$

and if the family invariant $GW^{\mathcal{H}}_{A_2,g_2}$ *is well-defined*

$$\begin{aligned} &I_{A_1+A_2}(\alpha_1,\cdots,\alpha_k) \\ &\quad = \sum_a GW_{A_1,g_1}(\alpha_1,\cdots,\alpha_{k_1},H_a)\, GW^{\mathcal{H}}_{A_2,g_2}(H^a,\alpha_{k_1+1},\cdots,\alpha_k). \end{aligned}$$

Here, GW *denotes the ordinary* GW *invariant of* $E(n)$*.*

(b) *For the gluing map* θ *in* (1.5)*, we have*

$$GW^{\mathcal{H}}_{A,g}(PD(\theta);\alpha_1,\cdots,\alpha_k) \;=\; \sum_a GW^{\mathcal{H}}_{A,g-1}(\alpha_1,\cdots,\alpha_k,H_a,H^a).$$

2. The Invariants of $E(n)$ — Outline.

By Proposition 1.4, the family GW invariants of $E(n)$ for the class $S+dF$ are unchanged under deformations of Kähler structure. Since the family moduli space with genus g and no marked points has dimension $2g$, we get numerical invariants by imposing g point constraints on the moduli space $\overline{\mathcal{M}}^{\mathcal{H}}_{g,g}(\,E(n),S+dF)$ — those are the numbers we aim to calculate. For convenience, we assemble them in the generating function

$$F_g(t) = \sum_{d\geq 0} GW^{\mathcal{H}}_{S+dF,g}(pt^g)\, t^d. \tag{2.1}$$

In this and the following six sections, we will derive the formula for $F_g(t)$ stated in Theorem 0.1. Thus, our aim is to prove:

Proposition 2.1. *For* $n\geq 1$*,*

$$F_g(t) = \big(tG'(t)\,\big)^g \prod_{d\geq 1}\left(\frac{1}{1-t^d}\right)^{12n} \tag{2.2}$$

This section shows how Proposition 2.1 follows from three formulas, Equations (2.4), (2.5) and (2.6) below, that are proved in later sections. Our proof parallels the proof of Ionel and Parker for GW invariants of $E(1)$ [9].

Here is the outline of the proof of (2.2). Consider the 'descendent' $\tau(F) = \psi_1\cup ev^*(F^*)$ where F^* denotes the Poincaré dual of the fiber class F and ψ_1 denotes the first Chern class of the line bundle $L\to \overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(\,E(n),S+dF\,)$ whose geometric fiber over $\big[\,f\,:\,(C;x)\,\to\,E(n),\alpha\big]$ is T^*_xC. We further introduce the generating function for a genus 1 invariant with the descendent constraint by the formula

$$H(t) \;=\; \sum_{d\geq 0} GW^{\mathcal{H}}_{S+dF,1}(\,\tau(F)\,)\, t^d. \tag{2.3}$$

We can compute $H(t)$ in two different ways. In Section 3, we show how to combine the composition law (Proposition 1.5) together with the TRR for genus 1 to obtain the formula

$$H(t) \;=\; \frac{1}{12}\, t\, F_0'(t) \;-\; \frac{1}{12}\, F_0(t) \;+\; (2-n)\, F_0(t)\, G(t) \tag{2.4}$$

Then, from Sections 4 to 8, we establish a family version of the sum formulas to show

$$H(t) = -\frac{1}{12}\, F_0(t) + 2\, F_0(t)\, G(t) \tag{2.5}$$

$$F_g(t) = F_{g-1}(t)\, t\, G'(t) \tag{2.6}$$

(see Proposition 8.5). Equations (2.4) and (2.5) give rise to the ODE

$$t\, F_0'(t) = 12\, n\, G(t)\, F_0(t) \tag{2.7}$$

and we show in Proposition 4.6 that the initial condition is $F_0(0) = 1$. Thus, the solution of this ODE is given by

$$F_0(t) = \prod_{d\geq 1} \left(\frac{1}{1-t^d} \right)^{12n}.$$

Now, (2.6) gives (2.2) by induction. That completes the proof of Proposition 2.1 and hence of the main Theorem 0.1 of the introduction. The heart of the matter, then, is to establish formulas (2.4), (2.5) and (2.6).

3. The Topological Recursion Relation (TRR).

This section shows the TRR formula (2.4). Following [1], we denote by $\mathcal{M}(G)$ the moduli space of all genus g stable curves with k marked points whose dual graph is G. We also denote by δ_G the orbifold fundamental class of $\overline{\mathcal{M}(G)}$, that is, the fundamental class divided by the order of the automorphisms of a general element of $\mathcal{M}(G)$. Graphs with one edge correspond to degree two classes. There are two types of such graphs, one of which is the graph G_{irr} with one vertex of genus $g-1$.

The following is the well-known genus 1 topological recursion relation:

$$\phi_1 \ (= c_1(\mathcal{L})) = \frac{1}{12}\,\delta_{G_{irr}} \quad \text{in} \quad H^2(\overline{\mathcal{M}}_{1,1};\mathbb{Q}) \tag{3.1}$$

where the line bundle $\mathcal{L} \to \overline{\mathcal{M}}_{1,1}$ has the geometric fiber T_x^*C at the point (C,x).

Proposition 3.1. *The generating function* (2.3) *satisfies*

$$H(t) \ = \ \frac{1}{12}\,t\,F_0'(t) \ - \ \frac{1}{12}\,F_0(t) \ + \ (2-n)\,F_0(t)\,G(t).$$

Proof. Let $\overline{\mathcal{M}}_{0,2}$ be the space of prestable curves of genus 0 with two marked points [6] and $\sigma : \overline{\mathcal{M}}_{0,2} \times \overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ be the gluing map as in (1.4). For any decomposition of $S + dF = A_1 + A_2$, we denote by

$$\overline{\mathcal{M}}(\,\sigma(A_1,A_2)\,) \ \subset \ \overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(S+dF) \tag{3.2}$$

the set of all (f,C,α) in $\overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(S+dF)$ such that (i) $C = \sigma(\,C_1, C_2\,)$ for some $C_1 \in \overline{\mathcal{M}}_{0,2}$ and $C_2 \in \overline{\mathcal{M}}_{1,1}$, (ii) the restriction of f to C_1 represents A_1, and (iii) the restriction of f to C_2 represents A_2. The machinery of Li and Tian [17] then yields a virtual fundamental class

$$[\,\overline{\mathcal{M}}(\,\sigma(A_1,A_2)\,)\,]^{\rm vir} \tag{3.3}$$

associated with (3.2).

Recall that ψ_1 is the first Chern class of the relative line bundle $L \to \overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(S+dF)$. Over the open dense set of $\overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(S+dF)$ consisting of maps with stable domains, the lative line bundles are related by $L = st^*\mathcal{L}$ where $st : \overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(S+dF) \to \overline{\mathcal{M}}_{1,1}$ is the stabilization map. That is not the case for maps with unstable domains. There are correction terms. We have

$$[\,\overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(\,S+dF\,)\,]^{\rm vir} \ \cap \ (\psi_1 - st^*\phi) \ = \ \sum \,[\,\overline{\mathcal{M}}(\,\sigma(A_1,A_2)\,)\,]^{\rm vir}$$

where the sum is over all $S + dF = A_1 + A_2$ (cf. (11) of [6]). Consequently, the coefficients $GW^{\mathcal{H}}_{S+dF,1}(\,\tau(F)\,)$ of $H(t)$ are

$$\begin{aligned}
&[\overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(S+dF)]^{\rm vir} \ \cap \ (\psi_1 \cup ev^*(F^*)) \\
&\quad = [\overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(S+dF)]^{\rm vir} \ \cap \ (st^*\phi_1 \cup ev^*(F^*)) \\
&\qquad + \ \sum\,[\overline{\mathcal{M}}(\sigma(A_1,A_2))]^{\rm vir} \ \cap \ ev^*(F^*).
\end{aligned} \tag{3.4}$$

Let $\{H^\gamma\}$ and $\{H_\gamma\}$ be bases of $H^*(E(n))$ dual by the intersection form. We have

$$
\begin{aligned}
[\overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(S+dF)]^{\mathrm{vir}} \cap (st^*\phi_1 \cup ev^*(F^*)) &= \frac{1}{12} GW^{\mathcal{H}}_{S+dF,1}(\delta_{G_{irr}}; F^*) \\
&= \frac{1}{24} \sum_\gamma GW^{\mathcal{H}}_{S+dF,0}(F^*, H^\gamma, H_\gamma) \\
&= \frac{2d-n}{24} GW^{\mathcal{H}}_{S+dF,0}
\end{aligned}
\tag{3.5}
$$

where the first equality follows from (3.1), the second follows from the fact $|\operatorname{Aut}(G_{irr})| = 2$ and Proposition 1.5 b, and the last follows from $\sum_\gamma (H^\gamma \cdot A)(H_\gamma \cdot A) = A^2$.

On the other hand, by Theorem 2.4 of [13], every (f,α) in $\overline{\mathcal{M}}^{\mathcal{H}}_{1,1}(S+dF)$ has $\alpha = 0$, i.e. f is truly holomorphic. The inclusion (3.2) thus means that the only possible decompositions of $S+dF$ with non-trivial virtual class (3.3) are $S+d_1F$ and d_2F with $d_1+d_2 = d$, and $d_1, d_2 \geq 0$. Proposition 1.5 a and routine dimension counts then imply that

$$
\sum [\overline{\mathcal{M}}(\sigma(A_1,A_2))]^{\mathrm{vir}} \cap ev^*(F^*) = \sum_{d=d_1+d_2} \sum_\gamma GW^{\mathcal{H}}_{S+d_1F}(F^*, H^\gamma) GW_{d_2F,1}(H_\gamma).
\tag{3.6}
$$

This can be further simplified by separating the $d_2 = 0$ term and simplifying using the facts (a) $\sum_\gamma (H^\gamma \cdot A)(H_\gamma \cdot B) = A \cdot B$, (b) $d_2\, GW_{d_2F,1} = (2-n)\sigma(d_2)$ (see [7]), and (c) $GW_{0,1}(H_\gamma) = \frac{1}{24}(K \cdot H_\gamma)$ where $K = (n-2)F$ is the canonical class (see 1.4.1 Proposition of [11]). The right-hand side of (3.6) then becomes

$$
(2-n) \sum_{1 \leq d_2 \leq d} GW^{\mathcal{H}}_{S+d_1F,0}\, \sigma(d_2) \; + \; \frac{n-2}{24} GW^{\mathcal{H}}_{S+dF,0}
\tag{3.7}
$$

The proof now follows from (3.4), (3.5), (3.7) and the definitions of $F_0(t)$ and $H(t)$. □

4. Ruan–Tian Invariants of $E(n)$.

Instead of constructing virtual fundamental class directly from the moduli space of stable J-holomorphic maps, Ruan and Tian [20, 21] perturbed J-holomorphic equation to $\overline{\partial} f = \nu$ where the inhomogeneous term ν can be chosen generically. For generic (J,ν), the moduli space of stable (J,ν)-holomorphic maps is then a compact smooth orbifold with all lower strata having codimension at least two. Ruan and Tian defined GW invariants from this (perturbed) moduli space.

We can follow a similar procedure for the family invariants by introducing an inhomogeneous term into the J_α-holomorphic equation and vary ν. This alternative definition of invariants is more geometric. In particular, using this definition of invariants, we can follow the analytic arguments of Ionel and Parker in [8, 9] to show sum formulas (2.5) and (2.6) for the case at hand: the class $S+dF$ in $E(n)$. To simplify notation in this section, we will set $A = S + dF$.

Using Prym structures defined as in [16], we can lift the Deligne–Mumford space $\overline{\mathcal{M}}_{g,k}$ to a finite cover

$$p_\mu : \overline{\mathcal{M}}^\mu_{g,k} \to \overline{\mathcal{M}}_{g,k}. \tag{4.1}$$

This finite cover is now a smooth manifold and has a universal family

$$\pi_\mu : \overline{\mathcal{U}}^\mu_{g,k} \to \overline{\mathcal{M}}^\mu_{g,k}$$

which is projective. Moreover, for each $b \in \overline{\mathcal{M}}^\mu_{g,k}$, $\pi_\mu^{-1}(b)$ is a stable curve isomorphic to $p_\mu(b)$.

We fix, once and for all, an embedding of $\overline{\mathcal{U}}^\mu_{g,k}$ into some $\mathbb{P}^N$. An inhomogeneous term ν is then defined as a section of the bundle $\mathrm{Hom}(\pi_1^*(T\mathbb{P}^N), \pi_2^* TE(n))$ which is anti-J-linear :

$$\nu(j_{\mathbb{P}}(v)) = -J(\nu(v)) \quad \text{for any} \quad v \in T\mathbb{P}^N \tag{4.2}$$

where $j_{\mathbb{P}}$ is the complex structure on $\mathbb{P}^N$.

For each stable map $f : C \to E(n)$, we can specify one element $j \in p_\mu^{-1}(st(C))$. Then, $\pi_\mu^{-1}(j)$ is isomorphic to the stable curve $st(C)$. In this way, we can define a map

$$\phi : C \to st(C) \cong \pi_\mu^{-1}(b) \subset \overline{\mathcal{U}}^\mu_{g,k} \hookrightarrow \mathbb{P}^N. \tag{4.3}$$

Definition 4.1. A stable (J, ν, α)-holomorphic map is a stable map $f : (C, \phi) \to E(n)$ satisfying

$$(df + J_\alpha df j_C)(p) = \nu_\alpha(\phi(p), f(p))$$

where ϕ is defined as in (4.3), and $\nu_\alpha = (I + JK_\alpha)^{-1}\nu$.

Instead of using $\mathcal{H}$, we will use an open neighborhood of $0 \in \mathcal{H}$ for parameter space to define family invariants. Let

$$\mathcal{D} = \{\alpha \in \mathcal{H} \,|\ \ |\alpha|_\infty < 1\}$$

and denote the moduli space of stable (J, α, ν)-holomorphic maps $((f, (\phi, C), \alpha)$ by

$$\overline{\mathcal{M}}_{g,k}(E(n), A, \nu, \mathcal{D}, \mu) \tag{4.4}$$

where α in $\mathcal{D}$ and $[f(C)] = A$ in $H_2(E(n); Z)$. We also denote by

$$\mathcal{M}_{g,k}(E(n), A, \nu, \mathcal{D}, \mu)$$

the set of $((f, (\phi, C)), \alpha)$ with a smooth domain C. We will often abuse notation by writing (f, C, α), (f, j, α) or simply (f, α), instead of $(f, (\phi, C), \alpha)$.

Lemma 4.2. *Let $|\nu|_\infty$ be sufficiently small. Then, for any (f, C, α) in $\overline{\mathcal{M}}_{g,k}(E(n), A, \nu, \mathcal{D}, \mu)$*

$$E(f) = \frac{1}{2}\int_C |df|^2 \leq E_A \quad \text{and} \quad |\alpha|_\infty < \frac{1}{2}$$

where E_A is a uniform constant depending only on the homology class A.

The proof of this lemma follows from the proof of Lemma 6.2 below.

There are stabilization and evaluation maps as in (1.1):

$$\mathcal{M}_{g,k}(E(n), A, \nu, \mathcal{D}, \mu) \xrightarrow{st^\mu \times ev^\mu} \overline{\mathcal{M}}^\mu_{g,k} \times E(n)^k. \tag{4.5}$$

Its Frontier is defined to be the set

$$\{r \in \overline{\mathcal{M}}^\mu_{g,k} \times E(n)^k \mid r$$
$$= \lim(st^\mu \times ev^\mu)(f_n, \alpha_n) \text{ and } (f_n, \alpha_n) \text{ has no convergent subsequence}\}.$$

We denote by $\mathcal{Y}_0$ the space of all ν with $|\nu|_\infty$ is sufficiently small.

Theorem 4.3. *For generic $\nu \in \mathcal{Y}_0$, the moduli space $\mathcal{M}_{g,k}(E(n), A, \nu, \mathcal{D}, \mu)$ is a smooth oriented manifold of dimension*

$$2\,c_1(A) + 2\,(g-1) + 2\,k + \dim(\mathcal{H}) = 2(g+k). \tag{4.6}$$

Furthermore, the frontier of the smooth map (4.5) *lies in dimension at most two less than* $2(g+k)$.

Sketch of Proof. The proof of this theorem is similar to that of Proposition 2.3 in [21]. The first statement follows from the standard argument using Sard–Smale Theorem and the bound of $|\alpha|_\infty$ in Lemma 4.2. To prove the second statement, we first consider the well-defined stabilization and evaluation map

$$\overline{\mathcal{M}}_{g,k}(\,E(n), A, \nu, \mathcal{D}, \mu\,) \xrightarrow{st^\mu \times ev^\mu} \overline{\mathcal{M}}^\mu_{g,k} \times E(n)^k. \tag{4.7}$$

It then follows from Gromov Convergence Theorem [10, 18, 19] and Lemma 4.2 that the stable moduli space (4.4) is compact and hence (4.5) extends (4.7) continuously.

As in [20, 21], we reduce the moduli space by (i) collapsing all ghost bubbles, (ii) replacing each multiple map from a bubble by its reduced map, and (iii) identifying those bubble components which have the same image. We denote this reduced moduli space by

$$\overline{\mathcal{M}}^r_{g,k}(\,E(n), A, \nu, \mathcal{D}, \mu\,).$$

The map (4.5) now descends to the reduced moduli space and by definition, we have

$$\mathrm{Fr}(st^\mu \times ev^\mu) \subset st^\mu \times ev^\mu \left(\overline{\mathcal{M}}^r_{g,k}(E(n), A, \nu, \mathcal{D}, \mu) \setminus \mathcal{M}_{g,k}(E(n), A, \nu, \mathcal{D}, \mu)\right).$$

It remains to show that those strata consisting of (f, α) with domain more than two components has a dimension at least two less than $2(g + k)$. Similarly, to the moduli space of (J, ν)-holomorphic maps, the strata corresponding to the domain with no bubble component has a dimension at least two less than $2(g + k)$ for generic ν.

On the other hand, it follows from compactness of stable moduli space (4.4) and Theorem 2.4 of [13] that the restriction of (f, α) to any component of domain should represent one of the following homology classes

$$S,\ S + d_1F,\ d_2F \quad \text{with} \quad 0 < d_1, d_2 \le d$$

Since the inhomogeneous term ν vanishes on bubble components, by Theorem 2.4 of [13] that each bubble component maps into either a section or a singular fiber.

Now, suppose (f, α) has some bubble components. Again by Theorem 2.4 of [13] either $\alpha \equiv 0$ or the zero divisor $Z(\alpha)$ contains some singular fibers. Since there's no fixed component in the complete linear system of a canonical divisor of $E(n)$, the parameter α lies in the proper subspace of $\mathcal{D}$. This reduces the dimension of the strata containing (f, α) at least two. □

Now, we are ready to define invariants. Instead of using intersection theory as in [20, 21], we will follow the approach in [8]. The above Structure Theorem and Proposition 4.2 of [KM2] assert that the image

$$st^\mu \times ev^\mu \left(\mathcal{M}_{g,k}(E(n), A, \nu, \mathcal{D}, \mu)\right)$$

gives rise to a rational homology class in $H_*(\overline{\mathcal{M}}^{\mu}_{g,k};\mathbb{Q})\otimes H_*(E(n)^k;\mathbb{Q})$. We denote it by

$$[\overline{\mathcal{M}}_{g,k}(E(n),A,\nu,\mathcal{D},\mu)]. \tag{4.8}$$

Definition 4.4. For $2g+k\geq 3$, we define invariants by

$$GW_{g,k}(E(n),A,\mathcal{D})(\beta;\alpha)=\frac{1}{\lambda_\mu}(\beta\otimes\alpha)\cap[\overline{\mathcal{M}}_{g,k}(E(n),A,\nu,\mathcal{D},\mu)]$$

where $\beta\in H^*(\overline{\mathcal{M}}_{g,k};\mathbb{Q})$, $\alpha=\alpha_1\otimes\cdots\otimes\alpha_k\in[H^*(E(n);\mathbb{Q})]^{\otimes k}$, and λ_μ is the order of the finite cover in (4.1).

Remark 4.5. By using the bound of α in Lemma 4.2 and by repeating the same arguments for ordinary GW invariants (cf. Proposition 2.3 of [17]), one can show that

$$(st\times ev)_*\big([\overline{\mathcal{M}}^{\mathcal{D}}_{g,k}(E(n),A)]^{\mathrm{vir}}\big)=[\overline{\mathcal{M}}_{g,k}(E(n),A,\nu,\mathcal{D})]$$

for sufficiently small $|\nu|_\infty$ – for any (f,α) in the moduli space $\overline{\mathcal{M}}_{g,k}(E(n),A,\nu,\mathcal{D})$ the parameter α is away from the boundary $\partial\mathcal{D}$. Here, the homology class in the left-hand side is the homology Euler class as in Proposition 1.1 obtained from the (J,α)-holomorphic map equation with $\alpha\in D$. Since $A=S+dF$ is of $(1,1)$ type, this class equals to the homology Euler class determined by the (J,α)-holomorphic map equation with $\alpha\in\mathcal{H}$ (cf. Remark 1.3). Namely,

$$[\overline{\mathcal{M}}^{\mathcal{D}}_{g,k}(E(n),A)]^{\mathrm{vir}}\ =\ [\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(E(n),A)]^{\mathrm{vir}}.$$

Therefore, for the class $A=S+dF$, Definition 1.2 and Definition 4.4 are equivalent;

$$GW^{\mathcal{H}}_{g,k}(E(n),A)\ =\ GW_{g,k}(E(n),A,\mathcal{D}).$$

In the below, we will not distinguish two invariants and use the same notation $GW^{\mathcal{H}}_{g,k}(E(n),A)$ for them. We end this section by showing $F_0(0)=1$ which provides the initial condition for (2.7).

Proposition 4.6. $GW^{\mathcal{H}}_{S,0}(E(n))=1$.

Proof. Fix $\nu=0$. Since the section class S is of type $(1,1)$, Theorem 2.4 of [13] implies that for any (J,α)-holomorphic map (f,α) with $[f]=S$, f

is holomorphic and $\alpha = 0$. In fact, there is a unique such map f since $S^2 = -n$.

Now, consider the linearization of (f, α)-holomorphic equation $L_f \oplus J\,df \oplus L_0$ as in appendix of [13]. Propositions A.1 and A.2 of the appendix of [13] show, quite generally, that L_f is a $\overline{\partial}$ operator and L_0 defines a map

$$L_0 : \mathcal{H} \to \mathrm{Coker}(L_f \oplus J\,df)$$

which is injective if and only if the family moduli space is compact. But we just showed the moduli space is a single point, and hence compact.

On the other hand, $\mathrm{Ker}(L_f \oplus Jdf)$ is same as $H^0(f^*N)$, where N is the normal bundle of the section in $E(n)$. It is trivial since the Chern number of N is $S^2 = -n < 0$. Therefore,

$$\dim \mathrm{Coker}(L_f \oplus Jdf) = -\mathrm{Index}(L_f \oplus Jdf) = -2\,(\,c_1(f^*TE(n)) - 1\,) = 2(n-1)$$

Since L_0 is injective and $\dim(\mathcal{H}) = 2(n-1)$, $L_f \oplus Jdf \oplus L_0$ is onto. That implies $\nu = 0$ is generic in the sense of Theorem 4.3. Consequently, the invariant is ± 1. In this case, the sign is determined by L_f and L_f is $\overline{\partial}$-operator, and thus the invariant is 1. $\square$

5. Degeneration of $E(n)$.

In this section, we describe a degeneration of $E(n)$ into a singular surface which is a union of $E(n)$ and $E(0)$ with $V = T^2$ intersection. We then define the parameter space and inhomogeneous terms corresponding to this degeneration. The sum formulas (2.5) and (2.6) will be formulated from this degeneration

Let $D \subset \mathbb{C}$ be a small disk and choose a smooth fibre V in $E(2)$. We denote by

$$p : Z \to E(n) \times D \tag{5.1}$$

the blow-up of $E(n) \times D$ along $V \times \{0\}$ and define $\lambda : Z \xrightarrow{p} E(n) \times D \to D$ to be the composition map, where the second map is the projection onto the second factor. The central fiber $Z_0 = \lambda^{-1}(0)$ is a singular surface $E(n) \cup_V E(0)$ and the fiber Z_λ with $\lambda \neq 0$ is isomorphic to $E(n)$ as a complex surface.

To save notation, we will use the same notation (ω, J, g) for the induced Kähler structure on Z and its restriction to Z_λ, $E(n)$, and $E(0)$.

Fix a normal neighborhood $N_{E(n)}$ of V in $E(n)$. It is then a product $V \times D'$, where $D' \subset \mathbb{C}$ is some disk. Let x be the holomorphic coordinate of D'. Then, the normal neighborhood N of V in Z is given by

$$N \;=\; \{(v, x, \lambda, [l_0; l_1]) \;|\; v \in V,\; x\, l_1 = \lambda\, l_0\} \;\subset\; N_{E(n)} \times D' \times D \times \mathbb{C}P^1$$

where $[l_0; l_1]$ is the homogeneous coordinates of $\mathbb{C}P^1$. It is covered by two patches $U_0 = (l_0 \neq 0)$ and $U_1 = (l_1 \neq 0)$. On U_0, we set $y = l_1/\,l_0$. Then, we have

$$N \;=\; \{\; (v, x, y) \;|\; v \in V \;\} \quad \text{with} \quad \lambda(v, x, y) = xy.$$

Clearly, $Z_\lambda \cap N$ is given by the equation $xy = \lambda$. Note that we can also think of y as a holomorphic normal coordinate of the normal neighborhood $N_{E(0)}$ of V in $E(0)$.

Definition 5.1. For some $\delta > 0$ and $|\lambda|$, we decompose Z_λ as a union of three pieces, two sides and a neck. The δ-*neck* is defined as

$$Z_\lambda(\delta) \;=\; \Big\{\; (v, x, y) \;\in\; Z_\lambda \cap N \;\Big|\; \big|\, |x|^2 - |y|^2 \,\big| \;\leq\; \delta \;\Big\}\,.$$

$Z_\lambda \setminus Z_\lambda(\delta)$ consists of two components. The $E(n)$-side is the component which contains the region $|x| > |y|$, while the component $E(0)$-side contains the region $|x| < |y|$.

On the neck region, there is a symplectic S^1-action with Hamiltonian $t = \frac{1}{2}(\,|y|^2 - |x|^2)$. We can thus decompose each Z_λ as $Z_\lambda = Z_\lambda^- \cup Z_\lambda^+$, where Z_λ^- is a union of $E(n)$-side and the part of $Z_\lambda(\delta)$ with $t \leq 0$. In fact, $E(n)$ (resp. $E(0)$) is the symplectic cut of Z_λ^- (resp. Z_λ^+) at $t = 0$. Therefore, we have a collapsing map

$$\pi_\lambda : Z_\lambda \to Z_0 \tag{5.2}$$

(cf. Section 2 of [9]).

Next, we define the parameter spaces. Let U be a neighborhood of V in $E(n)$ that does not contain any singular fibers. Choose a bump function β which satisfies $\beta = 1$ on $E(n) \setminus U$ and $\beta = 0$ near V in U.

Definition 5.2. We define the parameter spaces by

$$\mathcal{D}_\lambda = \{\alpha_\lambda = p_\lambda^* \,\beta\, \alpha \;\;|\;\; \alpha \in \mathcal{D}\} \quad \text{and} \quad \mathcal{D}_{E(n)} = \{\; \beta\, \alpha \;\;|\, \alpha \in \mathcal{D}\}$$

where p_λ is the restriction of (5.1) to Z_λ.

Note that $\mathcal{D}_{E(n)} = \{0\}$ for $n = 0, 1$. On the other hand, each $\alpha \in \mathcal{D}_{E(n)}$ (resp. $\alpha_\lambda \in \mathcal{D}_\lambda$) is J-anti-invariant and $\alpha = 0$ (resp. $\alpha_\lambda = 0$) near V by definition. Hence, $J_\alpha = J$ (resp. $J_{\alpha_\lambda} = J$) near V.

Lastly, following [9], we define inhomogeneous terms. An inhomogeneous term ν of the the fibration $\lambda : Z \to D$ is a section of the bundle $\mathrm{Hom}(T\mathbb{P}^N, TZ)$ over $\mathbb{P}^N \times Z$ for some $\mathbb{P}^N$, which satisfies Definition 2.2 of [9]. We denote by $\mathcal{J}_0(Z)$ the space of all such ν with sufficiently small $|\nu|_\infty$ and use the same notation ν for the restriction of ν to Z_λ, $E(n)$, and $E(0)$.

6. Splitting of Maps .

In this section, we show the uniform energy bound of maps. By Gromov Convergence Theorem, these lead to the compactness of family moduli spaces. The splitting arguments as in Section 3 of [9] then follows from the compactness and the choice of inhomogeneous terms and parameters α – we bumped α to 0 along V.

Let (X, ω, h, J) be a 4-dimensional almost Kähler manifold. Recall that α is a J-anti-invariant 2-form on X if $\alpha(Ju, Jv) = -\alpha(u, v)$. Fix a metric within the conformal class j on a Riemann surface (C, j) and let dv be the associated volume form.

Lemma 6.1. *Fix a point $z \in C$ and an orthogonal basis $\{e_1, e_2 = je_1\}$ of T_zC. For any C^1 map $f : C \to X$, if α is J-anti-invariant, then*

$$f^*\alpha(e_1, e_2) \le 2\,|\alpha|\,|df|\,|\overline{\partial}_J f|. \tag{6.1}$$

On the other hand, if the map f is (J, ν, α)-holomorphic, then we have

$$|\overline{\partial}_J f|^2\, dv = f^*\alpha + 2\,\langle \overline{\partial}_J f, \nu\rangle dv = f^*(|\alpha|^2)\,|\partial_J f|^2\, dv + 2\,\langle \overline{\partial}_J f + K_\alpha \partial_J f j, \nu\rangle dv \tag{6.2}$$

Proof. The proof of (6.2) is similar to the proof of Corollary 1.4 of [13]. We will prove (6.1) only. It follows that

$$\begin{aligned} \alpha\big(\, df(e_1), df(e_2)\,\big) &= \alpha\big(\, df(e_1), df(e_2) + Jdf(je_2) - Jdf(je_2)\,\big) \\ &= \alpha\big(\, df(e_1), 2\overline{\partial}_J f(e_2)\,\big) + \alpha\big(\, df(e_1), Jdf(e_1)\,\big). \end{aligned} \tag{6.3}$$

Since α is J-anti-invariant, $\alpha\big(\, df(e_1), Jdf(e_1)\,\big) = 0$. Therefore, (6.1) follows from (6.3). ∎

We denote the stable family moduli space of (J,ν,α_λ)-holomorphic maps (f,α_λ) by

$$\overline{\mathcal{M}}_{g,k}(Z_\lambda, S+dF, \nu, \mathcal{D}_\lambda), \quad \text{or simply} \quad \overline{\mathcal{M}}_{g,k}(\lambda, d)$$

where $\nu \in \mathcal{J}_0(Z)$ and $\alpha_\lambda \in \mathcal{D}_\lambda$. Compactness of the family moduli space follows from Gromov Convergence Theorem and the following lemma.

Lemma 6.2. *Let $|\nu|_\infty$ be sufficiently small. Then, for any (f, C, α_λ) in $\overline{\mathcal{M}}_{g,k}(\lambda, d)$*

$$E(f) = \frac{1}{2}\int_C |df|^2 \leq \ E_d \quad \textit{and} \quad |\alpha_\lambda|_\infty < \frac{1}{2}$$

where E_d is a uniform constant depending only on the homology class $S{+}dF$.

Proof. Observe that by shrinking $\mathcal{D}$, if necessary, one can assume that $|p_\lambda^*(\alpha)|_\infty < 1$ for any $\alpha \in \mathcal{D}$ and sufficiently small $|\lambda|$. We first show uniform bound of the Energy $E(f)$. By Definition 5.2, $\alpha_\lambda = p_\lambda^*(\beta\,\alpha)$ for some $\alpha \in \mathcal{D}$ and $p_\lambda^*(\alpha)$ is J-anti-invariant. We define C_- as the set of all z in C with $f^* p_\lambda^* \alpha(\, e_1(z), e_2(z)\,) \leq 0$, where $\{\, e_1(z), e_2(z) = j\, e_1(z)\,\}$ is an orthonormal basis of T_zC. Then, (6.2) implies that $|\overline{\partial}_J f| \leq 2\,|\nu|$ on C_- and hence, by (6.1)

$$0 \ \leq \ -f^*p_\lambda^*\alpha(\,e_1(z), e_2(z)) \leq 2|p_\lambda^*\alpha||df|\,|\overline{\partial}_J f| \leq 4|df|\,|\nu| \tag{6.4}$$

for any $z \in C_-$. Thus, we have

$$\begin{aligned}
\frac{1}{2}\int_C |df|^2 &= \int_C |\overline{\partial}_J f|^2 + \omega(S+dF) \\
&\leq \int_C f^*\, p_\lambda^*(\beta\alpha) + 2\int_C |df|\,|\nu| + \omega(S+dF) \\
&\leq \int_{C\setminus C_-} f^* p_\lambda^*(\alpha) + 2\int_C |df|\,|\nu| + \omega(S+dF) \\
&\leq -\int_{C_-} f^* p_\lambda^*(\alpha) + 2\int_C |df|\,|\nu| + \omega(S+dF) \\
&\leq 6\int_C |df|\,|\nu| + \omega(S+dF) \\
&\leq 3\left(\frac{1}{3^2}\int_C |df|^2 + 3^2\int_C |\nu|^2\right) + \omega(S+dF)
\end{aligned}$$

where the second inequality follows from (6.2), the fourth inequality follows from $p_\lambda^*\alpha\,(S+dF)=0$ and the fifth from (6.4). This implies the uniform energy bound independent of λ for sufficiently small $|\nu|_\infty$.

Next, we show the second inequality. Fix a small $|\lambda|$ and suppose the second inequality is not true. Then, there exists a sequence $\{(f_n,\alpha_\lambda^n)\}$ of $(J,\nu_n,\alpha_\lambda^n)$-holomorphic maps with $\nu_n\to 0$ as $n\to\infty$ and $1/2\le|\alpha_\lambda^n|_\infty<1$ for any n. By the uniform energy bound and Gromov Convergence Theorem, there exists a subsequence, still denoted by $\{(f_n,\alpha_\lambda^n)\}$, that converges to a (J,α_λ)-holomorphic map (f_0,α_λ^0) where $1/2\le|\alpha_\lambda^0|_\infty\le 1$. Let $\alpha_\lambda^0=p_\lambda^*(\beta\alpha^0)$ for some $\alpha^0\in\mathcal{D}$ and C_- be the set of all z in C with $f_0^*\,p_\lambda^*\,\alpha^0(\,e_1(z),e_2(z)\,)\le 0$, where $\{\,e_1(z),e_2(z)=j\,e_1(z)\,\}$ is an orthonormal basis of T_zC. Then, $|\overline{\partial}_Jf_0|=f_0^*\,p_\lambda^*\,\alpha^0\equiv 0$ on C_- by (6.2). Consequently, we have

$$\int_C|\overline{\partial}_Jf_0|^2=\int_C f_0^*p_\lambda^*(\beta\alpha^0)\le\int_C f_0^*p_\lambda^*(\alpha^0)=0$$

where the last equality follows from $\alpha^0(S+dF)=0$. Together with (6.2), this shows that

$$0\equiv|\overline{\partial}_Jf_0|^2\equiv f_0^*(|\alpha_\lambda^0|^2)\,|df_0|^2.$$

Since df_0 has at most finitely many zeros, the image of f_0 must lie in the zero set of α_λ^0 when $f_0^*(\beta)\ne 0$. Thus, we have a contradiction since f_0 represents the class $S+dF$. □

Remark 6.3. Repeating the same argument as in Section 4, one can show that the moduli spaces

$$\overline{\mathcal{M}}_{g,k}(E(n),S+dF,\mathcal{D}_{E(n)},\nu)\quad\text{and}\quad\overline{\mathcal{M}}_{g,k}(Z_\lambda,S+dF,\mathcal{D}_\lambda,\nu)$$

define family invariants $GW_{g,k}(E(n),A,\mathcal{D}_{E(n)})$ and $GW_{g,k}(Z_\lambda,A,\mathcal{D}_\lambda)$, respectively. Moreover, by applying the standard corbodism argument as in Lemma 4.9 of [21], one can show that

$$GW_{g,k}(E(n),S{+}dF,\mathcal{D}_{E(n)})=GW^{\mathcal{H}}_{g,k}(E(n),S{+}dF)=GW_{g,k}(Z_\lambda,S{+}dF,\mathcal{D}_\lambda).\tag{6.5}$$

The following shows how maps into $Z_\lambda=E(n)$ split along the degeneration of $E(n)$. It is also a key observation for gluing of maps into $E(n)$ and $E(0)$, which leads to the sum formulas (2.5) and (2.6).

Lemma 6.4. *Let $\nu\in\mathcal{J}_0(Z)$ and $\{(f_\lambda,C_\lambda,\alpha_\lambda)\}$ be any sequence with $(f_\lambda,C_\lambda,\alpha_\lambda)\in\overline{\mathcal{M}}_{g,k}(\lambda,d)$. Then as $\lambda\to 0$, f_λ converges to a limit $f_0:C_0\to Z_0$ and α_λ converges to α_0, after passing to some subsequences, such that* (*a*) *the limit map* f_0 *can be decomposed as*

$$f_1 : C_1 \to E(n),\ \ f_2 : C_2 \to E(0),\ \ \textit{and}\ \ f_3 : C_3 \to V$$

and f_1 (*resp.* f_2) *represents homology class* $S + d_1F$ *in* $E(n)$ (*resp.* $S + d_2F$ *in* $E(0)$) *and* f_3 *represents* $d_3[V]$ *in* V *with* $d_1 + d_2 + d_3 = d$,

(*b*) *for* $i = 1, 2$, *each* f_i *transverse to* V *with* $f_i^{-1}(V) = \{\, p_i \,\}$, *where* p_i *is a node of* C.

Proof. By Gromov Convergence Theorem and Lemma 6.2, f_λ converges to a limit $f_0 : C_0 \to Z_0$. Since $\alpha_\lambda = 0$ near $V \subset Z$, we have $J_{\alpha_\lambda} = J$ near V in Z. Therefore, (a) and (b) follows from Lemma 3.4 of [8] and Lemma 3.3 of [9]. □

7. Relative Invariants of $E(n)$.

In this section, following [8], we define relative invariants of $(E(n), V)$. In our case, the *rim tori* in $E(n) \setminus V$ disappear when we glue $E(n)$ and $E(0)$ along V. Together with the simple matching condition as in Lemma 6.4, that observation leads to the simple definition of relative invariants.

As in Section 4, we fix the complex structure on $E(n)$. We also assume that we always work with a finite good cover p_μ as in (4.1) without specifying it. Throughout this section, A always denotes the class $S + dF$.

For ν in $\mathcal{J}_0(Z)$, we define the relative moduli space by

$$\begin{aligned}&\mathcal{M}^V_{g,k+1}(E(n), A, \mathcal{D}_{E(n)}, \nu)\\ &\quad = \left\{ (\, f, \alpha\,) \in \overline{\mathcal{M}}_{g,k+1}(E(n), A, \mathcal{D}_{E(n)}, \nu) \mid f^{-1}(V) = \{x_{k+1}\} \right\}.\end{aligned}$$

As in [8], we compactify this moduli space by taking its closure

$$C\mathcal{M}^V_{g,k+1}(E(n), A, \mathcal{D}_{E(n)}, \nu)$$

in the space of stable maps $\overline{\mathcal{M}}_{g,k+1}(E(n), A, \mathcal{D}_{E(n)}, \nu)$. Note that for each $\alpha \in \mathcal{D}_{E(n)}$, $\alpha = 0$ in some neighborhood of $V \subset E(n)$ and hence, $J_\alpha = J$ on that neighborhood. Therefore, Proposition 7.1 below follows from the same arguments as in Lemma 4.2 and Proposition 6.1 of [8], and Theorem 4.3.

Proposition 7.1. *For generic* ν *in* $\mathcal{J}_0(Z)$

(*a*) $\mathcal{M}^V_{g,k+1}(E(n), A, \mathcal{D}_{E(n)}, \nu)$ *is an orbifold of dimension* $2 + 2(g + k)$ *for* $n = 0$ *and* $2(g + k)$ *for* $n \geq 1$, *and* (*b*) *the Frontier of the map*

$$(7.1)\qquad \mathcal{M}^V_{g,k+1}(E(n), A, \mathcal{D}_{E(n)}, \nu) \xrightarrow{st\times ev\times h} \overline{\mathcal{M}}_{g,k+1} \times E(n)^k \times V$$

is contained in codimension at least 2, where ev is the evaluation map of the first k marked points and h is the evaluation map of the last marked point.

Proposition 7.1 together with Proposition 4.2 of [12] asserts that the image of (7.1) gives rise to a rational homology class. We denote it by

$$[\,\mathcal{M}^V_{g,k+1}(E(n), A, \mathcal{D}_{E(n)})\,] \in H_*(\overline{\mathcal{M}}_{g,k+1};\mathbb{Q}) \otimes H_*(E(n)^k;\mathbb{Q}) \otimes H_*(V;\mathbb{Q}).$$

Definition 7.2. For $2g + k \geq 3$, we define the relative invariants of $(E(n), V)$ by

(7.2)

$$GW^V_{g,k+1}(E(n), A)(\,\beta;\alpha;C(\gamma)\,) = (\,\beta\otimes\alpha\otimes\gamma\,) \cap [\,\mathcal{M}^V_{g,k+1}(E(n), A, \mathcal{D}_{E(n)})\,]$$

where $\beta \in H^*(\overline{\mathcal{M}}_{g,k+1};\mathbb{Q})$, $\alpha = \alpha_1\otimes\cdots\otimes\alpha_k \in [\,H^*(E(n);\mathbb{Q})\,]^{\otimes k}$, and $\gamma \in H^*(V;\mathbb{Q})$.

The relative invariant (7.2) counts oriented number of (J,α)-holomorphic maps (f, C, α) satisfying (i) the genus of C is g, (ii) f represents the homology class A, (iii) $f^{-1}(V) = x_{k+1}$, (iv) $C \in K$ and $f(x_i) \in A_i$, $1 \leq i \leq k$, where K and A_i are representatives of the Poincaré dual of β and α_i, respectively, and (v) f has a contact of order one with V along a fixed representative of the Poincaré dual of γ.

The relative invariants of $E(0)$ and $E(1)$ defined as in Definition 7.2 are less finer than those in [8] (cf. Appendix in [9]). On the other hand, for another smooth fiber $U = T^2$ of $E(0)$, we can define relative invariants relative to both V and U as in Definition 7.2. In the below, we will denote ordinary and relative GW invariants for $E(0)$ by

$$\Phi_{A,g},\ \Phi^V_{A,g},\quad \text{and}\quad \Phi^{V,U}_{A,g},\quad \text{respectively.}$$

We end this section by relative invariants of $E(0)$ for the class $S + dF$. Recall that for positive integer d, $\sigma(d)$ is the sum of the divisors, namely $\sigma(d) = \sum_{k|d} k$. For convenience, we set $\sigma(0) = -1/24$, and $\delta_{d0} = 1$ if $d = 0$ and 0 otherwise. To save notation, we also denote by F the fundamental class $[V]$ of V.

Lemma 7.3 ([9]).

(*a*) $\Phi^V_{S+dF,0}(\,\tau(F); C(F)\,) = 0.$

(*b*) $\Phi^V_{S+dF,1}(\,\tau(F); C(pt)\,) = 2\sigma(d).$

(*c*) $\Phi^V_{S+dF,0}(\,pt; C(F)\,) = \Phi^V_{S+dF,0}(\,C(pt)\,) = \delta_{d0}.$

(*d*) $\Phi^V_{S+dF,1}(\,pt; C(pt)\,) = d\,\sigma(d).$

(*e*) $\Phi^{V,U}_{S+dF,1}(\,C(pt), C(pt)\,) = 0.$

8. Sum Formula .

This section shows the sum formulas (2.5) and (2.6) using a family version of Gluing Theorem – a map into Z_0 satisfying the matching condition as in Lemma 6.4 can be smoothed to produce a map into Z_λ. That smoothing relates invariants of $Z_\lambda = E(n)$ with relative invariants of $(E(n), V)$ and $(E(0), V)$.

Throughout this section, we fix $n \geq 1$. Recall the evaluation map of last marked point as in (7.1). There is an evaluation map

$$\begin{aligned} ev_V : \bigcup \big(\mathcal{M}^V_{g_1,k_1+1}\left(E(n), S+d_1F, \mathcal{D}_{E(n)}, \nu\right) \\ \times \mathcal{M}^V_{g_2,k_2+1}\left(E(0), S+d_2F, \nu\right)\big) \to V^2 \end{aligned}$$

which records the intersection points with V, where the union is over all $g_1+g_2 = g$, $k_1+k_2 = k$ and $d_1+d_2 = d$. We set

$$\mathcal{M}^V_{g,k}(d) = ev_V^{-1}(\triangle) \tag{8.1}$$

where $\triangle$ is the diagonal of V^2. This space is an orbifold of dimension $2(g+k)$ for generic ν in $\mathcal{J}_0(Z)$ and comes with stabilization and evaluation maps

$$\begin{aligned} \mathcal{M}^V_{g,k}(d) &\xrightarrow{st\times ev} \big(\overline{\mathcal{M}} \times E(n)\big)_{g,k} \\ &= \bigcup\ \big(\overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1}\big) \times \big(\,E(n)^{k_1} \times E(0)^{k_2}\,\big) \end{aligned}$$

where the union is over all $g = g_1+g_2$, and $k = k_1+k_2$.

Now, consider a sequence of maps $(f_\lambda, \alpha_\lambda)$ in $\overline{\mathcal{M}}_{g,k}(\,\lambda, d\,)$. By Lemma 6.4, as $\lambda \to 0$, the maps $(f_\lambda, \alpha_\lambda)$ converge to a limit (f_0, α_0), after passing to some subsequences. In general, the limit (f_0, α_0) might be not in $\mathcal{M}^V_{g,k}(\,d\,)$. That happens if some components of f_0 map entirely into V. On the other hand, by Lemma 1.5 of [8], there is a constant c_V, depending only on (J_V, ν_V) such that every stable (J_V, ν_V)-holomorphic maps have energy greater than c_V. This implies that for small $|\lambda|$ the energy of f_λ in the δ-neck

$$E^\delta(f_\lambda) \;=\; \frac{1}{2}\int |df_\lambda|^2 + |d\phi|^2 \tag{8.2}$$

is greater than c_V, where the integral is over $f_\lambda^{-1}(Z_\lambda(\delta))$ and $\phi : C_\lambda \to \mathbb{P}^N$ as in (4.3). Therefore, for each λ, if f_λ is δ-flat (see Definition 8.1 below), then f_0 is also δ-flat and hence the limit (f_0, α_0) is contained in $\mathcal{M}^V_{g,k}(\,d\,)$.

Following [9], we define δ-flat maps as follows:

Definition 8.1. A stable (J, ν, α)-holomorphic map (f, α) into Z_λ is δ-flat if

$$E^\delta(f) \le \frac{c_V}{2} \tag{8.3}$$

Note that any δ-flat map (f, α) into Z_0 has no component maps into V. We denote by

$$\overline{\mathcal{M}}^\delta_{g,k}(\lambda, d) \subset \overline{\mathcal{M}}_{g,k}(\lambda, d) \qquad (\text{resp. } \mathcal{M}^{V,\delta}_{g,k}(d) \subset \mathcal{M}^V_{g,k}(d)) \tag{8.4}$$

the set of all δ-flat maps in $\overline{\mathcal{M}}_{g,k}(\lambda, d)$ (resp. in $\mathcal{M}^V_{g,k}(d)$).

The following is a family version of Theorem 10.1 of [9]. It shows that a δ-flat map into Z_0 can be smoothed to produce a δ-flat map into Z_λ for small $|\lambda|$.

Theorem 8.2. *For generic $\nu \in \mathcal{J}_0(Z)$ and for small $|\lambda|$, there is a diagram*

$$\begin{tikzcd}
\mathcal{M}^{V,\delta}_{g,k}(\,d\,) \arrow[rr, "\Phi_\lambda"] \arrow[d, "st\times ev"] & & \overline{\mathcal{M}}^{\,\delta}_{g,k}(\,\lambda, d\,) \arrow[d, "st\times ev"'] \\
\big(\overline{\mathcal{M}} \times E(n)\big)_{g,k} \arrow[dr, "\sigma\times\pi_0"] & & \overline{\mathcal{M}}_{g,k} \times Z^k_\lambda \arrow[dl, "id\times\pi_\lambda"'] \\
& \overline{\mathcal{M}}_{g,k} \times Z^k_0 &
\end{tikzcd}$$

which commutes up to homotopy, where Φ_λ is an embedding, σ is the gluing map of the domain as in (1.4)*, π_λ is the collapsing map as in (*5.2*), and $\pi_0 : \bigcup (E(n)^{k_1} \times E(0)^{k_2}) \to Z^k_0$ defined by $\pi_0(x_1, \cdots, x_{k_1}, y_1, \cdots, y_{k_2}) = (x_1, \cdots, x_{k_1}, y_1, \cdots, y_{k_2})$.*

First, we use Theorem 8.2 to derive the sum formula (8.7) for certain constraints. Let $\beta = \beta_1 \otimes \cdots \otimes \beta_k \in H^{2r}(Z_0^k)$, where $r = g + k$. Denote by B^i a geometric representative of the Poincaré dual of β_i. We assume that for some $0 \le k_1 \le k$

(i) each B^i lies in $E(n)$-side if $i \le k_1$ and in $E(0)$-side if $i > k_1$, and

(ii) $\deg(\beta_1 \otimes \cdots \otimes \beta_{k_1}) = 2(g_1 + k_1)$ for some $0 \le g_1 \le g$.

Note that the assumption implies that there is a decomposition

$$\pi_0^*\beta = \beta' + \beta'' \quad \text{with} \quad \beta' \in H^{2(g_1+k_1)}(E(n)^{k_1}) \quad \text{and} \quad \beta'' \in H^{2(g_2+k_2)}(E(0)^{k_2})$$

where $g_2 = g - g_1$ and $k_2 = k - k_1$. On the other hand, the inverse image of B^i under π_λ gives a continuous family of geometric representatives B^i_λ of the Poincaré dual of $\pi_\lambda^*\beta_i$ in $H^*(Z_\lambda)$. We define the cut-down moduli spaces by

$$\overline{\mathcal{M}}_{g,k}(\lambda, d) \cap \pi_\lambda^*\beta = \big\{(f,\alpha) \in \overline{\mathcal{M}}_{g,k}(\lambda, d) \mid ev_i(f,\alpha) \in B^i_\lambda\big\} \tag{8.5}$$

$$\mathcal{M}^V_{g,k}(d) \cap \pi_0^*\beta = \big\{\big((f_1,\alpha), f_2\big) \in \mathcal{M}^V_{g,k}(d) \mid ev_i\big((f_1,\alpha), f_2\big) \in B^i\big\} \tag{8.6}$$

where ev_i is the evaluation map of the i-th marked point. Both cut-down moduli spaces (8.5) and (8.6) are finite. In particular, any map in (8.5) is δ-flat for some $\delta > 0$.

Proposition 8.3. *Let $\beta \in H^{2r}(Z_0^k)$ be a constraint as above. Then*

$$\begin{aligned} GW^{\mathcal{H}}_{S+dF,g}(\beta) = &\sum_{d_1+d_2=d} GW^V_{S+d_1F,g_1+1}(\beta'; C(pt))\Phi^V_{S+d_2F,g_2-1}(\beta''; C(F)) \\ &+ \sum_{d_1+d_2=d} GW^V_{S+d_1F,g_1}(\beta'; C(F))\Phi^V_{S+d_2F,g_2}(\beta''; C(pt)). \end{aligned} \tag{8.7}$$

Proof. Denote the set of limits of sequences of maps in (8.5) as $\lambda \to 0$ by

$$\lim_{\lambda\to 0} \big(\overline{\mathcal{M}}_{g,k}(\lambda, d) \cap \pi_\lambda^*\beta\big). \tag{8.8}$$

We first assume that the limit set (8.8) is contained in the space (8.1). Then for small $|\lambda|$, all maps in (8.5) should be δ-flat. In that case, Theorem 8.2 implies that

$$(id \times \pi_\lambda)_*\big[\overline{\mathcal{M}}_{g,k}(\lambda, d) \cap \pi_\lambda^*\beta\big] = (\sigma \times \pi_0)_*\big[\mathcal{M}^V_{g,k}(d) \cap \pi_0^*\beta\big] \tag{8.9}$$

as a homology class in $H_0\big(\overline{\mathcal{M}}_{g,k}\times Z_0^k\,;\mathbb{Q}\big)$. The left-hand side of (8.9) becomes

(8.10)
$$\big[\overline{\mathcal{M}}_{g,k}(\lambda,d)\cap\pi_\lambda^*\beta\,\big]=GW_{S+dF,g}\big(Z_\lambda,\mathcal{D}_\lambda\big)\big(\beta\big)=GW^{\mathcal{H}}_{S+dF,g}\big(E(n)\big)\big(\beta\big)$$

where the second equality follows from (6.5). On the other hand, by assumption on the constraint β and the routine dimension count, the right-hand side of (8.9) becomes

(8.11)
$$\begin{aligned}\big[\mathcal{M}^V_{g,k}(d)\cap\pi_0^*\beta\big]=&\sum_{d_1+d_2=d}GW^V_{S+d_1F,g_1+1}(\beta';C(pt))\Phi^V_{S+d_2F,g_2-1}(\beta'';C(F))\\&+\sum_{d_1+d_2=d}GW^V_{S+d_1F,g_1}(\beta';C(F))\Phi^V_{S+d_2F,g_2}(\beta'';C(pt)).\end{aligned}$$

Therefore, if the limit set (8.8) is contained in the space (8.1), we have the sum formula (8.7) from (8.9), (8.11) and (8.10).

In general, the limit set (8.8) is not contained in the space (8.1). In that case, there are maps f_λ in (8.5) that converge to a limit f_0 as $\lambda\to 0$ such that some components of f_0 map entirely into V. The contribution of those maps in (8.5) is called the *contribution from the neck* and enters into the sum formula (8.7) as a correction term. This correction term can be computed by using the S-matrix (cf. Section 12 of [9]). By the choice of the constraint β, we have the correction term

(8.12)
$$\sum GW^V_{S+d_1F,g_1}(\,\beta';C(F)\,)\,\Phi^{V,U}_{S+d_3F,1}(\,C(pt),C(pt)\,)\,\Phi^U_{S+d_2F,g_2-1}(\,\beta'';C(F)\,)$$

where the sum is over all $d_1+d_2+d_3=d$. The correction term (8.12) is zero by Lemma 7.3 e and hence, the proof is complete. □

Next, we use the sum formula (8.7) to compute relative invariants of $(E(n),V)$.

Lemma 8.4. *Let γ_1,γ_2 be a basis of $H^1(\,E(0);Z\,)\cong H^1(\,V;Z\,)$ such that $\gamma_1\cdot\gamma_2=1$ in V .*

(*a*) $\Phi^V_{S+dF,0}(\gamma_1,\gamma_2;C(F))=\delta_{d0}$.

(*b*) $\Phi^V_{S+dF,1}(\gamma_1,\gamma_2;C(pt))=0$

(*c*) $GW^V_{S+dF,g}(pt^{g-1};C(pt))=0$

(*d*) $GW^V_{S+dF,g}(pt^g; C(F)) = GW^{\mathcal{H}}_{S+dF,g}(pt^g)$

Proof. (a) follows from (i) $\Phi_{s,0}(E(0))(\gamma_1, \gamma_2) = 1$ (see Theorem 2 of [15]), (ii) for $g = 0$ relative invariants are same as absolute invariants (Proposition 14.9 of [9]), and (iii) there is no rational curve representing $S + dF$ with $d \neq 0$ on $E(0) = S^2 \times T^2$ with a product complex structure.

To prove (b), we will apply the sum formula (Theorem 12.4 of [9]) for the symplectic sum $E(0) = E(0)\#_V E(0)$. The only difference between that sum formula and (8.7) is the degree of constraints. We also note that as in the proof of Proposition 8.3, there is no contribution from the neck for our case. Split constraints γ_1 and γ_2 on one side and one point constraint on the other side. Then, we have

(8.13)
$$\begin{aligned}\Phi_{S+dF,1}(\gamma_1, \gamma_2, pt) = &\sum_{d_1+d_2=d} \Phi^V_{S+d_1F,1}(\gamma_1, \gamma_2; C(pt))\Phi^V_{S+d_2F,0}(pt; C(F)) \\ &+ \sum_{d_1+d_2=d} \Phi^V_{S+d_1F,0}(\gamma_1, \gamma_2; C(F))\Phi^V_{S+d_2F,1}(pt; C(pt)).\end{aligned}$$

Using Lemma 7.3 c,d, and (a), we can simplify (8.13) as

$$\Phi_{S+dF,1}(\gamma_1, \gamma_2, pt) = \Phi^V_{S+dF,1}(\gamma_1, \gamma_2; C(pt)) + d\,\sigma(d). \tag{8.14}$$

Now, (b) follows from (8.14) and $\Phi_{S+dF,1}(pt, \gamma_1, \gamma_2) = d\sigma(d)$ (see Theorem 2 of [15]).

Next, we use the sum formula (8.7), (a) and (b) to show (c) and (d). For the proof of (c), we split $g-1$ point constraints on $E(n)$-side and the constraints γ_1 and γ_2 on $E(0)$-side. Then,

$$\begin{aligned}&GW^{\mathcal{H}}_{S+dF,g}(pt^{g-1}, \pi^*_\lambda\gamma_1, \pi^*_\lambda\gamma_2) \\ &\quad= \sum_{d_1+d_2=d} GW^V_{S+d_1F,g}(pt^{g-1}; C(pt))\Phi^V_{S+d_2F,0}(\gamma_1, \gamma_2; C(F)) \\ &\quad\quad+ \sum_{d_1+d_2=d} GW^V_{S+d_1F,g-1}(pt^{g-1}; C(F))\Phi^V_{S+d_2F,1}(\gamma_1, \gamma_2; C(pt)).\end{aligned} \tag{8.15}$$

Since $E(n)$ is simply connected, the left-hand side of (8.15) is zero. Therefore, (c) follows from (8.15) together with (a) and (b).

For the proof of (c), we split g point constraints on $E(n)$-side to obtain

$$GW^{\mathcal{H}}_{S+dF,g}(pt^g) = \sum_{d_1+d_2=d} GW^V_{S+d_1F,g}(pt^g; C(F))\Phi^V_{S+d_2F,0}(C(pt)). \tag{8.16}$$

Now, (d) follows from (8.16) and Lemma 7.3 c. □

Finally, we are ready to show the sum formulas (2.5) and (2.6).

Proposition 8.5 (Sum Formulas).

(a) $H(t) = 2\,F_0(t)\left(G(t) \;-\; \frac{1}{24}\right)$

(b) $F_g(t) = F_{g-1}(t)\,t\,G'(t)$

Proof. If we choose a smooth fiber lying on the $E(0)$-side of Z_0, the constraint $\tau(F)$ splits so that $\tau(F)$ lies only on $E(0)$-side. Then, one can apply the same arguments as in the proof of Proposition 8.3 to obtain

(8.17)
$$\begin{aligned} GW^{\mathcal{H}}_{S+dF,1}(\tau(F)) = &\sum_{d_1+d_2=d} GW^V_{S+d_1F,1}(C(pt))\Phi^V_{S+d_2F,0}(\tau(F);C(F)) \\ &+ \sum_{d_1+d_2=d} GW^V_{S+d_1F,0}(C(F))\Phi^V_{S+d_2F,1}(\tau(F);C(pt)). \end{aligned}$$

By Lemma 7.3 a,b and Lemma 8.4 d, (8.17) becomes

$$GW^{\mathcal{H}}_{S+dF,1}(\tau(F)) = \sum_{d_1+d_2=d} 2\,GW^{\mathcal{H}}_{S+d_1F,0}\,\sigma(d_2). \tag{8.18}$$

Now, (a) follows from (8.18) and the definition of $F_0(t)$, $H(t)$, and $G(t)$.

To prove (b) we split $g-1$ points on $E(n)$-side and one point on $E(0)$-side. Then by (8.7) we have

$$\begin{aligned} GW^{\mathcal{H}}_{S+dF,g}(pt^g) = &\sum_{d_1+d_2=d} GW^V_{S+d_1F,g}(pt^{g-1};C(pt))\Phi^V_{S+d_2F,0}(\,pt;C(F)\,) \\ &+ \sum_{d_1+d_2=d} GW^V_{S+d_1F,g-1}(pt^{g-1};C(F))\Phi^V_{S+d_2F,1}(pt;C(pt)). \end{aligned}$$

By Lemmas 7.3 d and 8.4 c,d, this becomes

$$GW^{\mathcal{H}}_{S+dF,g}(pt^g) = \sum_{d_1+d_2=d} GW^{\mathcal{H}}_{S+d_1F,g-1}(pt^{g-1})d_2\sigma(d_2). \tag{8.19}$$

Together with the definition of $F_g(t)$ and $G(t)$, (8.19) implies (b). □

9. Acknowledgements.

I would like to thank most sincerely my advisor Thomas Parker for his guidance and helpful discussions. Without his help, this paper would not have been possible. I also wish to thank Naichung Conan Leung, Eleny Ionel and Jim Bryan for useful discussions and comments.

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