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Rigidity and Non-rigidity Results on the Sphere

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1. Introduction.

It is a simple consequence of the maximum principle that a superharmonic function u on $\mathbb{R}^n$(i.e. $\Delta u \leq 0$) which is 1 near infinity is identically 1 on $\mathbb{R}^n$ (throughout this paper, $n \geq 3$). Geometrically, this means that one cannot conformally deform the Euclidean metric in a bounded region without decreasing the scalar curvature somewhere. In fact, there is a much stronger result: one cannot have any compact deformation of the Euclidean metric without decreasing the scalar curvature somewhere, i.e., if g is a metric on $\mathbb{R}^n$ which has non-negative scalar curvature and is the Euclidean metric near infinity, then g is the Euclidean metric on $\mathbb{R}^n$. This is a simple version of the positive mass theorem [9, 12]. Another implication of the positive mass theorem is the following rigidity theorem for the unit ball in $\mathbb{R}^n$.

Theorem 1.1. *Let (M, g) be an n-dimensional compact Riemannian manifold with boundary and the scalar curvature $R \geq 0$. The boundary is isometric to the standard sphere S^{n-1} and has mean curvature $n-1$. Then (M, g) is isometric to the unit ball in R^n. (If $n > 7$, we also assume M is spin.)*

The proof uses a generalized version of the positive mass theorem, see Shi and Tam [11] and Miao [6]. On the other hand, there are non-trivial metrics on $\mathbb{R}^n$ which agree with the Euclidean metric near infinity and have non-positive scalar curvature by the work of Lohkamp [5].

One can establish parallel results for the hyperbolic space $\mathbb{H}^n$ by analogous methods. It is natural to wonder about the other space form S^n. The following conjecture was posed by Min-Oo in 1995.

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Conjecture 1.2. *Let (M, g) be an n-dimensional compact Riemannian manifold with boundary and the scalar curvature $R \geq n(n-1)$. The boundary is isometric to the standard sphere S^{n-1} and is totally geodesic. Then, (M, g) is isometric to the hemisphere S^n_+.*

This is an intriguing conjecture. It seems extremely difficult. The formulation given here is probably over-ambitious, but any progress under some extra assumptions would be interesting. In an unpublished manuscript [7], Min-Oo attempted to prove the conjecture by a Witten type argument under the assumption that M is spin. Unfortunately, his attempt has been unsuccessful. To the authors' knowledge, the conjecture is even open under the stronger assumption $\mathrm{Ric} \geq n-1$.

Inspired by this conjecture, we study some special cases and related questions. We first prove that on the standard sphere (S^n, g_{S^n}), we can even conformally deform g_{S^n} without decreasing the scalar curvature and with the deformation supported in any given open geodesic ball of radius $> \pi/2$. In other words, the corresponding rigidity for a geodesic ball of radius $> \pi/2$ fails even among conformal deformations. Without restricting to conformal deformation, we also construct a rotationally symmetric g on S^n such that its sectional curvature ≥ 1 and strict somewhere and near the north pole and south pole $g = g_{S^n}$. These results are interesting in view of the work of Corvino [3]. We then verify the rigidity for the hemisphere among conformal deformations. In fact, in this situation, we have some stronger results. In the last section, we establish the rigidity in the Einstein case.

2. Non-rigidity when the boundary is non-convex.

We first introduce some notations. Let (S^n, g_{S^n}) be the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$ with the induced metric. We denote the north pole by N and the south pole by S. For $r \in (0, \pi)$, let $B(N, r)$ be the open geodesic ball of radius r with center N. Its boundary is umbilic with principal curvatures all equal to $\frac{\cos r}{\sin r}$. Therefore, the boundary is non-convex if and only if $r > \pi/2$. The closed upper hemisphere is denoted by S^n_+.

Theorem 2.1. *For any $r \in (\frac{\pi}{2}, \pi)$ there is a smooth metric $g = e^{2\phi} g_{S^n}$ on S^n with the following properties*

- $R_g \geq n(n-1)$,
- $\mathrm{Supp}\,(\phi) \subset B(N, r)$.

- $\phi \not\equiv 0$.

Remark 2.2. Since $\phi \not\equiv 0$, the inequality $R_g \geq n(n-1)$ must be strict somewhere inside $B(N, r)$.

To put the above theorem in a context, we mention the following theorem due to Corvino [3] which has shed new light on the positive mass theorem.

Theorem 2.3 (Corvino). *Let Ω be a compactly contained smooth domain in a Riemannian manifold (M, g_0). Suppose the linearization L_{g_0} of the scalar curvature map $R : C^\infty(\Omega) \to C^\infty(\Omega)$ has an injective formal L^2-adjoint $L_{g_0}^*$ on Ω. Then $\exists \epsilon > 0$ such that for any smooth function f which equals $R(g_0)$ in a neighborhood of $\partial\Omega$ and $\|f - R(g_0)\|_{C^1} < \epsilon$, there is a smooth metric g on M with $R(g) = f$ and $g \equiv g_0$ outside Ω.*

The main point is that if g_0 is non-static (i.e. $\operatorname{Ker} L_{g_0}^* = 0$), then there are compact deformations of g_0 with the scalar curvature going either direction. This is in contrast with $\mathbb{R}^n$, which is static, where one cannot have compact deformations without decreasing the scalar curvature somewhere.

The sphere (S^n, g_{S^n}) is also static. In fact, $L_{g_{S^n}}^* f = -\Delta f \cdot g_{S^n} + D^2 f - (n-1) f \cdot g_{S^n}$ and its kernel is spanned by the $n+1$ coordinate functions $x^1, \ldots, x^{n+1}$ (also the first eigenspace). Theorem 2.1 shows that one still can deform g_{S^n} without decreasing the scalar curvature on any geodesic ball of radius $r > \pi/2$.

To prove Theorem 2.1, we need a technical lemma.

Lemma 2.4. *Assume $f_1, f_2 \in C^\infty([-1,1])$ and*

$$f_1(0) = f_2(0), \quad f_1'(0) < f_2'(0).$$

Let

$$g(x) = \begin{cases} f_1(x), & -1 \leq x \leq 0; \\ f_2(x), & 0 \leq x \leq 1. \end{cases}$$

Then, for any $\varepsilon > 0$ small, there exists a $g_\varepsilon \in C^\infty([-1,1], \mathbb{R})$ such that

$$\begin{aligned} g_\varepsilon(x) &= g(x) \text{ for } |x| \geq \varepsilon; \\ g(x) \leq g_\varepsilon(x) &\leq g(x) + \varepsilon \text{ for } |x| \leq \varepsilon; \\ g''(x) &\leq g_\varepsilon''(x) \text{ for } x \neq 0; \\ g''(x) &< g_\varepsilon''(x) \text{ for some } x \neq 0. \end{aligned}$$

Proof. Without loss of generality, we may assume $f_1 \equiv 0$. Then, $f_2(0) = 0, a = f_2'(0) > 0$. Let

$$f_3(x) = f_2(x) - f_2'(0)\,x,$$

then $f_3(0) = f_3'(0) = 0$. We may find some $M > 0$ such that

$$\left|f_3''(x)\right| \le M \text{ for } |x| \le 1.$$

Let

$$k(x) = \begin{cases} 0, & x \le 0; \\ ax, & 0 \le x; \end{cases} \qquad r(x) = \begin{cases} 0, & -1 \le x \le 0; \\ f_3(x), & 0 \le x \le 1. \end{cases}$$

Denote

$$\rho(x) = \begin{cases} c_0 e^{-\frac{1}{1-x^2}}, & |x| \le 1; \\ 0, & |x| > 1. \end{cases}$$

Here, c_0 is a positive constant such that $\int_{-\infty}^{\infty} \rho(x)\,dx = 1$.

Fix $\delta > 0$ small, then we let $k_\delta(x) = (\rho_\delta * k)(x)$, here $\rho_\delta(x) = \delta^{-1}\rho(x/\delta)$. It is clear that $k_\delta(x) = k(x)$ for $|x| \ge \delta$, $k(x) < k_\delta(x) \le k(x) + a\delta$ for $|x| < \delta$ and $k_\delta''(x) = a\rho_\delta(x)$. Fix a smooth function η on $\mathbb{R}$ such that $0 \le \eta \le 1$, $\eta(x) = 0$ for $x \le 0$ and $\eta(x) = 1$ for $x \ge 1$. For $0 < \tau < \delta/2$, we let $r_\tau(x) = \eta\left(\frac{x}{\tau}\right) f_3(x)$. Then for $0 \le x \le \tau$, we have

$$\left|r_\tau''(x)\right| = \left|\frac{1}{\tau^2}\eta''\left(\frac{x}{\tau}\right) f_3(x) + \frac{2}{\tau}\eta'\left(\frac{x}{\tau}\right) f_3'(x) + \eta\left(\frac{x}{\tau}\right) f_3''(x)\right| \le cM.$$

Here, c is an absolute constant. On the other hand, for $0 \le x \le \tau$,

$$\left|r_\tau(x) - r(x)\right| \le \frac{M}{2}\tau^2.$$

Hence, for $0 < |x| \le \tau$, we have

$$k_\delta''(x) + r_\tau''(x) \ge \frac{a}{\delta}\rho\left(\frac{1}{2}\right) - cM \ge 2M > g''(x)$$

if δ is small enough. For $|x| > \tau$, we have

$$k_\delta''(x) + r_\tau''(x) \ge g''(x).$$

Moreover, for $0 \le x \le \tau$, we have

$$k_\delta(x) > k(x) + \sigma$$

for some $\sigma > 0$, hence

$$g(x) + a\delta + \frac{M}{2}\tau^2 = k(x) + r(x) + a\delta + \frac{M}{2}\tau^2$$
$$\geq k_\delta(x) + r_\tau(x) \geq k(x) + r(x) + \sigma - \frac{M}{2}\tau^2 \geq g(x)$$

when τ is small enough. For other x, we clearly have

$$g(x) + a\delta = k(x) + r(x) + a\delta \geq k_\delta(x) + r_\tau(x) \geq k(x) + r(x) = g(x).$$

The lemma follows by taking $g_\varepsilon(x) = k_\delta(x) + r_\tau(x)$. □

We now present the proof of Theorem 2.1. The stereographic projection from the south pole is given by

$$\pi_S(y) = \frac{y'}{1 + y^{n+1}} \text{ for } y = \left(y', y^{n+1}\right) \in S^n.$$

On $\mathbb{R}^n$, we have standard coordinates $x^1, \cdots, x^n$, polar coordinates r, θ and cylindrical coordinates t, θ, where $r = e^{-t}$. We have

$$\left(\pi_S^{-1}\right)^* g_{S^n} = \frac{4}{\left(1 + |x|^2\right)^2} \sum_{i=1}^n dx^i \otimes dx^i = \frac{4}{\left(1 + r^2\right)^2}\left(dr \otimes dr + r^2 g_{S^{n-1}}\right)$$
$$= (\cosh t)^{-2}\left(dt \otimes dt + g_{S^{n-1}}\right).$$

Let g be the metric we are looking for, then

$$\left(\pi_S^{-1}\right)^* g = u^{\frac{4}{n-2}} \sum_{i=1}^n dx^i \otimes dx^i = v^{\frac{4}{n-2}}\left(dt \otimes dt + g_{S^{n-1}}\right).$$

The scalar curvature of g is given by

$$R = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \Delta u$$
$$= v^{-\frac{n+2}{n-2}}\left[-\frac{4(n-1)}{n-2}\left(v_{tt} + \Delta_{S^{n-1}} v\right) + (n-1)(n-2)v\right].$$

For $\lambda > 0$, let $d_\lambda x = \lambda x$ be the dilation, then

$$d_\lambda^* \left(\pi_S^{-1}\right)^* g_{S^n} = \frac{4\lambda^2}{\left(1 + \lambda^2 |x|^2\right)^2} \sum_{i=1}^n dx^i \otimes dx^i.$$

Denote

$$u_\lambda(x) = \frac{2^{\frac{n-2}{2}}}{\left(\frac{1}{\lambda} + \lambda |x|^2\right)^{\frac{n-2}{2}}},$$

then

$$-\Delta u_\lambda = \frac{n(n-2)}{4} u_\lambda^{\frac{n+2}{n-2}}.$$

We need to solve the following

$$\text{(2.1)} \qquad \left\{ \begin{array}{c} u \in C^\infty(\mathbb{R}^n), u > 0, u \not\equiv u_1, \\ u(x) = u_1(x) \text{ for } |x| > a; \\ -\Delta u \geq \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}. \end{array} \right.$$

Claim 2.5. *For any $a > 1$, (2.1) has at least one solution.*

Remark 2.6. It is interesting to note here that for $a \leq 1$, (2.1) has no solution. This is implied by the Theorem 3.1 below.

Proof. The rough idea is the following, let

$$u(x) = \min\{u_{a^{-2}}(x), u_1(x)\} = \left\{ \begin{array}{ll} u_{a^{-2}}(x), & |x| \leq a^{-2}; \\ u_1(x), & |x| \geq a^{-2}. \end{array} \right.$$

Then, clearly, $-\Delta u \geq \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}$ in weak sense. One may get a smooth u by suitable smoothing procedure.

More precisely, we may do the following, let $f(t) = (\cosh t)^{-\frac{n-2}{2}}$, then

$$-f'' = \frac{n(n-2)}{4} f^{\frac{n+2}{n-2}} - \frac{(n-2)^2}{4} f.$$

For $\delta > 0$ small, let $g(t) = f(t + 2\delta)$, then $f(-\delta) = g(-\delta)$, $f'(-\delta) > g'(-\delta)$. Let

$$h(t) = \left\{ \begin{array}{ll} g(t), & t \geq -\delta; \\ f(t), & t \leq -\delta. \end{array} \right.$$

By Lemma 2.4, for $\varepsilon > 0$ tiny, we may find a smooth function h_ε such that

$$\begin{gathered} h_\varepsilon(t) = h(t) \text{ for } |t + \delta| \geq \varepsilon; \\ h(t) - \varepsilon \leq h_\varepsilon(t) \leq h(t) \text{ for } |t + \delta| \leq \varepsilon; \\ h_\varepsilon''(t) \leq h''(t) \text{ for } t \neq -\delta. \end{gathered}$$

Hence, for $t \neq -\delta$,

$$-h_\varepsilon''(t) \geq -h''(t) = \frac{n(n-2)}{4} h(t)^{\frac{n+2}{n-2}} - \frac{(n-2)^2}{4} h(t)$$
$$\geq \frac{n(n-2)}{4} h_\varepsilon(t)^{\frac{n+2}{n-2}} - \frac{(n-2)^2}{4} h_\varepsilon(t),$$

observing that $h(t)$ is very close to 1 when $|t+\delta| \leq \varepsilon$.

Now $g = \pi_S^* \left(h_\varepsilon(t)^{\frac{4}{n-2}} (dt \otimes dt + g_{S^{n-1}}) \right)$ is the needed metric. □

If we do not restrict ourselves to conformal deformations, we can even construct a deformation without decreasing the sectional curvatures.

Claim 2.7. *For any $0 < a < b < \frac{\pi}{2}$, there exists a function $f \in C^\infty([0,2b], \mathbb{R})$ such that*

$$f(x) = \begin{cases} \sin x, & 0 \leq x \leq a, \\ \sin(2b-x), & 2b-a \leq x \leq 2b, \end{cases}$$
$$f(x) = f(2b-x) \text{ for } 0 < x < 2b,$$
$$-f'' \geq f > 0 \text{ on } (0,2b), \ -f'' > f \text{ somewhere},$$
$$1 \geq f'^2 + f^2 \text{ on } [0,2b], \ 1 > f^2 + f'^2 \text{ somewhere.}$$

Proof. Denote

$$\rho(x) = \begin{cases} ce^{-\frac{1}{1-x^2}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

here c is a positive constant such that $\int_{\mathbb{R}} \rho(x)\,dx = 1$. For $\delta > 0$, $\rho_\delta(x) = \frac{1}{\delta}\rho\left(\frac{x}{\delta}\right)$.

For $0 < \delta < \pi/2$, denote

$$c_\delta = \int_{-\delta}^{\delta} \rho_\delta(x) \cos x dx \in (0,1),$$

then

$$\int_{-\delta}^{\delta} \rho_\delta(y) \sin(x-y)\,dy = c_\delta \sin x.$$

Let

$$g(x) = \begin{cases} \sin x, & x \leq b, \\ \sin(2b-x), & b \leq x. \end{cases}$$

For $0<\delta<b-a$, let

$$f(x)=\frac{1}{c_\delta}\int_{-\delta}^{\delta}\rho_\delta(y)\,g(x-y)\,dy,$$

then f satisfies all the requirements. □

Theorem 2.8. *For any $a\in\left(0,\frac{\pi}{2}\right)$, there exists a smooth metric g on S^n such that $g=g_{S^n}$ on $B(S,a)\cup B(N,a)$ and the sectional curvature of g is at least 1 and larger than 1 somewhere.*

Proof. Fix a number $b\in\left(a,\frac{\pi}{2}\right)$. Let f be as in the Claim 2.7. Consider the metric

$$\widetilde{g}=dr\otimes dr+f(r)^2\,g_{S^{n-1}}.$$

Let $e_1,\cdots,e_{n-1}$ be a local orthonormal frame on S^{n-1}, then the curvature operator of $\widetilde{g}$ is given by

$$\widetilde{Q}(\partial_r\wedge e_i)=-\frac{f''}{f}\partial_r\wedge e_i,$$
$$\widetilde{Q}(e_i\wedge e_j)=\frac{1-f'^2}{f^2}e_i\wedge e_j,$$

for $1\le i,j\le n-1$. By Claim 2.7, we see the sectional curvature of $\widetilde{g}$ is at least 1.

Next we will construct a smooth function $\phi:[0,\pi]\to[0,2b]$ such that

$$\phi(r)=\begin{cases} r, & 0\le r\le a,\\ r+2b-\pi, & \pi-a\le r\le\pi,\end{cases}$$
$$\phi'(r)>0 \text{ and } \phi'(r)=\phi'(\pi-r).$$

Indeed, let

$$\alpha(x)=\begin{cases} ce^{-\frac{1}{x(1-x)}}, & 0<x<1,\\ 0, & x\le 0 \text{ or } x\ge 1,\end{cases}$$

here c is chosen such that $\int_0^1\alpha(x)\,dx=1$. Let $\beta(x)=\int_0^x\alpha(t)\,dt$. Fix a $\lambda>0$ such that $2a+\lambda(\pi-2a)<2b$. For $0<\varepsilon<\frac{\pi}{2}-a$, let $\delta=\min\left\{\varepsilon,\frac{\pi}{2}-a-\varepsilon\right\}$ and

$$g_\varepsilon(x)=\begin{cases} \lambda+(1-\lambda)\,\beta\left(\frac{\frac{\pi}{2}-\varepsilon-x}{\delta}\right), & 0\le x\le\frac{\pi}{2},\\ \lambda+(1-\lambda)\,\beta\left(\frac{x-\frac{\pi}{2}-\varepsilon}{\delta}\right), & \frac{\pi}{2}\le x\le\pi.\end{cases}$$

Then, for some ε, we have $\int_0^\pi g_\varepsilon(x)\,dx = 2b$. We may put $\phi(x) = \int_0^x g_\varepsilon(t)\,dt$ and ϕ satisfies all the requirements.

Let r be the distance function on S^n to N, then we may put

$$g = d\phi(r) \otimes d\phi(r) + f(\phi(r))^2\, g_{S^{n-1}}.$$

It satisfies all the requirements in the theorem. □

3. Conformal deformation on the hemisphere.

The assumption $r > \pi/2$ in Theorem 2.1 is optimal as it turns out that it is impossible to localize the deformation in the hemisphere.

Theorem 3.1. *Let $g = e^{2\phi} g_{S^n}$ be a C^2 metric on S^n_+ satisfying the assumptions*

- $R_g \geq n(n-1)$,
- *the boundary is totally geodesic and is isometric to the standard S^{n-1}.*

Then g is isometric to g_{S^n}.

Remark 3.2. This verifies Conjecture 1.2 among conformal deformations.

Proof. By the assumption $g|_{S^{n-1}} = e^{2\phi|_{S^{n-1}}} g_{S^{n-1}}$ is isometric to $g_{S^{n-1}}$. By the Obata theorem, there exist $\lambda \geq 1$ and $\zeta \in S^{n-1}$ such that $g|_{S^{n-1}} = \psi^*_{\lambda,\zeta} g_{S^{n-1}}$, where $\psi_{\lambda,\zeta}$ is the conformal transformation of S^n which is dilation by λ when we identify S^n with $\mathbb{R}^n$ by the stereographic projection from ζ. Replacing g by $(\psi^{-1}_{\lambda,\zeta})^* g$, we can assume $g|_{S^{n-1}} = g_{S^{n-1}}$, i.e. $\phi|_{S^{n-1}} \equiv 0$. We are to prove $\phi \equiv 0$ on S^n_+.

As in the proof the Theorem 2.1, we work on $\mathbb{R}^n$ via the stereographic projection from the south pole. We write

$$\left(\pi_S^{-1}\right)^* g = u^{\frac{4}{n-2}} \sum_{i=1}^n dx^i \otimes dx^i = v^{\frac{4}{n-2}} \left(dt \otimes dt + g_{S^{n-1}}\right).$$

Then $u \in C^2(\overline{B_1})$ is positive and satisfies

$$\begin{cases} -\Delta u \geq \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} & \text{in } B_1; \\ u = 1 & \text{on } \partial B_1 \\ \frac{\partial u}{\partial r} = -\frac{n-2}{2} & \text{on } \partial B_1. \end{cases} \tag{3.1}$$

(The Neumann boundary condition is the geometric assumption that the boundary is totally geodesic.)

Claim 3.3. *The only solution to* (3.1) *is* $u_1(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}}$.

We work with v in cylindrical coordinates.

$$-v_{tt} - \Delta_{S^{n-1}} v + \frac{(n-2)^2}{4} v \geq \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}}.$$

Let

$$f(t) = \frac{1}{n\omega_n} \int_{S^{n-1}} v(t,\theta)\, dS(\theta),$$

here ω_n is the volume of the unit ball in $\mathbb{R}^n$. By Holder's inequality, we have

$$\frac{1}{n\omega_n} \int_{S^{n-1}} v(t,\theta)^{\frac{n+2}{n-2}}\, dS(\theta) \geq f(t)^{\frac{n+2}{n-2}}. \tag{3.2}$$

Therefore, we have

$$\begin{cases} f \in C^2([0,\infty), \mathbb{R}), f > 0 \\ -f''(t) \geq \frac{n(n-2)}{4} f(t)^{\frac{n+2}{n-2}} - \frac{(n-2)^2}{4} f(t), \\ f(0) = 1, f'(0) = 0. \end{cases}$$

Denote

$$e(t) = -f''(t) - \frac{n(n-2)}{4} f(t)^{\frac{n+2}{n-2}} + \frac{(n-2)^2}{4} f(t) \geq 0.$$

Since $f''(0) < 0$, we see $f'(t) < 0$ for $t > 0$ small. Assume $b > 0$ such that $f'(t) < 0$ on $(0,b)$, then for $0 \leq t \leq b$, we have

$$f'(t)^2 = -2\int_0^t f'(s)\, e(s)\, ds + \frac{(n-2)^2}{4} \left(f(t)^2 - f(t)^{\frac{2n}{n-2}}\right).$$

In particular, $f'(b)^2 > 0$. This implies that $f'(t) < 0$ for any t.

Assume e is not identically zero, then for some $b > 0$, e is not identically zero on $(0,b)$, then for any $t > b$, we have

$$f'(t)^2 \geq c > 0.$$

This implies $f'(t) \leq -\sqrt{c}$ and hence, $\lim_{t\to\infty} f(t) = -\infty$, a contradiction. Hence, $e(t) \equiv 0$. This shows

$$f(t) = (\cosh t)^{-\frac{n-2}{2}} \text{ for } t \geq 0.$$

Moreover, the inequality (3.2) must be an equality. This implies that $v(t,\theta) = f(t) = (\cosh t)^{-\frac{n-2}{2}}$. Hence, $u = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}}$. □

With a little improvement of our argument, we can remove the assumption that the boundary is totally geodesic.

Theorem 3.4. *Let $g = e^{2\phi} g_{S^n}$ be a C^2 metric on S^n_+ satisfying the assumptions*

- $R_g \geq n(n-1)$,
- *the boundary is isometric to the standard S^{n-1}.*

Then, g is isometric to g_{S^n}.

By the same argument, we can reduce the problem to a partial differential inequality on $\overline{B_1} \subset \mathbb{R}^n$. In fact, we establish the following stronger result

Claim 3.5. *Assume $\overline{u} \in C^2\left(\overline{B_1}, \mathbb{R}\right)$, $\overline{u} > 0$ and*

$$\begin{cases} -\Delta \overline{u} \geq \frac{n(n-2)}{4} \overline{u}^{\frac{n+2}{n-2}} \text{ in } B_1; \\ \overline{u}|_{\partial B_1} \geq 1. \end{cases}$$

Then $\overline{u} = u_1$.

(The proof of Theorem 3.4 only requires the special case $u|_{\partial B_1} = 1$.)

To prove this claim, we take an approach different from our previous method. First, we observe that if we solve v such that

$$\begin{cases} -\Delta v = \frac{n(n-2)}{4} \overline{u}^{\frac{n+2}{n-2}} \text{ in } B_1; \\ v|_{\partial B_1} = 1. \end{cases}$$

Then $1 \leq v \leq \overline{u}$ and $-\Delta v \geq \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}}$. If we can prove $v = u_1$, then

$$\frac{n(n-2)}{4} \overline{u}^{\frac{n+2}{n-2}} = -\Delta v = -\Delta u_1 = \frac{n(n-2)}{4} u_1^{\frac{n+2}{n-2}}.$$

Hence, $\overline{u} = u_1$. Therefore, from now on, we may assume $\overline{u}|_{\partial B_1} = 1$.

We consider the following PDE

$$\begin{cases} -\Delta v = \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} \text{ in } B_1; \\ v|_{\partial B_1} = 1. \end{cases} \tag{3.3}$$

We claim the only *positive* solution is $v = u_1$. Indeed, it follows from the moving plane method of Gidas, Ni and Nirenberg [4] that $v(x) = f(|x|)$ for some f, moreover

$$\begin{cases} f'' + \frac{n-1}{r} f' + \frac{n(n-2)}{4} f^{\frac{n+2}{n-2}} = 0; \\ f(0) = a > 0, f'(0) = 0. \end{cases}$$

It is clear

$$g(r) = \frac{a}{\left(1 + \frac{a^{\frac{4}{n-2}}}{4} r^2\right)^{\frac{n-2}{2}}}$$

is a solution to the problem. On the other hand, since f satisfies

$$f(r) = a - \frac{n(n-2)}{4} \int_0^r dt \int_0^t \left(\frac{s}{t}\right)^{n-1} f(s)^{\frac{n+2}{n-2}} ds.$$

It follows from contraction mapping theorem that for some $\varepsilon > 0$, $f = g$ on $[0, \varepsilon]$. Hence, $f = g$. Since $f(1) = 1$, we see $a = 2^{\frac{n-2}{2}}$. Hence, $f(r) = \frac{2^{\frac{n-2}{2}}}{(1+r^2)^{\frac{n-2}{2}}}$ and $v = u_1$.

Since 1 is a subsolution for (3.3) and $\overline{u}$ is a supersolution with $\overline{u} \geq 1$, by the standard method of iteration, we may find a solution v for (3.3) and $1 \leq v \leq \overline{u}$. Since the only positive solution is u_1, we see $u_1 = v \leq \overline{u}$.

If $u_1 \neq \overline{u}$, then since $-\Delta(\overline{u} - u_1) \geq 0$ and $(\overline{u} - u_1)|_{\partial B_1} = 0$, we see $\overline{u} > u_1$ in B_1. Moreover, it follows from Hopf maximum principle that for some $c_1 > 0$, $\overline{u}(x) - u_1(x) \geq c_1(1 - |x|)$. This implies that for some $c > 0$,

$$\frac{\overline{u}(x)}{u_1(x)} \geq \left[1 + c\left(1 - |x|^2\right)\right]^{\frac{n-2}{2}}.$$

On the other hand, for $\lambda > 1$, we have

$$\begin{aligned} \left(\frac{u_\lambda(x)}{u_1(x)}\right)^{\frac{2}{n-2}} &= 1 + \frac{(\lambda - 1)\left(1 - \lambda\left|x^2\right|\right)}{1 + \lambda^2 |x|^2} \\ &\leq 1 + \frac{(\lambda - 1)\left(1 - \left|x^2\right|\right)}{1 + \lambda^2 |x|^2} \\ &\leq 1 + c\left(1 - |x|^2\right) \\ &\leq \left(\frac{\overline{u}(x)}{u_1(x)}\right)^{\frac{2}{n-2}} \end{aligned}$$

if $\lambda - 1$ is small enough. Hence, for some $\lambda > 1$, $u_\lambda \le \overline{u}$. Since u_λ is also a subsolution, we may find a solution v for (3.3) such that $u_\lambda \le v \le \overline{u}$. It follows from previous discussion that $v = u_1$. Hence, $u_\lambda \le u_1$, this contradicts with the fact $\lambda > 1$.

4. The Einstein case.

In this section, we prove the following uniqueness theorem:

Theorem 4.1. *Let (M, g) be a smooth n-dimensional compact Einstein manifold with boundary Σ. If Σ is totally geodesic and is isometric to S^{n-1} with the standard metric, then (M, g) is isometric to the hemisphere S^n_+ with the standard metric.*

This verifies Conjecture 1.2 in the special case that g is Einstein.

Given local coordinates $\xi^1, \dots, \xi^{n-1}$ on the boundary, we can introduce local coordinates on a collar neighborhood of Σ in M as follows. For $\xi \in \Sigma$, let $\gamma_\xi(t) = \gamma(t, \xi)$ be the normal geodesic starting at ξ with initial velocity $\nu(\xi)$, the unit inner normal vector at ξ. Then, $t, \xi^1, \dots, \xi^{n-1}$ form local coordinates on a collar neighborhood of Σ in M. Let $h_{ij} = \langle \frac{\partial \gamma}{\partial \xi^i}, \frac{\partial \gamma}{\partial \xi^j} \rangle$. By the Gauss lemma, the metric g takes the form

$$g = dt^2 + h_{ij}(t, \xi) d\xi^i d\xi^j,$$

where Latin indices $i, j, \dots$ run from 1 to $n-1$. Greek indices $\alpha, \beta, \dots$ will be used to run from 0 to $n-1$. We denote the curvature tensors of M and Σ by R and K, respectively. Since Σ is totally geodesic, by the Gauss equation, we have

$$R_{ikjl} = K_{ikjl}.$$

Then the Ricci tensor is given by

$$R_{ij} = R_{i0j0} + K_{ij}.$$

Taking trace, we get the scalar curvature

$$R = 2R_{00} + (n-1)(n-2) = 2R/n + (n-1)(n-2),$$

hence $R = n(n-1)$. Thus, $\mathrm{Ric}\,(g) = (n-1)g$.

The second fundamental form of the t-hypersurface is given by

$$A_{ij} = -\left\langle \frac{D}{\partial t} \frac{\partial \gamma}{\partial \xi^i}, \frac{\partial \gamma}{\partial \xi^j} \right\rangle = -\frac{1}{2} \frac{\partial h_{ij}}{\partial t}.$$

We also need to know the second derivative of h_{ij} in t.

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 h_{ij}}{\partial t^2} &= \left\langle \frac{D^2}{\partial t^2}\frac{\partial\gamma}{\partial\xi^i}, \frac{\partial\gamma}{\partial\xi^i}\right\rangle + \left\langle \frac{D}{\partial t}\frac{\partial\gamma}{\partial\xi^i}, \frac{D}{\partial t}\frac{\partial\gamma}{\partial\xi^j}\right\rangle \\ &= -R\left(\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial\xi^i}, \frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial\xi^j}\right) + \frac{1}{4}h^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t},\end{aligned}$$

where, in the last step, we use the fact that $\frac{\partial\gamma}{\partial\xi^i}$ is a Jacobi field along the geodesic $\gamma(t,\xi)$. As g is Einstein, the above equation can be written as

$$\frac{1}{2}\frac{\partial^2 h_{ij}}{\partial t^2} = -(n-1)h_{ij} + \frac{1}{4}h^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t} + h^{kl}R_{ikjl}. \tag{4.1}$$

Claim 4.2. *Infinitesimally* $h_{ij}(t,\xi)$ *equals* $\cos^2(t)h_{ij}(0,\xi)$.

Remark 4.3. It is clear that $g_{S^n} = dt^2 + \cos^2(t)g_{S^{n-1}}$

We prove by induction that

$$h_{ij}(t,\xi) = \cos^2(t)h_{ij}(0,\xi) + O(t^m), \text{ as } t \to 0$$

for any integer $m > 0$. The case $m = 1$ is trivial. Suppose it is true for m. We assume without loss of generality that $h_{ij}(0,\xi) = \delta_{ij}$. We first have $R_{ikjl} = \cos^4(t)(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}) + O(t^{m-1})$ (this is true because R_{ijkl} only involves differentiating the metric in t once). By (4.1), we get

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 h_{ij}}{\partial t^2} &= -(n-1)\cos^2(t)\delta_{ij} + \frac{1}{4}\delta^{kl}\frac{\sin^2(2t)}{\cos^2(t)}\delta_{ik}\delta_{jl} \\ &\quad + \cos^2(t)\delta^{kl}(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}) + O(t^{m-1}) \\ &= -(n-1)\cos^2(t)\delta_{ij} + \sin^2(t)\delta_{ij} + (n-2)\cos^2(t)\delta_{ij} + O(t^{m-1}) \\ &= -\cos(2t)\delta_{ij} + O(t^{m-1}).\end{aligned}$$

This implies that $h_{ij}(t,\xi) = \cos^2(t)h_{ij}(0,\xi) + O(t^{m+1})$.

Consider S^n_+ with the standard metric g_{S^n}. It is easy to see that $g_{S^n} = dt^2 + \cos^2(t)g_{S^{n-1}}$, where t is the distance to the boundary S^{n-1}. We form a closed manifold $\overline{M}$ by joining M and S^n_+ along their boundary. In view of Claim 4.2, we get a smooth Riemannian manifold with a totally geodesic hypersurface Σ which is isometric to S^{n-1}. The metric, also denoted by g, is of course Einstein.

By [2], g is real analytic in harmonic coordinates. We define Ω to be the set of points where g has constant curvature 1 in a neighborhood. This is an open set by definition. If it is not the whole manifold, we take a point p on its boundary and choose local harmonic coordinates $x^1, \ldots, x^n$ on a connected neighborhood U. The analytic functions $R_{ikjl} - g_{ij}g_{kl} + g_{il}g_{jk}$ vanish on an open subset of U for $U \cap \Omega \neq \emptyset$, hence vanish identically on U. Then $p \in \Omega$, a contradiction. Therefore, g has constant sectional curvature 1 everywhere. It is then easy to see that $(\overline{M}, g)$ is isometric to S^n and (M, g) is isometric to S^n_+.

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