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Growth of solutions to the minimal surface equation over domains in a half plane

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We consider minimal graphs $u = u(x, y) > 0$ over unbounded domains D with $u = 0$ on ∂D. We shall study the rates at which u can grow when D is contained in a half plane.

1. Introduction.

Let D be an unbounded domain in the half plane $H = \{(x, y) : x > 0, -\infty < y < \infty\}$ and $u(x, y)$ a positive solution to the minimal surface equation with vanishing boundary values

$$\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0, \qquad u > 0 \quad \text{in } D, \tag{1.1}$$
$$u = 0 \qquad \text{on } \partial D.$$

If H were replaced by a sector of opening less than π, then Nitsche [9, p. 256] observed that (1.1) would have no solutions. Thus, the first non-trivial sector is the half plane. In this case, we shall be concerned with upper and lower bounds for the growth of such solutions. In what follows, we shall use complex notation $z = x + iy$ for convenience.

Theorem 1.1. *Let D be a domain in a half plane whose boundary is a Jordan arc. If u satisfies* (1.1) *in D, then there exist positive constants M and R such that*

$$Mr \le \max_{|z|=r \;\; z\in D} u(z) \le e^{Mr} \qquad |z| > R. \tag{1.2}$$

One measure of growth for solutions to (1.1) is given by the *order* α of u,

$$\alpha = \limsup_{|z|\to\infty \;\; z\in D} \frac{\log u(z)}{\log |z|}.$$

Regarding the lower bound, the left side of (1.2) shows that u cannot have sublinear growth. Planes given by $u(x,y) = cx$ show that this is best possible. With the hypotheses of Theorem 1.1, the weaker condition on the order $\alpha \geq 1$ follows from [12, Theorem 1.1].

Using catenoid surfaces

$$u(x,y) = \left(\sqrt{\cosh^2 Cx - C^2y^2} - 1\right)/C,$$

in the subset of the right half plane where $u > 0$, we see that the upper bound in (1.2) is also sharp.

Theorem 1.1 indicates that there are severe limitations on the growth of solutions to the minimal surface equation. Some results have been obtained in recent years in this direction. In Section 5 we shall discuss this further as well as some open problems.

2. Preliminaries.

If S is a minimal graph over a simply connected region D, then S can be parametrized in isothermal coordinates by the Weierstrass functions $x(\zeta), y(\zeta), U(\zeta)$ with ζ in the right half plane H, $U(\zeta) = u(x(\zeta), y(\zeta))$ and (up to additive constants)

$$\begin{aligned} x(\zeta) &= \Re e \frac{1}{2}\int_{\zeta_0}^{\zeta} \omega(\tilde{\zeta})(1 - G^2(\tilde{\zeta}))\, d\tilde{\zeta} \\ y(\zeta) &= \Re e \frac{i}{2}\int_{\zeta_0}^{\zeta} \omega(\tilde{\zeta})(1 + G^2(\tilde{\zeta}))\, d\tilde{\zeta} \\ U(\zeta) &= \Re e \int_{\zeta_0}^{\zeta} \omega(\tilde{\zeta}) G(\tilde{\zeta})\, d\tilde{\zeta}. \end{aligned} \tag{2.1}$$

Here, $G(\zeta)$ is the stereographic projection of the Gauss map corresponding to the upper normal, ω is analytic for all values in H, and ω has zeros at the poles of G with multiplicity of the zero twice the order of the pole of G.

Since S is a graph, the function $z(\zeta) = x(\zeta) + iy(\zeta)$ is univalent and with the hypotheses of Theorem 1.1, we can normalize so that $z(\infty) = \infty$. Also, since $U(\zeta)$ is a positive harmonic function in H which is zero on the imaginary axis (cf. [5, Corollary 1]), it follows that

$$U(\zeta) = C \Re e\, \zeta \tag{2.2}$$

for some positive constant C. From the third equation in (2.1), we then have

$$\omega(\zeta) = C/G(\zeta). \tag{2.3}$$

In particular, $G(\zeta) \neq 0, \infty$.

Since $|G(\zeta)| > 1$, G has non-tangential limits a.e. on ∂H, so we may take $\zeta_0 = 0$, translating if necessary so that G has finite non-tangential limit at 0.

By (2.1) and (2.3), $z(\zeta)$ then satisfies

(2.4)

$$\begin{aligned}
z(\zeta) &= (C/2)\Re e \int_0^\zeta (1/G(\tilde\zeta) - G(\tilde\zeta))\, d\tilde\zeta \ + \ (C/2)\Re e\, i \int_0^\zeta (1/G(\tilde\zeta) + G(\tilde\zeta))\, d\tilde\zeta \\
&= (C/4)\Big(\int_0^\zeta (1/G(\tilde\zeta) - G(\tilde\zeta))\, d\tilde\zeta + \overline{\int_0^\zeta (1/G(\tilde\zeta) - G(\tilde\zeta))\, d\tilde\zeta} \\
&\quad - \int_0^\zeta (1/G(\tilde\zeta) + G(\tilde\zeta))\, d\tilde\zeta + \overline{\int_0^\zeta (1/G(\tilde\zeta) + G(\tilde\zeta))\, d\tilde\zeta}\ \Big) \\
&= (C/2)\left(- \int_0^\zeta G(\tilde\zeta)\, d\tilde\zeta + \overline{\int_0^\zeta 1/G(\tilde\zeta)\, d\tilde\zeta} \right).
\end{aligned}$$

3. Proof of the lower bound in (1.2).

With the hypotheses of Theorem 1.1, let $z(\zeta)$ be as in (2.4). Then $z(\zeta)$ is a harmonic function in H with $\Re e\, z(\zeta) > 0$, so if

$$F(\zeta) = \int_0^\zeta (1/G(\tilde\zeta) - G(\tilde\zeta))\, d\tilde\zeta,$$

then

(3.1)

$$\begin{aligned}
F(\zeta) &= - \int_0^\zeta G(\tilde\zeta)\, d\tilde\zeta + \int_0^\zeta 1/G(\tilde\zeta)\, d\tilde\zeta - \overline{\int_0^\zeta 1/G(\tilde\zeta)\, d\tilde\zeta} + \overline{\int_0^\zeta 1/G(\tilde\zeta)\, d\tilde\zeta} \\
&= (2/C) z(\zeta) - 2\,\Im m \int_0^\zeta 1/G(\tilde\zeta)\, d\tilde\zeta.
\end{aligned}$$

Thus, $F(\zeta)$ is analytic with $\Re e\, F(\zeta) > 0$ in H, and (cf [11, p. 152]) there exists a real constant k ($0 \le k < \infty$) such that in any sector $S_\beta = \{z :$

$|\arg z| \leq \beta < \pi/2\}$,

$$\lim_{|\zeta|\to\infty\ \zeta\in S_\beta} F'(\zeta) = k. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\lim_{|\zeta|\to\infty\ \zeta\in S_\beta} (1/G(\zeta) - G(\zeta)) = k.$$

Since G is the stereographic projection of the Gauss map of the surface given by $u > 0$, it follows that

$$\lim_{|\zeta|\to\infty\ \zeta\in S_\beta} G(\zeta) = K = -(k + \sqrt{k^2+4})/2. \tag{3.3}$$

In particular,

$$K \leq -1. \tag{3.4}$$

Returning to (2.4), this implies that

$$\begin{aligned} z(\zeta) =& (C/2)\left(-\int_0^\zeta (K + o(1))\, d\tilde{\zeta} + \overline{\int_0^\zeta (1/K + o(1))\, d\tilde{\zeta}}\right) \\ =& (C/2)(-K\zeta + (1/K)\overline{\zeta} + o(\zeta)) \\ =& (C/2)((-K + 1/K)\sigma - i(K + 1/K)\tau + o(\zeta)) \\ & (|\zeta| \to \infty,\ \zeta = \sigma + i\tau \in S_\beta). \end{aligned} \tag{3.5}$$

It follows from this that as $\zeta \to \infty$ along the real axis, we have

$$|z(\sigma)| < C_1\sigma \quad (\sigma > \sigma_0) \tag{3.6}$$

for some positive constants C_1 and σ_0. Using (2.2) and (3.6), we then have

$$\frac{U(\sigma)}{|z(\sigma)|} \geq \frac{C\sigma}{C_1\sigma} \qquad (\sigma > \sigma_0).$$

In the xy plane with $z = x + iy$, we then have for some $R > 0$,

$$\max_{|z|=r\ z\in D} \frac{u(z)}{|z|} \geq \frac{C}{C_1} \quad r > R.$$

□

4. Proof of the upper bound in (1.2).

We shall use the following result from [1, p. 826].

Theorem A. *Let* $\Omega \subseteq \Omega_1 = \{(x,y)|x > 0, -f(x) < y < f(x)\}$, *where* $f, g \in C[0,\infty)$, $f, g \geq 0$, $g(0) = 0$, $f(t)$, $g(t)/t$ *increases as* t *increases, and let* $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

Suppose that

i) $\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \geq 0$ *in* Ω,

ii) $u|_{\partial\Omega\cap\{x\}\times[-f(x),f(x)]} \leq g(x)$ *for* $x \in [0,\infty)$,

iii) $0 < \kappa(x) \equiv f(x)/(g(x)) < 1$ *for some* $x_1 > 0$ *and all* $x > x_1$,

iv) $\kappa(x)$ *decreases in* $[x_1,\infty)$.

Then $u(x,y) \leq g(x)/(1-\kappa(x))$ *for every* $(x,y) \in \Omega$ *with* $x > x_1$.

Continuing from Section 3 , we consider two cases in (3.3). Suppose first that $K \neq -1$. Then from (3.5), we find that in S_β,

$$\frac{U(\zeta)}{|z(\zeta)|} \leq \frac{C|\zeta|}{(C/2)(-K+1/K)|\zeta|\cos\beta}(1+o(1)) \quad (\zeta \to \infty,\ \zeta \in S_\beta). \tag{4.1}$$

Also from (3.5), it follows that $z(S_\beta)$ contains the portion of the x axis $x > R$ for some $R > 0$. So, from (4.1), we deduce that, in particular, on the x axis,

$$\limsup_{x\to 0}\frac{u(x)}{x} \leq \frac{1}{(1/2)(-K+1/K)\cos\beta}.$$

Let D^+ be the portion of D in the first quadrant and $e^{-i\pi/4}D^+$ the clockwise rotation of D^+ by $\pi/4$. Then $e^{-i\pi/4}D^+$ is contained in the sector $\Sigma = \{z : -\pi/4 < \arg z < \pi/4\}$. We now apply Theorem A with $\Omega = e^{-i\pi/4}D^+$ and $\Omega_1 = \Sigma$, so $f(t) = t$. We may take $g(t) = t^2$, since on one portion of the boundary of Ω u is 0, and on the other, (3.5) and (4.1) imply that it has at most linear growth. Taking $u - k$ for some constant k if need be so that (ii) above holds near the origin, Theorem A then shows that $u(z)$ cannot grow more rapidly than $|z|^2$ in D^+. The portion D^- of D in the fourth quadrant can be handled similarly.

Thus, we need only consider the case

$$\lim_{|\zeta|\to\infty\ \zeta\in S_\beta} G(\zeta) = -1. \tag{4.2}$$

Now, $|G(\zeta)| > 1$, so we may write

$$G(\zeta) = -e^{\phi(\zeta)}, \tag{4.3}$$

where $\phi(\zeta)$ is analytic in H, and

$$\Re e\, \phi > 0 \quad \text{and} \lim_{|\zeta|\to\infty\ \zeta\in S_\beta} \phi(\zeta) = 0. \tag{4.4}$$

Using (2.4), (4.2), (4.3), and (4.4) we may then write with $\zeta = \sigma + i\tau$,

$$\begin{aligned} z(\zeta) =&(C/2)\left(\sum_{j=0}^{\infty}\int_0^{\zeta}\frac{\phi(\tilde\zeta)^j}{j!}d\tilde\zeta - \sum_{j=0}^{\infty}\overline{\int_0^{\zeta}\frac{(-1)^j\phi(\tilde\zeta)^j}{j!}d\tilde\zeta}\right) \\ =&(C/2)\left(2i\sum_{j=0}^{\infty}\Im m\int_0^{\zeta}\frac{\phi(\tilde\zeta)^{2j}}{(2j)!}d\tilde\zeta + 2\sum_{j=0}^{\infty}\Re e\int_0^{\zeta}\frac{\phi(\tilde\zeta)^{2j+1}}{(2j+1)!}d\tilde\zeta\right). \end{aligned} \tag{4.5}$$

With $\phi = u + iv$, taking the path of integration along the real axis, we have

$$\sum_{j=0}^{\infty}\Re e\int_0^{\sigma}\frac{\phi(\tilde\sigma)^{2j+1}}{(2j+1)!}d\tilde\sigma = \int_0^{\sigma}u(\tilde\sigma)d\tilde\sigma + \sum_{j=1}^{\infty}\Re e\int_0^{\sigma}\frac{(u(\tilde\sigma)+iv(\tilde\sigma))^{2j+1}}{(2j+1)!}d\tilde\sigma.$$

Consider the term with $j = n$ in the sum. This is the sum of 2^{2n+1} terms of the form $u(\tilde\sigma)^k(iv(\tilde\sigma))^{2n+1-k}$. These are pure imaginary when k is even. In particular, this is the case when $k = 0$. Thus, we may factor one $u(\tilde\sigma)$ from all terms and obtain

$$\left|\Re e\frac{(u(\tilde\sigma)+iv(\tilde\sigma))^{2n+1}}{(2n+1)!}\right| \le \frac{2^{2n}}{(2n+1)!}u(\tilde\sigma)|\phi(\tilde\sigma)|^{2n}.$$

This with (4.4) then yields

$$\sum_{j=0}^{\infty}\Re e\int_0^{\sigma}\frac{\phi(\tilde\sigma)^{2j+1}}{(2j+1)!}d\tilde\sigma = \int_0^{\sigma}u(\tilde\sigma)d\tilde\sigma(1+o(1)) \tag{4.6}$$

as $\sigma \to \infty$.

We also need an estimate on the imaginary part. Here, we have on the real axis

$$\Im m\ z(\sigma) = (C/2)\sum_{j=0}^{\infty}\Im m\int_0^{\sigma}\frac{(u(\tilde\sigma)+iv(\tilde\sigma))^{2j}}{(2j)!}d\tilde\sigma$$

Again, in the $j = n$ term of the sum, there are 2^{2n} terms of the form $u(\tilde\sigma)^k(iv(\tilde\sigma))^{2n-k}$. These are pure real when k is even. Thus, the $j = 0$ does not appear, and every other term contains at least one $u(\tilde\sigma)$ factor. We then have

$$\left|\sum_{j=0}^{\infty}\Im m \int_0^\sigma \frac{(\phi(\tilde\sigma))^{2j}}{(2j)!}d\tilde\sigma\right| \le \sum_{j=1}^{\infty}\int_0^\sigma \frac{2^{2j}}{(2j)!}u(\tilde\sigma)|\phi(\tilde\sigma)|^{2j-1}d\tilde\sigma = o\left(\int_0^\sigma u(\tilde\sigma)d\tilde\sigma\right) \qquad (\sigma\to\infty). \tag{4.7}$$

Now, since u is a positive harmonic function in H, we may represent it [11, p. 149] by

$$u(\tilde\sigma) = \frac{\tilde\sigma}{\pi}\int_{-\infty}^{\infty}\frac{d\chi(s)}{|\tilde\sigma - is|^2} + c\tilde\sigma \tag{4.8}$$

where $\chi(s)$ is a non-decreasing function and $c \ge 0$ a constant. By (4.2) and (4.3), we have

$$c = 0. \tag{4.9}$$

Let $\delta > 0$ be large enough so that $\chi(\delta) - \chi(-\delta) = \chi_0 > 0$. Then, from (4.8) and (4.9)

$$\begin{aligned}\int_0^\sigma u(\tilde\sigma)d\tilde\sigma &\ge \int_0^\sigma \frac{\tilde\sigma}{\pi}\int_{-\delta}^{\delta}\frac{d\chi(s)}{\tilde\sigma^2+\delta^2}d\tilde\sigma\\ &= \frac{\chi_0}{\pi}\int_1^\sigma \frac{\tilde\sigma}{\tilde\sigma^2+\delta^2}d\tilde\sigma\\ &= \frac{\chi_0}{2\pi}\log(\sigma^2+\delta^2).\end{aligned}$$

Using this in (4.5) and (4.6), we obtain

$$\Re e\ z(\sigma) > \frac{C\chi_0}{2\pi}(\log(\sigma^2+\delta^2))(1+o(1)) \qquad (\sigma\to\infty). \tag{4.10}$$

We are now in a position again to use Theorem A. We first note that given $\varepsilon > 0$, then (4.5), (4.6), and (4.7) imply that on the real axis,

$$z(\sigma) \in S_\varepsilon \qquad \sigma > \sigma_0$$

for some $\sigma_0 > 0$.

We take $D^+ = z(\{0 < \arg\zeta < \frac{\pi}{2}\}) \cap \{-\varepsilon < \arg z < \frac{\pi}{2}\}$ $(0 < \epsilon < \pi/8)$. On one portion of ∂D^+, we have $u = 0$, and by (2.2) and (4.10) on the other, we have

$$\begin{aligned}\frac{u(z)}{e^{k|z|}} = \frac{U(z(\sigma))}{e^{|kz(\sigma)|}} &\le \frac{C\sigma}{\exp((kC\chi_0/2\pi)\log(\sigma^2+\delta^2)(1+o(1)))}\\ &= C\sigma\sigma^{-(kC\chi_0/\pi)(1+o(1)))}.\end{aligned}$$

Fixing $k > \pi/C\chi_0$. We then have for this portion of ∂D^+,

$$u(z) \le e^{k|z|} \qquad (|z| > R_0)$$

for some $R_0 > 0$.

Now, let $e^{-(\pi/4-\varepsilon/2)i}D^+$ be the clockwise rotation of D^+ through $\pi/4 - \varepsilon/2$ so that $e^{-(\pi/4-\varepsilon/2)i}D^+ \subset S_{\frac{\pi}{4}+\frac{\varepsilon}{2}}$. Using $\Omega_1 = S_{\frac{\pi}{4}+\frac{\varepsilon}{2}}$ so that $f(t) = (\tan(\frac{\pi}{4}+\frac{\varepsilon}{2}))t$, and $\Omega = e^{-(\pi/4-\varepsilon/2)i}D^+$ with $g(t) = t^2e^{2kt}$, the upper bound in (1.2) now follows for D^+ by Theorem A. The upper bound for $D\backslash D^+$ follows similarly. □

5. Survey of results and open questions.

In this section, we shall discuss general questions regarding solutions to

$$\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \qquad \text{in } D, u = 0 \quad \text{on } \partial D. \tag{5.1}$$

In Section 1, we assumed that $u > 0$ for convenience. For most questions, this can be assumed by separately considering components where u is positive and negative. Nitsche's theorem [9, p. 256] then says that (5.1) has only trivial solutions if D is contained in a sector of opening less than π.

The hypotheses of Theorem 1.1 include that topological condition that D be simply connected. It seems unlikely that this assumption is needed for the upper bound. In this regard, Hwang has studied the growth of solutions, but only in special regions contained in the half plane [2, 3, 4].

Problem 1. Is it true that for solutions to (5.1),

$$\max_{|z|=r\ z\in D} u(z) \le e^{Mr} \qquad |z| > R. \tag{5.2}$$

holds for any region contained in a half plane?

It seems likely that there should be an upper bound for the growth of solutions to (5.1) in any region.

Problem 1a. For solutions to (5.1), is there an upper bound for the rate of growth? In particular, does (5.2) hold for solutions over general regions?

Regarding the lower bound in (1.2), it seems likely that the condition on the boundary being a Jordan arc can be removed.

Problem 2. For solutions $u > 0$ to (5.1) in D, is the lower bound in (1.2) valid for any simply connected D?

Without the assumption that D lie in a half plane, the results in [12] show that the order must be at least 1/2 for regions bounded by a Jordan arc. The catenoid with axis of symmetry perpendicular to the xy plane shows that simple connectivity in this case is needed. However, it is easy to construct examples to show that 1/2 is the correct exponent.

Example. Let $z(\zeta) = x(\zeta) + iy(\zeta)$ be as in Section 2 be defined by

$$z(\zeta) = (\zeta + 1)^2/2 - \overline{\log(\zeta + 1)}.$$

Then $z(\zeta)$ maps H onto a region D. Its Jacobian is $|\zeta+1|^2 - |\zeta+1|^{-1} > 0$ in H, and its imaginary part on the boundary $\zeta = it$, $-\infty < t < \infty$ is $t + \tan^{-1} t$ which is monotone, so $z(\zeta)$ is univalent in H. The height function $U(\zeta)$ corresponding to $z(\zeta)$ is $2\Re e\,\zeta$. Thus, for any $z \in D$, there is a $\zeta \in H$ such that $z = z(\zeta)$ and we have

$$\frac{u(z)}{|z|^{1/2}} = \frac{u(z(\zeta))}{|z(\zeta)|^{1/2}} = \frac{2\Re e\zeta}{|(\zeta+1)^2/2 - \overline{\log(\zeta+1)}|^{1/2}}.$$

It seems likely that a conclusion stronger than order 1/2 should be true.

Problem 2a. Is it true that for solutions $u > 0$ to (5.1) in general simply connected domains D,

$$Mr^{1/2} \le \max_{|z|=r\ z\in D} u(z) \qquad |z| > R$$

holds for some positive constants M and R?

If something is known of the geometry of D, then further constraints are known to exist regarding the lower growth of u. If D is simply connected, then the *asymptotic angle* β is defined by

$$\beta = \limsup_{r\to\infty} \operatorname{meas}_\theta(D \cap \{|z| = r\})$$

where $0 < \operatorname{meas}_\theta \le 2\pi$ is the angular measure of the arc. For regions which are not simply connected, in classical potential theory, the quantity $\operatorname{meas}_\theta$ is taken to be $+\infty$ if D contains the whole circle $|z| = r$. In any case, partial results [8, Lemma 1], [10], and [12] raise the following question.

Problem 3. If D has asymptotic angle $\beta \geq \pi$, then must the order of any non-trivial solution u of (5.1) in D be at least π/β?

From Nitsche's theorem, it seems likely that the case $\beta < \pi$ is different.

Problem 3a. If D has asymptotic angle less than π is it true that (5.1) has only trivial solutions?

The estimates for asymptotic angle are useful in dealing with some conjectures of Meeks presented at his Clay Institute lectures.

Problem 4.(Meeks) Can there be at more than 2 disjoint domains D over which there are non-trivial solutions to (5.1)?

Meeks's conjecture to this is that there can be at most 2. Partial results to this are contained in [6], [10], and [8, Theorem 2]. In the case where u is of sublinear growth, ($|u(x)|/|x| \to 0$ as $x \to \infty$ in D) perhaps there is a stronger result.

Problem 4a.(Meeks) Can there be 2 disjoint domains D with non-trivial solutions to (5.1), and having sublinear growth?

It seems reasonable to expect that the answer is no.

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