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Schouten tensor and some topological properties

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In this paper, we prove a cohomology vanishing theorem on locally conformally flat manifold under certain positivity assumption on the Schouten tensor. And we show that this type of positivity of curvature is preserved under 0-surgeries for general Riemannian manifolds, and construct a large class of such manifolds.

1. Introduction.

The notion of positive curvature plays an important role in differential geometry. The existence of such a metric often implies some topological properties of the underlying manifold. A typical example is the Bochner vanishing theorem on manifolds of positive Ricci curvature. In this paper, we consider Riemannian metrics with certain type of positivity on the Schouten tensor. This notion of curvature was introduced by Viaclovsky [18] which extends the notion of scalar curvature.

Let (M, g) be an oriented, compact and manifold of dimension $n > 2$. And let S_g denote the Schouten tensor of the metric g, i.e.,

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right),$$

where Ric_g and R_g are the Ricci tensor and scalar curvature of g respectively. For any $n \times n$ matrix A and $k = 1, 2, \cdots, n$, let $\sigma_k(A)$ be the k-th elementary symmetric function of the eigenvalues of $n \times n$ matrix A, $\forall k = 1, 2, \cdots, n$. Define *$\sigma_k$-scalar curvature* of g by

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot S_g),$$

where $g^{-1} \cdot S_g$ is defined, locally by $(g^{-1} \cdot S_g)^i_j = g^{ik}(S_g)_{kj}$. When $k = 1$, σ_1-scalar curvature is just the scalar curvature R (up to a constant multiple). It is natural to consider manifolds with metric of positive k-scalar curvature.

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However, the surgery might not preserve this positivity. In fact, we consider a stronger positivity. Define

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\Lambda) > 0, \forall j \le k\}.$$

A metric g is said to be in Γ_k^+ if $\sigma_j(g)(x) > 0$ for $j \le k$ and $x \in M$. Such a metric is called a metric of *positive* Γ_k*-curvature*, or a Γ_k-positive metric. When $k = 1$, it is just the metric of positive scalar curvature. In this paper, we are only interested in the case $k \ge 2$.

It was proved in [6] that any metric g of positive Γ_k-curvature with $k \ge n/2$ is a metric of positive Ricci curvature. Hence, when the underlying manifold M is locally conformally flat, (M, g) is conformally equivalent to a spherical space form. We also proved in [5] that (M, g) is conformally equivalent to a spherical space form if $[g]$ has a metric with positive $\Gamma_{\frac{n}{2}-1}$-curvature and the Euler characteristic of M is positive. Here, we restrict our attention to the case $k < n/2$.

The first result of this paper the following vanishing theorem.

Theorem 1.1. *Let (M^n, g) be a compact, locally conformally flat manifold with $\sigma_1(g) > 0$.*

(*i*) *If $g \in \overline{\Gamma}_k^+$ for some $2 \le k < n/2$, then the qth Betti number $b_q = 0$ for*

$$\left[\frac{n+1}{2}\right] + 1 - k \le q \le n - \left(\left[\frac{n+1}{2}\right] + 1 - k\right).$$

(*ii*) *Suppose $g \in \Gamma_2^+$, then $b_q = 0$ for $\left[\frac{n-\sqrt{n}}{2}\right] \le q \le \left[\frac{n+\sqrt{n}}{2}\right]$. If $g \in \overline{\Gamma}_2^+$, $p = \frac{n-\sqrt{n}}{2}$ and $b_p \ne 0$, then (M, g) is a quotient of $\mathbb{S}^{n-p} \times H^p$.*

(*iii*) *If $k \ge \frac{n-\sqrt{n}}{2}$ and $g \in \Gamma_k^+$, then $b_q = 0$ for any $2 \le q \le n-2$. If $k = \frac{n-\sqrt{n}}{2}$, $g \in \overline{\Gamma}_k$, and $b_2 \ne 0$, then (M, g) is a quotient of $\mathbb{S}^{n-2} \times H^2$.*

Here, $\mathbb{S}^{n-p}$ is the standard sphere of sectional curvature 1 and H^p is a hyperbolic plane of sectional curvature -1.

A more precise and general statement will be given in Proposition 2.1 in the next section. When $k = 1$, the above was proved by Bourguignon [1] (see also [10, 13, 14]).

The most direct examples of Γ_k-positive metrics are Einstein manifolds with a positive scalar curvature (for instance $\mathbb{S}^n$ and $\mathbb{C}P^n$) and their small perturbations. Another example is the Hopf manifold (a quotient of

$\mathbb{S}^{n-1} \times \mathbb{S}^1$ with the product metric). It is easy to check that the product metric is Γ_k-positive if and only if $k < n/2$. It is implicitly proved in [7] that for $k < \frac{n}{2}$, the connected sum of two positive Γ_k-curved locally conformally flat manifolds can be assigned a locally conformally flat metric with positive Γ_k curvature. Here we modify the argument in [4] to construct more examples of manifolds with positive Γ_k-curvature without locally conformally flat assumption. The construction of manifolds by connected sums for positive scalar curvature was furnished in [4] and [16].

Theorem 1.2. *Let $2 \le k < n/2$, and let M_1^n and M_2^n be two compact manifolds (not necessarily locally conformally flat) of positive Γ_k-curvature. Then the connected sum $M_1 \# M_2$ also admits a metric of positive Γ_k-curvature. If in addition, M_1 and M_2 are locally conformally flat, then $M_1 \# M_2$ admits a locally conformally flat structure with positive Γ_k-curvature.*

It follows that the manifold of the form

$$L_1 \# \cdots \# L_i \# H_1 \# \cdots \# H_j, \tag{1.1}$$

carries a locally conformally flat structure of positive Γ_k-curvature ($k < n/2$), where $L_i's$ and H_j' are quotients of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and the standard sphere $\mathbb{S}^n$ respectively. Hence, any free product of finitely many copies of $\mathbb{Z}$ with finite many copies of the fundamental group of spherical space forms is the fundamental group of a manifold of positive Γ_k-curvature, for $k < n/2$. As mentioned that any locally conformally flat manifold with Γ_k-curvature for some $k \ge \frac{n}{2}$ is conformally equivalent to a spherical space form by [6]. For $k = [\frac{n-1}{2}]$, one would like to classify all such locally conformally flat manifolds with positive Γ_k-curvature. When $n = 3, 4$, results of Izeki [8] and Schoen-Yau [17] imply that if (M^n, g) is a compact Riemannian manifold with positive scalar curvature, then M has a form of (1.1).

The paper is organized as follows. In Section 2, we prove that the positivity of Γ_k curvature implies a positivity of a quantity arising in the Weitzenböck formula for p-forms. This leads to the application of the Bochner type technique to obtain Theorem 1.1. In Section 3, we present the construction of Γ_k-positive metrics on the connected sum.

2. A vanishing theorem.

We first introduce some notations. Let $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n$ be an n-tuple. For any $j = 1, 2, \cdots, n$, we set

$$\Lambda | j = (\lambda_1, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_n).$$

Assume that $2 \leq k < n/2$, $1 \leq p \leq n/2$. Define a function $G_{n,p} : \mathbb{R}^n \to \mathbb{R}$ by

$$G_{n,p}(\Lambda) = \min_{(i_1,\cdots,i_n)} \left\{ (n-p)\sum_{j=1}^{p} \lambda_{i_j} + p \sum_{j=p+1}^{n} \lambda_{i_j} \right\},$$

where $(i_1, \cdots, i_n)$ is a permutation of $(1, 2, \cdots, n)$ and the minimum is taken over all permutations. If we rearrange $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. It is obvious that for $p \leq n/2$,

$$G_{n,p}(\Lambda) = p\sum_{j=1}^{n-p} \lambda_j + (n-p) \sum_{j=n-p+1}^{n} \lambda_j.$$

In the rest of this section, we will always assume that Λ is so arranged. $G_{n,p}$ is related to a geometric quantity arising in the Weitzenböck form for p-forms (see (2.8)). Let $I_p = (1, 1, \cdots, 1) \in \mathbb{R}^p$ and $s > 0$. Define n-tuples by

$$E_{n,p} = (I_{n-p}, -I_p) \text{ and } E^s_{n,p} = (I_{n-p}, -sI_p).$$

It is trivial to see that $G_{n,p}(E_{n,p}) = 0$. A straightforward calculation shows that $E_{n,p}$ (up to a constant multiplier) is the Schouten tensor of the manifold $\mathbb{S}^{n-p} \times H^p$. It will become clear later that this manifold serves the "minimal" model in our vanishing theorem.

We want to find a condition on k, p under which $\Lambda \in \Gamma_k^+$ implies $G_{n,p}(\Lambda) > 0$. Our basic observation is that if $\sigma_k(E_{n,p}) = 0$, then $G_{n,p}(\Lambda)$ is a "linearization" of σ_k at $E_{n,p}$ in the direction of Λ. Namely,

$$\frac{\sigma_{k-1}(E_{n-1,p})}{p} G_{n,p}(\Lambda) = \frac{d}{dt}\sigma_k((1-t)E_{n,p} + t\Lambda)|_{t=0}.$$

Then by the convexity of Γ_k^+, if $E_{n,p} \in \overline{\Gamma}_k^+$ with $\sigma_k(E_{n,p}) = 0$, we have $\sigma_k(E_{n,p} + t\Lambda) \geq 0$ for $\Lambda \in \Gamma_k^+$. Note that $\overline{\Gamma}_k^+$ is the closure of Γ_k^+. It would imply that $G_{n,p}(\Lambda) \geq 0$. Roughly speaking, $\sigma_k(E_{n,p}) = 0$ is the condition we are seeking (the precise statements are given in Proposition 2.5 and Lemma 2.6).

The main objective of this section is to prove the following proposition.

Proposition 2.1. *Let (M^n, g) be a compact, locally conformally flat manifold and let $2 \leq k \leq n/2$ and $1 \leq p \leq n/2$. Suppose $g \in \overline{\Gamma}_k^+$ and $\sigma_1(g)$ is not identical to zero in M.*

(*i*) *If $E_{n,p} \in \Gamma_{k-1}^+$ and $E_{n,p} \notin \Gamma_k^+$, then $b_q = 0$ for all $p \leq q \leq n-p$.*

(*ii*) *Suppose* $E_{n,p} \in \overline{\Gamma}_k^+$, $\sigma_k(E_{n,p}) = 0$ *and* $\sigma_k(g) > 0$ *at some point in* M, *then* $b_q = 0$ *for all* $p \le q \le n-p$.

(*iii*) *Suppose* $E_{n,p} \in \overline{\Gamma}_k^+$, $\sigma_k(E_{n,p}) = 0$, *then* $b_p \neq 0$ *if and only if* (M, g) *is a quotient of* $\mathbb{S}^{n-p} \times H^p$.

We need some technical lemmas in the proof of Proposition 2.1.

Lemma 2.2. *For any* $s > 0$, *if* $E^s_{n,p} \in \overline{\Gamma}_k^+$, *then* $E^s_{n-1,p} \in \Gamma^+_{k-1}$ *and* $E^s_{n-2,p-1} \in \Gamma^+_{k-1}$. *If* $E_{n,p} \in \overline{\Gamma}_k^+$, *then* $E_{n-2,p-1} \in \Gamma_k^+$.

Proof. First, it is easy to check that $E^s_{n-1,p} \in \Gamma^+_{k-1}$ implies $E^s_{n-2,p-1} \in \Gamma^+_{k-1}$. If $E^s_{n,p} \in \Gamma^+_k$ (resp. $\overline{\Gamma}^+_k$), then $E^s_{n-1,p} \in \Gamma^+_{k-1}$ (resp. $\overline{\Gamma}^+_{k-1}$). Hence, we only need to deal with the case that $\sigma_k(E^s_{n,p}) = 0$. Assume by contradiction that $\sigma_{k-1}(E^s_{n-1,p}) = 0$. Since $\sigma_k(E^s_{n,p}) = \sigma_{k-1}(E^s_{n-1,p}) + \sigma_k(E^s_{n-1,p})$, we have $\sigma_k(E^s_{n-1,p}) = 0$. Together with $E^s_{n-1,p} \in \overline{\Gamma}^+_{k-1}$, it implies $E^s_{n-1,p} \in \overline{\Gamma}^+_k$. We may repeat this argument to produce a sequence of integers m such that $E^s_{m,p} \in \overline{\Gamma}^+_k$ and $\sigma_k(E^s_{m,p}) = \sigma_k(E^s_{m+1,p}) = 0$. This process must be stopped somewhere since $-sI_p$ is not in $\overline{\Gamma}^+_k$. We then obtain an integer m such that $\sigma_k(E^s_{m,p}) = \sigma_k(E^s_{m+1,p}) = 0$ and $E^s_{m,p} \in \Gamma^+_{k-1}$. Now

$$0 = \sigma_k(E^s_{m+1,p}) = \sigma_{k-1}(E^s_{m,p}) + \sigma_k(E^s_{m,p}) > 0,$$

this is a contradiction.

To prove the last assertion in the lemma, note that we already have $E_{n-2,p-1} \in \overline{\Gamma}^+_{k-1}$. Now,

$$0 \le \sigma_k(E_{n,p}) = \sigma_k(E_{n-2,p-1}) - \sigma_{k-2}(E_{n-2,p-1}).$$

It follows that

$$\sigma_k(E_{n-2,p-1}) \ge \sigma_{k-2}(E_{n-2,p-1}) > 0.$$

□

Lemma 2.3. *Let* $0 < s \le 1$ *and* $p \le n/2$. *If* $E^s_{n,p} \in \overline{\Gamma}^+_k$ *with* $\sigma_k(E^s_{n,p}) = 0$ *for some* $k \ge 2$, *then for any* $\Lambda \in \overline{\Gamma}^+_k$,

$$G_{n,p}(\Lambda) \ge 0.$$

If in addition $0 < s < 1$ *and* $\sigma_1(\Lambda) > 0$, *then*

$$G_{n,p}(\Lambda) > 0.$$

Proof. We first notice that if $\Lambda \in \overline{\Gamma}_2^+$ and $\sigma_1(\Lambda) = 0$. This follows from

$$\sum_{i=1}^{n} \lambda_i^2 = \sigma_1^2(\Lambda) - 2\sigma_2(\Lambda).$$

Thus, we now assume $\sigma_1(\Lambda) > 0$. For $0 \leq s \leq 1$, we note that $\sigma_1(E^s_{n,p}) \geq 0$. Since $E^s_{n,p} \in \overline{\Gamma}_2^+$ (note that $k \geq 2$ by assumption), we must have $\sigma_1(E^s_{n,p}) > 0$. By Lemma 2.2, we have $\sigma_{k-1}(E^s_{n-1,p-1}) > 0$. Using the identity $\sum_{j=1}^{n} \sigma_{k-1}(\Lambda|j)\lambda_j = k\sigma_k(\Lambda)$, we have

$$0 = k\sigma_k(E^s_{n,p}) = (n-p)\sigma_{k-1}(E^s_{n-1,p}) - sp\sigma_{k-1}(E^s_{n-1,p-1}). \tag{2.1}$$

We want to show that $G_{n,p}(\Lambda)$ is positive for $\Lambda \in \overline{\Gamma}_k^+$ with $\sigma_1(\Lambda) > 0$. Consider a function $f(t) = \sigma_k((1-t)E^s_{n,p} + t\Lambda)$. Denote $E^s_{n,p} = (e_1, e_2, \cdots, e_n)$. By the convexity of $\overline{\Gamma}_k$, we know $f(t) \geq 0$. Since $f(0) = 0$, we have $f'(0) \geq 0$ which implies

(2.2)

$$\begin{aligned}
0 \leq f'(0) &= \sum_{j=1}^{n} \sigma_{k-1}(E^s_{n,p}|j)(\lambda_j - e_j) = \sum_{j=1}^{n} \sigma_{k-1}(E^s_{n,p}|j)\lambda_j - \sigma_k(E^s_{n,p}) \\
&= \sigma_{k-1}(E^s_{n-1,p}) \sum_{j=1}^{n-p} \lambda_j + \sigma_{k-1}(E^s_{n-1,p-1}) \sum_{j=n-p+1}^{n} \lambda_j \\
&= \sigma_{k-1}(E^s_{n-1,p}) \left\{ \sum_{j=1}^{n-p} \lambda_j + \frac{n-p}{sp} \sum_{j=n-p+1}^{n} \lambda_j \right\} \qquad \text{(by (2.1))} \\
&= \frac{\sigma_{k-1}(E^s_{n-1,p})}{sp} \left\{ sp \sum_{j=1}^{n-p} \lambda_j + (n-p) \sum_{j=n-p+1}^{n} \lambda_j \right\}.
\end{aligned}$$

From Lemma 2.2, we have $\sigma_{k-1}(E^s_{n-1,p}) > 0$. Hence, (2.2) implies that

$$sp \sum_{j=1}^{n-p} \lambda_j + (n-p) \sum_{j=n-p+1}^{n} \lambda_j \geq 0. \tag{2.3}$$

If $s = 1$, this gives $G_{n,p}(\Lambda) \geq 0$. If $s < 1$, from assumption that $\sigma_1(\Lambda) = \sum_{j=1}^{n} \lambda_j > 0$, we have $\sum_{j=1}^{n-p} \lambda_j > 0$ by our arrangement of Λ. Therefore, (2.3) implies that $G_{n,p}(\Lambda) > 0$. □

Lemma 2.4. *Assume that for some $1 \le p < \frac{n}{2}$ and $2 \le k \le n/2$, $E_{n,p} \in \overline{\Gamma}_k^+$ with $\sigma_k(E_{n,p}) = 0$. If $\Lambda \in \overline{\Gamma}_k^+$, then $G_{n,p}(\Lambda) \ge 0$. The equality holds if and only if $\Lambda = \mu E_{n,p}$ for some $\mu \ge 0$. In particular, if $\Lambda \in \Gamma_k^+$, then $G_{n,p}(\Lambda) > 0$.*

Proof. As in the proof of Lemma 2.3, we may assume $\sigma_1(\Lambda) > 0$. Since the positivity of $G(\Lambda)$ does not change under a rescaling $\Lambda \to \mu\Lambda$, we may assume that $\sigma_1(\Lambda) = \sigma_1(E_{n,p})$. As in the previous lemma, we consider the function $f(t) = \sigma_k((1-t)E_{n,p} + t\Lambda)$. We have $f'(0) \ge 0$. The argument given in the previous Lemma implies that $G_{n,p}(\Lambda) > 0$ or $G_{n,p}(\Lambda) = 0$. Hence, we only need to examine the latter case. In this case, we also have $f'(0) = 0$. Since $f(0) = 0$ and $f(t) \ge 0$ for any $t \in [0,1]$, we have $f''(0) \ge 0$. By our choice of $E_{n,p}$, it is clear that $G_{n,p}(E_{n,p}) = 0$. This, together with $G_{n,p}(\Lambda) = 0$, gives

$$p\sum_{i=1}^{n-p}(e_i - \lambda_i) + (n-p)\sum_{i=n-p+1}^{n}(e_i - \lambda_i) = 0. \tag{2.4}$$

Here, we denote $E_{n,p}$ by $(e_1, e_2, \cdots, e_n)$. The normalization $\sigma_1(\Lambda) = \sigma_1(E_{n,p})$ gives

$$\sum_{i=1}^{n-p}(e_i - \lambda_i) + \sum_{i=n-p+1}^{n}(e_i - \lambda_i) = 0. \tag{2.5}$$

(2.4) and (2.5) imply

$$\sum_{i=1}^{n-p}(e_i - \lambda_i) = \sum_{i=n-p+1}^{n}(e_i - \lambda_i) = 0. \tag{2.6}$$

Let $\tilde\Lambda_1 = (e_1 - \lambda_1, \cdots, e_{n-p} - \lambda_{n-p})$ and $\tilde\Lambda_2 = (e_{n-p+1} - \lambda_{n-p+1}, \cdots, e_n - \lambda_n)$. (2.6) means that $\sigma_1(\tilde\Lambda_1) = \sigma_1(\tilde\Lambda_2) = 0$. Now, we compute $f''(0)$

$$\begin{aligned} 0 \le f''(0) &= \sum_{i \ne j} \sigma_{k-2}(E_{n,p}|ij)(\lambda_i - e_i)(\lambda_j - e_j) \\ &= 2\{\sigma_{k-2}(E_{n-2,p-1})\sigma_1(\tilde\Lambda_1)\sigma_1(\tilde\Lambda_2) \\ &\qquad + \sigma_{k-2}(E_{n-2,p-2})\sigma_2(\tilde\Lambda_1) + \sigma_{k-2}(E_{n-2,p})\sigma_2(\tilde\Lambda_2)\} \\ &= \sigma_{k-2}(E_{n-2,p-2})\left[\sigma_1^2(\tilde\Lambda_1) - \sum_{i=1}^{n-p}(e_i - \lambda_i)^2\right] \end{aligned} \tag{2.7}$$

$$
\begin{aligned}
&+ \sigma_{k-2}(E_{n-2,p}) \left[\sigma_1^2(\tilde{\Lambda}_2) - \sum_{n-p+1}^{n} (e_i - \lambda_i)^2 \right] \\
&= -\sigma_{k-2}(E_{n-2,p-2}) \sum_{i=1}^{n-p} (e_i - \lambda_i)^2 \\
&\quad - \sigma_{k-2}(E_{n-2,p}) \sum_{i=n-p+1}^{n} (e_i - \lambda_i)^2.
\end{aligned}
$$

By Lemma 2.2, we know that $\sigma_{k-2}(E_{n-2,p-2}) > 0$ and $\sigma_{k-2}(E_{n-2,p}) > 0$. Hence, (2.7) implies that

$$e_i = \lambda_i, \qquad \text{for any } i.$$

Hence, the equality holds implies $\Lambda = \mu E_{n,p}$ for some $\mu > 0$. □

Proposition 2.5. (*i*) *Suppose that* $\sigma_k(E_{n,p}) < 0$ *for some* $2 \le k < n/2$ *and* $2 \le p < n/2$. *If* $\Lambda \in \overline{\Gamma}_k^+$, *then* $G_{n,q}(\Lambda) \ge 0$ *for any* $p \le q \le n/2$. *If in addition* $\sigma_1(\Lambda) > 0$, *then* $G_{n,q}(\Lambda) > 0$ *for any* $p \le q \le n/2$.

(*ii*) *Suppose that* $\sigma_k(E_{n,p}) = 0$ *and* $E_{n,p} \in \overline{\Gamma}_k^+$ *for some* $2 \le k < n/2$ *and* $2 \le p < n/2$. *If* $\Lambda \in \overline{\Gamma}_k^+$, *then* $G_{n,q}(\Lambda) \ge 0$ *for any* $p \le q \le n/2$. *And if* $\Lambda \in \Gamma_k^+$, *then* $G_{n,q}(\Lambda) > 0$ *for any* $p \le q \le n/2$.

Proof. Set $0_p = (0, \dots, 0)$. Since $E^s_{n,p} = (I_{n-p}, 0_p) - s(0_{n-p}, I_p)$, $\sigma_k(E^s_{n,p})$, as a function of s, is decreasing. Hence, from $\sigma_k(E^0_{n,p}) > 0$ and the assumption that $\sigma_k(E^1_{n,p}) = \sigma_k(E_{n,p}) < 0$, there is a $s \in (0, 1)$ such that $\sigma_k(E^s_{n,p}) = 0$. And one can check that for any other integer $1 \le k' < k$, $\sigma_{k'}(E^s_{n,p}) > 0$. Hence, $E^s_{n,p} \in \bar{\Gamma}_k$ and we can apply Lemma 2.3 to show that $G_{n,p}(\Lambda) \ge 0$ for any $\Lambda \in \bar{\Gamma}_k^+$, and $G_{n,p}(\Lambda) > 0$ for any $\Lambda \in \bar{\Gamma}_k^+$ and $\sigma_1(\Lambda) > 0$. For $n/2 \ge q > p$, it is easy to see that $\sigma_k(E_{n,q}) < \sigma_k(E_{n,p}) < 0$. Therefore, the same argument applies for q. This proves (i). (ii) is proved by the same argument. □

We now prove Proposition 2.1.

Proof of Proposition 2.1. Recall the Weitzenböck formula for p-forms ω

$$\Delta\omega = \nabla^*\nabla\omega + \mathcal{R}\omega,$$

where

$$\mathcal{R}\omega = \sum_{j,l=1} \omega_j \wedge i(e_l) R(e_j, e_l)\omega.$$

Here, e_j is a local basis and $i(\cdot)$ denotes the interior product $\Delta = dd^* + d^*d$ is the Hodge-de Rham Laplacian and $\nabla^*\nabla$ is the (positive) Lapalacian. In local coordinates, let $\omega = \omega_1 \wedge \cdots \wedge \omega_p$. Then

$$\mathcal{R}\omega = \left((n-p) \sum_{i=1}^{p} \lambda_i + p \sum_{i=p+1}^{n} \lambda_i \right) \omega, \tag{2.8}$$

where λ's are eigenvalues of the Schouten tenser S_g. Under the conditions given in (i) or (ii), Proposition 2.5 implies that $\mathcal{R}$ is a non-negative operator and positive at some point. It is clear from the Weitzenböck formula that any q-harmonic form ω is parallel for such q considered in the Proposition. Since $\mathcal{R}$ is positive at some point, this forces $\omega = 0$ everywhere. So, $H^q(M) = \{0\}$.

Now, we prove (iii). By assumption, there is a non-zero harmonic p-form ω. In this case, $\mathcal{R}$ is non-negative by Proposition 2.5. Again, from the Weitzenböck formula, ω is parallel. Now, one can follow the argument given in [11] to prove that the restricted holonomy group of M is reducible and the universal cover $\tilde{M}$ of M is a Riemannian product. And we can conclude that $\tilde{M}$ is $\mathbb{S}^{n-p} \times H^p$. $\square$

Finally, we prove Theorem 1.1. We need to spell out the relationship of k and p such that the conditions in Proposition 2.5 are satisfied.

Lemma 2.6. *The followings are true.*

(*i*) *$k = 2$ and $\frac{n}{2} \geq p \geq [\frac{n-\sqrt{n}}{2}]$; then $E_{n,p} \notin \Gamma_2^+$. If $p = \frac{n-\sqrt{n}}{2}$ is an integer, then $E_{n,p} \in \overline{\Gamma}_2^+$ with $\sigma_2(E_{n,p}) = 0$.*

(*ii*) *$p = 2$ and $k \geq [\frac{n-\sqrt{n}}{2}]$, then $E_{n,2} \notin \Gamma_k^+$. If $k = \frac{n-\sqrt{n}}{2}$ is an integer, then $E_{n,2} \in \overline{\Gamma}_k^+$ with $\sigma_k(E_{n,2}) = 0$.*

(*iii*) *For the general case, $E_{n,p} \notin \overline{\Gamma}_k^+$, if $3 \leq p \leq n/2$, and*

$$k \geq \frac{n-2p+4-\sqrt{n-2p+4}}{2}; \tag{2.9}$$

or if $3 \leq k < n/2$, and

$$p \geq \frac{n-k+2-\sqrt{n-k+2}}{2}. \tag{2.10}$$

In particular, if $n > 4$ *and* $k = [\frac{n+1}{2}] + 1 - p$, *then* $E_{n,p} \notin \overline{\Gamma}_k^+$.

Proof. It is easy to compute that

$$\sigma_2(E_{n,p}) = \frac{(n-2p)^2 - n}{2}.$$

So, $E_{n,p} \notin \Gamma_2^+$ if $\frac{n}{2} \geq p \geq \frac{n-\sqrt{n}}{2}$.

Similarly, if $p = 2$, we compute

$$\begin{aligned} \sigma_k(E_{n,2}) &= \sigma_k(I_{n-2}) - 2\sigma_{k-1}(I_{n-2}) + \sigma_{k-2}(I_{n-2}) \\ &= \frac{(n-2)!}{k!(n-k)!}\{(n-2k)^2 - n\} \leq 0, \end{aligned}$$

if $k \geq \frac{n-\sqrt{n}}{2}$.

If $p > 2$ and $E_{n,p} \in \overline{\Gamma}_k$, applying Lemma 2.2 (the last assertion) repeatedly, we have $E_{n-2p+4,2} \in \Gamma_k^+$. However, one can compute

$$\begin{aligned} \sigma_k(E_{n-2p+4,2}) &= \sigma_k(I_{n-2p+2}) + \sigma_{k-2}(I_{n-2p+2}) - 2\sigma_{k-1}(I_{n-2p+2}) \\ &= \tfrac{(n-2p+2)!}{k!(n-2p+4-k)!}\{(n-2p+4-2k)^2 - (n-2p+4)\} \leq 0, \end{aligned}$$

for k satisfies (2.9). A contradiction.

And if $\frac{n}{2} > k > 2$ and $E_{n,p} \in \overline{\Gamma}_k$, by Lemma 2.2, $E_{n-k+2,p} \in \Gamma_2^+$. This implies that $n - k - 2p + 2 > 0$. By (2.10)

$$\sigma_2(E_{n-k+2,p}) = \frac{(n-k+2-2p)^2 - n + k - 2}{2} \leq 0.$$

Contradiction again. □

The statements (i) and (ii) in Lemma 2.6 are sharp, but (iii) is not sharp. When $p > 2$ and $k > 2$, the relationship of them is a combinatorial problem which involves polynomials of degree k. By Lemma 2.2, there is an optimal relation for each pair (k, p). For example, one may calculate that for $k \geq \frac{n-\sqrt{3n-2}}{2}$ and $p = 3$, $\sigma_k(E_{n,p}) \leq 0$. The dual relation is also true for $k = 3$. In (iii), we simply reduced it to the cases $p = 2$ or $k = 2$ to get relations (2.9) and (2.10). Regarding two relations (2.9) and (2.10), for relative small p, the first one is sharper, and for relative small k, the later is better.

Proof of Theorem 1.1. Theorem 1.1 follows from Proposition 2.1 and Lemma 2.6. □

3. Construction for the connected sums.

In this section, we first prove Theorem 1.2 for the case without the locally conformally flat structure. The proof follows closely the idea in [4]. Then we prove Theorem 1.2 for the case with the locally conformally flat structure.

Proof of Theorem 1.2 *for the case without the locally conformally flat structure.* Let M be an n-dimensional manifold with a Γ_k-positive metric and $n > 2k \geq 4$. Fix $p \in M$ and let $D = \{x \in \mathbb{R}^n \,|\, \|x\| \leq \bar{r}\}$ be a small normal coordinate ball centered at p of radius $\bar{r}$, where $\bar{r}$ is smaller than the injectivity radius of M. For any $\rho < \bar{r}$, let $S^{n-1}(\rho) = \{x \in D \,|\, \|x\| = \rho\}$ be the geodesic ball of radius ρ. Following [4], we consider the Riemannian product $D \times \mathbb{R}$ with coordinates (x, t) and its hypersurface

$$T = \{(x,t) \in D \times \mathbb{R} \,|\, (\|x\|, t) \in \gamma, \}$$

where γ is a curve in the (r,t)-plane. We will choose γ satisfying

1. γ begins at one end with a vertical line segment $t = 0$, $r_1 \leq r \leq \bar{r}$ and γ ends at another end with a horizontal line segment $r = r_\infty > 0$, with r_∞ sufficiently small, see figure in [4].
2. The resulted hypersurface T has a metric of Γ_k-positive.

The statement 2 is the crucial point of the construction. Now, we compute the Schouten tensor at $\bar{x} = (x,t) \in T$ with $\|x\| = r \neq 0$. Choose an orthonormal basis $e_1, e_2, \cdots, e_n$ of $T_{\bar{x}}$ such that e_1 is the tangent vector of the curve $\{(sx,t) | s, t \in \mathbb{R}_+\} \cap T$ parametrized by arc length. Moreover, one can choose this basis such that $e_1, e_2, \cdots, e_n$ are principal vectors of the second fundamental form of T in $D \times \mathbb{R}$. The corresponding principal curvature λ_i are

$$\begin{aligned} \lambda_1 &= \kappa, \\ \lambda_i &= (-\tfrac{1}{r} + O(r)) \sin\theta, \qquad i \geq 2, \end{aligned}$$

where κ is the geodesic curvature of the curve γ at (r,t) and θ is the angle between the normal to the hypersurface and the t-axis, see [4]. The Gauss equation is

$$\begin{aligned} R_{ijkl} &= \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk} \\ &= \bar{R}_{ijkl} + \lambda_i\lambda_j(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \end{aligned}$$

Here, R_{ijkl} ($\bar{R}_{ijkl}$ resp.) is the curvature tensor of T ($D \times \mathbb{R}$ resp.). It is clear that

$$\begin{array}{rcll} \bar{R}_{ijkl} & = & \bar{R}^D_{ijkl}, & i,j,k,l \geq 2, \\ \bar{R}_{1jkl} & = & \bar{R}^D_{\partial r jkl}\cos\theta, & j,k,l \geq 2, \\ \bar{R}_{1j1l} & = & \bar{R}^D_{\partial r j \partial r l}\cos^2\theta & j,l \geq 2. \end{array}$$

Hence, the Ricci tensor of T is given by $(\forall i,j \geq 2)$,

$$\begin{aligned} R_{ij} &= R^D_{ij} - R^D_{i\partial r j \partial r}\sin^2\theta - \kappa\delta_{ij}\left(\frac{1}{r} + O(r)\right)\sin\theta \\ &\quad + (n-2)\delta_{ij}\left(\frac{1}{r} + O(r)\right)^2 \sin^2\theta, \\ R_{11} &= R^D_{\partial r \partial r} - R^D_{\partial r\partial r}\sin^2\theta - (n-1)\kappa\left(\frac{1}{r} + O(r)\right)\sin\theta, \\ R_{1j} &= R^D_{1j} - R^D_{1j}(1-\cos\theta). \end{aligned}$$

We compute the scalar curvature R and Schouten tensor S:

$$\begin{aligned} R &= R^D - 2R^D_{\partial r\partial r}\sin^2\theta + (n-1)(n-2)\left(\frac{1}{r^2} + O(1)\right)\sin^2\theta \\ &\quad - 2(n-1)\kappa\left(\frac{1}{r} + O(r)\right)\sin\theta, \end{aligned}$$

$$\begin{aligned} S &= S^D + V\sin^2\theta \\ &\quad + \frac{1}{2}\left(\frac{1}{r^2} + O(1)\right)\sin^2\theta \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \\ &\quad - \kappa\left(\frac{1}{r} + O(r)\right)\sin\theta \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \\ &= S^D + V\sin^2\theta + \frac{1}{2}\left(\frac{1}{r^2} + O(1)\right)\sin^2\theta G_0 - \kappa\left(\frac{1}{r} + O(r)\right)\sin\theta G_1, \end{aligned} \tag{3.1}$$

where all entries of the matrix V are bounded (depending only on the metric on D).

It is easy to check that the $n \times n$ matrix G_0 is in Γ_k^+ if and only if $n > 2k$. From this fact, one can see that when r is small, the third term in

(3.1) dominates others. Since Γ_k^+ is a open cone and $S^D \in \Gamma_k^+$, there is an positive constant $c_0 > 0$ such that $S^D - c_0 I \in \Gamma_k^+$. Here, I is the identity matrix, In order to satisfy 2, from the convexity of Γ_k^+, we only need to find a curve such that the matrix

$$F := c_0 I - c_1 \sin^2 \theta I + \frac{1}{2}\left(\frac{1}{r^2} - c_2\right)\sin^2 \theta G_0 - \kappa \left(\frac{1}{r} + c_3\right)\sin\theta G_1 \tag{3.2}$$

is in Γ_k^+, Where $c_1 > 0$ is chosen such that $(V_{ij}) + c_1 I$ is positive definite and c_2, c_3 are positive constants independent of r. Near the starting point $(0, r_1)$, by the openness of Γ_k^+, we can choose a small θ_0 such that for $0 \le \theta < \theta_0$, the matrix F is in Γ_k^+. Hence, we can bend γ in a small region around the point $(0, r_1)$ and end the "first bend" at (t_2, r_2) with $\theta = \theta_0$, see [4]. Now, we continue the curve γ by a straight line segment with angle θ_0 and end at a point (t_3, r_3) where $r_3 > 0$ is very small which will be chosen later. Since on (t_2, t_3) the geodesic curvature $\kappa = 0$, F is in Γ_k^+. We find r_3 small so that for any $r \le r_3$

$$(c_0 - c_1 \sin^2\theta) I + \frac{1}{2}\left(\frac{1}{r^2} - c_2\right)\sin^2\theta G_0 - \kappa\left(\frac{1}{r} + c_3\right)\sin\theta G_1 \in \Gamma_k^+,$$

We compute

$$\sigma_k(F) = a^k \frac{(n-1)!}{k!(n-k)!}\left(n - 2k - \frac{b}{a}\right),$$

where

$$a = \frac{1}{2}\left(\frac{1}{r^2} - c_2\right)\sin^2\theta + (c_0 - c_1\sin^2\theta),$$

$$b = \kappa\left(\frac{1}{r} + c_3\right)\sin\theta - 2(c_0 - c_1 \sin^2\theta).$$

Since $n > 2k$, to keep the k-positivity, we only need $b/a < 1/2$, i.e.,

$$2\kappa\left(\frac{1}{r} + c_3\right)\sin\theta < \frac{1}{2}\left(\frac{1}{r^2} - c_2\right)\sin^2\theta + 3(c_0 - c_1\sin^2\theta).$$

Now, we can choose γ as in [15] to finish the proof. $\square$

The proof of the last statement in Theorem 1.2 is inspired by arguments in [12], we need a lemma which was proven in [7].

Lemma 3.1. *Let D be the unit disk in $\mathbb{R}^n$ and ds^2 the standard Euclidean metric. Let $g_0 = e^{-2u_0} ds^2$ be a metric on D of positive Γ_k-curvature with $k < n/2$. Then there is a conformal metric $g = e^{-2u} ds^2$ on $D \backslash \{0\}$ of positive Γ_k-curvature with the following properties:*

1) $\sigma_k(g) > 0$ *in* $D\backslash\{0\}$.

2) $u(x) = u_0(x)$ *for* $r = |x| \in (r_0, 1]$.

3) $u(x) = a + \log r$ *for* $r = |x| \in (0, r_3)$ *and some constant* a.

for some constant r_0 *and* r_3 *with* $0 < r_3 < r_0 < 1$.

Proof of Theorem 1.2 *in the case of locally conformally flat.* Let (M_1, g_1) and (M_2, g_2) be two compact locally conformally flat manifolds of Γ_k-positive. Let $p_i \in M_i$. The locally conformally flatness of M_i implies that there is a neighborhood U_i of p_i such that $(U_i, g_i) = (D, e^{-2u_i}|dx|^2)$. Applying Lemma 5, we obtain a new conformal metric $\tilde{g}_i$ on $M_i\backslash\{p_i\}$ satisfying in D conditions (1)–(3) with constants r_0^i, r_3^i and a^i. By scaling of metrics, we may assume that $a^1 = a^2 = 0$. Let $r^0 = \min\{r_3^1, r_3^2\}$. Hence, in $\{0 < |x| \le r^0\}$ two metrics $\tilde{g}_1$ and $\tilde{g}_2$ are the same, namely, $\tilde{g}_1 = \tilde{g}_2 = \frac{1}{|x|^2}|dx|^2$. The inversion map $\phi(x) = \frac{(r^0)^2}{2}\frac{x}{|x|^2}$, maps $\{\frac{r^0}{2} \le |x| \le r^0\}$ into itself. Now, we can glue M_1 and M_2 by identifying $\{\frac{r^0}{2} \le |x| \le r^0\}$ by ϕ. Note that ϕ is conformal. It is clear that the glued manifold is a locally conformally flat manifold of positive Γ_k. $\square$

Added-in-proof. After this paper was completed, we learned two recent papers by Chang-Hang-Yang [2] and González [3] on further topological implications of Γ_k positivity for locally conformally flat manifolds.

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