communications in
analysis and geometry
Volume 13, Number 5, 845-885, 2005

Phase Space for the Einstein Equations

Robert Bartnik[1]

A Hilbert manifold structure is described for the phase space $\mathcal{F}$ of asymptotically flat initial data for the Einstein equations. The space of solutions of the constraint equations forms a Hilbert submanifold $\mathcal{C} \subset \mathcal{F}$. The ADM energy-momentum defines a function which is smooth on this submanifold, but which is not defined in general on all of $\mathcal{F}$. The ADM Hamiltonian defines a smooth function on $\mathcal{F}$ which generates the Einstein evolution equations only if the lapse-shift satisfies rapid decay conditions. However a regularised Hamiltonian can be defined on $\mathcal{F}$ which agrees with the Regge-Teitelboim Hamiltonian on $\mathcal{C}$ and generates the evolution for any lapse-shift appropriately asymptotic to a (time) translation at infinity. Finally, critical points for the total (ADM) mass, considered as a function on the Hilbert manifold of constraint solutions, arise precisely at initial data generating stationary vacuum spacetimes.

1. Introduction.

It has long been known that the Einstein equations can be expressed as a Hamiltonian field theory, at least formally. Our aim is to justify these calculations by providing Hilbert space structures in which important quantities such as the constraint map and the total energy-momentum, become smooth functions. We work with a phase space $\mathcal{F}$ consisting of pairs (g, π) of $H^2 \times H^1$ local regularity with decay appropriate for asymptotically flat spacetimes. Our main results imply in particular:

- the constraint set $\mathcal{C}$ is a Hilbert submanifold of $\mathcal{F}$ (Theorem 3.12);
- the ADM energy-momentum is a C^∞ function on $\mathcal{C}$ (Theorem 4.1);

[1]I would like to thank Vince Moncrief, John Hutchinson, Leon Simon and Rick Schoen for valuable discussions. This research was supported in part by the Australian Research Council.

- a regularization $\mathcal{H}(g, \pi; \xi)$ of the Regge–Teitelboim (RT) Hamiltonian is C^∞ on $\mathcal{F}$ and generates the correct equations of motion (Theorem 5.2); and
- constrained critical points of the regularized Hamiltonian $\mathcal{H}$ on $\mathcal{C}$ correspond to Killing initial data (Theorem 6.1).

In Section 3, we show that the set of asymptotically flat solutions to the constraint equations $\Phi(g, \pi) = 0$ is a smooth Hilbert submanifold of the phase space $\mathcal{F}$. This is the property of *linearization stability* [18], so-called because it implies that any solution of the linearized Einstein equations corresponds to a curve of solutions of the non-linear equations, provided a suitable local existence result is available for the regularity class in question.

However, the best local existence and uniqueness results for the vacuum Einstein evolution at present require slightly more: $(g, \pi) \in H^s \times H^{s-1}$ with $s > 2$ [4, 32, 22]. Interestingly, it has been conjectured that this can be improved to $s = 2$, the case examined here, and possibly even to $s > 3/2$, but the calculations here rely heavily on $s = 2$. Maxwell [23] has shown that the conformal method can be used to construct $\mathrm{tr}_g K = 0$ constraint solutions with $s > 3/2$ data, which suggests at least some of the results here can be improved. Also, an alternative approach based on the Corvino–Schoen perturbation technique [15, 16] has been used [14] to obtain Banach manifold structures for the constraint set under a wide range of asymptotic condition, but with much more stringent regularity conditions.

The ADM total mass and energy momentum [2] are defined by limits at infinity of coordinate-dependent integrals. The consistency of these definitions, and their independence of the coordinate framing, is established in Section 4; this extends previous results [5, 13, 26]. Furthermore, the ADM energy-momentum is a smooth function on the constraint manifold $\mathcal{C}$; however, it is not finite in general on $\mathcal{F}$.

The Einstein evolution equations may be written in Hamiltonian form [3], with the lapse-shift ξ freely specifiable. In Section 5, we show that the ADM Hamiltonian is also smooth on $\mathcal{F}$, provided ξ decays suitably, and its derivative on $\mathcal{F}$ generates the evolution equations. To extend this result to ξ asymptotic to a time translation at infinity, we modify the RT Hamiltonian [28] to construct a regularized Hamiltonian which is smooth on all $\mathcal{F}$ and agrees with the ADM energy-momentum on $\mathcal{C} \subset \mathcal{F}$.

It is appealing to conjecture that, although the Hamiltonian flow vector field is only densely defined on $\mathcal{F}$, it might still be possible to construct integral curves directly, perhaps by a judicious choice of lapse-shift ξ to smooth the tangent vectors. This would amount to a direct proof of local existence for $s = 2$ and is unlikely to succeed, because it does not take into account the characteristic structure of the Einstein equations, which plays an important role in other approaches to the low-regularity local existence problem [22, 32].

The lapse-shift ξ in the regularized Hamiltonian may be regarded as a Lagrange multiplier for constrained variations, and in Section 6, we use this to establish rigorously an identity of Brill–Deser–Fadeev [10], which equates constrained critical points of the ADM energy with Killing initial data, i.e. $D\Phi(g,\pi)^*\xi = 0$.

Critical points of energy arise naturally from the mass-minimizing definition of *quasi-local mass* [6, 8], which motivates the conjecture that mass-minimizing extensions of a given region (Ω, g, π) are stationary. The static case has been established in [15] by a different method, but a direct variational proof, based on extending the results of Section 6 to data sets with boundary, would be more natural. This question will be addressed in future work.

2. Notation and Formulae.

In this section we introduce the basic framework and notations used in the paper and recall some well-known formulae concerning the constraint equations.

Let $\mathcal{M}$ be a connected, oriented and non-compact 3-dimensional manifold, and suppose there is a compact subset $\mathcal{M}_0 \subset \mathcal{M}$ such that there is a diffeomorphism $\phi : \mathcal{M}\backslash\mathcal{M}_0 \to E_1$, where $E_R \subset \mathbb{R}^3$ is an exterior region, $E_R = \{x \in \mathbb{R}^3 : |x| > R\}$. We also use the notation B_R for the open ball of radius R centred at $0 \in \mathbb{R}^3$, $A_R = B_{2R}\backslash\overline{B_R}$ for the annulus and $S_R = \partial B_R$ for the sphere of radius R. Although we assume $\partial M = \emptyset$ for simplicity, most of the earlier results are valid also when $\partial\mathcal{M}$ is non-empty and consists of a finite collection of disjoint compact 2-surfaces. Let $\mathring{g}$ be a fixed Riemannian metric on $\mathcal{M}$ satisfying $\mathring{g} = \phi^*(\delta)$ in $\mathcal{M}\backslash\mathcal{M}_0$, where δ is the natural flat metric on $\mathbb{R}^3$. In the terminology of [5], ϕ is a *structure of infinity* on $\mathcal{M}$. Let $r \in C^\infty(\mathcal{M})$ be some function satisfying $r(x) \geq 1\ \forall x \in \mathcal{M}$ and $r(x) = |x|\ \forall x \in \mathcal{M}\backslash\mathcal{M}_0$. Using r and $\mathring{g}$, we define the weighted Lebesgue and Sobolev spaces [5] L^p_δ, $W^{k,p}_\delta$, $1 \leq p \leq \infty$, $\delta \in \mathbb{R}$, as the completions of $C^\infty_c(\mathcal{M})$ under the norms

$$\|u\|_{p,\delta} = \left(\int_{\mathcal{M}} |u|^p r^{-\delta p - 3} dv_o\right)^{1/p},$$

$$\|u\|_{k,p,\delta} = \sum_{j=0}^{k} \|\mathring{\nabla}^j u\|_{p,\delta-j},$$

if $p < \infty$, and the appropriate supremum norm if $p = \infty$. Here $dv_o, \mathring{\nabla}$ are respectively the volume measure and connection determined by the metric $\mathring{g}$. The weighted Sobolev space of sections of a bundle E over $\mathcal{M}$ is defined similarly and denoted $W^{k,p}_{\delta}(E)$. We distinguish especially the spaces

$$\begin{array}{rclcrcl} \mathcal{G} & = & W^{2,2}_{-1/2}(\mathcal{S}), & \qquad & \mathcal{K} & = & W^{1,2}_{-3/2}(\tilde{\mathcal{S}}), \\ \mathcal{L} & = & L^{2}_{-1/2}(\mathcal{T}), & & \mathcal{L}^* & = & L^{2}_{-5/2}(\mathcal{T}^* \otimes \Lambda^3), \end{array}$$

where $\mathcal{S} = S^2T^*\mathcal{M}$ is the bundle of symmetric bilinear forms on $\mathcal{M}$, $\tilde{\mathcal{S}} = S^2T\mathcal{M} \otimes \Lambda^3T^*\mathcal{M}$ is the bundle of symmetric tensor-valued 3-forms (densities) on $\mathcal{M}$ and $\mathcal{T}$ is the bundle of spacetime tangent vectors. Thus for example, $\mathcal{L}$ is a class of spacetime tangent vector fields on $\mathcal{M}$, and $\mathcal{L}$ and $\mathcal{L}^*$ are dual spaces with respect to the natural integration pairing. The following Hilbert manifolds modelled on $\mathcal{G}$ are natural domains for asymptotically flat metrics:

$$\begin{aligned} &\mathcal{G}^+ = \{g : g - \mathring{g} \in \mathcal{G}, g > 0\}, \\ &\mathcal{G}^+_\lambda = \{g \in \mathcal{G}^+, \lambda\mathring{g} < g < \lambda^{-1}\mathring{g}\}, \quad 0 < \lambda < 1. \end{aligned}$$

We note that by virtue of the Sobolev inequality and the Morrey lemma [5], tensors in $\mathcal{G}$ are Hölder continuous (with Hölder exponent 1/2) and thus the matrix inequality conditions on g in the definitions of $\mathcal{G}^+, \mathcal{G}^+_\lambda$ are satisfied in the pointwise sense. The Hilbert manifold we shall consider as the phase space for the Einstein equations is then

$$\mathcal{F} = \mathcal{G}^+ \times \mathcal{K}. \tag{2.1}$$

Theorem 4.7 shows that $\mathcal{F}$ is independent of the choice of structure of infinity ϕ.

If we suppose that $\mathcal{M}$ is a spacelike submanifold of a 4-dimensional Lorentzian manifold (spacetime), then the second fundamental form or extrinsic curvature tensor K is the bilinear form defined by

$$K(u, v) = g^{(4)}(u, \nabla^{(4)}_v n), \tag{2.2}$$

where $g^{(4)}$, $\nabla^{(4)}$ are the spacetime metric and connection, u, v are tangent vectors to $\mathcal{M}$ and $n \in \mathcal{T}$ is the future unit normal to $\mathcal{M}$. It is often convenient to use the conjugate momentum π as a reparameterisation of K — we adopt the definition

$$\pi^{ij} = (K^{ij} - \mathrm{tr}_g K\, g^{ij})\sqrt{g}, \tag{2.3}$$

where $\sqrt{g} = \sqrt{\det g}/\sqrt{\det \mathring{g}}$ denotes the volume form of the induced metric g, so π is a section of the bundle $\tilde{\mathcal{S}} = S^2 T\mathcal{M} \otimes \Lambda^3 T^*\mathcal{M}$. Either (g, K) or (g, π) can be used as coordinates on $\mathcal{F}$, and we will move freely between these two parameterisations in the following formulæ.

For sufficiently smooth metric g and second fundamental form K (or π), the constraint functions $\Phi = (\Phi_0, \Phi_i) = \Phi(g, \pi)$ are defined by

$$\begin{aligned} \Phi_0(g,\pi) &= \left(R(g) - |K|^2 + (\mathrm{tr}_g K)^2\right)\sqrt{g}, \\ &= R(g)\sqrt{g} - (\,|\pi|^2 - \tfrac{1}{2}(\mathrm{tr}_g \pi)^2)/\sqrt{g} \end{aligned} \tag{2.4}$$

$$\begin{aligned} \Phi_i(g,\pi) &= 2\,\left(\nabla^j K_{ij} - \nabla_i(\mathrm{tr}_g K)\right)\sqrt{g} \\ &= 2\, g_{ij} \nabla_k \pi^{jk}, \end{aligned} \tag{2.5}$$

where $R(g)$, ∇, tr_g are respectively the Ricci scalar, covariant derivative and trace of the metric g, and $|K|^2 = g^{ik} g^{jl} K_{ij} K_{kl}$. Notice that Φ takes values in $\mathcal{T}^* \otimes \Lambda^3 T^*\mathcal{M}$, the bundle of density-valued spacetime cotangent vectors on $\mathcal{M}$. If the Einstein equations are satisfied, then the normalisation chosen ensures that Φ and the stress-energy tensor are related by $\Phi_\alpha = 16\pi\kappa T_{n\alpha}\sqrt{g}$, where $n = e_0$ is the future unit normal to $\mathcal{M}$, κ is Newton's gravitational constant, and $T_{n\alpha}\sqrt{g}$ is the local energy-momentum density 4-covector as seen by an observer with world vector n. Consequently our sign conventions vary slightly from those used in [24, 18].

The functional derivative $D\Phi$ is given formally by

(2.6)

$$\begin{aligned} D\Phi_0(g,\pi)(h,p) &= (\delta_g\delta_g h - \Delta_g \mathrm{tr}_g h)\sqrt{g} - h_{ij}\left(Ric^{ij} - \tfrac{1}{2}R(g) g^{ij}\right)\sqrt{g} \\ &\quad + h_{ij}\left(\mathrm{tr}_g\pi\,\pi^{ij} - 2\pi^i_k \pi^{kj} + \tfrac{1}{2}\,|\pi|^2 g^{ij} - \tfrac{1}{4}(\mathrm{tr}_g\pi)^2 g^{ij}\right)/\sqrt{g} \\ &\quad + p^{ij}\left(\mathrm{tr}_g\pi g_{ij} - 2\pi_{ij}\right)/\sqrt{g}, \end{aligned}$$

(2.7)

$$D\Phi_i(g,\pi)(h,p) = \pi^{jk}\left(2\nabla_j h_{ik} - \nabla_i h_{jk}\right) + 2h_{ij}\nabla_k \pi^{jk} + 2g_{ik}\nabla_j p^{jk},$$

where $\delta_g\delta_g h = \nabla^i\nabla^j h_{ij}$. Multiplying by (N, X^i) and integrating by parts and ignoring boundary terms gives formulæ for the formal $L^2(dv_o)$-adjoint operator $D\Phi^*$,

(2.8)
$$(h,p)\cdot D\Phi_0(g,\pi)^*N =$$
$$h_{ij}\left(\nabla^i\nabla^j N - \Delta_g N g^{ij}\right)\sqrt{g} - N h_{ij}\left(Ric^{ij} - \tfrac{1}{2}R(g)g^{ij}\right)\sqrt{g}$$
$$+ N h_{ij}\left(\mathrm{tr}_g\pi\,\pi^{ij} - 2\pi^i_k\pi^{kj} + \tfrac{1}{2}|\pi|^2 g^{ij} - \tfrac{1}{4}(\mathrm{tr}_g\pi)^2 g^{ij}\right)/\sqrt{g}$$
$$+ N p^{ij}\left(\mathrm{tr}_g\pi g_{ij} - 2\pi_{ij}\right)/\sqrt{g},$$

(2.9)
$$(h,p)\cdot D\Phi_i(g,\pi)^*X^i \quad =$$
$$h_{ij}\left(X^k\nabla_k\pi^{ij} + \nabla_k X^k\pi^{ij} - 2\nabla_k X^{(i}\pi^{j)k}\right) - 2\,p^{ij}\nabla_{(i}X_{j)}.$$

These calculations are carefully described in [18]. Adopting some natural shorthand notations, the adjoint operator can be rewritten

(2.10)
$$(h,p)\cdot D\Phi(g,\pi)^*(N,X) \quad =$$
$$h\bullet\Big\{\left(\nabla^2 N - \Delta_g N g - N\left(Ric - \tfrac{1}{2}R(g)g\right)\right)^{\#}\sqrt{g}$$
$$- N\left(K\pi + \pi K - \tfrac{1}{2}\pi\bullet K\, g\right)^{\#} + \mathcal{L}_X\pi\Big\} - p\bullet(2KN + \mathcal{L}_X g)$$

where $^{\#}$ signifies the indexed-raised tensor, $\mathcal{L}_X$ is the Lie derivative in the direction X, $(K\pi)^{ij} = K^i_k\pi^{kj}$ and $\bullet$ is the natural contraction between 2-tensors, eg. $\pi\bullet K = \pi^{ij}K_{ij}$. Defining

$$S^{ij} = g^{-1}(\mathrm{tr}_g\pi\pi^{ij} - 2\pi^i_k\pi^{jk} + \tfrac{1}{2}|\pi|^2 g^{ij} - \tfrac{1}{4}(\mathrm{tr}_g\pi)^2)g^{ij}), \tag{2.11}$$

and $E^{ij} = Ric^{ij} - \frac{1}{2}R(g)g^{ij}$, we may express $D\Phi$ in matrix form as

$$D\Phi(g,\pi)(h,p) = \begin{bmatrix} \sqrt{g}(\delta_g\delta_g - \Delta_g\mathrm{tr}_g + S - E) & -2K \\ \hat{\pi}\nabla + 2\delta_g\pi & 2\delta_g \end{bmatrix}\begin{bmatrix} h \\ p \end{bmatrix}, \tag{2.12}$$

where
$$\hat{\pi}\nabla h = \hat{\pi}_i^{jkl}\nabla_j h_{kl} = (\pi^{jk}\delta^l_i + \pi^{jl}\delta^k_i - \pi^{kl}\delta^j_i)\nabla_j h_{kl}.$$

Similarly the adjoint may be written as
(2.13)
$$D\Phi(g,\pi)^*(N,X) = \begin{bmatrix} \sqrt{g}(\nabla^2 - g\Delta_g + S - E) & \nabla\pi - \hat{\pi}\nabla \\ -2K & -\epsilon_g \end{bmatrix}\begin{bmatrix} N \\ X \end{bmatrix},$$

where
$$(\nabla\pi - \hat{\pi}\nabla)X = \mathcal{L}_X\pi = \nabla_X\pi^{ij} - \hat{\pi}_l^{kij}\nabla_k X^l$$

and $\epsilon_g(X) = \mathcal{L}_X g = 2\nabla_{(i}X_{j)}$ is the strain operator. Let $D\Phi(g,\pi)^*_a(N,X)$, $a = 1,2$ denote the two components of $D\Phi^*$ in (2.13).

We will also use the notation $\xi = (\xi^\alpha) = (N, X^i)$, where ξ has a natural interpretation as the lapse-shift of the spatial slicing of the evolved spacetime. If g and (N, X^i) depend on an evolution parameter t and $N > 0$, then the Lorentzian metric

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + X^i dt)(dx^j + X^j dt) \tag{2.14}$$

describes a spacetime satisfying some form of the Einstein equations, and $\xi = Nn + X^i\partial_i$ coincides with the time evolution vector ∂_t.

Greek letters $\alpha, \beta, \ldots$ will be used for spacetime indices, with range $0, \ldots, 3$, and Latin letters $i, j, \ldots,$ will indicate spatial indices (on $\mathcal{M}$), with range $1, 2, 3$. Index-free and indexed expressions will be intermixed as convenient. The letters c, C will be used to indicate constants which may vary from line to line, with c generally denoting a constant depending only on the background metric $\mathring{g}$ and the ellipticity λ, and C denoting a constant whose dependence on significant parameters will be explicitly indicated.

3. The constraint manifold.

In this section we show that the constraint map

$$\Phi : \mathcal{F} \to \mathcal{L}^* \tag{3.1}$$

is a smooth map between Hilbert manifolds, and that the level sets $\mathcal{C}(\epsilon, S) = \Phi^{-1}(\epsilon, S)$ are Hilbert submanifolds. In particular, the space $\mathcal{C} = \Phi^{-1}(0)$ of asymptotically flat vacuum initial data is a Hilbert manifold. The proof is based on the implicit function theorem method used in previous studies [18, 24, 1] of the constraint set over a compact manifold. In fact, the main result of this section may be considered as the logical extension of those results to the case of asymptotically flat manifolds. We note in particular that the quadratic Taub constraints on the linearised solutions which arise in the case where the underlying spacetime admits a symmetry, do not occur in the asymptotically flat case — as was observed by Moncrief [25] — and consequently, the cone-like singularities which occur in the space of solutions of the constraints over a compact manifold (at data sets generating vacuum spacetimes admitting a Killing vector), are absent in the asymptotically flat constraint manifold. The space of asymptotically flat (vacuum) constraint data is a smooth Hilbert manifold, at all points.

However, the result shown here, that the space of solutions of the constraint equations forms a Hilbert manifold, does not prove that the Einstein equations with asymptotically flat data are linearization stable, in the sense of [18, 24], because the regularity condition $(g, \pi) \in \mathcal{F}$ is too weak to be able to apply known local existence and uniqueness theorems for the Einstein equations. It is interesting, therefore, that it has been conjectured that the minimal regularity conditions for the well-posedness of the Einstein equations exactly correspond to $(g, \pi) \in \mathcal{F}$. If this conjecture is correct, then linearization stability will hold under the conditions considered here as well.

Alternatively, linearization stability may be obtained by requiring higher differentiability in the spaces $\mathcal{G}^+, \mathcal{K}$ and $\mathcal{L}^*$, and then observing that the results about the boundedness and smoothness of Φ and the triviality of the kernel of $D\Phi^*$ remain valid — the result is a phase space of initial data with sufficient regularity for known existence and uniqueness theorems to apply. The details of this extension are left to the interested reader.

Proposition 3.1. *Suppose $g \in \mathcal{G}^+_\lambda$ for some $\lambda > 0$ and $\pi \in \mathcal{K}$. Then there is a constant $c = c(\lambda)$ such that*

$$\|\Phi_0(g,\pi)\|_{2,-5/2} \leq c\left(1 + \|g - \mathring{g}\|^2_{2,2,-1/2} + \|\pi\|^2_{1,2,-3/2}\right), \tag{3.2}$$

$$\|\Phi_i(g,\pi)\|_{2,-5/2} \leq c\left(\|\mathring{\nabla}\pi\|_{2,-5/2} + \|\mathring{\nabla} g\|_{1,2,-3/2}\,\|\pi\|_{1,2,-3/2}\right). \tag{3.3}$$

Proof. Since $g \in \mathcal{G}^+_\lambda$, g is Hölder-continuous with Hölder exponent $1/2$ and we have the global pointwise bounds

$$\lambda \mathring{g}_{ij}(x) v^i v^j < g_{ij}(x) v^i v^j < \lambda^{-1} \mathring{g}_{ij}(x) v^i v^j \quad \forall\, x \in \mathcal{M},\ v \in \mathbb{R}^3. \tag{3.4}$$

For later use, we note the following consequence of the weighted Hölder and Sobolev inequalities [5], valid for any function or tensor field u,

$$\begin{aligned} \|u^2\|_{2,-5/2} = \|u\|^2_{4,-5/4} &\leq c\,\|u\|^2_{4,-3/2} \\ &\leq c\,\|u\|^{3/2}_{6,-3/2}\,\|u\|^{1/2}_{2,-3/2} \\ &\leq c\,\|u\|^2_{1,2,-3/2}. \end{aligned} \tag{3.5}$$

The $g, \mathring{g}$ connections are related by the difference tensor $A^k_{ij} = \Gamma^k_{ij} - \mathring{\Gamma}^k_{ij}$, which may be defined invariantly by

$$A^k_{ij} = \tfrac{1}{2} g^{kl}\left(\mathring{\nabla}_i g_{jl} + \mathring{\nabla}_j g_{il} - \mathring{\nabla}_l g_{ij}\right). \tag{3.6}$$

The scalar curvature can be expressed in terms of $\mathring{\nabla}$ and A^k_{ij} by

$$
\begin{aligned}
R(g) &= g^{jk} Ric(\mathring{g})_{jk} + g^{jk}\big(\mathring{\nabla}_i A^i_{jk} - \mathring{\nabla}_j A^i_{ik} + A^l_{jk}A^i_{il} - A^i_{jl}A^l_{ki}\big) \\
&= g^{ik}g^{jl}\big(\mathring{\nabla}^2_{ij} g_{kl} - \mathring{\nabla}^2_{ik} g_{jl}\big) + Q(g^{-1}, \mathring{\nabla} g) + g^{jk} Ric(\mathring{g})_{jk},
\end{aligned} \tag{3.7}
$$

where $Q(g^{-1}, \mathring{\nabla} g)$ denotes a sum of terms quadratic in $g^{-1}, \mathring{\nabla} g$. Using (3.4), (3.5), (3.7), we may estimate

$$
\begin{aligned}
\|R(g)\|^2_{2,-5/2} &\le c \int_{\mathcal{M}} \big(|\mathring{\nabla}^2 g|^2 + |\mathring{\nabla} g|^4 + |Ric(\mathring{g})|^2\big) r^2 \, dv_o \\
&\le c\big(1 + \|\mathring{\nabla}^2 g\|^2_{2,-5/2} + \|\mathring{\nabla} g\|^4_{4,-5/4}\big) \\
&\le c\big(1 + \|\mathring{\nabla} g\|^4_{1,2,-3/2}\big),
\end{aligned}
$$

and since

$$
\| \, |\pi|^2 \|_{2,-5/2} \le c \, \|\pi\|^2_{1,2,-3/2},
$$

the estimate (3.2) follows and $\Phi_0(g,\pi) \in L^2_{-5/2}$.

The proof of the corresponding estimates for the momentum constraint is similar, but somewhat simpler. Since

$$
\nabla_j \pi^{ij} = \mathring{\nabla}_j \pi^{ij} + A^i_{jk}\pi^{jk}, \tag{3.8}
$$

we have

$$
\Phi_i(g,\pi) = 2 g_{ij} \left(\mathring{\nabla}_k \pi^{jk} + A^j_{kl}\pi^{kl} \right), \tag{3.9}
$$

and Hölder's inequality, (3.4) and (3.5) give

$$
\|\Phi_i(g,\pi)\|^2_{2,-5/2} \le c \left(\|\mathring{\nabla}\pi\|^2_{2,-5/2} + \|\mathring{\nabla} g\|^2_{1,2,-3/2} \, \|\pi\|^2_{1,2,-3/2} \right).
$$

□

Thus Φ is a quadratically bounded map between the Hilbert manifolds $\mathcal{F} = \mathcal{G}^+ \times \mathcal{K}$ and $\mathcal{L}^* = L^2_{-5/2}(\mathcal{T})$; together with the polynomial structure of the constraint functionals, this enables us to show that Φ is smooth, in the sense of infinitely many Frechét derivatives.

Corollary 3.2. *$\Phi : \mathcal{F} \to \mathcal{L}^*$ is a smooth map of Hilbert manifolds.*

Proof. Proposition 3.1 shows that $\|\Phi(g,\pi)\|_{\mathcal{L}^*} \le c(1 + \|g - \mathring{g}\|^2_{\mathcal{G}} + \|\pi\|^2_{\mathcal{K}})$, so Φ is locally bounded on $\mathcal{F}$. To show Φ is smooth, we note from the representations (3.6), (3.7), (3.9) that Φ can be expressed as the composition

$$\Phi(g,\pi) = F(g, g^{-1}, \sqrt{g}, 1/\sqrt{g}, \mathring{\nabla} g, \mathring{\nabla}^2 g, \pi, \mathring{\nabla}\pi),$$

where $F = F(a_1, \dots, a_8)$ is a polynomial function which is quadratic in the parameters a_5 and a_7 and linear in the remaining parameters. The map $g \mapsto (g, g^{-1}, \sqrt{g}, 1/\sqrt{g})$ is analytic on the space of positive definite matrices, and the maps $g \mapsto \mathring{\nabla} g$, $g \mapsto \mathring{\nabla}^2 g$ and $\pi \mapsto \mathring{\nabla}\pi$ are bounded linear, hence smooth, from the Hilbert manifolds $\mathcal{G}^+$ and $\mathcal{K}$ to $\mathcal{L}^*$. Results of Zorn and Hille [21, Section 3, Section 26] on locally bounded polynomial functionals show Φ has continuous Frechét derivatives of all orders. $\square$

The constraint set $\mathcal{C} = \Phi^{-1}(0) \subset \mathcal{F}$ is of particular interest, since it gives the class of initial data for the vacuum Einstein equations. To show that $\mathcal{C}$ is a Hilbert manifold using the implicit function method, we study the kernel of the adjoint operator $D\Phi(g,\pi)^*$.

The first step establishes coercivity of $D\Phi(g,\pi)^*$.

Proposition 3.3. *$D\Phi^*$ satisfies the ellipticity estimate, for all $\xi \in W^{2,2}_{-1/2}$,*

(3.10)
$$\|\xi\|_{2,2,-1/2} \le c(\|D\Phi(g,\pi)^*_1(\xi)\|_{2,-5/2} + \|D\Phi(g,\pi)^*_2(\xi)\|_{1,2,-3/2}) + C\,\|\xi\|_{1,2,0}$$

where C depends on $\mathring{g}$, λ and $\|(g,\pi)\|_{\mathcal{F}}$.

Proof. Rearranging the first component of (2.8) gives

$$\begin{aligned} \nabla^2 N &= Q - \tfrac{1}{2}\mathrm{tr}_g Q\, g, \\ Q &= D\Phi(g,\pi)^*_1(\xi)/\sqrt{g} + (E - S)N - \mathcal{L}_X\pi/\sqrt{g}, \end{aligned} \tag{3.11}$$

and thus $|\nabla^2 N|^2 \le \frac{5}{4}|Q|^2$. This leads to the estimate

(3.12)
$$\begin{aligned} \|\mathring{\nabla}^2 N\|_{2,-5/2} \le c\Big(&\|D\Phi_0(g,\pi)^*_1(\xi)\|_{2,-5/2} \\ &+ \|N\|_{\infty,0}(\|E\|_{2,-5/2} + \|S\|_{2,-5/2}) + \|A\mathring{\nabla} N\|_{2,-5/2} \\ &+ \|X\|_{\infty,0}\|\mathring{\nabla}\pi\|_{2,-5/2} + \|\mathring{\nabla} X\|_{3,-1}\|\pi\|_{6,-3/2}\Big). \end{aligned}$$

Using a combination of the weighted Sobolev and Hölder inequalities, we can establish estimates which control the various right-hand terms in (3.12). For example,

$$\begin{aligned} \|u\|_{\infty,0} &\le c\,\|u\|_{1,4,0} \\ &\le c\,\|u\|_{1,2,0}^{\lambda}\,\|u\|_{1,6,0}^{1-\lambda}, \qquad \lambda = \tfrac{1}{4}, \\ &\le c\,\|u\|_{1,2,0}^{\lambda}\,\|u\|_{2,2,0}^{1-\lambda} \\ &\le \epsilon\,\|\mathring{\nabla}^2 u\|_{2,-2} + c\epsilon^{-3}\,\|u\|_{1,2,0}, \end{aligned} \tag{3.13}$$

for any $\epsilon > 0$. Similarly we find, for any $\delta \in \mathbb{R}$,

$$\|u\|_{3,\delta} \le \epsilon\,\|\mathring{\nabla} u\|_{2,\delta-1} + c\epsilon^{-1}\,\|u\|_{2,\delta}.$$

Consequently there is a constant C, depending only on λ, $\mathring{g}$, ϵ and $\|(g,\pi)\|_{\mathcal{F}}$, such that

$$\|\mathring{\nabla}^2 N\|_{2,-5/2} \le c\,\|D\Phi_0(g,\pi)_1^*(\xi)\|_{2,-5/2} + \epsilon\,\|\mathring{\nabla}^2\xi\|_{2,-2} + C\,\|\xi\|_{1,2,0}. \tag{3.14}$$

From the identity (using the metric $g \in \mathcal{G}^+$)

$$X_{i|jk} = -R_{ijkl}X^l + X_{(i|j)k} + X_{(i|k)j} - X_{(j|k)i}, \tag{3.15}$$

which is valid for any sufficiently smooth X_i, we may write

$$X_{i|jk} = -R_{ijkl}X^l - \tfrac{1}{2}\left(H_{ij|k} + H_{ik|j} - H_{jk|i}\right) - (NK_{ij})_{|k} - (NK_{ik})_{|j} + (NK_{jk})_{|i}, \tag{3.16}$$

where

$$H_{ij} = H_{ij}(X) = -2(NK_{ij} + X_{(i|j)}) = D\Phi(g,\pi)_2^*(\xi)$$

and (N, X^i) are assumed sufficiently smooth. The various terms of (3.16) can be controlled using the Sobolev, Hölder and interpolation inequalities in a similar fashion, leading to the estimate

$$\|\mathring{\nabla}^2 X\|_2 \le c\,\|D\Phi_0(g,\pi)_2^*(\xi)\|_{1,2,-3/2} + \epsilon\,\|\mathring{\nabla}^2\xi\|_{2,-2} + C\,\|\xi\|_{1,2,0}. \tag{3.17}$$

Since $\|u\|_{k,p,\delta_1} \le \|u\|_{k,p,\delta_2}$ if $\delta_1 \ge \delta_2$, ϵ may be chosen such that (3.14), (3.17) combine to give

$$\|\mathring{\nabla}^2\xi\|_{2,-5/2} \le c(\,\|D\Phi_0(g,\pi)_1^*(\xi)\|_{2,-5/2} + \|D\Phi_0(g,\pi)_2^*(\xi)\|_{1,2,-3/2}) + C\,\|\xi\|_{1,2,0}, \tag{3.18}$$

for smooth ξ. Since C_c^∞ is dense in $W^{2,2}_{-1/2}$, it follows that (3.18) holds for all $\xi \in W^{2,2}_{-1/2}$. Now (3.10) follows from the weighted Poincaré inequality [5, Theorem 1.3]

$$\|u\|_{p,\delta} \le c\,\|\mathring{\nabla} u\|_{p,\delta-1} \le c\,\|\mathring{\nabla}^2 u\|_{p,\delta-2},$$

for any $\delta < 0$ and $u \in W^{2,p}_\delta$. □

It will be useful to restructure $D\Phi^*$ into the operator P^* defined by

$$P^*(\xi) = \begin{bmatrix} g^{1/4}\left(\nabla^i\nabla_j N - \delta^i_j \Delta_g N + (S^i_j - E^i_j)N\right) + g^{-1/4}\mathcal{L}_X \pi^i_j \\ -\, g^{1/4}\nabla_p(2K^i_j N + \mathcal{L}_X g^i_j) \end{bmatrix} \tag{3.19}$$

$$= \rho \circ \begin{bmatrix} 1 & 0 \\ 0 & \nabla \end{bmatrix} \circ D\Phi(g,\pi)^*\xi,$$

where $g^{1/4} = (\det g / \det \mathring{g})^{1/4}$ is a density of weight $\frac{1}{2}$, and

$$\rho = \rho(g) = \begin{bmatrix} g^{-1/4} g_{jk} & 0 \\ 0 & g^{1/4} g^{ik} \end{bmatrix}.$$

The $L^2(dv_0)$-adjoint of P^* is then

$$P = D\Phi(g,\pi) \circ \begin{bmatrix} 1 & 0 \\ 0 & -\delta_g \end{bmatrix} \circ \rho, \tag{3.20}$$

where $\delta_g q = \nabla^p(q_p^{ij})$, so $P(f_i^j, q_{pi}^{\ j}) = D\Phi(f_{ij}, -\nabla^p(q_p{}^{ij}))$, and the composition PP^* is well-defined.

Proposition 3.4. *$P^* : W^{2,2}_{-1/2}(\mathcal{T}) \to L^2_{-5/2}$ is bounded and satisfies*

$$\|\xi\|_{2,2,-1/2} \le c\, \|P^*\xi\|_{2,-5/2} + C\, \|\xi\|_{1,2,0}, \tag{3.21}$$

where C depends on $\|(g,\pi)\|_{\mathcal{F}}$, and $P^* = P^*_{(g,\pi)}$ has Lipschitz dependence on $(g,\pi) \in \mathcal{F}$,

$$\|(P^*_{(g,\pi)} - P^*_{(\tilde{g},\tilde{\pi})})\xi\|_{2,-5/2} \le C_1\, \|(g-\tilde{g}, \pi-\tilde{\pi})\|_{\mathcal{F}}\, \|\xi\|_{2,2,-1/2}, \tag{3.22}$$

where C_1 depends on $\|(g,\pi)\|_{\mathcal{F}}$, $\|(\tilde{g},\tilde{\pi})\|_{\mathcal{F}}$.

Proof. That P^* is bounded,

$$\|P^*_{(g,\pi)}\xi\|_{2,-3/2} \le C\, \|\xi\|_{2,2,-1/2}, \tag{3.23}$$

follows from estimates similar to but simpler than those of Proposition 3.3. The elliptic estimate (3.21) follows directly from (3.10). $(P^*_{(g,\pi)} - P^*_{(\tilde{g},\tilde{\pi})})\xi$ is controlled by breaking it up. Since $\|g-\tilde{g}\|_\infty$, $\|(N,X)\|_\infty$ are bounded by $\|g-\tilde{g}\|_{2,2,-1/2}$, $\|\xi\|_{2,2,-1/2}$ respectively, terms such as

$$\begin{bmatrix} g^{-1/4}-\tilde{g}^{-1/4} & 0 \\ 0 & g^{1/4}-\tilde{g}^{1/4} \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 0 & -\delta_1 \end{bmatrix} \circ D\Phi(g,\pi)^*\xi$$

are controlled by $C\,\|g-\tilde{g}\|_{2,2,-1/2}\,\|\xi\|_{2,2,-1/2}$. Since $\nabla-\tilde{\nabla}\sim\mathring{\nabla}(g-\tilde{g})$, we may use (3.5) to estimate, for example,

$$\|(\nabla-\tilde{\nabla})D\Phi_2^*\xi\|_{2,-5/2}\le c\,\|\mathring{\nabla}(g-\tilde{g})\|_{1,2,-3/2}\,\|D\Phi_2^*\xi\|_{1,2,-3/2}.$$

Using $D\Phi_2^*\xi=-2(NK_{ij}+\nabla_{(i}X_{j)})$ shows

$$\begin{aligned}&\|D\Phi(g,\pi)_2^*\xi-D\Phi(\tilde{g},\tilde{\pi})_2^*\xi\|_{1,2,-3/2}\\ &\quad\le c\,\|N(K-\tilde{K})\|_{1,2,-3/2}+c\,\|\mathring{\nabla}(g-\tilde{g})X\|_{1,2,-3/2},\end{aligned}$$

which is controlled by

$$\|N\|_\infty\,\|K-\tilde{K}\|_{1,2,-3/2}+\,\|\mathring{\nabla}N(K-\tilde{K})\|_{2,-5/2}$$

for the first, and similarly for the second term. Again using the L^∞ bound and (3.5) controls the difference by $C\,\|\xi\|_{2,2,-5/2}$ as required; the terms in $D\Phi(g,\pi)_1^*\xi-D\Phi(\tilde{g},\tilde{\pi})_1^*\xi$ are controlled by very similar estimates, giving (3.22). □

We now show that the elliptic estimate is also satisfied by weak solutions, which are *a priori* only in L^2. We say that $\xi\in\mathcal{L}$ is a *weak solution* of $D\Phi(g,\pi)^*(\xi)=(f_1,f_2)$ for $(f_1,f_2)\in L^2_{-3/2}(\tilde{\mathcal{S}})\times W^{1,2}_{-3/2}(\mathcal{S})$ if

$$\int_{\mathcal{M}}\xi\cdot D\Phi(g,\pi)(h,p)=\int_{\mathcal{M}}(f_1,f_2)\cdot(h,p),\quad\forall\ (h,p)\in\mathcal{G}\times\mathcal{K}. \tag{3.24}$$

In this definition, it suffices to test with just $(h,p)\in C_c^\infty(\mathcal{S}\times\tilde{\mathcal{S}})$, since this space is dense in $\mathcal{G}\times\mathcal{K}$.

Proposition 3.5. *Suppose $(g,\pi)\in\mathcal{F}$, $(f_1,f_2)\in L^2_{-3/2}(\tilde{\mathcal{S}})\times W^{1,2}_{-3/2}(\mathcal{S})$, and $\xi=(N,X^i)\in\mathcal{L}=L^2_{-1/2}(\mathcal{T})$ is a weak solution of $D\Phi(g,\pi)^*(\xi)=(f_1,f_2)$. Then $\xi\in W^{2,2}_{-1/2}(\mathcal{T})$ is a strong solution and ξ satisfies (*3.10*).*

Proof. We first show $\xi\in W^{2,2}_{\mathrm{loc}}$, so restrict to a coordinate neighbourhood Ω. In local coordinates, $P^*(\xi)=f$ is equivalent to relations of the form

$$A\cdot\partial^2\xi+B\cdot\partial\xi+C\xi=f,$$

where $A : \mathbb{R}^{36} \to \mathbb{R}^{36}$ is invertible and determined solely by g; see (3.11), (3.16). Furthermore, $A \in W^{2,2}$, $B \in W^{1,2}$, $C \in L^2$ in Ω, so this is equivalent to

$$\partial^2_{ij}\xi^\alpha + \partial_k(b^{k\alpha}_{ij\beta}\xi^\beta) + c^\alpha_{ij\beta}\xi^\beta = f^\alpha_{ij} \tag{3.25}$$

for suitable $b \in W^{1,2}$, $c, f \in L^2$. Thus $\xi \in L^2$ satisfies the weak form of (3.25),

$$\int_\Omega (\partial^2_{ij}\phi^{ij}_\alpha + b^{k\beta}_{ij\alpha}\partial_k\phi^{ij}_\beta + c^\beta_{ij\alpha}\phi^{ij}_\beta)\xi^\alpha \, dx = \int_\Omega \phi^{ij}_\alpha f^\alpha_{ij} \, dx,$$

for all $\phi \in W^{2,2}_c(\Omega)$. Replacing ϕ by $J_\epsilon\phi$ where J_ϵ is a Friedrichs mollifier with mollification parameter $\epsilon > 0$, we see that $\xi_\epsilon = J_\epsilon\xi$ is smooth and satisfies

$$\partial^2\xi_\epsilon + \partial J_\epsilon(b\xi) + J_\epsilon(c\xi) = J_\epsilon f.$$

Following a suggestion of L. Simon, we let $u = \chi\xi_\epsilon$ where $\chi \in C^\infty_c(\Omega)$ is any cutoff function. Then taking a trace shows that $u \in C^\infty_c(\Omega)$ satisfies an equation of the form

$$\Delta_0 u = F + \partial G,$$

where $F = F_1 + F_2 + F_3$, $G = G_1 + G_2$ and $F_1 = \chi''\xi_\epsilon + \chi J_\epsilon f$, $F_2 = \chi' J_\epsilon(b\xi)$, $F_3 = \chi J_\epsilon(c\xi)$, $G_1 = \chi'\xi_\epsilon$, $G_2 = \chi J_\epsilon(b\xi)$. The terms F, G are smooth with compact support, so u has a representation

$$u(x) = \Gamma * (F + \partial G) = \int_\Omega \Gamma(x - y)(F(y) + \partial G(y)) \, dy,$$

where $\Gamma(x - y) = (4\pi|x - y|)^{-1}$ is the fundamental solution of Laplace's equation. Let $D = (-\Delta_0)^{1/2}$ be the Riesz potential [31, Ch. V]. The operators $K_{ij} = \partial^2_{ij}\Gamma$ and $K_i = \partial_i\Gamma D$ are Calderon–Zygmund kernels in the sense of [31, Ch. II], and hence satisfy

$$\|K_{ij} * w\|_{L^p(\Omega)} + \|K_i * w\|_{L^p(\Omega)} \le c\,\|w\|_{L^p(\Omega)}. \tag{3.26}$$

We now use these bounds to control the various terms in $\Gamma * (F + \partial G)$ and thereby bootstrap the estimates for u up to a $W^{2,2}$ bound which is independent of ϵ.

Since $K_{ij} * F_1 = \partial^2_{ij}(\Gamma * F_1)$, (3.26) with $p = 2$ shows that

$$\|\Gamma * F_1\|_{2,2} \le c\,\|F_1\|_2 \le c(\,\|\xi\|_2 + \|f\|_2),$$

where the norms here are over Ω. In particular, $\Gamma * F_1$ is uniformly bounded in $W^{2,2}(\Omega)$, independent of ϵ. Since $b\xi \in W^{1,2} \cdot L^2 \subset L^6 \cdot L^2 \subset L^{3/2}$, F_2 is uniformly bounded in $L^{3/2}$ and thus

$$\|\Gamma * F_2\|_{2,3/2} \leq c\,\|F_2\|_{3/2} \leq c\,\|b\|_{1,2}\,\|\xi\|_2.$$

Now $F_3 \in L^1(\Omega)$ only, so we instead note that $\partial_i u = K_i * (Du)$ where D satisfies

$$\|Dw\|_p \leq c\,\|w\|_q,$$

for either $1 < q < n$ with $1/p = 1/q - 1$, or if $q = 1$, with any $1 < p < n/(n-1) = 3/2$. With $q = 1$ and $p < 3/2$, we thus have

$$\|\partial\Gamma * F_3\|_p \leq c\,\|D * F_3\|_p \leq c\,\|F_3\|_1,$$

and the Sobolev inequality now shows that $\|\Gamma * F_3\|_{3-\delta}$ is uniformly bounded in terms of $\|c\|_2\,\|\xi\|_2$, for any small $\delta > 0$. Now, $\|G_1\|_2 \leq c\,\|\xi\|_2$, so we use the identity $\Gamma * (\partial_k G_1^k) = \partial_k\Gamma * G_1^k$ and the Sobolev inequality to estimate

$$\|\Gamma * (\partial G_1)\|_6 \leq c\,\|K_k * G_1^k\|_2 \leq c\,\|G_1\|_2 \leq c\,\|\xi\|_2.$$

Likewise, since $G_2 \in L^{3/2}$ uniformly, we find by a similar argument that $\|\Gamma * (\partial g_2)\|_3 \leq c\,\|b\|_{1,2}\,\|\xi\|_2$.

Assembling all the pieces now shows that $\xi \in L^{3-\delta}_{\mathrm{loc}}$, for any $\delta > 0$, and we now repeat the above arguments with this stronger bound on ξ. Bootstrapping in this way shows eventually that $\xi \in W^{2,2}_{\mathrm{loc}}$. Thus $\chi_R\xi \in W^{2,2}_{-1/2}$ for any cutoff function $\chi_R \in C^\infty_c(\mathcal{M})$, $\chi_R(x) = \chi(x/R)$ with $\chi_R(x) = 1$ on B_R. Now (3.10) shows that $\chi_R\xi$ is uniformly bounded in $W^{2,2}_{-1/2}$ since $\xi \in L^2_{-1/2}$ and $\chi_R\xi \to \xi$, so $\xi \in W^{2,2}_{-1/2}$ as required. $\square$

We next show that the kernel of $D\Phi^*$ is trivial in the space of lapse-shift pairs decaying at infinity. We may interpret this result as saying there are no generalised Killing vectors decaying to zero at infinity, where by *generalised Killing vector* ξ of $(g,\pi) \in \mathcal{F}$, we mean that $\xi \in W^{2,2}_{\mathrm{loc}}(\mathcal{T})$ satisfies $D\Phi(g,\pi)^*\xi = 0$. Likewise, if there exists a non-trivial vector field ξ satisfying $D\Phi(g,\pi)^*\xi = 0$ then (g,π) is a *Killing initial data* set, where the terminology is motivated by a result of Moncrief [25] which shows that if (g,π) satisfies the constraint equations, then a generalised Killing vector determines a standard Killing vector field in the spacetime generated from the initial data (g,π) by solving the vacuum Einstein equations. Of course, this requires that (g,π) has enough regularity that a local existence and uniqueness theorem for the Einstein evolution can be applied, which is not the case at present for general $(g, \pi) \in \mathcal{F}$. However, if local existence and uniqueness could be established for $s = 2$, then it would be possible to identify a generalised Killing vector (ie. $\xi \in \ker D\Phi^*$) with the spatial restriction of a true vacuum spacetime Killing vector. A similar technique was used [23] to show non-existence of spatial conformal Killing vectors.

Theorem 3.6. *Suppose $\Omega \subset \mathcal{M}$ is a connected domain and $E_R \subset \Omega$ for some exterior domain E_R, fix $(g, \pi) \in \mathcal{F}$ and suppose $\xi \in L^2_{-1/2}(\mathcal{T})$ satisfies $D\Phi(g, \pi)^*\xi = 0$ in Ω. Then $\xi \equiv 0$ in Ω.*

Proof. By Proposition 3.5, $\xi \in W^{2,2}_{-1/2}(\mathcal{T})$ and (3.16), the equation $D\Phi(g, \pi)^*\xi = 0$ shows that ξ satisfies an equation of the form

$$\mathring{\nabla}^2\xi = b_1\nabla\xi + b_0\xi, \tag{3.27}$$

with coefficients $b_0 \in L^2_{-5/2}$, $b_1 \in W^{1,2}_{-3/2}$. We must now show that a solution of (3.27) which decays as $\xi = o(r^{-1/2})$, must vanish. The structure of the argument to follow is well-known: the difficulty here lies in the absence of the continuity assumptions used essentially in [12].

If $u \in W^{1,2}_0(\mathbb{R}^n)$, then the Sobolev inequality is true in the sharp form $\|u\|_{n/(n-1)} \le c\|Du\|_1$. Such an inequality remains valid without the hypothesis of compact support, provided u vanishes on a sufficiently large set.

Lemma 3.7. *Suppose $n \ge 3$, $B_R \subset \mathbb{R}^n$, $1 \le p < \infty$, and $q \le np/(n-p)$ if $p < n$, $q < \infty$ if $p = n$, $q \le \infty$ if $p > n$. If $u \in W^{1,p}(B_R)$ satisfies $u \equiv 0$ in $B_{\eta R}$ for some $0 < \eta \le 1$, then*

$$\|u\|_{q;B_R} \le c\eta^{2(1-n/p)}R^{1+n/q-n/p}\|Du\|_{p;B_R}. \tag{3.28}$$

Proof. By rescaling, we may assume $R = 1$. Let $\tilde{u}(x) = u(\psi(x))$, where $\psi : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^n\backslash\{0\}$ is the inversion map, $\psi(x) = x/|x|^2$. Since $1 \le |d\psi(x)| \le \eta^{-2}$ for $\eta \le |x| \le 1$, we see that $\tilde{u} \in W^{1,p}(\mathbb{R}^n\backslash B_1)$ and $\tilde{u}(x) = 0$ for $|x| \ge \eta^{-1}$. The usual argument for the Sobolev inequality in $\mathbb{R}^n$ applies also to $\mathbb{R}^n\backslash B_1$ (see [19, Chapter 7]) and shows that for $p < n$,

$$\|\tilde{u}\|_{np/(n-p)} \le c\|D\tilde{u}\|_p;$$

it is not necessary that $\tilde{u}$ be defined in B_1. Now $\|\tilde{u}\|_q \le c\,\|\tilde{u}\|_{np/(n-p)}$ gives

$$\|\tilde{u}\|_q \le c\,\|D\tilde{u}\|_p;$$

if $p \ge n$, then this estimate follows similarly from Sobolev embedding. The result now follows from the bounds $1 \le |d\psi(x)| \le \eta^{-2}$ and rescaling. $\square$

Lemma 3.8. *Suppose $\Omega \subset \mathbb{R}^3$ and $u = (u^1, \dots, u^K) \in W^{2,2}(\Omega, \mathbb{R}^K)$ satisfies*

$$D^2_{ij} u^A = a^{AB}_{ij} u^B + b^{AB}_{ijk} D_k u^B, \tag{3.29}$$

where $a \in L^2(\Omega, \mathbb{R}^{9K^2})$, $b \in L^6(\Omega, \mathbb{R}^{27K^2})$. Then there is a constant $R_1 > 0$, depending on $\|a\|_2$, $\|b\|_6$, such that if $R \le R_1$, $B_R(x_0) \subset \Omega$, and $u \equiv 0$ in $B_{R/2}(x_0)$, then $u \equiv 0$ in $B_R(x_0)$.

Proof. Since $u = 0$ in $B_{R/2}$, Lemma 3.7 may be applied with $q = \infty$ and $q = 6$ to give

$$\begin{aligned} \|D^2 u\|_{2;B_R} &\le \|a\|_{2;\Omega}\,\|u\|_{\infty;B_R} + \|b\|_{6;\Omega}\,\|Du\|_{3/2;B_R} \\ &\le cR^{1/2}\,\|a\|_{2;\Omega}\,\|D^2 u\|_{2;B_R} + cR^{3/2}\,\|b\|_{6;\Omega}\,\|Du\|_{6;B_R} \\ &\le cR^{1/2}\left(\|a\|_{2;\Omega} + R\,\|b\|_{6;\Omega}\right)\|D^2 u\|_{2;B_R}. \end{aligned} \tag{3.30}$$

Thus if $R \le R_1 = \frac{1}{2}\min\{1, c^{-2}(\|a\|_{2;\Omega} + \|b\|_{6;\Omega})^{-2}\}$, then $\|D^2 u\|_{2;B_R} = 0$ and hence $u \equiv 0$ as claimed. $\square$

Proposition 3.9. *Suppose $(g,\pi) \in \mathcal{F}$, $\xi = (N, X^i)$ satisfies $D\Phi(g,\pi)^*\xi = 0$ in a connected subset $\Omega \subset \mathcal{M}$, and $\xi \equiv 0$ in some open set $U \subset \Omega$. Then $\xi \equiv 0$ in Ω.*

Proof. We may cover $\Omega \subset \mathcal{M}$ by a finite set of coordinate neighbourhoods in which $C^{-1}|v|^2 \le g_{ij}v^i v^j \le c\,|v|^2$, $\forall\ v = v^i\partial_i$, where $|v|^2 = \Sigma(v^i)^2$. Since $\nabla^2_{ij} = D^2_{ij} - \Gamma^k_{ij}D_k$, after moving some Christoffel terms into b_1 the equation (3.27) in a given coordinate chart Ω' may be written symbolically as

$$D^2\xi = a\xi + bD\xi, \tag{3.31}$$

where $a \in L^2(\Omega')$, $b \in W^{1,2}(\Omega') \subset L^6(\Omega')$ and

$$\|a\|_{2;\Omega'} + R\,\|b\|_{6;\Omega'} \le C\left(\|g - \mathring{g}\|_{2,2-1/2} + \|\pi\|_{1,2,-3/2}\right).$$

We can apply the previous lemma in each coordinate chart: in particular, if $\xi \equiv 0$ in some open set $U \subset \Omega$ but $\xi \neq 0$ at some point of Ω, then there is a coordinate chart Ω' and a ball $B_{R_2}(x_0) \subset \Omega'$ such that $\xi \equiv 0$ in $B_{R_2}(x_0)$ but $\xi \not\equiv 0$ in $B_{R_3}(x_0)$ for every $R_3 \geq R_2$. But Lemma 3.8, applied to $B_{R/2}(x_0 + (R_2 - R/2)e)$ for any unit vector $e \in \mathbb{R}^3$ and $R \leq R_1$, shows that $\xi \equiv 0$ in $B_{R_2+R/2}(x_0)$, which is a contradiction. Thus ξ vanishes in the coordinate set Ω', and hence in all Ω since it is connected. $\square$

To complete the proof of Theorem 3.6, we must show ξ vanishes near infinity. To do this, we establish a weighted Poincaré inequality about the point at infinity.

Lemma 3.10. *Suppose p, δ satisfy $p \geq 1$, $|\delta p/n + 1| < 1$ and $u \in W^{1,p}_{\delta}(E_R)$, $E_R \subset \mathbb{R}^n$, then there is $c = c(n,p,\delta)$ such that*

$$\|u\|_{p,\delta;E_R} \leq c\,\|Du\|_{p,\delta-1;E_R}. \tag{3.32}$$

Proof. Since $C^\infty_c(E_R)$ is dense in $W^{1,p}_\delta(E_R)$, it suffices to prove (3.32) for smooth, compactly supported u. For $\lambda \in \mathbb{R}^+, f \in C^\infty_c(E_R)$, $\frac{d}{d\lambda} f(\lambda x) = |x| D_r f(\lambda x)$ implies

$$f(x) = -\int_1^\infty |x| D_r f(\lambda x)\, d\lambda,$$

because $f(x) = 0$ for $r = |x|$ sufficiently large. Hence

$$\begin{aligned}\int_{E_R} |f(x)|\, dx &\leq \int_1^\infty \int_{E_R} |x|\, |D_r f(\lambda x)|\, dx\, d\lambda \\ &\leq \int_1^\infty \int_{E_{\lambda R}} |x|\, |D_r f(x)|\, dx\, \lambda^{-n-1} d\lambda \\ &\leq \frac{1}{n} \int_{E_R} |x|\, |D_r f(x)|\, dx.\end{aligned}$$

Now substituting $f(x) = |u(x)|^p |x|^{-\delta p - n}$, whence

$$|D_r f| \leq p\, |u|^{p-1}\, |Du|\, |x|^{-\delta p-n} + |\delta p + n|\, |u|^p\, |x|^{-\delta p - n - 1},$$

we find that

$$\frac{1}{n} \int_{E_R} |x|\, |D_r f|\, dx \leq |1 + \delta p/n| \int_{E_R} |u|^p\, |x|^{-\delta p - n} dx$$

$$+ \frac{p}{n} \left(\int_{E_R} |u|^p \, |x|^{-\delta p - n} dx \right)^{1-1/p} \left(\int_{E_R} |Du|^p \, |x|^{-(\delta-1)p-n} dx \right)^{1/p} .$$

Thus if $|1 + \delta p/n| < 1$, then

$$\|u\|_{p,\delta;E_R} \le \frac{p/n}{1 - |1+\delta p/n|} \|Du\|_{p,\delta-1;E_R}, \tag{3.33}$$

as required. □

In particular, in $\mathbb{R}^3$ and in $E_R \subset \mathcal{M}$, we have the estimates

$$\|Du\|_{2,-3/2;E_R} \le \tfrac{2}{3} \|D^2 u\|_{2,-5/2;E_R}, \tag{3.34}$$

$$\|u\|_{2,-1/2;E_R} \le 2 \|Du\|_{2,-3/2;E_R}, \tag{3.35}$$

for $R \ge R_0$, valid whenever both sides of the inequalities are finite. Using the weighted Hölder and Sobolev inequalities [5], we have in E_R,

$$\|D^2\xi\|_{2,-5/2} \le \left(\|b_0\|_{2,-5/2} \|\xi\|_\infty + c \|b_1\|_{6,-3/2} \|D\xi\|_{3,-1} \right) .$$

But from (3.34),

$$\begin{aligned} \|D\xi\|_{3,-1} &\le \|D\xi\|_{1,2,-1} \\ &\le R^{-1/2} \left(\|D\xi\|_{2,-3/2} + \|D^2\xi\|_{2,-5/2} \right) \\ &\le cR^{-1/2} \|D^2\xi\|_{2,-5/2}, \end{aligned}$$

since $\|u\|_{p,\delta;E_R} \le R^{\eta-\delta} \|u\|_{p,\eta;E_R}$ for $\eta < \delta$. Thus there is $R_1 = R_1(\|b_1\|_{1,2,-3/2})$ such that for any $R \ge R_1$, we have

$$\|D^2\xi\|_{2,-5/2;E_R} \le c \|b_0\|_{2,-5/2} \|\xi\|_{\infty,0;E_R}. \tag{3.36}$$

Since for any $u \in W^{2,2}_{-1/2}$,

$$\begin{aligned} \|u\|_{\infty,0;E_R} &\le R^{-1/2} \|u\|_{\infty,-1/2;E_R} \\ &\le cR^{-1/2} \|u\|_{2,2,-1/2;E_R} \\ &\le cR^{-1/2} \|D^2 u\|_{2,-5/2,E_R}, \end{aligned}$$

by the Sobolev inequality and Lemma 3.10, it follows that

$$\|D^2\xi\|_{2,-5/2;E_R} \le CR^{-1/2} \|D^2\xi\|_{2,-5/2;E_R} \tag{3.37}$$

and thus ξ vanishes in E_R for R sufficiently large. Combining this result with Proposition 3.9 completes the proof of Theorem 3.6, since $\Omega \subset \mathcal{M}$ is assumed to be connected. □

Corollary 3.11. *There is a constant C_2 depending on $\|(g,\pi)\|_{\mathcal{F}}$ such that for all $\xi \in W^{2,2}_{-1/2}$,*

$$\|\xi\|_{2,2,-1/2} \le C_2 \, \|P^*\xi\|_{2,-5/2}. \tag{3.38}$$

Proof. This follows from a standard Morrey contradiction argument. Suppose not, so there is a sequence ξ_k, $k = 1, 2, \ldots$ such that $\|\xi_k\|_{2,2,-1/2} = 1$, $\|P^*\xi_k\|_{2,-5/2} \le 1/k$. Then $P^*\xi_k \to 0$ strongly in $L^2_{-5/2}$. Now $W^{2,2}_{-1/2}$ embeds compactly in $W^{1,2}_0$, so ξ_k converges strongly in $W^{1,2}_0$, to ξ say. Applying (3.21) to $\xi_j - \xi_k$ shows that ξ_k is a Cauchy sequence in $W^{2,2}_{-1/2}$ and hence converges strongly to ξ in $W^{2,2}_{-1/2}$. Then $\|\xi\|_{2,2,-1/2} = 1$ and $P^*\xi = 0$, which contradicts the triviality of $\ker P^*$ (Theorem 3.6). □

The Implicit Function Theorem method is used to conclude that $\mathcal{C}$ is a smooth Hilbert submanifold of $\mathcal{F}$ — in fact we show that *all* level sets of Φ are smooth submanifolds.

Theorem 3.12. *For each $(\varepsilon, S_i) \in \mathcal{L}^*$, the constraint set*

$$\mathcal{C}(\varepsilon, S_i) = \{(g,\pi) \in \mathcal{F} : \Phi(g,\pi) = (\varepsilon, S_i)\} \tag{3.39}$$

is a Hilbert submanifold of $\mathcal{F}$. In particular, the space of solutions of the vacuum constraint equations, $\mathcal{C} = \Phi^{-1}(0) = \mathcal{C}(0,0)$, is a Hilbert manifold.

Proof. By the implicit function theorem, it suffices to show that $D\Phi : \mathcal{G} \times \mathcal{K} \to \mathcal{L}^*$ is surjective and splits. Since $D\Phi$ is bounded, its kernel is closed and hence splits. We have shown in Proposition 3.5 and Theorem 3.6 that $\ker\{D\Phi(g,\pi)^* : \mathcal{L} \to (\mathcal{G} \times \mathcal{K})^*\} = \{0\}$, so the cokernel of $D\Phi$ is trivial. It remains to show that $D\Phi$ has closed range, which we show by a direct argument. Note that the argument of Fischer–Marsden [18] based on the ellipticity of PP^* encounters some difficulties, arising from the low regularity of some low order coefficients (such as $\nabla^2 Ric$) of PP^*, and we have not been able to overcome these problems. This difficulty appears to restrict the Fischer–Marsden elliptic method to neighbourhoods of data (g,π) which are 2 derivatives smoother, ie. $H^4 \times H^3$.

Instead, we consider particular variations (h,p) of (g,π) determined from fields (y, Y^i), of the form (cf. [16])

$$h_{ij} = 2y g_{ij}, \quad p^{ij} = (\nabla^i Y^j + \nabla^j Y^i - \nabla_k Y^k g^{ij})\sqrt{g}$$

and define

(3.40)

$$F(y,Y) = D\Phi(h,p) = \left[\begin{array}{c} -4\sqrt{g}\Delta y + \Phi_0(g,\pi)y + \mathrm{tr}_g\pi\nabla_k Y^k - 4\pi\bullet\nabla Y \\ 2\sqrt{g}(\Delta Y_i + Ric_{ij}Y^j) + 2\Phi_i(g,\pi)y + (4\pi_i^j - 2\mathrm{tr}_g\pi\delta_i^j)\nabla_j y \end{array} \right].$$

We see that if $y \in W^{2,2}_{-1/2}(\mathcal{M})$, $Y \in W^{2,2}_{-1/2}(T\mathcal{M})$, then $(h,p) \in \mathcal{G} \times \mathcal{K}$, and it is straightforward to check that

$$F : W^{2,2}_{-1/2}(\mathcal{M}) \times W^{2,2}_{-1/2}(T\mathcal{M}) \to L^2_{-5/2}(\mathcal{T}^* \otimes \Lambda^3) = \mathcal{L}^* \tag{3.41}$$

is bounded. Moreover, the general scale-broken elliptic estimate [5]

$$\|u\|_{2,2,-1/2} \leq c\,\|\Delta u\|_{2,-5/2} + C\,\|u\|_{2,0}$$

shows that

$$\begin{aligned} \|(y,Y)\|_{2,2,-1/2} \leq{}& c\,\|F(y,Y)\|_{2,-5/2} + C\,\|(y,Y)\|_{2,0} \\ &+ \|\Phi y\|_{2,-5/2} + \|\pi\nabla(y,Y)\|_{2,-5/2} + \|Ric(Y)\|_{2,-5/2}, \end{aligned}$$

and the last terms are estimated by Hölder, Sobolev and interpolation inequalities, eg:

$$\begin{aligned} \|\pi\nabla u\|^2_{2,-5/2} &\leq c\,\|\nabla u\|_{3,-1}\,\|\pi\|_{6,-3/2} \\ &\leq c\,\|\pi\|_{1,2,-3/2}\,\|\nabla u\|_{3,-1} \\ &\leq \epsilon\,\|u\|_{2,2,-1/2} + C\,\|u\|_{2,0}, \end{aligned}$$

where C depends on ϵ, λ and $\|(g,\pi)\|_{\mathcal{F}}$ as usual. Thus, F satisfies the scale-broken estimate

$$\|(y,Y)\|_{2,2,-1/2} \leq c\,\|F(y,Y)\|_{2,-5/2} + C\,\|(y,Y)\|_{2,0}. \tag{3.42}$$

Now the adjoint F^* has a similar structure and the same argument shows F^* also satisfies an estimate (3.42). It follows that F has closed range (from (3.42)) with finite dimensional cokernel (since F^* has finite dimensional kernel by the elliptic estimate for F^*). Since clearly $\mathrm{ran}\,F \subset \mathrm{ran}\,D\Phi$, we have shown that $D\Phi$ has closed range and the proof of Theorem 3.12 is complete. □

4. ADM energy-momentum.

The ADM total energy-momentum $\mathbb{P}(g,\pi) = (\mathbb{P}_\alpha) = (E, p_i)$ is usually defined by the formal expressions

$$16\pi E = \oint_{S_\infty} (\partial_i g_{ij} - \partial_j g_{ii})\, dS^j \tag{4.1}$$

$$16\pi p_i = 2 \oint_{S_\infty} \pi_{ij}\, dS^j \tag{4.2}$$

where dS^j is the normal element of the sphere at infinity S_∞, the indices refer to a suitable rectangular coordinate system near infinity, and the integral over S_∞ is understood as a limit of integrals over finite coordinate spheres. The expression for the total energy E was investigated in [5] and shown to be well-defined (that is, independent of the limiting process used to define S_∞ and of the choice of structure at infinity), for metrics satisfying $g - \mathring{g} \in W^{2,q}_{-1/2}$ for some $q > 3$, and $R(g) \in L^1$. In this section, we reformulate (4.1), (4.2) and show that the redefined $\mathbb{P}$ is well-defined under weaker regularity conditions, which are better adapted to the Hilbert manifold structure of $\mathcal{C}$. It is not immediately clear the formal definitions (4.1), (4.2) can be made sensible under the weaker conditions; that this can be done, with result agreeing with the definitions (4.10), (4.11) below, is shown in Proposition 4.5. We also show that $\mathbb{P}$ is independent of the choice of structure of infinity, thereby extending the mass uniqueness result of [5].

The first result implies in particular that $\mathbb{P}$ (after suitable reformulation) defines a bounded function from the (vacuum) constraint manifold $\mathcal{C}$ to $\mathbb{R}^4$, which is smooth with respect to the Hilbert manifold structure of $\mathcal{C}$. However, it turns out that the definition of $\mathbb{P}$ cannot be extended to all $(g,\pi) \in \mathcal{F}$ as a bounded (well-defined) function. This restriction is not an artifact of the rather weak regularity conditions of $\mathcal{F}$; rather it reflects the need for additional decay conditions in defining $\mathbb{P}$. In the usual physics framework, where (g,π) satisfy the decay conditions (with $r = |x|$),

$$|g_{ij} - \delta_{ij}| + r\,|\partial_i g_{jk}| + r^2\,|\partial_i\partial_j g_{kl}| = O(1/r), \tag{4.3}$$

$$|K_{ij}| + r\,|\partial_i K_{jk}| = O(1/r^2), \tag{4.4}$$

the nature of the additional decay conditions is usually expressed by the requirements [35]

$$R(g) = O(r^{-4}), \qquad \partial_i K_{ij} - \partial_j K_{ii} = O(r^{-4}). \tag{4.5}$$

These may be reformulated more invariantly (and more generally) as

$$R(g) \in L^1(\mathcal{M}), \qquad \nabla_j \pi^{ij} \in L^1(T\mathcal{M}), \tag{4.6}$$

and we emphasise that these conditions are *not* satisfied by general $(g, \pi) \in \mathcal{F}$. Indeed, they are equivalent to requiring $\Phi(g, \pi) \in L^1(\mathcal{T}^*)$, and exactly this condition turns out to be sufficient for $\mathbb{P}(g, \pi)$ to be well-defined.

In order to define $\mathbb{P}$ in all of $\mathcal{F}$, we first need a suitable definition of translation vector at infinity. Fix a 4-vector $\xi_\infty = (\xi^\alpha_\infty) = (\xi^0_\infty, \xi^i_\infty) \in \mathbb{R}^4$ (where the indices take the ranges $\alpha = 0, 1, \ldots, 3,\ i = 1, \ldots, 3$); using the metric $\mathring{g}$ near infinity, which we consider as defining a connection on the spacetime tangent bundle $\mathcal{T}$ which is flat near infinity, we may identify ξ_∞ with a parallel vector field $\tilde{\xi}_\infty$ defined in an exterior region E_{R_1} for some $R_1 \geq R_0$. We say that a vector field $\hat{\xi}_\infty \in C^\infty(\mathcal{T})$ is a *constant translation near infinity* representing $\xi_\infty \in \mathbb{R}^4$ if $\hat{\xi}_\infty = \tilde{\xi}_\infty$ in E_{2R_1} and $\hat{\xi}_\infty = 0$ in $\mathcal{M} \setminus E_{R_1}$. Obviously $\hat{\xi}_\infty$ is not uniquely determined by its constant value ξ_∞; however two representatives of ξ_∞ differ only by a smooth, compactly supported, vector field.

A vector field $\xi = (\xi^\alpha)$ is then said to be an *asymptotic translation* if there is $\xi_\infty \in \mathbb{R}^4$ with a corresponding constant translation vector at infinity $\hat{\xi}_\infty$, such that $\xi - \hat{\xi}_\infty \in L^2_{-1/2}(\mathcal{T}) = \mathcal{L}$. Note that if $\xi^{(1)}, \xi^{(2)}$ are two asymptotic translations (representing the same translation vector ξ_∞), then $\xi^{(1)} - \xi^{(2)} \in \mathcal{L}$; hence we may define the class

$$\xi_\infty + \mathcal{L} = \{\xi : \xi - \hat{\xi}_\infty \in \mathcal{L}\}, \tag{4.7}$$

of asymptotic translation vector fields representing ξ_∞. By replacing $\mathcal{L}$ with $W^{k,2}_{-1/2}(\mathcal{T})$, $k \geq 1$, we may similarly define classes of asymptotically constant vectors with better regularity properties.

Rather than work with the asymptotic boundary integrals (4.1), (4.2), it is more convenient (although logically equivalent, as we shall show) to work with spatial integrals of exact divergences. Therefore we introduce the density-valued linear operators $\mathcal{R}_o(g), \mathcal{P}_o(\pi)$ by

$$\mathcal{R}_o(g) = \left(\mathring{\nabla}^{ij} g_{ij} - \Delta_o \mathrm{tr}_{\mathring{g}} g\right)\sqrt{\mathring{g}}, \tag{4.8}$$

$$\mathcal{P}_{oi}(\pi) = \mathring{g}_{ij} \mathring{\nabla}_k \pi^{jk}. \tag{4.9}$$

The ADM (total) energy-momentum vector $\mathbb{P}(g, \pi) = (E, p)$ is then defined by describing the pairing with a vector at infinity $\xi_\infty \in \mathbb{R}^{3,1}$; let $\hat{\xi}_\infty$ be a corresponding representative translation vector field at infinity, then we define $\xi^\alpha_\infty \mathbb{P}_\alpha(g,\pi)$ by

$$(4.10)\qquad 16\pi\xi^0_\infty \mathbb{P}_0(g,\pi) = \int_{\mathcal{M}} \left(\hat{\xi}^0_\infty \mathcal{R}_o(g) + \mathring{\nabla}^i \hat{\xi}^0_\infty \left(\mathring{\nabla}^j g_{ij} - \mathring{\nabla}_i \mathrm{tr}_{\mathring{g}} g\right)\sqrt{\mathring{g}},\right)$$

$$(4.11)\qquad 16\pi\xi^i_\infty \mathbb{P}_i(g,\pi) = 2\int_{\mathcal{M}} \left(\hat{\xi}^i_\infty \mathcal{P}_{oi}(\pi) + \pi^{ij}\mathring{\nabla}_i \hat{\xi}_{\infty j}\right),$$

where indices are raised and lowered using the background metric $\mathring{g}$. The physical interpretation of $\xi^\alpha \mathbb{P}_\alpha$ is as the energy of $(\mathcal{M}, g, \pi)$ as observed by the asymptotic time vector ξ_∞, and $\mathbb{P}_\alpha$ is the total energy-momentum covector of $(\mathcal{M}, g, \pi)$. Since $\mathcal{R}_o(g), \mathcal{P}_o(\pi), \sqrt{\mathring{g}}, \pi$ are all tensor densities, the volume elements in (4.10),(4.11) are present implicitly, and it is readily seen that the right-hand sides depend only on ξ_∞ and not on the specific choice of representative asymptotic translation vector $\hat{\xi}_\infty$, since a change of $\hat{\xi}_\infty$ changes the integrands only by an exact divergence of compact support.

Theorem 4.1. *If $(\varepsilon, S) \in L^1(\mathcal{T}^*)$, then $\mathbb{P}$ defined by (4.10),(4.11) defines a smooth function on the Hilbert manifold $\mathcal{C}(\varepsilon, S)$,*

$$\mathbb{P} \in C^\infty(\mathcal{C}(\varepsilon, S), \mathbb{R}^{3,1}).$$

Proof. We begin by proving an analogue of the L^2 bounds (3.2), (3.3).

Proposition 4.2. *Suppose $g \in \mathcal{G}^+_\lambda$ for some $\lambda > 0$ and $\pi \in \mathcal{K}$. There is a constant $c = c(\lambda)$ such that*

$$(4.12)\qquad \|\Phi_0(g,\pi) - \mathcal{R}_o(g)\|_{L^1(\mathcal{M})} \le c\big(1 + \|\mathring{\nabla} g\|^2_{2,-3/2} + \|\pi\|^2_{2,-3/2} + \|g - \mathring{g}\|_{2,-1/2}\, \|\mathring{\nabla}^2 g\|_{2,-5/2}\big),$$

$$(4.13)\qquad \|\Phi_i(g,\pi) - \mathcal{P}_{oi}(\pi)\|_{L^1(\mathcal{M})} \le c\,\Big(\|g - \mathring{g}\|_{2,-1/2}\ \|\mathring{\nabla}\pi\|_{2,-5/2} + \|\mathring{\nabla} g\|_{2,-3/2}\, \|\pi\|_{2,-3/2}\Big).$$

Proof. From (3.7), we may express the scalar curvature in terms of $\mathcal{R}_o(g)$ by

$$(4.14)\qquad R(g) = \mathcal{R}_o(g)/\sqrt{\mathring{g}} + Q(g^{-1}, \mathring{\nabla} g) + g^{jk} Ric(\mathring{g})_{jk} + \left(\left(g^{ik} - \mathring{g}^{ik}\right) g^{jl} + \mathring{g}^{ik}\left(g^{jl} - \mathring{g}^{jl}\right)\right)\left(\mathring{\nabla}^2_{ij} g_{kl} - \mathring{\nabla}^2_{ik} g_{jl}\right),$$

The individual terms may be easily estimated as before, giving
(4.15)
$$\|R(g)\sqrt{g} - \mathcal{R}_o(g)\|_{L^1(\mathcal{M})} \leq c(\lambda)\left(1 + \|g - \mathring{g}\|_{2,-1/2}\,\|\mathring{\nabla}^2 g\|_{2,-5/2} + \|\mathring{\nabla} g\|^2_{2,-3/2}\right),$$

from which (4.12) follows, since

$$\|\,|\pi|^2_g\|_{L^1(\mathcal{M})} \leq c(\lambda)\,\|\pi\|^2_{2,-3/2}.$$

From (3.9), it follows that

$$\Phi_i(g,\pi) - \mathcal{P}_{oi}(\pi) = (g_{ij} - \mathring{g}_{ij})\,\mathring{\nabla}_k \pi^{jk} + g_{ij} A^j_{kl}\pi^{kl},$$

which can be bounded easily,

$$\begin{aligned} \text{(4.16)} \qquad & \|\Phi_i(g,\pi) - \mathcal{P}_{oi}(\pi)\|_{L^1(\mathcal{M})} \\ & \leq c\left(\|g - \mathring{g}\|_{2,-1/2}\,\|\mathring{\nabla}\pi\|_{2,-5/2} + \|\mathring{\nabla} g\|_{2,-3/2}\,\|\pi\|_{2,-3/2}\right), \end{aligned}$$

as required. □

Since $\mathbb{P}(g,\pi)$ depends linearly on (g,π), to complete the proof of Theorem 4.1, it will suffice to show that $\mathbb{P}$ is bounded on $\mathcal{C}(\varepsilon, S)$. From (4.12), we see that

$$\begin{aligned} \|\mathcal{R}_o(g)\|_{L^1} \leq\ & \|\Phi_0(g,\pi) - \mathcal{R}_o(g)\|_{L^1} + \|\Phi_0(g,\pi)\|_{L^1} \\ & \leq c(g)\big(1 + \|\mathring{\nabla} g\|^2_{2,-3/2} + \|\pi\|^2_{2,-3/2} \\ & \qquad + \|g - \mathring{g}\|_{2,-1/2}\,\|\mathring{\nabla}^2 g\|_{2,-5/2}\big) + \|\varepsilon\|_{L^1}, \end{aligned}$$

and hence $\mathcal{R}_o(g)$ is integrable. Since $\mathring{\nabla}\hat{\xi}_\infty$ has compact support, it follows that the integrand of (4.10) is integrable and $\mathbb{P}_0(g,\pi)$ is finite on $\mathcal{C}(\varepsilon, S)$. Similarly we estimate using (4.13), assuming $|\hat{\xi}^i_\infty| \leq 1$ for simplicity,

$$\begin{aligned} \|\hat{\xi}^i_\infty \mathcal{P}_{oi}(\pi)\|_{L^1} \leq\ & \|\hat{\xi}^i_\infty(\Phi_i(g,\pi) - \mathcal{P}_{oi}(g))\|_{L^1} + \|\hat{\xi}^i_\infty \Phi_i(g,\pi)\|_{L^1} \\ & \leq c\left(\|g - \mathring{g}\|_{2,-1/2}\,\|\mathring{\nabla}\pi\|_{2,-5/2} + \|\mathring{\nabla} g\|_{2,-3/2}\,\|\pi\|_{2,-3/2}\right) \\ & + \|S\|_{L^1}, \end{aligned}$$

whereupon the integrand of (4.11) is integrable and thus $\mathbb{P}_i(g,\pi)$ is finite. □

We now show that the definitions (4.10), (4.11) adopted for $\mathbb{P}$ agree with the formal definitions (4.1), (4.2), when suitably interpreted, under the general conditions of the mass existence Theorem 4.1, and that the value of $\mathbb{P}(g,\pi)$ does not depend on the choice of structure of infinity ϕ and its associated background metric $\mathring{g} = \phi^*(\delta)$ cf. [5, Theorem 4.2].

The following two elementary lemmas will take care of the major technical details of the proof, and will be useful elsewhere. The first lemma reviews the validity of integration by parts, and is valid under considerably more general circumstances than required here.

Lemma 4.3. *Suppose $\mathcal{M} = \bigcup_{k\geq 1} \mathcal{M}_k$ is an exhaustion of a non-compact, n-dimensional manifold $\mathcal{M}$ by compact subsets with smooth boundaries $\partial\mathcal{M}_k$, and suppose $\beta \in W^{1,2}_{loc}(\Lambda^{n-1}T^*\mathcal{M})$ satisfies $d\beta \in L^1(\Lambda^n T^*\mathcal{M})$. Then*

(i)

$$\oint_{\partial\mathcal{M}_k} \beta \quad \textit{exists for } k \geq 1;$$

(ii)

$$\oint_{\partial\mathcal{M}_\infty} \beta := \lim_{k\to\infty} \oint_{\partial\mathcal{M}_k} \beta \textit{ exists.}$$

Proof. Since $\partial\mathcal{M}_k$ is smooth, the trace theorem [31, 33] shows that $\beta \in W^{1/2,2}(\partial\mathcal{M}_k) \subset L^2(\partial\mathcal{M}_k) \subset L^1(\partial\mathcal{M}_k)$, where the fractional Sobolev space is defined using the Fourier transform in the usual manner. This shows that the finite boundary integrals are well-defined. The definition of weak derivative allows us to apply Stokes' theorem to $d\beta$ over any compact region; in particular, for $1 \leq q \leq p$, we have

$$\oint_{\partial\mathcal{M}_p} \beta - \oint_{\partial\mathcal{M}_q} \beta = \int_{\mathcal{M}_p\setminus\mathcal{M}_q} d\beta.$$

Since $d\beta \in L^1(\Lambda^n T^*\mathcal{M})$, the right-hand side is $o(1)$ as $q = \min(p,q) \to \infty$ and hence $\{\oint_{\partial\mathcal{M}_k} \beta\}_{k=1}^{\infty}$ is a Cauchy sequence and convergent as claimed. $\square$

Likewise, the second lemma is valid with more general values for the indices, but this will not be needed here.

Lemma 4.4. *Suppose $E_{R_0} \subset \mathbb{R}^3$, $R_0 \geq 1$ and $u \in W^{1,2}_{-3/2}(E_{R_0})$. Then $u \in L^4(S_R)$ for every $R \geq R_0$, and there is a constant c, independent of R, such that*

$$\oint_{S_R} |u|\, dS \leq cR^{1/2}\, \|u\|_{1,2,-3/2;A_R}, \tag{4.17}$$

(where the notation indicates the norm over the annular domain A_R); hence

$$\|u\|_{1;S_R} = o(R^{1/2}) \quad \text{as } R \to \infty. \tag{4.18}$$

Proof. As in [5], we define $u_R(x) = u(Rx)$ and recall the uniform comparison

$$\|u_R\|_{k,p;A_1} \approx R^\delta \|u\|_{k,p,\delta;A_R}, \quad \text{for any } R \geq R_0.$$

Since $u_R \in W^{1,2}(A_1)$, the trace theorem again implies $u_R \in W^{1/2,2}(S_1)$, and

$$\|u_R\|_{1/2,2;S_1} \leq \|u_R\|_{1,2;A_1}. \tag{4.19}$$

It readily follows that

$$\|u_R\|_{1;S_1} \leq c\,\|u_R\|_{4;S_1} \leq c\,\|u_R\|_{1/2,2;S_1} \leq \|u_R\|_{1,2;A_1}$$

and thus

$$\|u\|_{1;S_R} \leq cR^2 \|u_R\|_{1,2;A_1} \leq cR^{1/2} \|u\|_{1,2,-3/2;A_R}.$$

In fact, using the Sobolev inequality in $W^{1/2,2}(S_1)$ gives

$$\|u\|_{4;S_R} \leq cR^{-1} \|u\|_{1,2,-3/2;A_R};$$

a stronger inequality which we will not need here. The conclusion (4.18) follows as in [5], since $u \in W^{k,p}_\delta(E_{R_0})$ implies both $\|u\|_{k,p,\delta;A_R} = o(1)$ and $\|u\|_{k,p;A_R} = o(R^\delta)$ as $R \to \infty$. □

It follows easily that the formal asymptotic definition of (E, p) agrees with the integral definition of $\mathbb{P}$. This generalises and extends Proposition 4.1 of [5].

Proposition 4.5. *Suppose $(g, \pi) \in \Phi^{-1}(L^1(\mathcal{T}^* \otimes \Lambda^3))$. Then (E, p) from (4.1), (4.2) are defined, in the sense of Lemma 4.3, and satisfy $(E, p) = \mathbb{P}$.*

Proof. After noting that the integrals of (4.10), (4.11) may be written as exact divergences, respectively of

$$\begin{aligned} &\mathring{\nabla}^i \left(\hat{\xi}^0_\infty \left(\mathring{\nabla}^j g_{ij} - \mathring{\nabla}_i \mathrm{tr}_{\mathring{g}} g\right)\right) \sqrt{g}, \\ &2\mathring{\nabla}_i \left(\hat{\xi}^k_\infty \mathring{g}_{jk} \pi^{ij}\right), \end{aligned} \tag{4.20}$$

which both satisfy the integrability condition of Lemma 4.3, by Proposition 4.2 and the hypothesis $\Phi(g, \pi) \in L^1(\mathcal{T}^* \otimes \Lambda^3)$, we see that (E, p) is well-defined. The equality of the two definitions is now a tautology. □

Corollary 4.6. *The definition (4.10), (4.11) of $\xi_\infty^\alpha \mathbb{P}_\alpha(g,\pi)$ remains valid (and unchanged) if the constant translation at infinity ξ_∞ is replaced by any asymptotic translation $\xi \in \xi_\infty + W^{2,2}_{-1/2}(\mathcal{T})$.*

Proof. The difference between the two definitions of $\mathbb{P}$ (using $\hat{\xi}_\infty, \xi$ respectively) is a sum of divergences of the form (4.20), with $\hat{\xi}_\infty$ replaced by $\xi - \hat{\xi}_\infty \in W^{2,2}_{-1/2}(\mathcal{T})$. The weighted Sobolev inequality implies $\xi - \hat{\xi}_\infty$ is Hölder continuous and decays as $o(R^{-1/2})$, so by Lemma 4.4, the boundary integral of (4.20) is defined and decays as $o(R^{-1/2})o(R^{1/2}) = o(1)$. □

The proof that the value of $\mathbb{P}$ is independent of the choice of structure of infinity ϕ follows [5, Theorem 4.2].

Theorem 4.7. *Suppose $\phi : \mathcal{M}\backslash\mathcal{M}_0 \to \mathbb{R}^3$, $\psi : \mathcal{M}\backslash\mathcal{M}_1 \to \mathbb{R}^3$ are two structures of infinity such that $(g,\pi) \in \mathcal{F}(\phi) \cap \mathcal{F}(\psi)$, where the notation indicates the phase space (and weighted Sobolev spaces) defined with respect to the indicated structure of infinity. Then $\mathcal{F}(\phi) = \mathcal{F}(\psi)$, the underlying Hilbert Sobolev spaces have comparable norms, and $\mathbb{P}(g,\pi;\phi) = \mathbb{P}(g,\pi;\psi)$.*

This justifies the notation used elsewhere in this paper, where we do not indicate the choice of structure of infinity.

Proof. If $g \in \mathcal{G}^+$ then by the Sobolev inequality, $\phi_* g - \delta \in W^{1,6}(E_R)$, and ϕ, ψ satisfy the conditions of [5, Section 3]. Hence the transition function $\psi \circ \phi^{-1} : E_{R_2} \to \mathbb{R}^3$ for some $R_2 \geq 1$, after possibly moving ψ by a rigid motion of $\mathbb{R}^3$, satisfies $\psi \circ \phi^{-1} - \mathrm{Id} \in W^{2,6}_{1/2}(E_{R_2})$. A trivial modification shows $\psi \circ \phi^{-1} - Id \in W^{3,2}_{1/2}(E_{R_2})$, whereupon the background metrics satisfy $\phi^*\delta - \psi^*\delta \in W^{2,2}_{-1/2}(\mathcal{S}(\mathcal{M}_0 \cap \mathcal{M}_1))$ and it follows that the spaces $\mathcal{G}^+, \mathcal{K}$ are in fact independent of the choice of structure of infinity ϕ.

To show invariance of the ADM energy-momentum, let $\tilde{g}$ be a background metric for ψ, so $\tilde{g} = \psi^*\delta$ in $\mathcal{M}\backslash\mathcal{M}_1$, and let $y(x) = \psi \circ \phi^{-1}(x)$, $x \in E_{R_2}$ be the coordinate transition function. Let $\tilde{\nabla}$ and $\tilde{\mathbb{P}}$, respectively, be the connection and total ADM energy-momentum operators of $\tilde{g}$. By Corollary 4.6 and the uniqueness of $W^{2,2}_{-1/2}$, we may use the same vector field $\xi \in \xi_\infty + W^{2,2}_{-1/2}(\mathcal{T})$ to define both $\mathbb{P}, \tilde{\mathbb{P}}$.

The divergence expression (4.20) for the integrand for $\xi_\infty^\alpha \mathbb{P}_\alpha$ may be written in arbitrary coordinates in the form

$$\partial_p \left(\xi^0 \mathring{g}^{ip} \mathring{g}^{jk} \left(\mathring{\nabla}_k g_{ij} - \mathring{\nabla}_i g_{jk} \right) \sqrt{\mathring{g}} \right) + 2\partial_i \left(\xi^k \mathring{g}_{jk} \pi^{iij} \right),$$

with a similar expression being valid for $\xi^\alpha_\infty \tilde{\mathbb{P}}_\alpha$. Since

$$\mathring{\nabla}_i g_{jk} - \tilde{\nabla}_i g_{jk} = \tilde{A}^p_{ij} g_{pk} - \tilde{A}^p_{ik} g_{jp},$$

where $\tilde{A}^p_{ij} = \tilde{\Gamma}^p_{ij} - \mathring{\Gamma}^p_{ij}$, after a certain amount of calculation, we find that the difference between the energy-momentum integrands may be written in the form

$$\begin{aligned} &\partial\Big(\ \xi(\mathring{g} - \tilde{g})(\pi + \mathring{\nabla} g) + \xi(g - \mathring{g})\mathring{\nabla}\tilde{g}\Big) \\ + \quad &\partial_j \left(\xi^0 \left(\mathring{\nabla}^j \mathrm{tr}_{\mathring{g}}\tilde{g} - \mathring{g}^{jk}\mathring{\nabla}^l \tilde{g}_{kl}\right)\sqrt{\mathring{g}}\right). \end{aligned}$$

The precise form of the first term is of no account, since by the argument of Corollary 4.6 and the decay conditions on $g, \tilde{g}, \xi$, the first term integrates to zero. Integrating, we arrive at the relation

$$\xi^\alpha \mathbb{P}_\alpha(g,\pi) - \xi^\alpha \tilde{\mathbb{P}}_\alpha(g,\pi) = \xi^\alpha \mathbb{P}_\alpha(\tilde{g}, 0), \tag{4.21}$$

and it remains to show that $(\tilde{g}, 0)$ has vanishing energy-momentum (notice that the fact $\tilde{g} = \psi^*(\delta)$ has not yet been used, so (4.21) is valid more generally).

Working in the $\mathring{g}$ rectangular coordinates x^i, in which $\mathring{g}_{ij} = \delta_{ij}$, the metric $\tilde{g}$ is given in terms of the transition functions $y(x)$ by $\tilde{g}_{ij} = \partial_i y^p \partial_j y^p$, where $\partial_i = \partial/\partial x^i$. Since $\mathring{g}$ is explicitly flat in the coordinates x^i, $\xi^\alpha \mathbb{P}_\alpha(\tilde{g}, 0) = \xi^0 \mathbb{P}_0(\tilde{g}, 0)$ is the integral of the $\mathbb{R}^3$-divergence of

$$\xi^0(\partial_i \tilde{g}_{ij} - \partial_j \tilde{g}_{ii}) = \xi^0(\partial^2_{ii} y^p \partial_j y^p - \partial_i y^p \partial^2_{ij} y^p).$$

After a rotation, we may assume $\partial y^p/\partial x^i - \delta^p_i \in W^{2,2}_{-1/2}$ in the exterior region, and therefore, by the argument of Corollary 4.6 again, the above expression may be reduced to

$$\partial^2_{ii} y^j - \partial^2_{ij} y^i.$$

Expressing this explicitly as a 2-form gives

$$\begin{aligned} \left(\partial^2_{ii} y^j - \partial^2_{ij} y^i\right) * dx^j &= \partial_i \left(\partial_i y^j - \partial_j y^i\right) * dx^j \\ &= -d\left(\epsilon_{ijk} \partial_i y^j \, dx^k\right), \end{aligned}$$

which is a closed 2-form and therefore it does not contribute to any boundary integral. It follows that $\mathbb{P}(\tilde{g}, 0) = 0$. $\square$

5. Hamiltonians.

The formal variational structure of the Einstein equations is well-known and due originally to Hilbert and Einstein [17, 20]: the Euler–Lagrange equations of the Lagrangian functional

$$\mathcal{L}_{EH}(g^{(4)}) := \int_V R(g^{(4)})\sqrt{g^{(4)}}\, d^4x, \tag{5.1}$$

are obtained in the usual manner, by making a compactly supported variation of the spacetime metric $g^{(4)}$ and once integrating by parts, and are just the (vacuum) Einstein equations. In this respect, the Einstein equations are similar to the equations of motion of most other Lagrangian field theories, such as the classical wave equation. However, it differs in that although the resulting equations are second order in the metric, the Lagrangian contains explicit second derivatives. As is well known, the Gauss–Bonnet formula shows that the Einstein–Hilbert integrand can be written in a local coordinate system in the form

$$R(g^{(4)})\sqrt{g^{(4)}}\, d^4x = d(A_1(g^{(4)}, \partial g^{(4)})) + A_2(g^{(4)}, \partial g^{(4)}),$$

where A_1 is linear and A_2 is quadratic in $\partial g^{(4)}$, and thus the Euler–Lagrange equations are determined by A_2 since compactly supported variations of the divergence terms dA_1 will not contribute to the equations. However, A_2 depends on a choice of frame (cf. (3.7)) and thus is neither unique nor a geometrically invariant quantity. We therefore have the curious situation of a non-unique, non-geometric (coordinate dependent), integrand giving rise to a geometric (tensorial) Euler-Lagrange equation.

The Hamiltonian interpretation of the Einstein–Hilbert Lagrangian was provided by Arnowitt, Deser and Misner [2], who decomposed $\mathcal{L}_{EH}$ by imposing a $3+1$ splitting of the spacetime V and after an integration by parts in the time direction and dropping the resulting boundary integral, arrived at the ADM form of the Lagrangian

$$\mathcal{L}_{EH} \simeq \int_V (\pi\bullet\partial_t g - \xi^\alpha \Phi_\alpha(g,\pi)) \tag{5.2}$$

where $\xi = (N, X^i)$ is the (unspecified) lapse and shift of the $3+1$ decomposition. This decomposition, incidentally, is the origin of the form (2.3) for the conjugate momentum π. Now introducing the ADM Hamiltonian,

$$\mathcal{H}_{ADM}(g,\pi;\xi) = -\int_{\mathcal{M}} \xi^\alpha \Phi_\alpha(g,\pi), \tag{5.3}$$

the Einstein–Hilbert variational computation, with compactly supported variations, may be re-expressed as Hamilton's equations of motion for $\mathcal{H}_{ADM}$,

$$\frac{d}{dt}\begin{pmatrix} g \\ \pi \end{pmatrix} = -J\ D\Phi(g,\pi)^*(\xi) \tag{5.4}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : T\mathcal{F} \to T\mathcal{F}$ is the implied symplectic form, $J\begin{pmatrix} -p \\ h \end{pmatrix} = \begin{pmatrix} h \\ p \end{pmatrix}$, in the (g,π) coordinates on $\mathcal{F}$ [18, 11].

We note parenthetically that this $3+1$ reduction involves two geometric (gauge) choices; that of a timeflow vector field (reducing to ξ on the hypersurface) and a choice of spacelike hypersurface. Rather remarkably, it turns out that the spacelike integrand may be considered as the restriction to the hypersurface of a 3-form defined globally on the spacetime, and depending only on the spacetime metric and the choice of timeflow vector field — see [27] for the computations involved.

In this section we are instead concerned with formulating the above equations in the context of the phase space $\mathcal{F}$. The aim is to construct Hamiltonian functionals which, together with apropriately chosen decay and boundary conditions for the lapse-shift ξ, lead to the evolution equations (5.4). The existence and uniqueness for the Einstein evolution equations is a separate and rather difficult question in analysis which will not be considered here — in particular, it does not seem possible to deduce this on general grounds from the Hamiltonian structure on the phase space.

Since one of the primary difficulties is the control of boundary terms, we record the complete form of the boundary terms arising from the integration by parts relating the variational derivative $D\Phi(g,\pi)$ to the adjoint operator $D\Phi(g,\pi)^*$. This follows directly from the expressions (2.8–2.10).

$$\begin{aligned} &\xi^\alpha D\Phi_\alpha(g,\pi)(h,p) - D\Phi(g,\pi)^*\xi\cdot(h,p) \\ &\quad = \nabla^i\left\{\xi^0\left(\nabla^j h_{ij} - \nabla_i \mathrm{tr}_g h\right) - \left(h_{ij}\nabla^j\xi^0 - \mathrm{tr}_g h\nabla_i\xi^0\right)\right\}\sqrt{g} \\ &\qquad + \nabla_i\left\{2\xi_j p^{ij} + 2\xi^j\pi^{ik}h_{jk} - \xi^i\pi^{jk}h_{jk}\right\} \end{aligned} \tag{5.5}$$

In the following, we assume that $\partial\mathcal{M}$ is empty.

Theorem 5.1. *The ADM Hamiltonian* (5.3) *with lapse-shift $\xi \in \mathcal{L}$ defines a smooth map of Hilbert manifolds*

$$\mathcal{H}_{ADM} : \mathcal{F}\times\mathcal{L} \to \mathbb{R}.$$

If $\xi \in W^{2,2}_{-1/2}(\mathcal{T})$*, then for all* $(h,p) \in T_{(g,\pi)}\mathcal{F}$*,*

$$D_{(g,\pi)}\mathcal{H}_{ADM}(g,\pi;\xi)(h,p) = -\int_{\mathcal{M}} (h,p)\cdot D\Phi(g,\pi)^*(\xi). \tag{5.6}$$

Proof. Hölder's inequality and the decay condition $\xi \in \mathcal{L} = L^2_{-1/2}(\mathcal{T})$ shows that $\mathcal{H}_{ADM}$ is defined and bounded on $\mathcal{F}\times\mathcal{L}$, hence the linearity in ξ implies smoothness with respect to ξ. Likewise, smoothness with respect to (g,π) follows from the smoothness of the map $(g,\pi)\mapsto\Phi(g,\pi)$.

To show (5.6), we must control the boundary terms in (5.5). For this, we use the trace theorem, in the form of Lemma 4.4. The individual components of the boundary term

$$\begin{aligned} B^i = {} & \xi^0\left(\nabla^j h_{ij} - \nabla_i \mathrm{tr}_g h\right)\sqrt{g} - \left(h_{ij}\nabla^j\xi^0 - \mathrm{tr}_g h\nabla_i\xi^0\right)\sqrt{g}, \\ & + \xi_j p^{ij} + \xi^j\pi^{ik}h_{jk} - \tfrac{1}{2}\xi^i\pi^{jk}h_{jk} \end{aligned} \tag{5.7}$$

have well-defined traces on the spheres S_R (and on any other smooth hypersurface in $\mathcal{M}$), thus integration of the adjoint operator formula (5.5) yields the expected boundary integrals. We may now estimate the boundary contribution over S_R in the limit as $R\to\infty$,

$$\begin{aligned} \oint_{S_R} |B|\,dS \le {} & \|\xi\|_{\infty;S_R}\left(\|\nabla h\|_{1;S_R} + \|p\|_{1;S_R}\right) \\ & + \|h\|_{\infty;S_R}\left(\|\nabla\xi\|_{1;S_R} + \|\xi\|_{\infty;S_R}\|\pi\|_{1;S_R}\right) \\ \le {} & o(1)\left(\|\nabla h\|_{1,2,-3/2;A_R} + \|\nabla\xi\|_{1,2,-3/2;A_R}\right. \\ & \left. + \|\pi\|_{1,2,-3/2;A_R} + \|p\|_{1,2,-3/2;A_R}\right), \end{aligned}$$

which shows the boundary integral is $o(1)$ as $R\to\infty$. Integrating (5.5) over $\mathcal{M}_R := \{x\in\mathcal{M} : \sigma(x) < R\}$ and letting $R\to\infty$ establishes (5.6) and completes the proof of Theorem 5.1. Note that the use of the spherical exhaustion $\mathcal{M}_R, R\ge R_0$, is merely a convenience; since the integrands $\xi D\Phi(h,p)$, $(h,p)\cdot D\Phi^*\xi$ are integrable, the improper integrals in (5.6) are independent of the choice of exhaustion used to define them, and the boundary integrals evaluated with any other smooth exhaustion of $\mathcal{M}$ will also vanish in the limit. □

We emphasise that an identity of the form (5.6) is necessary if the Hamiltonian is to generate the correct equations of motion. The restriction above to lapse-shift ξ decaying at infinity is essential, both in defining $\mathcal{H}_{ADM}$

(since $\Phi(g,\pi)$ is not integrable for generic $(g,\pi)\in\mathcal{F}$) and in ensuring that asymptotic boundary terms are absent in (5.6). However, we would like to be able to choose ξ asymptotic to a translation at infinity in the evolution equations and retain the validity of (5.6); this necessitates a modification of the Hamiltonian functional $\mathcal{H}_{ADM}$, as suggested in [28].

The underlying principle here is that adding a divergence to the Hamiltonian (or the Einstein–Hilbert Lagrangian) will not change the formal equations of motion, but such a term will affect the phase space (domain of definition) of the Hamiltonian and the resulting equations of motion. In particular, to extend the definition of the ADM Hamiltonian to permit lapse-shift asymptotic to a (non-zero) translation at infinity, we should add a divergence which cancels the dominant contribution from the asymptotic translation — from (4.10), (4.11), we recognise that the ADM energy $\xi^\alpha_\infty \mathbb{P}_\alpha$ is an appropriate choice. Thus we arrive at the Regge–Teitelboim Hamiltonian [28]

$$\mathcal{H}_{RT}(g,\pi;\xi) = 16\pi\xi^\alpha_\infty \mathbb{P}_\alpha(g,\pi) - \int_{\mathcal{M}} \xi^\alpha \Phi_\alpha(g,\pi), \tag{5.8}$$

where $\xi\in\xi_\infty+\mathcal{L}$. This expression is well-defined on $\mathcal{C}$, where it has the value $\xi^\alpha_\infty\mathbb{P}_\alpha(g,\pi)$, and more generally on $\Phi^{-1}(L^1(\mathcal{T}^*))$, but for general $(g,\pi)\in\mathcal{F}$, the individual terms are not defined, and thus (5.8) does not provide a definition valid on all $\mathcal{F}$. We circumvent this problem by inserting the definition of $\mathbb{P}$ and rearranging terms — thus for general $(g,\pi)\in\mathcal{F}$ and $\xi\in\xi_\infty+\mathcal{L}$, we define the regularised Hamiltonian $\mathcal{H}(g,\pi,\xi)$ by

$$\begin{aligned}\mathcal{H}(g,\pi;\xi) = &\int_{\mathcal{M}} \left(\hat{\xi}^0_\infty - \xi^0\right)\Phi_0(g,\pi) + \int_{\mathcal{M}} \left(\hat{\xi}^i_\infty - \xi^i\right)\Phi_i(g,\pi) \\ &+ \int_{\mathcal{M}} \hat{\xi}^0_\infty \left(\mathcal{R}_o(g) - \Phi_0(g,\pi)\right) + \int_{\mathcal{M}} \mathring{\nabla}^i \hat{\xi}^0_\infty \left(\mathring{\nabla}^j g_{ij} - \mathring{\nabla}_i \mathrm{tr}_{\mathring{g}} g\right)\sqrt{\mathring{g}} \\ &+ \int_{\mathcal{M}} \hat{\xi}^i_\infty \left(\mathcal{P}_{oi}(\pi) - \Phi_i(g,\pi)\right) + \int_{\mathcal{M}} 2\pi^{ij}\mathring{\nabla}_i \hat{\xi}_{\infty j},\end{aligned} \tag{5.9}$$

where $\xi\in\xi_\infty+\mathcal{L}$ and $\hat{\xi}_\infty$ is constant at infinity with value ξ_∞. For $(g,\pi)\in\Phi^{-1}(L^1(\mathcal{T}^*))$, this agrees with (5.8). As in Section 4, the sum of the integrals in (5.9) is independent of the particular choice of constant at infinity vector field $\hat{\xi}_\infty$ representing the translation ξ_∞, although the pointwise values of the integrands are not invariant. We emphasise that for generic $(g,\pi)\in\mathcal{F}$, $\mathcal{H}$ does not have a simple geometric interpretation such as (5.8). Nevertheless, it does have some useful properties.

Theorem 5.2. *The functional $\mathcal{H}(g,\pi;\xi)$ defined by (5.9) is bounded on $\mathcal{F}\times(\mathbb{R}^{3,1}+\mathcal{L})$ and smooth with respect to the Hilbert structure on this space. If $\xi\in\xi_\infty+W^{2,2}_{-1/2}(\mathcal{T})$, then for all $(g,\pi)\in\mathcal{F}$ and $(h,p)\in T_{(g,\pi)}\mathcal{F}$, we have*

$$D_{(g,\pi)}\mathcal{H}(g,\pi;\xi)(h,p) = -\int_{\mathcal{M}}(h,p)\cdot D\Phi(g,\pi)^*\xi. \tag{5.10}$$

Proof. As before, for smoothness it suffices to show that $\mathcal{H}$ is bounded on $\mathcal{F}\times(\mathbb{R}^{3,1}+\mathcal{L})$. Since $\|\xi-\hat{\xi}_\infty\|_{2,-1/2}\le C$ for $\xi\in\xi_\infty+\mathcal{L}$, the first two integrals of (5.9) may be estimated by

$$\left|\int_{\mathcal{M}}\left(\hat{\xi}^\alpha_\infty-\xi^\alpha\right)\Phi_\alpha(g,\pi)\right|\le\ \|\xi-\hat{\xi}_\infty\|_{2,-1/2}\,\|\Phi(g,\pi)\|_{2,-5/2},$$

which is bounded, by Theorem 3.1. The fourth and sixth integrals are bounded because $\mathring{\nabla}\hat{\xi}_\infty$ has compact support, and the third and fifth integrals are bounded, since Proposition 4.2 shows that $\mathcal{R}_o(g)-\Phi_0(g,\pi)$ and $\mathcal{P}_{oi}(\pi)-\Phi_i(g,\pi)$ are both integrable ($L^1(\mathcal{M})$). Hence $\mathcal{H}$ is bounded and therefore smooth, by the same arguments as used in Proposition 3.1. To show (5.10), we must separately consider the variational derivatives of the individual terms of (5.9). Since $\xi-\hat{\xi}_\infty\in W^{2,2}_{-1/2}(\mathcal{T})$, Theorem 5.1 may be applied to the variation of the first two integrals, which may then be rewritten as

$$\int_{\mathcal{M}}(h,p)\cdot D\Phi(g,\pi)^*(\hat{\xi}_\infty-\xi). \tag{5.11}$$

The variational derivative of the third and fourth terms of (5.9) may be rearranged using (4.8), (2.6), (5.5) to give

$$\begin{aligned}&\int_{\mathcal{M}}\left\{\mathring{\nabla}^i\left(\hat{\xi}^0_\infty\left(\mathring{\nabla}^j h_{ij}-\mathring{\nabla}_i\mathrm{tr}_{\mathring{g}}h\right)\right)\sqrt{\mathring{g}}-\hat{\xi}^0_\infty D\Phi_0(g,\pi)(h,p)\right\}\\ &\quad=\ \int_{\mathcal{M}}\left\{\mathring{\nabla}^i\left(\hat{\xi}^0_\infty\left(\mathring{\nabla}^j h_{ij}-\mathring{\nabla}_i\mathrm{tr}_{\mathring{g}}h\right)\right)\sqrt{\mathring{g}}-\nabla^i\left(\hat{\xi}^0_\infty\left(\nabla^j h_{ij}-\nabla_i\mathrm{tr}_g h\right)\right)\sqrt{g}\right.\\ &\qquad\qquad\left.+\nabla^i\left(\nabla^j\hat{\xi}^0_\infty h_{ij}-\nabla_i\hat{\xi}^0_\infty\mathrm{tr}_g h\right)\sqrt{g}-(h,p)\cdot D\Phi_0(g,\pi)^*(\hat{\xi}^0_\infty)\right\}.\end{aligned}$$

The dominant terms of the first two divergences in this expression cancel, and the remaining parts of the boundary term may therefore be written symbolically as $\hat{\xi}_\infty(g-\mathring{g})(\mathring{\nabla}h+h\mathring{\nabla}g)$. Now, $|\hat{\xi}_\infty|=O(1)$ and $g-\mathring{g}=o(R^{-1/2})$, and Lemma 4.4 serves to show that the remaining terms have well-defined traces, hence the boundary integral is $o(1)$ as $R\to\infty$. Consequently the variation of the third and fourth terms of (5.9) is just

$$-\int_{\mathcal{M}} (h,p)\cdot D\Phi_0(g,\pi)^*(\hat{\xi}^0_\infty).$$

The argument controlling the variational derivative of the final two terms of (5.9) is very similar, and results in the expression

$$-\int_{\mathcal{M}} (h,p)\cdot D\Phi_i(g,\pi)^*(\hat{\xi}^i_\infty),$$

from which the final identity (5.10) follows. □

6. Critical points of the ADM mass.

The results of the previous section, particularly Theorem 5.2, have an elegant interpretation in terms of critical points of the ADM mass. The fundamental observation is that stationary metrics are critical points of the ADM energy functional on the constraint manifold; and an argument implying the converse was suggested in [10]. In this section, we show that the phase space $\mathcal{F}$ and the regularised Hamiltonian functional $\mathcal{H}$ allow a rigorous presentation of the previously heuristic arguments relating stationary metrics and criticality properties of the ADM mass. The main result establishes the equivalence between critical points of the total energy and generalised Killing vectors.

Theorem 6.1. *Suppose $(g,\pi)\in\mathcal{F}$ satisfies $\Phi(g,\pi)=(\varepsilon,S_i)\in L^1(\mathcal{T}^*\otimes\Lambda^3)$, let $\xi_\infty\in\mathbb{R}^{3,1}$ be a fixed future timelike vector and define the energy functional $E\in C^\infty(\mathcal{C}(\varepsilon,S_i))$ by*

$$E(g,\pi)=\xi^\alpha_\infty\mathbb{P}_\alpha(g,\pi),\quad \forall\ (g,\pi)\in\mathcal{C}(\varepsilon,S_i). \tag{6.1}$$

Then the following two statements are equivalent:

(i) For all $(h,p)\in T_{(g,\pi)}\mathcal{C}(\varepsilon,S_i)$ we have

$$DE(g,\pi)(h,p)=0;$$

(ii) There is $\xi\in\xi_\infty+W^{2,2}_{-1/2}(\mathcal{T})$ satisfying

$$D\Phi(g,\pi)^*\xi=0.$$

If the energy-momentum covector $\mathbb{P}$ is timelike or null, then the ADM (total) mass can be defined,

$$m_{ADM} = \sqrt{-\mathbb{P}^\alpha \mathbb{P}_\alpha},$$

and in many applications, such as the quasi-local mass definition of [6], it is more natural to use m_{ADM} rather than the energy $E(g,\pi)$ with respect to the direction ξ_∞. The following corollary shows how Theorem 6.1 can be used to relate critical points of m_{ADM} to stationary metrics. The hypothesis that $\mathbb{P}$ be timelike follows from the extension in [7] of the spinorial proof [34] of the Positive Mass Theorem [29, 30, 34] to the decay and regularity condition $(g,\pi) \in \mathcal{C}(\varepsilon, S_i)$, assuming that the local energy-momentum density (ε, S_i) satisfies the Dominant Energy Condition

$$\xi^0 \varepsilon + \xi^i S_i \geq 0, \quad \text{for all future timelike vector fields } \xi \in C_c^\infty(\mathcal{T}).$$

Similarly, it is well-known that if ξ is a Killing vector, timelike near infinity, then $\mathbb{P}^\alpha$ and ξ_∞^α are proportional [9].

Corollary 6.2. *Suppose* $(g,\pi) \in \mathcal{F}$, $\Phi(g,\pi) = (\varepsilon, S_i) \in L^1(\mathcal{T}^*)$ *and* $\mathbb{P} = \mathbb{P}(g,\pi)$ *is a future timelike vector. If* $Dm_{ADM}(g,\pi)(h,p) = 0$ *for all* $(h,p) \in T\mathcal{C}(\varepsilon, S_i)$*, then* (g,π) *is a generalised stationary initial data set, with generalised Killing vector* ξ *such that* ξ_∞^α *is proportional to* $\mathbb{P}^\alpha = \eta^{\alpha\beta}\mathbb{P}_\beta(g,\pi)$*. Conversely, if* (g,π) *is a generalised stationary initial data set, with generalised Killing vector* ξ *such that* ξ_∞^α *is proportional to* $\mathbb{P}^\alpha$*, then* $Dm_{ADM}(g,\pi)(h,p) = 0$ *for all* $(h,p) \in T\mathcal{C}(\varepsilon, S_i)$*.*

Proof. If $\mathbb{P}_\alpha$ is a timelike vector, then we may choose $m_{ADM}\xi_\infty^\alpha = -\eta^{\alpha\beta}\mathbb{P}_\beta$, thereby normalising ξ_∞ to be a future unit timelike vector. Defining $E = \xi_\infty^\alpha \mathbb{P}_\alpha$, we have $Dm_{ADM} = \xi_\infty^\alpha D\mathbb{P}_\alpha = DE$, and m_{ADM} is critical on $\mathcal{C}(\varepsilon, S_i)$ exactly when E is critical also. Thus if (g,π) is a critical point for m_{ADM} on $\mathcal{C}(\varepsilon, S_i)$, then Theorem 6.1 shows that (g,π) admits a generalised Killing vector $\xi \in \xi_\infty + W^{2,2}_{-1/2}$, with ξ_∞ proportional to $(\mathbb{P}^\alpha)$.

Conversely, if (g,π) admits a generalised Killing vector ξ with ξ_∞ proportional to $(\mathbb{P}^\alpha)$, then defining $E(g',\pi') = \xi_\infty^\alpha \mathbb{P}_\alpha(g',\pi')$ with ξ normalised so ξ_∞ is a unit timelike vector, it follows that $DE(g,\pi) = 0$ on $\mathcal{C}(\varepsilon, S_i)$; since $Dm_{ADM} = DE$, we then have $Dm_{ADM}(g,\pi) = 0$ on $\mathcal{C}(\varepsilon, S_i)$. □

The proof of Theorem 6.1 is based on a generalisation of the classical method of Lagrange multipliers to Banach spaces, which we now recall. I am indebted to John Hutchinson for the following elegant proof.

Theorem 6.3. *Suppose $K : B_1 \to B_2$ is a C^1 map between Banach spaces, such that $DK(u) : B_1 \to B_2$ is surjective and splits (ie. $DK(u)$ has closed kernel, with closed complementary subspace), for every $u \in K^{-1}(0)$, and suppose $f \in C^1(B_1)$. Let $u \in K^{-1}(0)$ be given, then the following are equivalent:*

(*i*) *For all $v \in \ker DK(u)$, we have*

$$Df(u)v = 0;$$

(*ii*) *There is $\lambda \in B_2^*$ such that for all $v \in B_1$,*

$$Df(u)v = \langle \lambda, DK(u)v \rangle,$$

where $\langle\ ,\ \rangle$ denotes the dual pairing;

(*iii*) *Defining $F : B_1 \times B_2^* \to \mathbb{R}$, $F(u, \lambda) = f(u) - \langle \lambda, K(u) \rangle$, there is $\lambda \in B_2^*$ such that $DF(u, \lambda)(v, \mu) = 0$, for all $v \in B_1, \mu \in B_2^*$.*

We can paraphrase (*i*) by saying that "u is a critical point of f on $K^{-1}(0)$". The conditions on DK ensure that $K^{-1}(0)$ is a Banach submanifold of B_1, by the Implicit Function Theorem, and thus $T_u(K^{-1}(0)) = \ker DK(u)$. Clearly, λ is the infinite dimensional Lagrange multiplier.

Proof. The equivalence of (*ii*) and (*iii*) is obvious, as is the implication (*ii*) $\Rightarrow$ (*i*). If u is a critical point of f on $K^{-1}(0)$, then $\ker DK(u) \subset \ker Df(u) \subset B_1$, with both subspaces closed and having closed complements. It follows that there is a natural projection

$$\pi : B_1 / \ker DK(u) \to B_1 / \ker Df(u)$$

which is a bounded map of Banach (quotient) spaces. Since $Df(u) \in B_1^*$, we have a homomorphism $j_1 : B_1 / \ker Df(u) \to \mathbb{R}$. Since $DK(u)$ is surjective and splits, it factors as $DK(u) = j_2 \circ \pi_2$, where $\pi_2 : B_1 \to B_1 / \ker DK(u)$ and $j_2 : B_1 / \ker DK(u) \to B_2$ is an isomorphism. Then $\lambda = j_1 \circ \pi \circ j_2^{-1} : B_2 \to \mathbb{R}$ is a bounded linear map, ie. $\lambda \in B_2^*$, and $\lambda \circ DK(u) = j_1 \circ \pi \circ \pi_2 = Df(u)$, which gives (*ii*). □

To show (*ii*) $\Rightarrow$ (*i*) in Theorem (6.1), notice that for $(g, \pi) \in \mathcal{C}(\varepsilon, S)$, we have $\mathcal{H}(g, \pi; \xi) = E(g, \pi) - \int_{\mathcal{M}} (\xi^0 \varepsilon + \xi^i S_i)$ and thus

$$D_{(g,\pi)} \mathcal{H}(g, \pi; \xi)(h, p) = DE(g, \pi)(h, p), \quad \forall (h, p) \in T_{(g,\pi)} \mathcal{C}(\varepsilon, S).$$

But (ii) and Theorem 5.2 together imply that

$$D_{(g,\pi)}\mathcal{H}(g,\pi;\xi)(h,p) = 0 \quad \forall (h,p) \in \mathcal{G}\times\mathcal{K},$$

and (i) follows. To show the converse $(i) \Rightarrow (ii)$, choose any $\tilde{\xi} \in \xi_\infty + W^{2,2}_{-1/2}(\mathcal{T})$ and consider the functional

$$\tilde{H}(g',\pi') := \mathcal{H}(g',\pi';\tilde{\xi}), \quad (g,\pi) \in \mathcal{F}.$$

From (i), it follows that (g,π) is a critical point for both $\tilde{H}$ and $E = \xi^\alpha_\infty \mathbb{P}_\alpha$ on the submanifold $\mathcal{C}(\varepsilon, S)$. We may apply Theorem 6.3 with $B_1 = \mathcal{G}\times\mathcal{K} \supset \mathcal{F}$, $B_2 = \mathcal{L}^*$, $K = \Phi - (\varepsilon, S)$ and $f = \tilde{H}$; since (i) holds, there is $\lambda \in \mathcal{L} = L^2_{-1/2}(\mathcal{T})$ such that

$$D\tilde{H}(g,\pi)(h,p) = \int_{\mathcal{M}} \lambda^\alpha D\Phi_\alpha(g,\pi)(h,p) \tag{6.2}$$

for all $(h,p) \in \mathcal{G}\times\mathcal{K} = T_{(g,\pi)}\mathcal{F}$. Defining $\xi = \tilde{\xi} + \lambda \in \xi_\infty + L^2_{-1/2}(\mathcal{T})$ and inserting the definition of $\tilde{H}$ into (6.2) shows that $D_{(g,\pi)}\mathcal{H}(g,\pi;\xi) = 0$; Theorem 5.2 then implies $D\Phi(g,\pi)^*\xi = 0$ (weakly) and thus (by Proposition 3.5), it follows that $\xi \in \xi_\infty + W^{2,2}_{-1/2}(\mathcal{T})$ is a generalised Killing vector, as required. This completes the proof of Theorem 6.1.

Observe that under the conditions of Theorem 6.1, alternative (iii) of Theorem 6.3 shows that $(g,\pi;\xi)$ is a critical point in all $\mathcal{F}\times\mathcal{L}$ for the functional

$$\mathcal{H}(g,\pi;\xi) - \int_{\mathcal{M}} (\xi^0\varepsilon + \xi^i S_i).$$

References.

[1] J. Arms, J. E. Marsden, and V. Moncrief. The structure of the space of solutions of Einstein's equations II: Several Killing fields and the Einstein-Yang-Mills equations. *Annals Phys.*, 144(1):81–106, 1982.

[2] R. Arnowitt, S. Deser, and C. Misner. Coordinate invariance and energy expressions in general relativity. *Phys. Rev.*, 122:997–1006, 1961.

[3] R. Arnowitt, S. Deser, and C. Misner. The dynamics of general relativity. In L. Witten, editor, *Gravitation*, pages 227–265. Wiley, N.Y., 1962.

[4] H. Bahouri and J.-Y. Chemin. équations d'ondes quasilinéaires et estimations de Strichartz. *Am. J. Math.*, 121:1337–1377, 1999.

[5] R. Bartnik. The mass of an asymptotically flat manifold. *Comm. Pure Appl. Math.*, 39:661–693, 1986.

[6] R. Bartnik. New definition of quasilocal mass. *Phys. Rev. Lett.*, 62(20):2346–2348, May 1989.

[7] R. Bartnik and P. Chruściel. Boundary value problems for Dirac-type equations. *J. reine u. angewandte Math.*, 579:13–73, 2005.

[8] R. Bartnik. Mass and 3-metrics of non-negative scalar curvature. In Li Tatsien, editor, *Proceedings, ICM 2002*, volume II, pages 231–240. HEP Beijing, 2002. math.DG/0304259.

[9] R. Beig and P. T. Chruściel. Killing vectors in asymptotically flat space–times: I. Asymptotically translational Killing vectors and the rigid positive energy theorem. *J. Math. Phys.*, 37:1939–1961, 1996.

[10] D. Brill, S. Deser, and L. Fadeev. Sign of gravitational energy. *Phys. Lett.*, 26A(11):538–539, 1968.

[11] Y. Choquet-Bruhat and J. W. York. The Cauchy problem. In A. Held, editor, *General Relativity and Gravitation – the Einstein Centenary*, chapter 4, pages 99–160. Plenum, 1979.

[12] D. Christodoulou and N. Ó Murchadha. The boost problem in general relativity. *Comm. Math. Phys.*, 80:271–300, 1981.

[13] P. T. Chruściel. Boundary conditions at spatial infinity from a Hamiltonian point of view. In P. G. Bergmann and V. de Sabbata, editors, *Topological properties and global structure of space-time*. Plenum, New York, 1986.

[14] P. T. Chrusciel and E. Delay. Manifold structures for sets of solutions of the general relativistic constraint equations. *Jour. Geom. Phys.* 51:442–472, 2004.

[15] J. Corvino. Scalar curvature deformation and a gluing construction for the Einstein constraint equations. *Commun. Math. Phys.*, 214:137–189, 2000.

[16] J. Corvino and R. Schoen. On the asymptotics for the vacuum Einstein constraint equations. gr-qc/0301071.

[17] A. Einstein. Die Grundlage der allgemeinen Relativitätstheorie. *Annalen d. Phys.*, 49:769, 1916.

[18] A. E. Fischer and J. E. Marsden. Topics in the dynamics of general relativity. In J. Ehlers, editor, *Structure of Isolated Gravitating Systems*, pages 322–395. 1979.

[19] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer Verlag, 2nd edition, 1977.

[20] D. Hilbert. Grundlagen der Physik. *Nachr. Ges. Wiss. Göttingen*, page 395, 1915.

[21] E. Hille and R. Phillips. *Functional analysis and semi-groups*, volume 31 of *Colloquium Series*. Am. Math. Soc., 2 edition, 1957.

[22] S. Klainerman and I. Rodnianski. Rough solutions to the Einstein vacuum equations. math.AP/0109173.

[23] D. Maxwell. Rough solutions of the Einstein constraint equations. gr-qc/0405088, 2004.

[24] V. Moncrief. Spacetime symmetries and linearization stability of the Einstein equations I. *J. Math. Phys.*, 16(3):493–498, 1975.

[25] V. Moncrief. Spacetime symmetries and linearization stability of the Einstein equations II. *J. Math. Phys.*, 17(10):1893–1902, 1976.

[26] N. Ó Murchadha. Total energy momentum in general relativity. *J. Math. Phys.*, 27:2111–2128, 1986.

[27] J. M. Nester. The gravitational Hamiltonian. In F. J. Flaherty, editor, *Asymptotic behaviour of mass and space-time geometry (Oregon 1983)*, Lecture Notes in Physics 212, pages 155–163. Springer Verlag, 1984.

[28] T. Regge and C. Teitelboim. Role of surface integrals in the Hamiltonian formulation of general relativity. *Ann. Phys.*, 88:286–318, 1974.

[29] R. Schoen and S.-T. Yau. Proof of the positive mass theorem. *Comm. Math. Phys.*, 65:45–76, 1979.

[30] R. Schoen and S.-T. Yau. Proof of the positive mass theorem II. *Comm. Math. Phys.*, 79:231–260, 1981.

[31] E. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton UP, 1970.

[32] Daniel Tataru. Nonlinear wave equations. In Li Tatsien, editor, *Proceedings, ICM 2002*, volume III, pages 209–220. HEP Beijing, 2002. arXiv/math.AP/0304397.

[33] M. E. Taylor. *Pseudodifferential Operators.* Princeton UP, 1981.

[34] E. Witten. A simple proof of the positive energy theorem. *Comm. Math. Phys.*, 80:381–402, 1981.

[35] J. York. Energy and momentum of the gravitational field. In F. Tipler, editor, *Essays in general relativity in honour of A. Taub*, pages 39–58. Academic Press, 1980.

School of Mathematical Sciences
Monash University
Clayton, Vic. 3800
Australia
E-mail address: robert.bartnik@sci.monash.edu.au