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General finite type IFS and M-matrix

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In [14], Ngai and Wang introduced the concept of *finite type* IFS to study the Hausdorff dimension of self-similar sets without open set condition. In this paper, by applying the M-matrix theory([15]), we generalize the notion of finite type IFS to the *general finite type IFS*.

A family of IFS with 3 parameters, but without open set condition is presented. The Hausdorff dimension of the associated attractors can be calculated by both the M-matrix method and the general finite type IFS method. But these IFS are not finite type except for those parameters lying in a set of measure zero.

1. Introduction.

For any map $f : \mathbb{R}^n \to \mathbb{R}^d$ define

$$u(f) = \inf\{a \in \mathbb{R}; |f(v_1) - f(v_2)| \le a|v_1 - v_2| \text{ for all } v_1, v_2 \in \mathbb{R}^d\}. \tag{1.1}$$

Then, $|f(v_1) - f(v_2)| \le u(f)|v_1 - v_2|$ for all $v_1, v_2 \in \mathbb{R}^d$. f is called a contraction if $u(f) < 1$ and f is called a similarity if $|f(v_1) - f(v_2)| = u(f)|v_1 - v_2|$ for all $v_1, v_2 \in \mathbb{R}^d$. A finite set of contractions on $\mathbb{R}^d$ is called an IFS (iterated function system) on $\mathbb{R}^d$. For any IFS Φ on $\mathbb{R}^d$, there exists an unique compact subset $E \subset \mathbb{R}^d$ such that $\cup_{f\in\Phi} f(E) = E$. E is called the attractor of Φ and is denoted as $A(\Phi)$. Attractor of IFS is one of the most important kinds of fractals.

If Φ consists of contractive similarities, the attractor $A(\Phi)$ is called a self-similar set. When an IFS Φ on $\mathbb{R}^d$ consists of similarities and satisfies open set condition, in 1981 Hutchinson [4] proved that $\dim_H A(\Phi) = \alpha$, where α is the unique solution of Moran's equation[13].

$$\sum_{f\in\Phi} u(f)^\alpha = 1. \tag{1.2}$$

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In 1982, Dekking [2] constructed a new kind of fractals, called recurrent sets, which can be described as the limit set of an orbit on $\mathbb{R}^n$ of an semigroup endomorphism, and gave a formula to calculate its Hausdorff dimension. In 1986, Bedford [1] generalized the concept and the dimension formula of recurrent sets such that some kinds of self-similar set can be expressed as a recurrent set.

In 1988, Mauldin and Williams [9] studied the Hausdorff dimension of graph directed construction objects, which generalized the concept of recurrent sets. Their results include those of Hutchinson and Dekking, and has some parts similar to those of Bedford. They used a directed graph $G = (V, E)$ to construct the fractal set. A collection of contractive similarities $\{\phi_e, e \in E\}$ indexed by E is associated with the graph G. They proved that when the graph is strongly connected, there exists a unique collection of non-empty compact sets $\{F_v, v \in V\}$ such that $F_u = \cup_{v\in V, e\in E_{uv}} \phi_e(F_v)$, where E_{uv} is the set of edges from u to v. The graph directed construction objects is defined as $\cup_{v\in V} E_v$.

Attractors of IFS and recurrent sets are two different kinds of fractals. However, some attractors of IFS can be described as recurrent sets. The most interesting fact is that there are some special attractors of IFS, without open set condition, hence, we cannot calculate their Hausdorff dimension by Hutchinson–Moran formula, but these attractors can be described as recurrent sets and their Hausdorff dimension can be gotten by the methods given at [1, 2, 9].

In [15], we introduced the concepts of M-matrix and c-vector. An M-matrix means a matrix whose entries are sets of mappings. It is a generalization of IFS and graph directed construction. Meanwhile, the concepts of M-matrix are also generalization of Markov partition and Iterated Function Scheme which are used to study fractals in complex dynamic system (cf. [5, 10, 11, 12]). An IFS can be considered as an M-matrix of size 1×1. If E is a recurrent set generated by m contractions $\phi_i, i = 1, \ldots, m$, and recurrent positions $a_{ij} \in \mathbb{R}^n$ (see [1, 2]), we let $\phi_{ij}(x) = \phi_j(x) + a_{ij}$ and define M-matrix $M = (M_{ij})$ with $M_{ij} = \{\phi_{ij}\}$, then E is the union of the components of the maximal invariant c-vector of M. For the graph directed constructions in [9], it can be described as an M-matrix with each entry consists of contractive similarities and each row contains at least one non-empty entry.

In [14], Ngai and Wang introduced the notation of finite type IFS. The attractor of finite type IFS can be realized as recurrent sets. Hence, their Hausdorff dimensions are computable. A lot of IFS without open set condition are finite type. So, their Hausdorff dimension can be calculated according to Ngai and Wang's results and recurrent method. But as we shall see in Section 4 of this paper, there are some restrictions for an IFS to be finite type (see Theorems 4.1 and 4.2). In this paper, we remove these restrictions and define the *general finite type* IFS, which includes more IFS which are not finite type in Ngai and Wang's sense. A general finite type IFS can be equipped with an M-matrix. Then, we can calculate the Hausdorff dimension of a general finite type IFS by M-matrix theory.

A brief review of the M-matrix theory [15] is given in Section 2. In Section 3, we recall the definition of finite type IFS in [14]. In Section 4, we give a necessary condition for an IFS to be finite type and we find the relation between finite type IFS and M-matrix. In Section 5, we define the general finite type IFS. We prove that the attractor of a general finite type IFS can be realized by an invariant c-vector of an M-matrix which satisfies open set condition. Hence, we can calculate the Hausdorff dimension of the attractor of general finite type IFS by M-matrix theory. As an example, we study a family of general finite type IFS which are not finite type except some occasional cases.

2. M-matrix and c-vector.

Motivating from the self-similar property of fractals, we introduced the concepts of M-matrix and c-vector in [15]. An M-matrix means a matrix whose entries are sets of mappings. It is a generalization of IFS and graph directed construction. Meanwhile, the concepts of M-matrix and c-vector are also generalization of Markov partition and Iterated Function Scheme which are used to study fractals in complex dynamic system (cf. [5, 10, 11, 12]). An IFS can be considered as an M-matrix of size 1×1. If E is a recurrent set generated by m contractions $\phi_i, i = 1, \dots, m$, and recurrent positions $a_{ij} \in \mathbb{R}^n$ (see [1, 2]), we let $\phi_{ij}(x) = \phi_j(x) + a_{ij}$ and define M-matrix $M = (M_{ij})$ with $M_{ij} = \{\phi_{ij}\}$, then E is the union of the components of the maximal invariant c-vector of M. For the graph directed constructions in [9], it can be described as an M-matrix where each entry consists of contractive similarities and each row contains at least one non-empty entry.

In this section, we give a brief description of the M-matrix and c-vector theory ([15]).

Definition 2.1. An *M-matrix* is a matrix whose entries are finite sets of mappings on $\mathbb{R}^d$. The set of all M-matrices of size $m \times n$ is denoted as $\mathfrak{M}(m, n)$. We use $\mathfrak{M}_c(m, n)$ (or $\mathfrak{M}_s(m, n)$) to express the set of all M-matrices whose entries are finite sets of contractions (or similarities, respectively).

For any two sets of mappings Φ and Ψ, define $\Phi\Psi = \{f \circ g | f \in \Phi, g \in \Psi\}$. We shall frequently identify $\{f\}$ with f for a mapping f and write fg for $f \circ g$. For a set of mappings Φ and a subset $X \subseteq \mathbb{R}^d$, define $\Phi(X) = \cup_{f \in \Phi} f(X)$. For any M-matrix $M = (M_{ij}) \in \mathfrak{M}(m,n)$ and $N = (N_{ij}) \in \mathfrak{M}(n,p)$, define $MN \in \mathfrak{M}(m,p)$ with (i,j) entry $\cup_q M_{iq} N_{qj}$. For M-matrices $(M_{ij}), (N_{ij}) \in \mathfrak{M}(m,n)$, we define $(M_{ij}) \cup (N_{ij}) = (M_{ij} \cup N_{ij})$ and $(M_{ij}) \cap (N_{ij}) = (M_{ij} \cap N_{ij})$. Then, $(\mathfrak{M}(m,m), \cup, \cap, \cdot)$ satisfies some algebraic law with $\emptyset$ as "zero" element and I as "unit" element, where $\emptyset$ is the M-matrix whose entries are all empty sets and $I = \text{diag}(\{1\}, \ldots, \{1\})$, 1 is the identity mapping. We call $\mathfrak{M}(m,m)$ *M-algebra* for the time being. $\mathfrak{M}_c(m,m)$ and $\mathfrak{M}_s(m,m)$ form two M-subalgebras of $\mathfrak{M}(m,m)$. An M-matrix $M \in \mathfrak{M}(m,m)$ is called invertible if there exists $N \in \mathfrak{M}(m,m)$ such that $MN = NM = I$ and we denote N by M^{-1}. A permutation M-matrix is an M-matrix $P = (P_{ij}) \in \mathfrak{M}(m,m)$ such that each column and row has precisely one non-empty entry $P_{ij} = \{1\}$, where 1 is the identity mapping, all the other entries are empty set. An M-matrix $M \in \mathfrak{M}(m,m)$ is invertible if and only if $M = P\text{diag}(\{\phi_1\}, \ldots, \{\phi_m\})$ for some permutation M-matrix P and $M^{-1} = \text{diag}(\{\phi_1^{-1}\}, \ldots, \{\phi_m^{-1}\})P^t$.

We write $M^1 = M$ and $M^q = MM^{q-1}$ for $q \geq 2$ when $M \in \mathfrak{M}(m,m)$. For an M-matrix $M = (M_{ij}) \in \mathfrak{M}(m,m)$, we call M is *irreducible* if for any $1 \leq i,j \leq m$, there exist some $k_1, \ldots, k_s \in \{1, \ldots, m\}$ such that $M_{ik_1} M_{k_1k_2} \ldots M_{k_sj} = \{f_0 \circ f_1 \circ \ldots \circ f_s | f_0 \in M_{ik_1}, f_i \in M_{k_ik_{i+1}}, f_s \in M_{k_sj}\} \neq \emptyset$. The *standard form* of M-matrix is given by the following theorem:

Theorem 2.2 ([15]). For any $M \in \mathfrak{M}(m,m)$, there exists a permutation M-matrix $P \in \mathfrak{M}(m,m)$ such that

$$PMP^t = \begin{pmatrix} H_1 & B_{12} & \cdots & B_{1s} \\ \emptyset & H_2 & \cdots & B_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \emptyset & \emptyset & \cdots & H_s \end{pmatrix}$$

where, for each $i = 1, \ldots, s$, either $H_i \in \mathfrak{M}(m_i, m_i)$ is irreducible with $m_i > 0$ or $H_i = (\{\emptyset\})$.

Now, we consider the vector of subsets of $\mathbb{R}^d$. We shall call a vector whose components are compact subsets of $\mathbb{R}^d$ to be a c-vector. Let $\overline{\mathcal{K}}^m$ (and $\mathcal{K}^m$) be the set of all c-vectors with m entries (or m non-empty entries respectively):

$$\begin{aligned}\overline{\mathcal{K}}^m &= \{(E_1, \ldots, E_m)^t | E_i \subset X \text{ is compact } \},\\ \mathcal{K}^m &= \{(E_1, \ldots, E_m)^t | E_i \subset X \text{ is compact }, E_i \neq \emptyset\}.\end{aligned}$$

Define operations "$\cup$" and "$\cap$" on $\overline{\mathcal{K}}^m$ by

$$(E_1, \ldots, E_m)^t \cup (F_1, \ldots, F_m)^t = (E_1 \cup F_1, \ldots, E_m \cup F_m)^t$$

and

$$(E_1, \ldots, E_m)^t \cap (F_1, \ldots, F_m)^t = (E_1 \cap F_1, \ldots, E_m \cap F_m)^t.$$

For an M-matrix $M = (M_{ij}) \in \mathfrak{M}(m, n)$ and a vector $X = (X_1, \ldots, X_n)^t$ $(X_i \subseteq \mathbb{R}^d)$, we define

$$M(X) = \begin{pmatrix} \cup_q M_{1q}(X_q) \\ \cup_q M_{2q}(X_q) \\ \vdots \\ \cup_q M_{mq}(X_q) \end{pmatrix}. \tag{2.1}$$

For $M \in \mathfrak{M}(m, m)$ and $E \in \mathcal{K}^m$, if $M(E) = E$, we call E to be an invariant c-vector of M. Denote the maximal invariant c-actor of M by $A(M)$.

The following theorem gives a complete description about the invariant c-vectors of an M-matrix M when $M^k \in \mathfrak{M}_c(m, m)$ for some k.

Theorem 2.3 ([15]). Let $M \in \mathfrak{M}(m, m)$. Suppose there is some integer $k > 0$ such that $M^k \in \mathfrak{M}_c(m, m)$.

(1) $A(M) \in \overline{\mathcal{K}}^m$ exists and

$$\begin{cases} \overline{\cup_{i\geq 1} M^i(F)} = \cup_{i\geq 1} M^i(F) \cup A(M) \in \overline{\mathcal{K}}^m, & \forall F \in \mathcal{K}^m, \\ \cap_{k\geq 1}\overline{\cup_{i\geq k} M^i(F)} = A(M), \forall F \in \mathcal{K}^m, \\ \overline{\cup_{i\geq 1} M^i(F)} \subseteq \cup_{i\geq 1} M^i(F) \cup A(M) \in \overline{\mathcal{K}}^m, & \forall F \in \overline{\mathcal{K}}^m, \\ \cap_{k\geq 1}\overline{\cup_{i\geq k} M^i(F)} \subseteq A(E), \forall F \in \overline{\mathcal{K}}^m. \end{cases}$$

(2) If $E \in \mathcal{K}^m$ and $M(E) = E$, then $A(M) = E$.

(3) For the empty M-matrix $\emptyset$, $A(\emptyset) = \emptyset$.

(4) If M is irreducible, then there are only two M-invariant c-vectors: $A(M) \in \mathcal{K}^m$ and $\emptyset$.

(5) Suppose $P \in \mathfrak{M}(m,m)$ is an invertible M-matrix. Then, E is an invariant c-vector of M if and only if $P(E)$ is invariant under PMP^{-1}. In particular, $A(PMP^{-1}) = P(A(M))$.

(6) Suppose $M = \begin{pmatrix} H_1 & B \\ \emptyset & H_2 \end{pmatrix}$ with $H_i \in \mathfrak{M}(m_i, m_i)$, $B \in \mathfrak{M}(m_1, m_2)$, $m_1 + m_2 = m$, and $m_1, m_2 > 0$. Then, any invariant c-vector E of M must has the form

$$\begin{pmatrix} E_1 \cup \overline{(\cup_{i=0}^{\infty} H_1^i B(E_2))} \\ E_2 \end{pmatrix},$$

E_1 and E_2 are invariant c-vectors of H_1 and H_2 respectively. In particular,

$$A(M) = \begin{pmatrix} A(H_1) \cup (\cup_{i=0}^{\infty} H_1^i B(A(H_2))) \\ A(H_2) \end{pmatrix}.$$

Furthermore, $A(M) = \begin{pmatrix} A(H_1) \\ \emptyset \end{pmatrix}$ if $H_2 = \emptyset$ and $A(M) = \begin{pmatrix} A(H_1) \\ A(H_2) \end{pmatrix}$ if $B = \emptyset$.

Similar with the IFS case, we define the open set condition for M-matrix as follows.

Definition 2.4. Let $M = (M_{ij}) \in \mathfrak{M}(m,m)$ and $U = (U_1, \ldots, U_m)^t$, where $U_i \subseteq \mathbb{R}^d$ are non-empty bounded open sets. M satisfies *open set condition* with respect to U if

(1) $\cup_j M_{ij}(U_j) \subset U_i$, $i = 1, \ldots, m$;

(2) $M_{ij}(U_j) \cap M_{ij'}(U_{j'}) = \emptyset$ if $j \neq j'$;

(3) $\phi(U_j) \cap \psi(U_j) = \emptyset$ if $\phi, \psi \in M_{ij}$ and $\phi \neq \psi$.

When M satisfies open set condition with respect to U, M^k satisfies open set condition with respect to U and M satisfies open set condition with respect to $M^k(U)$ for any $k = 1, 2, \ldots$. While we use M-matrices to describe recurrent sets [1, 2], graph directed objects [9], conformal iterated function schemes [5] and Markov partition for conformal dynamic systems equipped with invariant densities [10, 11, 12], it can be proved that all these M-matrices must satisfy the open set condition for M-matrix.

For an M-matrix $M = (M_{ij}) \in \mathfrak{M}(m,m)$ and a real number $x \geq 0$, we define a numerical matrix $F(M,x)$ by

$$F(M,x) = (\sum_{f \in M_{ij}} u(f)^x)_{1 \leq i,j \leq m}, \tag{2.2}$$

where $u(f)$ is defined by (1.1). The eigenvalue of $F(M,x)$ has closed relation with the Hausdorff dimension of the components of invariant c-vector of M.

Theorem 2.5 ([15]). Let $M \in \mathfrak{M}(m,m)$ and $M^k \in \mathfrak{M}_c(m,m)$ for some k. Suppose P is a permutation M-matrix such that

$$PMP^t = \begin{pmatrix} H_1 & B_{12} & \cdots & B_{1s} \\ \emptyset & H_2 & \cdots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \emptyset & \emptyset & \cdots & H_s \end{pmatrix}$$

where $H_i \in \mathfrak{M}(m_i, m_i)$, $i = 1, \ldots, s$, are either irreducible or empty with size 1×1.

(1) $A(M)$, $A(PMP^t)$ and each $A(H_i)$, $i = 1, \ldots, s$, exist.

$$A(M) = P^t A(PMP^t) = \cup_{i=0}^{\infty} M^i P^t \begin{pmatrix} A(H_1) \\ \vdots \\ A(H_s) \end{pmatrix}. \tag{2.3}$$

(2) Write $A(PMP^t) = P(A(M))$ as $(\mathcal{E}^{(1)}, \ldots, \mathcal{E}^{(s)})^t$, where $\mathcal{E}^{(i)} \in \mathcal{K}(m_i)$, $i = 1, \ldots, s$. Then, for any fixed i, each component of $\mathcal{E}^{(i)}$ has the same Hausdorff dimension. Write this dimension as $\dim_H \mathcal{E}^{(i)}$, then

$$\dim_H \mathcal{E}^{(i)} = \max\{\dim_H A(H_i), \dim_H \mathcal{E}^{(j)} | B_{ij} \neq \emptyset\}.$$

(3) Let D_i be the unique number such that the biggest real eigenvalue of $F(H_i, D_i)$ is 1. Then, $\dim_H A(H_i) \leq D_i$. In particular, if $H_i \in \mathfrak{M}_s(m_i, m_i)$ and the open set condition holds for H_i, then $\dim_H A(H_i) = D_i$.

Let $M = (M_{ij}) \in \mathfrak{M}(m,m)$ be a M-matrix. Let k_{ij} be the cardinality of M_{ij}. The integer matrix $T = (k_{ij})$ is called the incidence matrix of M. The following result will be used to compare with the results of [14].

Corollary 2.6. Let $M = (M_{ij}) \in \mathfrak{M}(m,m)$ be an irreducible M-matrix and $0 < \rho < 1$. Suppose that each mapping $\phi \in \cup_{ij} M_{ij}$ is a similarity with contractive ratio ρ. If the open set condition holds for M, then the Hausdorff dimension of each component of $A(M)$ is

$$-\frac{\ln \lambda}{\ln \rho} \quad , \tag{2.4}$$

where λ is the maximal real eigenvalue of the incidence matrix $T = (k_{ij})$ of M, where $k_{ij} = |M_{ij}|$.

3. IFS of finite type (Ngai and Wang [14]).

In this section, we recall the notion of finite type IFS of Ngai and Wang[14]. Let $\Phi = \{\phi_j; 1 \le j \le q\}$ be an IFS on $\mathbb{R}^d$, where each ϕ_j is a contractive similarity with similar ratio $u(\phi_j) = \rho_j$, $0 < \rho_j < 1$. The code space Σ_q concerned with Φ is defined as the set of all finite sequence in $\{1, \dots, q\}$:

$$\Sigma_q = \{(j_1, \dots, j_k) | 1 \le j_1, \dots, j_k \le q, k = 1, 2, \dots\}.$$

An element $\mathbf{j} = (j_1, \dots, j_k) \in \Sigma_q$ is called a word of length k. We denote the length of $\mathbf{j}$ by $|\mathbf{j}|$. In particular, we can define an *empty word* $\emptyset$ which has length 0. And we define $\Sigma_q^* = \{\emptyset\} \cup \Sigma_q$. For $\mathbf{i} \in \Sigma_q^*$ and $\mathbf{j} \in \Sigma_q^*$, let $\mathbf{ij} \in \Sigma_q^*$ be the concatenation of $\mathbf{i}$ and $\mathbf{j}$, and call $\mathbf{i}$ an *initial (proper) segment* of $\mathbf{ij}$ (if $\mathbf{j} \ne \emptyset$).

For $\mathbf{j} = (j_1, \dots, j_m) \in \Sigma_q^*$, define

$$\phi_{\mathbf{j}} := \begin{cases} \phi_{j_1} \circ \dots \circ \phi_{j_m}, & \text{if } |\mathbf{j}| \ge 1 \\ 1, \text{ the identity mapping}, & \text{if } \mathbf{j} = \emptyset \end{cases} \tag{3.1}$$

and

$$\rho_{\mathbf{j}} := \begin{cases} \rho_{j_1} \cdots \rho_{j_m} & \text{if } |\mathbf{j}| \ge 1 \\ 1, & \text{if } \mathbf{j} = \emptyset \end{cases} \tag{3.2}$$

Then, $\phi_{\mathbf{j}}$ is a contractive similarity with similar ratio $u(\phi_{\mathbf{j}}) = \rho_{\mathbf{j}}$.

Let $\rho = \min\{\rho_j, j = 1 \dots, q\}$. Define

$$\begin{cases} \Lambda_0 = \{\emptyset\}, \\ \Lambda_k = \{\mathbf{j} \in \Sigma_q^* | \begin{array}{l} \rho_{\mathbf{j}} \le \rho^k \text{ but } \rho_{\mathbf{i}} > \rho^k \text{ if } \mathbf{i} \text{ is a} \\ \text{proper initial segment of } \mathbf{j} \end{array} \}, \quad k = 1, 2, \dots. \\ \\ \mathrm{V}_k = \{\phi_{\mathbf{j}} | \mathbf{j} \in \Lambda_k\}, \quad k = 0, 1, 2, \dots \\ \mathrm{V} = \cup_{k \ge 0} \mathrm{V}_k. \end{cases} \tag{3.3}$$

A non-empty bounded open set $\Omega \subset \mathbb{R}^d$ is invariant under IFS Φ if $\phi_j(\Omega) \subseteq \Omega$ for all j. Such an Ω always exists. We say two mappings $f_1, f_2 \in \mathrm{V}_k$ are neighbors (with respect to Ω) if $f_1(\Omega) \cap f_2(\Omega) \neq \emptyset$. For $f \in \mathrm{V}_k$, the set

$$\Omega(f) := \{g \in \mathrm{V}_k | g \text{ is a neighbor of } f\}$$

is called the neighborhood of f (with respect to Ω). Two mappings $f_1 \in \mathrm{V}_k$ and $f_2 \in \mathrm{V}_{k'}$ are said to have the same neighborhood type if there exists a similarity τ with similar ratio $\rho^{k-k'}$ such that

$$\Omega(f_1) = \tau\Omega(f_2) \quad \text{and } f_1 = \tau \circ f_2. \tag{3.4}$$

We denote it by $f_1 \sim f_2$ (or $f_1 \overset{\tau}{\sim} f_2$ to indicate the mapping τ). Then, "$\sim$" is an equivalence relation on V. The IFS Φ is said to be of finite type if there are finite many distinct neighborhood types.

Using V as the set of vertices, we can define a graph G as follows. Give two mappings $f, g \in V$, if $f = \phi_{\mathbf{i}}$ with $\mathbf{i} \in \Lambda_k$, $g = \phi_{\mathbf{j}}$ with $\mathbf{j} \in \Lambda_{k+1}$ and there exists an word $\mathbf{l} \in \Sigma_q^*$ such that $\mathbf{j} = \mathbf{il}$, then we connect a directed edge $\mathbf{l} : f \mapsto g$. We call f a parent of g and g an offspring of f.

Notice that a vertex in G might have several parents. We will remove some edges from G so that every vertex has at most one parent. To do this, we use the lexicographical order on Σ_q^*. For each vertex $f \in \mathrm{V}$, let $\mathbf{l}_1, \ldots, \mathbf{l}_p$ be all the directed edges going from some vertices to f. Suppose that $\mathbf{l}_1 < \ldots < \mathbf{l}_p$ in the lexicographical order. Then, we keep $\mathbf{l}_1$ and remove all other edges. Thus, we obtain G_R. If f is a parent of g (i.e., g is an offspring of f) in the reduced graph G_R, we denote it as $f \gg g$.

The incidence matrix $S = (s_{\alpha\beta})$ for the IFS Φ is defined as follows. Suppose that there are m neighborhood types. Then, the size of S is $m \times m$. Choose any mappings $v \in \mathrm{V}$ that has neighborhood type α. Its offspring in $\mathcal{G}_R$ will have various neighborhood types β. The entry $s_{\alpha\beta}$ denotes the number of offsprings that have neighborhood type β. From the following Lemma, it is proved that S is well defined (cf. [14]).

Lemma 3.1. Suppose $f_1, f_2 \in \mathrm{V}$ and $f_1 \overset{\tau}{\sim} f_2$. Let g be an offspring of f_2 with edge $\mathbf{l} : f_2 \mapsto g$. Then, $\tau \circ g \overset{\tau}{\sim} g$ and $\tau \circ g$ is an offspring of f_1 with edge $\mathbf{l} : f_1 \mapsto \tau \circ g$.

The following is the main theorem of [14]

Theorem 3.2. Let $\Phi = \{\phi_j; 1 \leq j \leq q\}$ be an IFS on $\mathbb{R}^d$, where each ϕ_i is a contractive similarity with similar ratio ρ_i, $0 < \rho_i < 1$. Suppose that Φ is of finite type with respect to a bounded invariant open set Ω, and let S be the corresponding incidence matrix. Then, the Hausdorff dimension of the attractor $F = A(\Phi)$ is

$$\dim_H(F) = -\frac{\log \lambda}{\log \rho}, \tag{3.5}$$

where $\rho = min_j \rho_j$ and λ is the maximal real eigenvalue of S.

Remark 3.3. The incidence matrix S of Φ depends on the open set Ω. It can be proved that if Φ is a finite type IFS with respect to open set Ω with incidence matrix S and Ω' is another open set such that $\Phi(\Omega') \subseteq \Omega'$, then Φ is also finite type with respect to Ω'. Suppose the incidence matrix of Φ with respect to Ω' is S', then $S' \neq S$ in most cases.

4. Necessary conditions for an IFS to be finite type.

In this section, we will discuss the conditions for an IFS to be finite type.

Theorem 4.1. Let $\Phi = \{\phi_i; 1 \leq i \leq q\}$ be an IFS on $\mathbb{R}^d$, where each ϕ_i is a contractive similarity with similar ratio ρ_i, $0 < \rho_i < 1$. Suppose that Φ is of finite type. Then, there exists a real number $0 < \rho_0 < 1$ such that each $\rho_i = \rho_0^{k_i}$ for some positive integer k_i, $1 \leq i \leq q$.

Proof. Assume that $\rho_1 \leq \rho_2 \leq \ldots \leq \rho_q$. Let $\mathcal{P} = \{\rho_1^a | a \in \mathbb{Q}, a > 0\}$. If $\{\rho_1, \ldots, \rho_q\} \not\subset \mathcal{P}$, then there exists an i_0, $2 \leq i_0 \leq q$, such that

$$\rho_{i_0} \notin \mathcal{P}. \tag{4.1}$$

For any $k > 0$, there exists an positive integer n_k such that

$$\rho_{i_0}^{n_k} \leq \rho_1^k \text{ and } \rho_{i_0}^{n_k - 1} > \rho_1^k$$

So, by (3.3),

$$\mathbf{j}_k := (\underbrace{i_0 \ldots i_0}_{n_k}) \in \Lambda_k.$$

Hence, $\phi_{\mathbf{j}_k} \in \mathrm{V}_k$, $k = 1, 2, \ldots$. As Φ is of finite type, so there exist $k \neq k'$ such that $\phi_{\mathbf{j}_k}$ and $\phi_{\mathbf{j}_{k'}}$ have the same neighborhood type. Thus, there exists a similarity τ with similar ratio $\rho_1^{k'-k}$ such that

$$\tau \circ \phi_{\mathbf{j}_k} = \phi_{\mathbf{j}_{k'}}$$

The similar ratio of $\tau \circ \phi_{\mathbf{j}_k}$ is $\rho_1^{k'-k}\rho_{i_0}^{n_k}$. The similar ratio of $\phi_{\mathbf{j}_{k'}}$ is $\rho_{i_0}^{n_{k'}}$. Hence,

$$\rho_1^{k'-k}\rho_{i_0}^{n_k} = \rho_{i_0}^{n_{k'}}$$

This implies

$$\rho_{i_0} = \rho_1^{(k'-k)/(n_{k'}-n_k)} \in \mathcal{P}$$

This contradict with (4.1). Thus, we have $\{\rho_1, \dots, \rho_q\} \subset \mathcal{P}$.

As $\{\rho_1, \dots, \rho_q\} \subset \mathcal{P}$, so there exist $\frac{s_i}{t_i} \in \mathbb{Q}$, $i = 1, \dots, q$, where s_i and t_i are positive integers, such that

$$\rho_i = \rho_1^{s_i/t_i}, i = 1, \dots, q$$

Let T be the minimal common multiple of $t_1, \dots, t_q$. Then, $\frac{Ts_i}{t_i}$ are integers. Let $\rho_0 = \rho_1^{1/T}$ and $k_i = \frac{Ts_i}{t_i}$, then $\rho_i = \rho_0^{k_i}$. □

In view of Theorem 4.1, one can see that even though there exist some finite type IFS without open set condition, finite type condition is quite restrictive. Furthermore, we have the following result.

Theorem 4.2. An attractor of any finite type IFS must be a component of an invariant c-vector of an M-matrix $M = (M_{ij})$ with open set condition such that each mapping $\phi \in \cup_{ij} M_{ij}$ is a similarity with a fixed contractive ratio $0 < \rho < 1$.

Proof. Let $\Phi = \{\phi_j; 1 \le j \le q\}$ be a finite type IFS on $\mathbb{R}^d$ respect to open set Ω with m neighborhood types and incidence matrix $S = (s_{ij})$. Let $\rho = \min_{1 \le j \le q}\{u(\phi_j)\}$. Let $F = A(\Phi)$.

For any $f \in \mathrm{V}$, define

$$\begin{aligned}
&C^0(f) = \{f\},\\
&C(f) = C^1(f) := \{g \in \mathrm{V} | f \gg g\}, \text{ the set of all offsprings of } f,\\
&C^{i+1}(f) = \{g \in \mathrm{V} | h \gg g \text{ for some } h \in C^i(f)\}, i = 1, 2, \dots.
\end{aligned}$$

Then, it is easy to see that

$$C^i(1) = \mathrm{V}_i, i = 1, 2, \dots, \tag{4.2}$$

where V_i is defined by (3.3). For each neighborhood type $i = 1, \dots, m$, we choose one mappings $v_i \in \mathrm{V}_{k_i}$ that has neighborhood type i. In particular, we can assume that $v_1 = 1 = \phi_\emptyset \in \mathrm{V}_0$, the identity mapping. Suppose

$$C(v_i) = \{g_{ijt} | j = 1, \dots, m, 1 \le t \le s_{ij}\}, \tag{4.3}$$

where g_{ijt} has neighborhood type j. As $v_i \in \mathrm{V}_{k_i}$, so $g_{ijt} \in \mathrm{V}_{k_i+1}$. But g_{ijt} and v_j has the same neighborhood type, so there exists a similarity τ_{ijt} with similar ratio $\rho^{k_i-k_j+1}$ such that (see (3.4))

$$g_{ijt} \overset{\tau_{ijt}}{\sim} v_j. \tag{4.4}$$

Hence,

$$g_{ijt} = \rho^{k_j-k_i}\tau_{ijt} \circ (\rho^{k_i-k_j} v_j). \tag{4.5}$$

Write $\rho^{k_j-k_i}\tau_{ijt}$ as $\hat{\tau}_{ijt}$. Then, $\hat{\tau}_{ijt}$ is a similarity with similar ratio ρ and

$$\rho^{-k_i} g_{ijt} = \hat{\tau}_{ijt} \circ \rho^{-k_j} v_j. \tag{4.6}$$

Let $M_{ij} = \{\hat{\tau}_{ijt} | 1 \leq t \leq s_{ij}\}$. Define M-matrix $M = (M_{ij})$. Then, by (4.3) and (4.6), we have

$$\begin{pmatrix} \rho^{-k_1} C(v_1) \\ \vdots \\ \rho^{-k_m} C(v_m) \end{pmatrix} = M \begin{pmatrix} \{\rho^{-k_1} v_1\} \\ \vdots \\ \{\rho^{-k_m} v_m\} \end{pmatrix}. \tag{4.7}$$

Using Lemma 3.1, by induction on p, we can prove that

$$\begin{pmatrix} \rho^{-k_1} C^p(v_1) \\ \vdots \\ \rho^{-k_m} C^p(v_m) \end{pmatrix} = M^p \begin{pmatrix} \{\rho^{-k_1} v_1\} \\ \vdots \\ \{\rho^{-k_m} v_m\} \end{pmatrix}. \tag{4.8}$$

Now, M is a M-matrix. Its incidence matrix T is the same as the incidence matrix S of Φ. We shall prove that F is a component of $A(M)$. Let

$$\mathcal{F} = \begin{pmatrix} \{\rho^{-k_1} v_1\} \\ \vdots \\ \{\rho^{-k_m} v_m\} \end{pmatrix} (F) = \begin{pmatrix} \rho^{-k_1} v_1(F) \\ \vdots \\ \rho^{-k_m} v_m(F) \end{pmatrix}.$$

Then, by (1) of Theorem 2.3,

$$A(M) = \cap_{i \geq 1} \overline{\cup_{p \geq i} M^p(\mathcal{F})} \tag{4.9}$$

But

$$F = \mathrm{V}_0(F) = \ldots = \mathrm{V}_p(F) = \ldots.$$

So, by (4.2),

$$F = C^0(1)(F) = \ldots = C^p(1)(F) = \ldots. \tag{4.10}$$

Notice that v_1 is the identity mapping and $\rho^{-k_1} = 1$, (4.8) and (4.10) imply that the first component of $M^p(\mathcal{F})$ is always F. Thus, by (4.9), the first component of $A(M)$ is F.

Finally, let

$$(4.11)\quad U_i = \rho^{-k_i}[\cup_{p\geq 0}(C^p(v_i)(\Omega) - \cup_{f\in\Omega(v_i), f\neq v_i} C^p(f)(\overline{\Omega}))], i = 1, \ldots, m,$$

where Ω is the open set mentioned at the begining of the proof. Then, U_i are open sets and it is easy to check that M satisfies open set condition with respect to $U_1, \ldots, U_m$ (see Definition 2.4). $\square$

Remark 4.3. According to Theorem 4.2, we can apply Corollary 2.6 to finite type IFS. This will yield Theorem 3.2. In other words, we give a new proof of Theorem 3.2.

Remark 4.4. There are more than one way to represent the attractor of a IFS as a component of an invariant c-vector of a M-matrix. Theorem 4.2 shows that the neighborhood type method of Ngai and Wang is an useful method for getting a suitable M-matrix for finite type IFS. In the next section, we will generalize this method to study more IFS.

5. General finite type IFS.

In this section, we will define the (general) finit type IFS. Let $\Phi = \{\phi_j; 1 \leq j \leq q\}$ be an IFS on $\mathbb{R}^d$ with attractor $E = A(\Phi)$, where each ϕ_i is a contractive similarity with similar ratio $u(\phi_j) = \rho_i$, $0 < \rho_i < 1$. Let Σ_q be the code space concerned with Φ and $\Sigma_q^* = \{\emptyset\} \cup \Sigma_q$. Let $\mathcal{G} = \{1\} \cup \Phi \cup \Phi^2 \cup \ldots = \{\phi_{\mathbf{i}} | \mathbf{i} \in \Sigma_q^*\}$.

We define a finite set of mappings $V \subset \mathcal{G}$ to be a section if: (1) For any mapping $f \in \mathcal{G}$, there exist $h, g, \psi \in \mathcal{G}$ such that $\psi \in V$ and $fh = \psi g$; (2) For any two mappings $f, g \in V$, if $f = gh$ for some $h \in \mathcal{G}$, then $f = g$. Sections always exist (For example, $\Psi = \Phi \setminus \{f \in \Phi | \exists g \in \Phi \textit{ and } h \in \mathcal{G}, h \neq 1, \ni f = gh\}$ is a section). If V is a section, then $V(E) = E$.

For two sections $V_1, V_2 \subset \mathcal{G}$, we write $V_1 \succ V_2$ if for each $f \in V_2$, there exists $g \in V_1$ and $h \in \mathcal{G}$, $h \neq 1$, such that $f = gh$. We call a sequence of sections $(V_i, i = 0, 1, 2, \ldots)$ to be a *flag* if $V_0 = \{1\}$ and $V_i \succ V_{i+1}$ for each $i = 0, 1, 2, \ldots$. (For example, (3.3) defines a flag.)

Let $(V_i, i = 0, 1, 2, \ldots)$ be a flag and $\Omega \subset \mathbb{R}^d$ be a non-empty bounded open set which is invariant under Φ. We say two mappings $f_1, f_2 \in V_k$ are neighbors (with respect to Ω) if $f_1(\Omega) \cap f_2(\Omega) \neq \emptyset$. For $f \in V_k$, the set

$\Omega(f) := \{g \in \mathrm{V}_k | g$ is a neighbor of $f\}$ is called the neighborhood of f (with respect to Ω). Two mappings $f_1 \in V_k$ and $f_2 \in V_{k'}$ are said to have the same neighborhood type (in general sense) if there exists a similarity τ such that

$$\Omega(f_1) = \tau\Omega(f_2) \quad \text{and } f_1 = \tau \circ f_2. \tag{5.1}$$

(Notice that, here, we omit the restriction on the similar ratio of τ in (3.4).) We denote it by $f_1 \sim_2 f_2$ (or $f_1 \overset{\tau}{\sim}_2 f_2$ to indicate the mapping τ). Let $V = \cup_k V_k$. It is clear that for any $f_1, f_2, f_3 \in V$, (1). $f_1 \overset{1}{\sim}_2 f_1$, where 1 is the identity mappings; (2). $f_1 \overset{\tau}{\sim}_2 f_2 \Rightarrow f_2 \overset{\tau^{-1}}{\sim}_2 f_1$; (3). $f_1 \overset{\tau}{\sim}_2 f_2$ and $f_2 \overset{\tau'}{\sim}_2 f_3$ $\Rightarrow f_1 \overset{\tau\tau'}{\sim}_2 f_3$. So "$\sim_2$" is an equivalence relation on V.

Using V as the set of vertices, same with Ngai and Wang's [14] process recalled in Section 3 of this paper, we define a graph G as follows. Give two mappings $f, g \in V$, if $f \in V_k$, $g \in V_{k+1}$ and there exists a mapping $h \in G$ such that $g = fh$, then, we connect a directed edge $h : f \mapsto g$. We call f a parent of g and g an offspring of f.

Similar to the case in Section 3, a vertex in G might have several parents and we use the lexicographical order on Σ_q^* to remove extra edges from G. For each vertex $f \in V$, let $\mathbf{l}_1, \ldots, \mathbf{l}_p \in \Sigma_q^*$ be all the words such that $\phi_{\mathbf{l}_k}$ are directed edges going from some vertices to f. Suppose that $\mathbf{l}_1 < \ldots < \mathbf{l}_p$ in the lexicographical order. Then, we keep $\phi_{\mathbf{l}_1}$ and remove all other edges. Thus, we obtain G_2. If f is a parent of g (i.e., g is an offspring of f) in the reduced graph $\mathcal{G}_2$, we denote it as $f \gg_2 g$.

If for any $f_1, f_2, g \in V$ and $h \in \mathcal{G}$ such that $f_1 \overset{\tau}{\sim}_2 f_2$ and g is an offspring of f_2 with edge $h : f_2 \mapsto g$ in the graph G_2, we always have that $\tau g \overset{\tau}{\sim}_2 g$ and τg is an offspring of f_1 with edge $h : f_1 \mapsto \tau g$ in the graph G_2, then, we say (V_k) is a *recurrentable flag*. For example, by Lemma 3.1, we know that (3.3) defines a recurrentable flag.

The IFS Φ is said to be of general finite type if there exist a flag $\mathcal{F} = (V_k | k = 0, 1, 2, \ldots)$ and a Φ-invariant non-empty bounded open set Ω such that $\mathcal{F}$ is a recurrentable flag and there are finite many distinct neighborhood types in the sense of (5.1).

Let Φ be a general finite type IFS Φ with respect to open set Ω and recurrentable flag $\mathcal{F} = (V_k | k = 0, 1, 2, \ldots)$. The incidence M-matrix $S = (s_{\alpha\beta})$ for the Φ is defined as follows. Suppose that there are m neighborhood types $\alpha_1, \ldots, \alpha_m$. Then, the size of S is $m \times m$. For each $i = 1, \ldots, m$, choose a mapping $f_i \in V$ that has neighborhood type α_i. Let $C_2(f_i) = \{g \in V | f_i \gg_2 g\}$. Each mapping $g \in C_2(f_i)$ will belong to one of the neighborhood types α_j. So, there exists a similar mapping τ_g such that

$g \overset{\tau_g}{\sim}_2 f_j$. We collect these mappings τ_g to form the (i,j)-entry of S, i.e., define

$$\left\{ \begin{array}{l} S = (S_{ij})_{1\le i,j\le m}, \\ \text{where } S_{ij} = \{\tau | \exists g \in C(f_i) \ni g \overset{\tau}{\sim}_2 f_j\}. \end{array} \right. \tag{5.2}$$

Notice that $\tau_g = g f_j^{-1}$ if $g \overset{\tau_g}{\sim}_2 f_j$, so τ_g is unique determined by g. Consequently, S_{ij} is unique determined by f_i and f_j. Now, we study what will happen on S if we choose different f_i. Suppose $\{\hat{f}_i, i = 1, \ldots, m\} \subset V$ is another collection of mappings such that $\hat{f}_i$ has neighborhood type α_i and $\hat{S} = (\hat{S}_{ij})$ is the M-matrix defined by $\hat{S}_{ij} = \{\tau | \exists g \in C_2(\hat{f}_i) \ni g \overset{\tau}{\sim}_2 \hat{f}_j\}$, where $C_2(\hat{f}_i) = \{g \in V | \hat{f}_i \gg_2 g\}$. As f_i and $\hat{f}_i$ have the same neighborhood type, so there exist similar mappings σ_i such that $f_i \overset{\sigma_i}{\sim}_2 \hat{f}_i$. Let $\tau \in S_{ij}$. Then, $\exists g \in C_2(f_i)$ such that $g \overset{\tau}{\sim}_2 f_j$. Because $\mathcal{F}$ is a recurrentable flag, $f_i \overset{\sigma_i}{\sim}_2 \hat{f}_i$ and $g \in C_2(f_i)$ imply $g \overset{\sigma_i}{\sim}_2 \sigma_i^{-1} g$ and $\sigma_i^{-1} g \in C_2(\hat{f}_i)$. Now, $\sigma_i^{-1} g \overset{\sigma_i^{-1}}{\sim}_2 g \overset{\tau}{\sim}_2 f_j \overset{\sigma_j}{\sim}_2 \hat{f}_j$. So, $\sigma_i^{-1} g \overset{\sigma_i^{-1}\tau\sigma_j}{\sim}_2 \hat{f}_j$. Thus, $\sigma_i^{-1}\tau\sigma_j \in \hat{S}_{ij}$. So, $\sigma_i^{-1} S_{ij} \sigma_j \subseteq \hat{S}_{ij}$. Similarly, it can be proved that $\sigma_i \hat{S}_{ij} \sigma_j^{-1} \subseteq S_{ij}$. So, $\sigma_i^{-1} S_{ij} \sigma_j = \hat{S}_{ij}$. Let $P =$diag$(\{\sigma_1\}, \ldots, \{\sigma_m\})$. Then, we have $\hat{S} = P^{-1} S P$.

The following theorem gives the relation of the invariant c-vector of S and the attractor E of Φ. As a consequence, we can get the Hausdorff dimension of E by applying Theorem 2.5 on S.

Theorem 5.1. Let Φ be a general finite type IFS Φ with respect to recurrentable flag $\mathcal{F} = (V_k | k = 0, 1, 2, \ldots)$ and open set Ω. Suppose that there are m neighborhood types $\alpha_1, \ldots, \alpha_m$ and that the identity mapping 1 has type α_1. Choose mappings $f_i \in V$ such that f_i has neighborhood type α_i and $f_1 = 1$. Let S be the associated incidence M-matrix. Then, $S^k \in \mathfrak{M}_c(m, m)$ for some k and S satisfies open set condition. Furthermore, the attractor E of Φ is the first component of the maximal invariant c-vector $A(S)$ of S.

Proof. For any $f \in V = \cup_k V_k$, define

$$\begin{array}{l} C_2^0(f) := \{f\}, \\ C_2^1(f) := C_2(f) = \{g \in \mathrm{V} | f \gg_2 g\}, \\ C_2^{i+1}(f) = \{g \in \mathrm{V} | h \gg_2 g \text{ for some } h \in C_2^i(f)\}, i = 1, 2, \ldots. \end{array}$$

Then, it is easy to see that

$$C_2^i(1) = \mathrm{V}_i, i = 1, 2, \ldots. \tag{5.3}$$

By (5.2),

$$
\begin{aligned}
S_{ij} &= \{\tau \mid \exists g \in C_2(f_i), \ni g \overset{\tau}{\sim}_2 f_j\} \\
&= \{g f_j^{-1} \mid g \in C_2(f_i), g \sim_2 f_j\}.
\end{aligned} \tag{5.4}
$$

So

$$
S_{ij} f_j = \{g \in C_2(f_i) \mid g \sim_2 f_j\}. \tag{5.5}
$$

Every mapping in $C(f_i)$ must belong to one of the neighberhood types of $\alpha_1, \ldots, \alpha_m$. So $C_2(f_i) = \cup_j \{g \in C(f_i) \mid g \sim_2 f_j\} = \cup_j S_{ij} f_j$. Hence,

$$
S \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} C_2(f_1) \\ \vdots \\ C_2(f_m) \end{pmatrix}. \tag{5.6}
$$

Denote the (ij)-entry of S^p as $S_{ij}^{(p)}$. Now, we shall prove

$$
\begin{cases}
(A). \quad S^p \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} C_2^p(f_1) \\ \vdots \\ C_2^p(f_m) \end{pmatrix}, \\
(B). \quad g \in S_{ij}^{(p)} \ and \ h \in C_2^q(f_j) \Rightarrow gh \in C_2^{p+q}(f_i) \ and \ gh \overset{g}{\sim}_2 h,
\end{cases} \tag{5.7}
$$

for $p = 1, 2, 3, \ldots$ and $q = 0, 1, 2, \ldots$ by induction.

If $p = 1$ and $q = 0$, by (5.5) and (5.6), we know that (5.7) holds. Suppose $K \geq 1$ is an integer. Assume that (5.7) holds if $p + q \leq K$.

Let $g \in C_2^{K+1}(f_i)$. Then, there exists $h \in C_2^K(f_i)$ such that $h \gg_2 g$. By induction assumption, (A) of (5.7) holds when $p = K$ an $p = K - 1$. So, $C_2^K(f_i) = \cup_j S_{ij}^{(K)} f_j = \cup_j (\cup_t S_{it} S_{tj}^{(K-1)} f_j) = \cup_t S_{it} C_2^{K-1}(f_t)$. Hence, $h \in S_{it_0} C_2^{K-1}(f_{t_0})$ for certain $t_0 \in \{1, \ldots, m\}$. So, we can suppose that $h = \alpha\beta$, where $\alpha \in S_{it_0}$ and $\beta \in C_2^{K-1}(f_{t_0})$. By induction assumption, (B) of (5.7) holds when $p = 1$ and $q = K - 1$. So, $h = \alpha\beta \overset{\alpha}{\sim}_2 \beta$. Hence, $\beta \overset{\alpha^{-1}}{\sim}_2 h$. But $\mathcal{F}$ is a recurrentable flag, so $\beta \gg_2 \alpha^{-1} g$. Therefore, $\alpha^{-1} g \in C_2^K(f_{t_0})$. So, $g = \alpha\alpha^{-1} g \in S_{it_0} C_2^K(f_{t_0})$. Thus, we have

$$
\begin{aligned}
C_2^{K+1}(f_i) &\subseteq \cup_t S_{it} C_2^K(f_t) \\
&= \cup_t S_{it} (\cup_j S_{tj}^{(K)} f_j) \\
&= \cup_j S_{ij}^{(K+1)} f_j.
\end{aligned} \tag{5.8}
$$

Conversly, let $g \in \cup_j S_{ij}^{(K+1)} f_j$. Because $\cup_j S_{ij}^{(K+1)} f_j = \cup_j \cup_t S_{it} S_{tj}^{(K)} f_j = \cup_t S_{it} C_2^K(f_t)$, there exists $t_0 \in \{1, \ldots, m\}$ such that $g \in S_{it_0} C_2^K(f_{t_0})$. Choose

$\alpha \in S_{it_0}$ and $\beta \in C_2^K(f_{t_0})$ such that $g = \alpha\beta$. By the definition of $C_2^K(f_{t_0})$, there exists $\gamma \in C_2^{K-1}(f_{t_0})$ such that $\gamma \gg_2 \beta$. By induction assumption, (B) of (5.7) holds if $p = 1$ and $q = K-1$. So, $\alpha \in S_{it_0}$ and $\gamma \in C_2^{K-1}(f_{t_0})$ imply $\alpha\gamma \in C_2^K(f_i)$ and $\alpha\gamma \overset{\alpha}{\sim}_2 \gamma$. Because $\mathcal{F}$ is recurrentable, we have $\alpha\gamma \gg_2 \alpha\beta$. So, $g = \alpha\beta \in C_2^{K+1}(f_i)$. Thus, we proved that

$$C_2^{K+1}(f_i) \supseteq \cup_j S_{ij}^{(K+1)} f_j \tag{5.9}$$

Combining (5.8) and (5.9), we have

$$C_2^{K+1}(f_i) = \cup_j S_{ij}^{(K+1)} f_j \tag{5.10}$$

Thus, (A) of (5.7) holds if $p \le K+1$. Consequently, we also get that if $p+q \le K+1$, $\alpha \in S_{ij}^{(p)}$ and $\beta \in C_2^q(f_j)$ imply $\alpha\beta \in C_2^{p+q}(f_i)$. To prove that (B) of (5.7) holds in this case, we only need to prove that $\alpha\beta \overset{\alpha}{\sim}_2 \beta$ if $p+q = K+1$.

Suppose $1 \le p \le K$ and $q = K-p+1$. Choose $\gamma \in C_2^{K-p}(f_j)$ such that $\gamma \gg_2 \beta$. By induction assumption, $\alpha\gamma \overset{\alpha}{\sim}_2 \gamma$. As $\mathcal{F}$ is recurrentable, we have $\alpha\beta \overset{\alpha}{\sim}_2 \beta$.

Now, suppose $p = K+1$ and $q = 0$. Then, $\alpha \in S_{ij}^{(K+1)}$ and $\beta = f_j$. But $S_{ij}^{(K+1)} = \cup_t S_{it} S_{tj}^{(K)}$, so $\alpha = \alpha_1\alpha_2$ for some $\alpha_1 \in S_{it_0}$ and $\alpha_2 \in S_{t_0 j}^{(K)}$. By induction assumption, $\alpha_2\beta \overset{\alpha_2}{\sim}_2 \beta$ and $\alpha_2\beta \in C_2^K(f_t)$. By previous paragraph, $\alpha_1\alpha_2\beta \overset{\alpha_1}{\sim}_2 \alpha_2\beta$. So $\alpha_1\alpha_2\beta \overset{\alpha_1\alpha_2}{\sim}_2 \beta$, i.e. $\alpha\beta \overset{\alpha}{\sim}_2 \beta$.

Thus, we proved that (5.7) holds when $p+q \le K+1$. Hence, (5.7) holds for all $p = 1, 2, 3, \ldots$ and $q = 0, 1, 2, \ldots$.

Now, we shall find a k such that $S^k \in \mathfrak{M}_c(m,m)$. Let $\rho_{max} = \max\{u(\phi) | \phi \in \Phi\}$, $u_{max} = \max\{u(f_i) | 1 \le i \le m\}$ and $u_{min} = \min\{u(f_i) | 1 \le i \le m\}$. Then, $0 \le \rho_{max} < 1$. For any $f \in C_2^p(f_i)$, $u(f) \le u(f_i)\rho_{max}^{p-1} \le u_{max}\rho_{max}^{p-1}$. Choose k such that $u_{max}\rho_{max}^{k-1} \le \frac{1}{2}u_{min}$. For any $f \in S_{ij}^{(k)}$, $ff_j \in C_2^k(f_i)$. So, $u(ff_j) \le \frac{1}{2}u_{min} \le \frac{1}{2}u(f_j)$. Thus, $u(f) \le \frac{1}{2}$. Therefore, $S^k \in \mathfrak{M}_s(m,m)$.

Now, we shall prove that E is a component of $A(S)$. Let

$$\mathcal{E} := \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} (E) = \begin{pmatrix} f_1(E) \\ \vdots \\ f_m(E) \end{pmatrix}.$$

Then, $\mathcal{E} \in \mathcal{K}^m$. So by (1) of Theorem 2.3,

$$A(S) = \cap_{i \ge 1} \overline{\cup_{p \ge i} S^p(\mathcal{E})} \tag{5.11}$$

But

$$E = \mathrm{V}_0(E) = \ldots = \mathrm{V}_p(E) = \ldots.$$

So, by (5.3),

$$E = C_2^0(1)(E) = \ldots = C_2^p(1)(E) = \ldots. \tag{5.12}$$

Noticing that $f_1 = 1$ is the identity mapping, (5.12) and (A) of (5.7) implies that the first component of $S^p(\mathcal{E})$ is always E. Thus, by (5.11), the first component of $A(S)$ is E.

Finally, let

$$U_i = [\cup_{p\geq 0}(C_2^p(f_i)(\Omega) - \cup_{f\in\Omega(f_i), f\neq f_i} C_2^p(f)(\overline{\Omega}))], i = 1 \ldots, m. \tag{5.13}$$

Then, U_i are open sets and it is easy to check that S satisfies open set condition with respect to $U_1, \ldots, U_m$ (see Definition 2.4). $\square$

6. Examples.

In this section, we study a class of general finite type IFS as examples. Suppose $0 < \beta < 1/2$, $0 < \alpha < (1-2\beta)/(1-\beta)$ and $\gamma \neq 0$. Let $\Phi = \{\phi_1, \phi_2, \phi_3\}$, where $\phi_1(x) = \alpha x$, $\phi_2(x) = \beta x + \alpha\gamma$, and $\phi_3(x) = \beta x + \gamma$. Then, Φ is an IFS on $\mathbb{R}$. According to Theorem 4.1, $\alpha = \beta^p$ for some $p \in \mathbb{Q}$ is a necessary condition for Φ to be finite type. So, these IFS are not finite type in Ngai and Wang's sense [14] except some occasional cases. It is easy to check that Φ does not satisfy the open set condition. Let $E = A(\Phi)$ be the attractor of Φ.

Firstly, we shall find the Hausdorff dimension of E by M-matrix theory. Let $W_1 = \{1\}$, where 1 is the identity mapping. Let $W_2 = \{\phi_1, \phi_2\}$. Then, it is easy to check

$$\begin{cases} W_1\Phi = \phi_3 W_1 \cup W_2 \\ W_2\Phi = \phi_2\phi_3 W_1 \cup \{\phi_1, \phi_2\} W_2 \end{cases} \tag{6.1}$$

Define

$$M = \begin{pmatrix} \{\phi_3\} & \{1\} \\ \{\phi_2\phi_3\} & \{\phi_1, \phi_2\} \end{pmatrix}. \tag{6.2}$$

Then, (6.1) becomes

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \Phi = M \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \tag{6.3}$$

Let $\mathcal{E} = (W_1(E), W_2(E))^t = (E, \phi_1(E) \cup \phi_2(E))^t$. Then, $\mathcal{E} \in \mathcal{K}^2$.

$$\begin{aligned}
M(\mathcal{E}) &= \begin{pmatrix} \{\phi_3\} & \{1\} \\ \{\phi_2\phi_3\} & \{\phi_1, \phi_2\} \end{pmatrix} \begin{pmatrix} W_1(E) \\ W_2(E) \end{pmatrix} \\
&= \begin{pmatrix} (\phi_3 W_1 \cup W_2)(E) \\ (\phi_2\phi_3 W_1 \cup \{\phi_1, \phi_2\} W_2)(E) \end{pmatrix} \\
&= \begin{pmatrix} W_1 \Phi(E) \\ W_2 \Phi(E) \end{pmatrix} \\
&= \begin{pmatrix} W_1(E) \\ W_2(E) \end{pmatrix} = \mathcal{E}.
\end{aligned}$$

So, $\mathcal{E}$ is invariant under M. It is clear that M is irreducible and $M^2 \in \mathfrak{M}_c(2,2)$. But $\mathcal{E} \in \mathcal{K}^2$, so by Theorem 2.3, we know that $\mathcal{E} = A(M)$.

Define open intervals

$$U_1 := \begin{cases} (0, \gamma/(1-\beta)) & \text{if } \gamma > 0 \\ (\gamma/(1-\beta), 0) & \text{if } \gamma < 0 \end{cases} \tag{6.4}$$

and

$$U_2 := W_2(U_1) = \begin{cases} (0, (\frac{1}{1-\beta} + \alpha)\gamma) & \text{if } \gamma > 0 \\ ((\frac{1}{1-\beta} + \alpha)\gamma, 0) & \text{if } \gamma < 0 \end{cases} \tag{6.5}$$

Then, one can check that

$$\begin{cases} \phi_3(U_1) \cap U_2 \subseteq U_1, \\ \phi_2\phi_3(U_1) \cup \phi_1(U_2) \cup \phi_2(U_2) \subseteq U_2, \\ \phi_3(U_1) \cap U_2 = \emptyset, \\ \phi_2\phi_3(U_1) \cap (\phi_1(U_2) \cup \phi_2(U_2)) = \emptyset, \\ \phi_1(U_2) \cap \phi_2(U_2) = \emptyset. \end{cases} \tag{6.6}$$

So, M satisfies the open set condition with respect to U_1 and U_2. By Theorem 2.5, the Hausdorff dimension $d = \dim E$ (notice that E is the first component of $\mathcal{E}$) is the unique number such that $F(M,d)$ has 1 as the biggest real eigenvalue. Now

$$F(M,d) = \begin{pmatrix} \beta^d & 1 \\ \beta^{2d} & \alpha^d + \beta^d \end{pmatrix}.$$

It has two eigenvalues. The bigger one is

$$\lambda = \frac{\alpha^d + 2\beta^d + \sqrt{\alpha^{2d} + 4\beta^{2d}}}{2}.$$

Then, $\lambda = 1$ implies

$$\alpha^d + 2\beta^d = 1 + \alpha^d \beta^d. \tag{6.7}$$

So, the Hausdorff dimension $\dim_H A(\Phi)$ is the solution of (6.7).

Now, we shall prove that Φ is a general finite type IFS and get the Hausdorff dimension of E by the incidence M-matrix. We can prove that $\Phi^n, n = 1, 2, \ldots$, are sections. To do this, we only need to prove that for any $f, g \in \Phi^n$, $f = gh$ for some $h \in \mathcal{G}$ will give $f = g$. Otherwise, suppose $f \neq g$. Then, $f(U_1) = gh(U_1) \subseteq g(U_1)$. So

$$f(U_1) \cap g(U_1) \neq \emptyset. \tag{6.8}$$

By (6.3), we have

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \Phi^n = M^n \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}. \tag{6.9}$$

Denote the (i,j)-th entry of M^n by $M_{ij}^{(n)}$. Then,

$$\Phi^n = W_1 \Phi^n = M_{11}^{(n)} W_1 \cup M_{12}^{(n)} W_2 = M_{11}^{(n)} \cup M_{12}^{(n)} W_2. \tag{6.10}$$

Because M satisfies the open set condition with respect to $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}(U_1)$, we know that M^n also satisfies the open set condition with respect to $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}(U_1)$. Hence,

$$\begin{cases} h_1, h_2 \in M_{11}^{(n)}, h_1 \neq h_2 \Rightarrow h_1(U_1) \cap h_2(U_1) = \emptyset, \\ h_1 \in M_{11}^{(n)}, h_2 \in M_{12}^{(n)} \Rightarrow h_1(U_1) \cap h_2 W_2(U_1) = \emptyset, \\ h_1, h_2 \in M_{12}^{(n)}, h_1 \neq h_2 \Rightarrow h_1 W_2(U_1) \cap h_2 W_2(U_1) = \emptyset, \end{cases} \tag{6.11}$$

So, (6.8) holds only in the case $\{f, g\} \in \psi W_2$ for some $\psi \in M_{12}^{(n)}$. Notice that $f = gh$ and $W_2 = \{\phi_1, \phi_2\}$. So, $\phi_1 = \phi_2 h$ or $\phi_2 = \phi_1 h$. Thus,

$$\phi_1(U_1) = \phi_2 h(U_1) \subseteq \phi_2(U_1) \textit{ or } \phi_2(U_1) \subseteq \phi_1(U_1). \tag{6.12}$$

But $\phi_1(U_1) = (0, \frac{\alpha\gamma}{1-\beta})$ and $\phi_2(U_1) = (\alpha\gamma, \frac{\beta\gamma}{1-\beta} + \alpha\gamma)$. Hence, (6.12) cannot hold. So, the assumption $f \neq g$ is impossible. Thus, we proved $f = g$.

Now, let $V_0 = \{1\}$, $V_n = \Phi^n$ for $n = 1, 2, \ldots$. Then, (V_n) is a flag. Using U_1 as the open set Ω. From (6.10) and (6.11), noticing that $\phi_1(U_1) \cap \phi_2(U_1) \neq \emptyset$, we know that the neighborhood (in the sense of (5.1)) of any $f \in \Phi^n$ is

$$\Omega(f) = \begin{cases} \{f\} & \text{if } f \in M_{11}^{(n)} \\ \psi W_2 & \text{if } f \in \psi W_2 \text{ for some } \beta \in M_{12}^{(n)}. \end{cases} \tag{6.13}$$

Thus, we have

$$\begin{cases} f \overset{f}{\sim}_2 1 & \text{if } f \in M_{11}^{(n)} \\ f \overset{\psi}{\sim}_2 \phi_1 & \text{if } f = \psi\phi_1 \text{ for some } \psi \in M_{12}^{(n)} \\ f \overset{\psi}{\sim}_2 \phi_2 & \text{if } f = \psi\phi_2 \text{ for some } \psi \in M_{12}^{(n)} \end{cases} \tag{6.14}$$

It is clear that $\phi_1 \not\sim_2 \phi_2$. So, there are three neighborhood types.

From (6.14), one can check that $(V_n; n = 0, 1, 2, \ldots)$ is a recurrent flag. Select $1, \phi_1, \phi_2$ to represent the three different neighborhood types. Consider the offsprings of $1, \phi_1$ and ϕ_2 in the graph G_2. 1 has offsprings ϕ_1, ϕ_2 and $\phi_3 \overset{\phi_3}{\sim}_2 1$. ϕ_1 has offsprings $\phi_1^2 \overset{\phi_1}{\sim}_2 \phi_1$, $\phi_1\phi_2 \overset{\phi_1}{\sim}_2 \phi_2$ and $\phi_1\phi_3 \overset{\phi_1\phi_3}{\sim}_2 1$. ϕ_2 has offsprings $\phi_2^2 \overset{\phi_2}{\sim}_2 \phi_2$ and $\phi_2\phi_3 \overset{\phi_2\phi_3}{\sim}_2 1$. Thus, the associated incidence M-matrix

$$S = \begin{pmatrix} \phi_3 & 1 & 1 \\ \phi_1\phi_3 & \phi_1 & \phi_1 \\ \phi_2\phi_3 & \emptyset & \phi_2 \end{pmatrix}. \tag{6.15}$$

The Hausdorff dimension $d = \dim E$ (notice that E is the first component of $A(S)$) is the unique number such that $F(S, d)$ has 1 as the biggest real eigenvalue. Now

$$F(S, d) = \begin{pmatrix} \beta^d & 1 & 1 \\ \alpha^d\beta^d & \alpha^d & \alpha^d \\ \beta^{2d} & 0 & \beta^d \end{pmatrix}.$$

It has there eigenvalues:

$$0, \quad \frac{\alpha^d + 2\beta^d \pm \sqrt{\alpha^{2d} + 4\beta^{2d}}}{2}.$$

The fact that the biggest eigenvalue is one implies

$$\alpha^d + 2\beta^d = 1 + \alpha^d\beta^d. \tag{6.16}$$

This is same as (6.7).

Recently we were informed by Sze–Man Ngai that he and Ka–Sing Lau have done similar works in our Section 5 by different method.

References.

[1] Bedford, Tim, Dimension and dynamics for fractal recurrent sets, J. London Math. Soc. (2), **33** (1986), 89–100.

[2] Dekking, F. M. , Recurrent sets, Adv. in Math. **44** (1982), 78–104.

[3] Falconer, K. J., The Geometry of Fractal sets, Cambridge tracts in Math., New York, 1985

[4] Hutchinson, J. E., *Fractals and self-similarity*, Indiana Univ. Math. J., **30**(1981), 713–747.

[5] Jenkinson, Oliver and Pollicott, Mark, Calculating Hausdorff dimensions of Julia sets and Kleinian limit sets, Amer. J. Math. **124** (2002), 495–545.

[6] Lau, Ka-Sing and Rao, Hui, On one-dimensional self-similar tilings and pq-tiles. Trans. Amer. Math. Soc. **355** (2003), 1401–1414

[7] Massopust, P. R., Fractal functions, fractal surfaces, and wavelets, Academic Press, Inc., New York, 1994.

[8] Mandelbrot, B., Fractals: Form, Chance, and Dimension, Freeman, San Francisco 1977

[9] Mauldin, R. D. and Williams, S. C. Hausdorff dimension in graph directed constructions, Trans. of AMS, **309** (1988), 811–829.

[10] McMullen, C., Hausdorff dimension and conformal dynamics. I. Strong convergence of Kleinian groups, J. Differential Geom. **51** (1999), 471–515.

[11] McMullen, C., Hausdorff dimension and conformal dynamics. II. Geometrically finite rational maps, Comment. Math. Helv. **75** (2000), 535–593.

[12] McMullen, C., Hausdorff dimension and conformal dynamics. III. Computation of dimension, Amer. J. Math. **120** (1998), 691–721.

[13] Moran, P.A.P., Additive functions of intervals and Hausdorff measure, Proc. Camb. Phil. Soc. **42** (1946), 15–23.

[14] Ngai, Sze-Man and Wang, Yang, *Hausdorff dimension of self-similar sets with overlaps*, J. London Math. Soc.(2) **63** (2001), 655–672

[15] Ning Jin and Stephen S.-T. Yau, M-matrix and c-vectors, submitted for publication.

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