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Stability of Gradient Kähler–Ricci Solitons

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We study the stability of non-compact gradient Kähler–Ricci solitons under the Kähler–Ricci flow. Our main result is that appropriate perturbations of Cao's steady soliton metric on $\mathbb{C}^n$ will converge to the original soliton under the Kähler–Ricci flow as time tends to infinity. These perturbations correspond to appropriately decaying perturbations of the soliton potential function; in particular, this includes any compactly supported perturbation. To obtain this result, we construct appropriate barriers and introduce an L^p-norm that decays for these barriers with non-negative Ricci curvature.

1. Introduction.

In [17], Hamilton introduced the Ricci flow on a compact Riemannian manifold and used this to produce an Einstein metric on a compact Riemannian three manifold with positive Ricci curvature. The flow is a geometric evolution equation which evolves a Riemannian metric on a smooth manifold by its negative Ricci curvature. On a Kähler manifold, this becomes the Kähler–Ricci flow

$$\frac{d}{dt} g_{i\bar{\jmath}} = -R_{i\bar{\jmath}}. \tag{1.1}$$

In [6], Cao used (1.1) to produce Kähler–Einstein metrics on compact Kähler manifolds with zero or negative first Chern class, thus re-establishing the results of Yau in [24]. In the special case of a Kähler metric $g_{i\bar{\jmath}}$ on $\mathbb{C}^n$ with a smooth global Kähler potential U (thus $g_{i\bar{\jmath}} = U_{i\bar{\jmath}}$), (1.1) is equivalent to the following evolution equation for U

$$\frac{d}{dt} U = \log\det(U_{i\bar{\jmath}}). \tag{1.2}$$

(For the notations and conventions used in this paper, we refer to Section 2.) In general Ricci and Kähler–Ricci flow may develop singularities well before converging to an Einstein metric, and blow-up analysis gives rise to complete non-compact solitons for these flow equations [16, 4]. Thus, an understanding of the flow on non-compact manifolds is essential. The general theory for the Ricci flow on complete non-compact Riemannian manifolds was established by Shi. See [19] for a reference to this, and in particular, to the parabolic maximum principles on non-compact manifolds which we will use later. For a survey article concerning the Ricci flow and singularity analysis in particular see [7, 15], and for a list of solitons for the Kähler–Ricci flow see [11]. The stability of solitons and special metrics under the Ricci flow is also a topic of special interest. In particular, Stability questions for the Ricci flow have been considered near compact Ricci flat metrics and near complete metrics on $\mathbb{R}^2$ in [13, 23, 14].

In this paper, we focus on the non-compact complete gradient Kähler–Ricci solitons found in [5]. They are rotationally symmetric with positive holomorphic bisectional curvature and their existence has been proved by solving an ordinary differential equation. It turns out that these solutions are unique (up to scaling and dilatations) in the class of rotationally symmetric gradient solitons with positive holomorphic bisectional curvature. To learn more about these solitons, it is desirable to know whether they are stable under appropriate perturbations. In this paper, we answer this question in the affirmative. We will show that the gradient solitons in [5] on $\mathbb{C}^n$ are stable under appropriately decaying perturbations of the Kähler potential.

A further question in this direction is, whether there exist other solitons without rotational symmetry. We wish to remark that recently, Xu-Jia Wang found strictly convex translating solutions to the mean curvature flow without rotational symmetry in [22]. The problem of stability for these solutions seems to be unsolved.

We wish to give a heuristic argument why we impose the condition that the perturbation should decay at infinity. In our situation, we do not expect to get stronger results than for the standard heat equation on $\mathbb{R}^n$. In this case, however, we can take a bounded perturbation of the stationary solution $u = 0$ that satisfies for $t = 0$

$$u((x_1, x_2, \ldots, x_n), t) \to \begin{cases} 1 & \text{as } x_1 \to \infty, \\ -1 & \text{as } x_1 \to -\infty, \end{cases} \tag{1.3}$$

uniformly in $(x_2, \ldots, x_n)$. It follows directly from the heat kernel representation of a solution that (1.3) remains true during the evolution, i.e. for $t > 0$. Similarly, we expect that for general bounded perturbations of the potential in our equation, the oscillation of a perturbation will not tend to zero. Of course, in such special cases as the one above, one can show that the solution flattens out on compact sets. This explains why it is natural to have decay assumptions in our stability theorem. To simplify its formulation, we give the following

Definition 1.1. A function $u_0 : \mathbb{C}^n \to \mathbb{R}$ is called a C-potential, if it is rotationally symmetric, $(u_0)_{i\bar{j}}$ is a Kähler metric with positive holomorphic bisectional curvature, and gives rise to a gradient Kähler–Ricci soliton.

We refer to Theorem 1 in [5], where C-potentials are shown to exist and to be unique up to scaling and holomorphic transformations. In particular, using Notation 2.3, it is shown that C-potentials are characterized by the Equations (5.1) and (5.2).

Theorem 1.2. *Let u_0 be a C-potential in complex dimension $n \geq 2$ and $\tilde{u}$ a smooth perturbation such that*

1. *$\tilde{u}_{i\bar{j}}$ defines a complete Kähler metric on $\mathbb{C}^n$ equivalent to $(u_0)_{i\bar{j}}$ with bounded curvature.*
2. *$u = \tilde{u} - u_0$ satisfies $|u(x)| \leq K$ and*

$$|u(x)| \leq K \cdot (2 \log |x|)^{-\alpha} \quad \text{for } |x| > 1 \tag{1.4}$$

for some $0 < K$, $0 < \alpha < 1$.

Then, with u as initial condition, (2.3) *has a long time smooth solution converging to* 0 *as time tends to infinity. As a consequence, the Kähler metric defined by $\tilde{u}_{i\bar{j}}$ converges back to the soliton metric $(u_0)_{i\bar{j}}$ under the following reparametrization of* (1.1)

$$\frac{d}{dt} g_{i\bar{j}} = -R_{i\bar{j}} + f_{i\bar{j}},$$

where f is a smooth potential for the Ricci tensor of $(u_0)_{i\bar{j}}$.

In the special case of a compactly supported perturbation, we get

Corollary 1.3. *Let u_0 be a C-potential in complex dimension $n \geq 2$ and $\tilde{u}$ a smooth perturbation such that $\tilde{u}_{i\bar{j}}$ defines a complete Kähler metric on $\mathbb{C}^n$ equivalent to $(u_0)_{i\bar{j}}$ and $u = \tilde{u} - u_0$ is compactly supported. Then, with u as initial condition,* (2.3) *has a long time smooth solution converging to* 0 *as time tends to infinity.*

A geometric interpretation of the decay condition (1.4) is as follows. If the eigenvalues of the perturbed metric minus the soliton metric $\tilde{u}_{i\bar{\jmath}} - (u_0)_{i\bar{\jmath}}$ with respect to the soliton metric $(u_0)_{i\bar{\jmath}}$ decay like $(\log|x|)^{-2-\alpha}$ and u tends to 0 at infinity, then (1.4) is fulfilled for an appropriate value of K. This is obtained by integrating u radially. Note that along a ray, Du, evaluated radially, has to become small on a sequence of points tending to infinity. That can be used to anchor the integration. We wish to mention that the barriers introduced in (5.3) have the same decay in terms of the metric.

To give an overview of the method used here to prove Theorem 1.2, we describe our proof in words. It will be convenient to transform our flow equation such that the gradient solitons we are interested in become stationary solutions of the new equations, namely (2.1). In Section 4, we obtain smooth longtime existence and get uniform estimates for the perturbation of a C-potential. Then, we want to apply the maximum principle to deduce that the oscillation of the perturbation is strictly decreasing in time (or zero). Due to the non-compactness of $\mathbb{C}^n$, however, we have to make sure that the supremum is attained somewhere. We do not know how to prove this directly. Instead, we enclose our perturbation from above and from below by radially symmetric barriers that decay at infinity (as a function of $r = |x|$) and correspond to Kähler metrics with positive holomorphic bisectional curvature. During the evolution, the upper barrier stays positive, monotone in r, and rotationally symmetric. This ensures that it attains its maximum at the origin and we can apply the strong maximum principle to deduce that the oscillation, as a function of time t, is strictly decreasing. However, this is not enough to show that the barrier converges to zero. It might happen (and seems to be an interesting question, for which equations it actually happens), that the fact that the perturbation tends to zero at infinity is destroyed during the evolution as $t \to \infty$. This would imply that the perturbation would converge to a positive constant as $t \to \infty$. For the standard heat equation, however, it is quite easy to exclude this phenomenon as the L^2-norm of a solution is non-increasing, so it remains finite during the evolution, at least for $H^{1,2}$ initial data. The argument extends to any smooth solution. In our situation, we can find a quantity (Lemma 6.3) that is equivalent to the intrinsic L^p-norm of the perturbation and is also decaying. This property relies heavily on the construction of special barriers with positive holomorphic bisectional curvature (this is preserved during the evolution [19]). As the total (intrinsic) volume of our soliton is infinite, this excludes the possibility that the perturbation tends to a positive constant. Similar considerations apply to the lower barrier and thus the original perturbation, enclosed in between these two barriers during the evolution, converges to zero. Once C^0-convergence is established, smooth convergence follows immediately from our *a priori* estimates.

In Section 2, we introduce the notations that we will use throughout the paper and transform the evolution equation (1.2) to other coordinate systems. We explain in Section 3, why we have short time existence of solutions, and prove uniform *a priori* estimates that guarantee long time existence in Section 4. We sketch how to construct barriers in Section 5 and refer to the Appendices B and C for details. In the proof of Theorem 1.2, we use the evident Lemma 6.2 and give it's proof in Appendix A. Finally, Section 6 contains the proof, that our barrier converges to zero, the crux of the proof of Theorem 1.2.

2. Preliminaries and Transformations.

2.1. Preliminaries.

Notation 2.1. We use indices to denote partial derivatives,

$$u_i = \frac{\partial}{\partial z^i} u, \quad u_{i\bar{\jmath}} = \frac{\partial^2}{\partial z^i \partial z^{\bar{\jmath}}} u, \quad \ldots.$$

If for a function $u : \mathbb{C}^n \to \mathbb{R}$, the matrix $(u_{i\bar{\jmath}})$ is positive definite, we call u a Kähler potential. Then, $(u_{i\bar{\jmath}})$ is a Kähler metric and we denote its inverse by $\left(u^{i\bar{\jmath}}\right)$. Lower case Latin indices range from 1 to n. We use the Einstein summation convention with a special convention for Latin capitals, e.g.

$$u^{i\bar{\jmath}} w_{i\bar{\jmath}} := \sum_{i=1}^{n} \sum_{\bar{\jmath}=1}^{n} u^{i\bar{\jmath}} w_{i\bar{\jmath}}, \quad z^I u_I := \sum_{i=1}^{n} z^i u_i + \sum_{\bar{\jmath}=1}^{n} z^{\bar{\jmath}} u_{\bar{\jmath}}.$$

We will use $\left(z^i\right)$ and $(z^{\bar{\jmath}})$ to denote standard flat coordinates on $\mathbb{C}^n$. Sometimes, it will be appropriate to use standard Euclidean coordinates $\left(x^i\right)$. The Laplace operator with respect to the metric $(u_{i\bar{\jmath}})$ is defined by

$$\Delta_u w = u^{i\bar{\jmath}} w_{i\bar{\jmath}}.$$

In the estimates that follow, we will use c to denote a fixed positive constant that does not depend on time, but may change its value from line to line. Indices preceded by a comma, e.g. $u_{,i\bar{\jmath}k}$, indicate covariant differentiation with respect to the background metric $(u_0)_{i\bar{\jmath}}$ introduced in (2.3). As usually, we use $R_{i\bar{\jmath}}$ to denote the Ricci tensor, Rm for the Riemannian curvature tensor, $\|\cdot\|$ to denote a (pointwise) norm with respect to the induced metric, and ∇_g to indicate covariant differentiation with respect to the metric $g_{i\bar{\jmath}}$. We do not use different notations for the initial value $u : \mathbb{C}^n \to \mathbb{R}$ and for the corresponding solution to Kähler–Ricci flow $u : \mathbb{C}^n \times [0, \infty) \to \mathbb{R}$.

Notation 2.2. For a Kähler metric $(g_{i\bar{\jmath}})$, we obtain Christoffel symbols as follows

$$\Gamma^i_{jk} = g^{i\bar{l}} \frac{\partial g_{j\bar{l}}}{\partial z^k}, \quad \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} = \overline{\Gamma^i_{jk}},$$

and other components are identically zero. Covariant differentiation is defined by

$$\omega^A_{,B} = \frac{\partial \omega^A}{\partial z^B} + \Gamma^A_{BC}\omega^C, \quad X_{A,B} = \frac{\partial X_A}{\partial z^B} - \Gamma^C_{AB} X_C.$$

We can interchange covariant derivatives,

$$\begin{aligned} X_{c,ab} &= X_{c,ba}, & X_{\bar{c},ab} &= X_{\bar{c},ba}, \\ X_{c,a\bar{b}} &= X_{c,\bar{b}a} - R_{a\bar{b}c\bar{d}} g^{\bar{d}e} X_e, & X_{\bar{c},a\bar{b}} &= X_{\bar{c},\bar{b}a} + R_{a\bar{b}d\bar{c}} g^{d\bar{e}} X_{\bar{e}}. \end{aligned}$$

In these formulae, the (holomorphic) Riemannian curvature tensor appears, which is defined by

$$R_{i\bar{\jmath}k\bar{l}} = -\frac{\partial^2 g_{i\bar{\jmath}}}{\partial z^k \partial z^{\bar{l}}} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{\jmath}}}{\partial z^{\bar{l}}}.$$

Contracting with respect to the metric yields, the (holomorphic) Ricci tensor

$$R_{i\bar{\jmath}} = g^{k\bar{l}} R_{i\bar{\jmath}k\bar{l}} = -(\log\det(g_{k\bar{l}}))_{i\bar{\jmath}}.$$

Finally, a Kähler manifold has positive biholomorphic sectional curvature, if

$$R_{i\bar{\imath}j\bar{\jmath}} > 0.$$

Note, that we do not sum here.

Notation 2.3. In the proofs, we will switch between equivalent forms of the evolution equation. If we assume in the following that a function U fulfills (1.2), we will also assume that the function u is obtained from U by applying the transformations leading to (2.3), so it solves (2.3). Similarly, we assume that $\tilde{u}$ fulfills (2.1). When we consider a rotationally symmetric solution of (2.1) depending on the variable $s = \log |z|^2$, we denote this by $\hat{u}$. Analogous notations are used for other functions solving (1.2). Given a C-potential u_0, we will denote by U_0 the corresponding solution to (1.2).

2.2. Transformations.

We will now fix a C-potential u_0. Here, and in the rest of the paper, all evolution equations are defined on $\mathbb{C}^n$.

For further considerations, it will be convenient, to change coordinates such that the evolution of a C-potential in time is as simple as possible. This can be obtained by introducing

$$\tilde{u}(x,\,t) := U\left(e^{\frac{1}{2}t}x,\,t\right) + \tfrac{1}{2}nt^2,$$

where U is a solution to (1.2). We immediately get $\frac{d}{dt}\tilde{u} = \frac{1}{2}U_I z^I e^{\frac{1}{2}t} + \frac{d}{dt}U + nt$ and $\det(\tilde{u}_{i\bar{\jmath}}) = e^{nt}\det(U_{i\bar{\jmath}})$. Using (1.2), we see that the evolution equation for $\tilde{u}$ is given by

$$\frac{d}{dt}\tilde{u} = \log\det(\tilde{u}_{i\bar{\jmath}}) + \tfrac{1}{2}\left(z^i\tilde{u}_i + z^{\bar{\jmath}}\tilde{u}_{\bar{\jmath}}\right), \tag{2.1}$$

where we write again z^i and $z^{\bar{\jmath}}$ for $e^{\frac{1}{2}t}z^i$ and $e^{\frac{1}{2}t}z^{\bar{\jmath}}$, respectively, i.e. we evaluate $\tilde{u}$ at $\left(\left(z^i\right),\,\left(z^{\bar{\jmath}}\right),\,t\right)$. The initial value is clearly unchanged, $\tilde{u}(x,0) = U(x,0)$.

We now show that the C-potentials introduced in [5] are in fact stationary for (2.1). These potentials are characterized in [5] by radially symmetric functions $\hat{u}(s)$ in the variable $s = \log|z|^2$ for which the following conditions are satisfied. For $\hat{u}'(s) \equiv \varphi(s)$, it is required that $\varphi(s) \to 0$ for $s \to -\infty$ and φ fulfills (when normalized appropriately) the ordinary differential equation

$$\varphi^{n-1}\varphi' e^{\varphi} = e^{ns}. \tag{2.2}$$

By differentiating $\hat{u}\left(\log|z|^2\right) = \tilde{u}$, we obtain

$$\begin{aligned}\tfrac{1}{2}\left(z^i\tilde{u}_i + z^{\bar{\jmath}}\tilde{u}_{\bar{\jmath}}\right) =& \hat{u}',\\ \tilde{u}_{i\bar{\jmath}} =& \hat{u}''\frac{z_{\bar{\imath}}z_j}{|z|^4} + \hat{u}'\frac{1}{|z|^2}\left(\delta_{i\bar{\jmath}} - \frac{z_{\bar{\imath}}z_j}{|z|^2}\right).\end{aligned}$$

In appropriate coordinates, it is easily seen that the eigenvalues of $\tilde{u}_{i\bar{\jmath}}$ with respect to the flat metric are $\frac{\hat{u}''}{|z|^2}$ and $\frac{\hat{u}'}{|z|^2}$ with multiplicity $(n-1)$. We get

$$\begin{aligned}\log\det(\tilde{u}_{i\bar{\jmath}}) + \tfrac{1}{2}\left(z^i\tilde{u}_i + z^{\bar{\jmath}}\tilde{u}_{\bar{\jmath}}\right) &= \log\hat{u}'' + (n-1)\log\hat{u}' - n\log|z|^2 + \hat{u}'\\ &= \log\varphi' + (n-1)\log\varphi - ns + \varphi.\end{aligned}$$

Compare this to (2.2). Thus, we deduce that a C-potential is a stationary solution to (2.1). To show stability of (2.1) at u_0, it will be convenient to write $\tilde{u} = u_0 + u$. As $\tilde{u}$ and u_0 solve (2.1), we get directly from the definition of u its evolution equation

$$\frac{d}{dt}u = \log \frac{\det((u_0)_{i\bar{j}} + u_{i\bar{j}})}{\det((u_0)_{i\bar{j}})} + \tfrac{1}{2}\left(z^i u_i + z^{\bar{j}} u_{\bar{j}}\right). \tag{2.3}$$

The advantage of this evolution equation is that u_0 is time-independent and it will turn out that it allows to consider functions $u(x, t)$ that are uniformly bounded in time.

2.3. Hölder spaces.

We now define the parabolic and elliptic Hölder spaces of a non-compact Kähler manifold $(M, g_{i\bar{j}})$. We will use these spaces to apply Schauder estimates in proving *a priori* estimates for (2.3). These are parabolic versions of the elliptic spaces defined in [8, 20, 21].

Definition 2.4. Let m be a positive integer and $\alpha \in (0,1)$. A complex n-dimensional Kähler manifold $(M, g_{i\bar{j}})$ is said to have bounded geometry of order $m+\alpha$ if there are numbers $r_1, r_2, k_1, k_2, C > 0$ such that for every $p \in M$:

1. There is a neighborhood U_p of p and a non-degenerate holomorphic map $\xi_p : V_p \to U_p$ where $V_p \subset \mathbb{C}^n$, $B_{r_1}(0) \subseteq V_p \subseteq B_{r_2}(0)$ and $\xi_p(0) = p$.
2. $k_1 \delta_{a\bar{b}} \le \xi_p^* g_{a\bar{b}} \le k_2 \delta_{a\bar{b}}$ on V_p,
3. For all a, b, we have $\|\xi_p^* g_{a\bar{b}}\|_{p,m+\alpha} \le C$ where $\|\cdot\|_{p,m+\alpha}$ is the standard $C^{m+\alpha}$ Hölder norm on $V_p \subset \mathbb{C}^n$.

$(M, g_{i\bar{j}})$ is said to have bounded geometry of order ∞ if it has bounded geometry of order $m+\alpha$ for every m. Let $(M, g_{i\bar{j}})$ have bounded geometry of order $m+\alpha$ and let $[0,T)$ be an arbitrary time interval. For some choice of maps ξ_p as in Definition 2.4, consider the following norm for any smooth function u on $M \times [0,T)$:

$$\|u\|_{m+\alpha, m/2+\alpha/2} := sup_{p\in M}\{\|\xi_p^* u\|_{p,m+\alpha,m/2+\alpha/2}\}, \tag{2.4}$$

where $\xi_p^* u$ is the pull back of u to V_p and $\|\cdot\|_{p,m+\alpha,m/2+\alpha/2}$ is the standard parabolic Hölder norm on $V_p \times [0,T)$. The following definition is independent of the choice of $\xi_p' s$.

Definition 2.5. : Let $(M, g_{i\bar{j}})$ be a complete Kähler manifold with bounded geometry of order $m+\alpha$. With respect to (2.4), we define the parabolic Hölder spaces $C^{m+\alpha, m/2+\alpha/2}(M \times [0,T))$ to be the closure of the set of all smooth functions $u(x,t) : M \times [0,T) \longmapsto \mathbb{R}$ for which (2.4) is finite. Also, given $(M, g_{i\bar{j}})$ above, one can define the elliptic Hölder spaces $C^{m+\alpha}(M)$ in an obvious way.

$C^{m+\alpha, m/2+\alpha/2}(M \times [0,T))$ with the norm (2.4) for some choice of maps ξ_p, and $C^{m+\alpha}(M)$ with the analogous elliptic norm, are easily checked to be Banach spaces.

3. Short Time Existence.

We now establish the following general short time existence result.

Lemma 3.1. *Let $(M, g_{i\bar{j}})$ be a complete non-compact Kähler manifold such that $\|Rm\| \leq c_0$ and $f : M \to \mathbb{R}$ is a smooth potential of the Ricci tensor, i.e. $R_{i\bar{j}} = -f_{i\bar{j}}$. Then, for some $T > 0$ depending only on c_0, the following initial value problem has a smooth solution $u(x,t)$ for $t \in (0,T]$.*

$$\begin{aligned} \frac{du}{dt} &= \log \frac{\det(g_{i\bar{j}} + u_{i\bar{j}})}{\det(g_{i\bar{j}})} + f \\ u(x,0) &= 0 \end{aligned} \tag{3.1}$$

Moreover, for any $t \in (0,T]$, the Kähler metric $g_{i\bar{j}}(x) + u_{i\bar{j}}(x,t)$ is equivalent to $g_{i\bar{j}}$ and has bounded geometry of order ∞ and $f(x) + (\log\det(u_{i\bar{j}}))(x,t)$ is a potential for $R_{i\bar{j}}(x,t)$.

This is the Kähler potential version of the following theorem of Shi [19].

Theorem 3.2. *Let $(M, g_{i\bar{j}})$ be a complete non-compact Kähler manifold such that $\|Rm\| \leq c_0$. Then, for some constant $T > 0$ depending only on c_0, there is a smooth short time solution $\tilde{g}_{i\bar{j}}(x,\, t)$ to the Kähler Ricci flow equation*

$$\begin{aligned} \frac{d\tilde{g}_{i\bar{j}}}{dt} &= -\tilde{R}_{i\bar{j}} \\ \tilde{g}_{i\bar{j}}(x,0) &= g_{i\bar{j}}. \end{aligned} \tag{3.2}$$

for $t \in (0,\, T]$. Moreover, for all $t \in (0,T]$, $\tilde{g}_{i\bar{j}}(x,\, t)$ is a complete Kähler metric on M equivalent to $g_{i\bar{j}}$ and we have the following estimates for the *covariant derivatives of the curvature tensor of* $\tilde{g}_{i\bar{j}}(x,\,t)$*.*

$$\left\|\nabla^m_{\tilde{g}} \tilde{Rm}(x,\,t)\right\|^2 \le C(n,m,c_0)(1/t)^m.$$

Proof of Lemma 3.1. Under the hypothesis of the lemma, Theorem 3.2 guarantees a short time solution $\tilde{g}_{i\bar{j}}$ to the Kähler–Ricci flow (3.2). Using this solution, we solve the following ordinary differential equation on $[0,\,T]$ for $x \in M$

$$\begin{aligned} \frac{du}{dt} =& \log \frac{\det(\tilde{g}_{i\bar{j}})}{\det(g_{i\bar{j}})} + f \\ u(x,0) =& 0 \end{aligned}$$

for a smooth function $u(x,t)$. It is then straightforward to verify that we must have $\tilde{g}_{i\bar{j}}(x,t) = g_{i\bar{j}}(x) + u_{i\bar{j}}$ and thus, $u(x,t)$ is a smooth solution to (3.1). The details of this verification can be found in [9]. To complete the proof of Lemma 3.1, we need to show that for any $t \in (0,\,T]$ the Kähler metric $\tilde{g}_{i\bar{j}}(x,t) = g_{i\bar{j}}(x)+u_{i\bar{j}}(x,t)$ has bounded geometry of order ∞. In [21], the authors prove that on a non-compact Kähler manifold, one has bounded geometry of order $2+\alpha$ provided one has bounded curvature and gradient of scalar curvature. Their proof can in fact be extended to show that one has bounded geometry of infinite order provided one has all covariant derivatives of curvature bounded. Thus, since $\tilde{g}_{i\bar{j}}(x,t)$ has all covariant derivatives of its curvature bounded by Theorem 3.2, we see that $\tilde{g}_{i\bar{j}}(x,t)$ in fact, has bounded geometry of order infinity. This completes the proof of Lemma 3.1. □

4. A Priori Estimates And Longtime Existence.

In this section, we prove *a priori* estimates for solutions of (2.3). We follow the approach first used by Cao [6] which is to adapt the elliptic estimates proved by Yau [24] and Aubin [3] for the elliptic complex Monge–Ampère equation to the parabolic case. As we may transform (2.3) to an equation of the form (3.1), we get short time existence. We may assume that we have a smooth solution $v \in C^\infty(\mathbb{C}^n \times [0,\,T])$ to (2.3), that is, if v is not smooth at $t = 0$, we use $t - \varepsilon$, $1 \gg \varepsilon > 0$, instead of t. Choosing T smaller if necessary, we may also assume that $v(\cdot,\,t)$ gives rise to a complete Kähler metric uniformly equivalent to $(u_0)_{i\bar{j}}$.

4.1. Lower Order Estimates.

Lemma 4.1. *A solution v to* (2.3) *satisfies*

$$|v(\cdot,t)|_{C^0} \leq |v(\cdot,0)|_{C^0} =: K_0$$

and

$$\left|\frac{d}{dt}v(\cdot,\, t)\right|_{C^0} \leq \left|\frac{d}{dt}v(\cdot,\, 0)\right|_{C^0} =: K_{\frac{d}{dt}}$$

for all $t \in [0,\, T]$.

Proof. This follows directly from the maximum principle in [10]. □

Lemma 4.2. *A solution v to* (2.3) *satisfies $\left|z^I v_I\right| \leq c =: K_{z\nabla v}$ uniformly in t.*

Proof. We estimate

$$\begin{aligned}\tfrac{1}{2}\left|z^I v_I\right| \leq& \left|\tfrac{1}{2}z^I v_I - \frac{d}{dt}v\right| + \left|\frac{d}{dt}v\right| \leq \left|\log\frac{\det(\tilde{v}_{i\bar{\jmath}})}{\det((u_0)_{i\bar{\jmath}})}\right| + K_{\frac{d}{dt}}\\ =& \left|\log\frac{\det(V_{i\bar{\jmath}})}{\det((U_0)_{i\bar{\jmath}})}\right| + K_{\frac{d}{dt}} = \left|\frac{d}{dt}V - \frac{d}{dt}U_0\right| + K_{\frac{d}{dt}}.\end{aligned}$$

As $(V - U_0)\left(e^{\frac{1}{2}t}x,\, t\right) = (\tilde{v} - \tilde{u}_0)(x,\, t) = v(x,\, t)$ is uniformly bounded in C^0, it suffices to prove that

$$\left|\frac{d^2}{dt^2}V - \frac{d^2}{dt^2}U_0\right| \leq c.$$

Then, interpolation gives the claimed inequality. We differentiate (1.2) and obtain

$$\begin{aligned}\frac{d^2}{dt^2}V - \frac{d^2}{dt^2}U_0 =& \frac{d}{dt}\log\det(V_{i\bar{\jmath}}) - \frac{d}{dt}\log\det((U_0)_{i\bar{\jmath}})\\ =& V^{i\bar{\jmath}}\left(\frac{d}{dt}V\right)_{i\bar{\jmath}} - U_0^{i\bar{\jmath}}\left(\frac{d}{dt}U_0\right)_{i\bar{\jmath}}\\ =& V^{i\bar{\jmath}}(\log\det(V_{k\bar{l}}))_{i\bar{\jmath}} - U_0^{i\bar{\jmath}}(\log\det((U_0)_{k\bar{l}}))_{i\bar{\jmath}}\\ =& -R_V + R_{U_0},\end{aligned}$$

where R_V and R_{U_0} are the scalar curvatures of the metrics $V_{i\bar{j}}$ and $(U_0)_{i\bar{j}}$, respectively. As V and U_0 give rise to solutions of Kähler–Ricci flow, the corresponding scalar curvatures are uniformly bounded [19]. □

We are not able to prove gradient estimates directly. Instead, we have

Lemma 4.3. *Let v be a solution to* (2.3)*. Then, there exists a constant $K_{1+\alpha}$ that depends only on the C-soliton such that*

$$\|v(\cdot,\, t)\|_{1+\alpha} \le K_{1+\alpha} \cdot \left(K_0 + \left(n + \sup_{\mathbb{C}^n} \Delta_{u_0} v(\cdot,\, t) \right) \right)$$

for all $t \in [0,\, T]$.

Proof. We apply L^p-estimates [12, Theorem 9.11] to $0 < n + \Delta_{u_0} v$ and obtain spatial $H^{2,p}$-bounds for v. Then, the Sobolev imbedding theorem implies the result. Note that we only used $0 < n + \Delta_{u_0} v$ and the C^0-bound. □

4.2. Second Order Estimates.

Consider the quantity

$$A = \log(n + \Delta v) - kv, \tag{4.1}$$

where Δv denotes the Laplacian of v with respect to $(u_0)_{i\bar{j}}(x)$ and the constant $k \gg 1$ is to be chosen later. Clearly, a bound on $|A|$ implies a bound on $|\Delta v|$. We will bound A from above using the maximum principle. The bound from below will follow directly from some simple inequalities.

Lemma 4.4. *Assume that* $\sup_{\mathbb{C}^n} \Delta v \ge 1$*. Then,*

$$\frac{1}{n + \Delta v} \left(\Delta \left(z^I v_I \right) \right) \le z^I A_I + k \left(z^I v_I \right) + c$$

holds in $\Omega := \left\{ x \in \mathbb{C}^n : \Delta v(x) \ge \tfrac{1}{2} \sup_{\mathbb{C}^n} \Delta v \right\}$.

Proof. We compute

$$\begin{aligned} \Delta \left(z^I v_I \right) =& (u_0)^{l\bar{k}} \left(z^i v_i + z^{\bar{j}} v_{\bar{j}} \right)_{l\bar{k}} \\ =& (u_0)^{l\bar{k}} \left(z^i_{,l} v_i + z^i v_{,il} + z^{\bar{j}} v_{,\bar{j}l} \right)_{\bar{k}} \\ \le& (u_0)^{l\bar{k}} \left(z^i_{,l} v_{,i\bar{k}} + z^{\bar{j}}_{,\bar{k}} v_{,\bar{j}l} + z^i v_{,l\bar{k}i} + z^{\bar{j}} v_{,l\bar{k}\bar{j}} \right) + c \cdot \|\nabla v\|_0 \end{aligned} \tag{4.2}$$

and

$$z^I A_I = \frac{1}{n+\Delta v} z^i \left((u_0)^{l\bar{k}} v_{,l\bar{k}} \right)_{,i} + \frac{1}{n+\Delta v} z^{\bar{j}} \left((u_0)^{l\bar{k}} v_{,l\bar{k}} \right)_{,\bar{j}} - k z^I v_I. \tag{4.3}$$

Note that z^i, $z^{\bar{j}}$, and the Riemannian curvature tensor induced by u_0 are bounded with respect to the metric $((u_0)_{i\bar{j}})$, see Remark C.3. This allows to estimate the terms obtained by interchanging the order of covariant differentiation.

We combine (4.2) and (4.3), and get in Ω

$$\begin{aligned} &\frac{1}{n+\Delta v} \Delta \left(z^I v_I \right) - z^I A_I \\ &\le k z^I v_I + c \frac{\|\nabla v\|_0}{n+\Delta v} + \frac{1}{n+\Delta v} (u_0)^{l\bar{k}} \left(z^i_{,l} v_{i\bar{k}} + z^{\bar{j}}_{,\bar{k}} v_{\bar{j}l} \right) \\ &\le k z^I v_I + c \frac{\Delta v}{n+\Delta v} + c \frac{1}{n+\Delta v}. \end{aligned} \tag{4.4}$$

Here, we have used that at a fixed point, we can always choose holomorphic coordinates such that $(u_0)^{l\bar{k}} = \delta^{l\bar{k}}$ and $v_{i\bar{k}} = 0$ for $i \neq k$, so we get

$$(u_0)^{l\bar{k}} \left(z^i_{,l} v_{i\bar{k}} + z^{\bar{j}}_{,\bar{k}} v_{\bar{j}l} \right) \le c\Delta v + c$$

and deduce the second inequality in (4.4). In such coordinates, the terms $1 + v_{i\bar{\imath}}$ are positive for each i and are simply the eigenvalues of the tensor $(u_0)_{i\bar{j}} + v_{i\bar{j}}$ with respect to metric $(u_0)_{i\bar{j}} = \delta_{i\bar{j}}$. Finally, in passing to the last line of (4.4), we have used Lemma 4.3. □

We are now in a position to prove the following

Lemma 4.5. *There is a constant $K_2 > 0$ such that $|\Delta v(x, t)| \le K_2$ for all $(x, t) \in \mathbb{C}^n \times \in [0, T]$.*

Proof. For the proof of the upper bound for Δv, we will only consider those $(x, t) \in \mathbb{C}^n \times [0, T]$ such that $\sup\limits_{\mathbb{C}^n \times \{t\}} \Delta v > 1$ and $\Delta v(x, t) > \frac{1}{2} \sup\limits_{\mathbb{C}^n} \Delta v(\cdot, t)$. Thus, we can use Lemmata 4.3 and 4.4. We compute the evolution equation for A, interchange fourth covariant derivatives and use [3, p. 264] to estimate third derivatives

$$\begin{aligned} \frac{dA}{dt} - \tilde{\Delta} A \le & \frac{1}{n+\Delta v} \frac{d\Delta v}{dt} - k \frac{dv}{dt} - (k-c) \tilde{v}^{i\bar{j}} (u_0)_{i\bar{j}} \\ & - \frac{1}{n+\Delta v} \left(\Delta \frac{dv}{dt} - \tfrac{1}{2} \Delta \left(z^I v_I \right) \right) + nk, \end{aligned}$$

where $\tilde{\Delta}$ denotes the Laplacian in the metric $\tilde{v}_{i\bar{\jmath}}(x,t) = (u_0)_{i\bar{\jmath}}(x) + v_{i\bar{\jmath}}(x,t)$ with inverse $\tilde{v}^{i\bar{\jmath}}$. We use the geometric–arithmetic means inequality and

$$\left(\sum_{i=1}^n \frac{1}{\lambda_i}\right)^{n-1} \ge \tfrac{1}{n}\sum_{i=1}^n \lambda_i \cdot \prod_{i=1}^n \frac{1}{\lambda_i}, \quad \lambda_i > 0,$$

which is proved easily as we may assume that $1 = \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$, to obtain

$$\begin{aligned}(n+\Delta v) \ge & \left[\frac{\det(\tilde{v}_{i\bar{\jmath}})}{\det((u_0)_{i\bar{\jmath}})}\right]^{\frac{1}{n}} = e^{\frac{1}{n}\left(-\frac{1}{2}z^I v_I + \frac{dv}{dt}\right)},\\ \tilde{v}^{i\bar{\jmath}}(u_0)_{i\bar{\jmath}} \ge & \frac{1}{c(n)}\left[(n+\Delta v)\cdot e^{\frac{1}{2}z^I v_I - \frac{dv}{dt}}\right]^{\frac{1}{n-1}}.\end{aligned}$$

We apply Lemma 4.4 and estimate

$$\begin{aligned}\frac{dA}{dt} - \tilde{\Delta}A \le & \frac{1}{n+\Delta v}\frac{d\Delta v}{dt} - k\frac{dv}{dt} - (k-c)\tilde{v}^{i\bar{\jmath}}(u_0)_{i\bar{\jmath}}\\ & - \frac{1}{n+\Delta v}\Delta\frac{dv}{dt} + \tfrac{1}{2}z^I A_I + \tfrac{1}{2}kz^I v_I + nk + c\\ \le & -(k-c)\tilde{v}^{i\bar{\jmath}}(u_0)_{i\bar{\jmath}} + \tfrac{1}{2}z^I A_I + \tfrac{1}{2}kz^I v_I + nk - k\frac{dv}{dt} + c\\ \le & -\frac{k-c}{c(n)} e^{\frac{1}{n-1}\left(\frac{1}{2}z^I v_I - \frac{dv}{dt}\right)}(n+\Delta v)^{\frac{1}{n-1}}\\ & + \tfrac{1}{2}z^I A_I + \tfrac{1}{2}kz^I v_I + nk - k\frac{dv}{dt} + c.\end{aligned}$$

Fixing $k \gg 1$ so large that $k-c$ is bounded below by some positive constant, the maximum principle can be applied to the evolution equation

$$\frac{d}{dt}A - \tilde{\Delta}A \le -\tfrac{1}{c}e^{\frac{A-c}{n-1}} + \tfrac{1}{2}z^I A_I + c,$$

implying the upper bound.

To prove a lower bound for Δv, we use coordinates as in Lemma 4.4. Our lower order estimates (Lemmata 4.1 and 4.3) imply that $\prod_{i=1}^n (1+v_{i\bar{\imath}})$ is bounded below by a positive constant. The function v gives rise to a Kähler metric. So, all the factors are positive. As we have seen that $v_{i\bar{\imath}}$ is uniformly bounded above for each i, the lower bound follows. □

Corollary 4.6. *There is a constant $K_1 > 0$ depending only on K_0 and K_2 such that $\|v(\cdot,t)\|_1 < K_1$ for all $t \in [0,T]$.*

Proof. Use Lemmata 4.1, 4.3 and 4.5. □

Corollary 4.7. *The metric* $w_{i\bar{j}}(x,t) = v_{i\bar{j}}(x,t) + (u_0)_{i\bar{j}}(x)$ *is equivalent to* $(u_0)_{i\bar{j}}(x)$ *for all* $t \in [0,T]$*. Moreover, the equivalence factor depends only on* K_0*,* K_1 *and* K_2*.*

Proof. This follows from the proof of Lemma 4.5. □

4.3. Higher Order Estimates and Long Time Existence.

Consider the quantities

$$Q_3 = \tilde{v}^{i\bar{j}}\tilde{v}^{k\bar{l}}\tilde{v}^{r\bar{s}} v_{,i\bar{l}r} v_{,\bar{j}k\bar{s}},$$

and

$$Q_4 = \tilde{v}^{i\bar{j}}\tilde{v}^{k\bar{l}}\tilde{v}^{r\bar{s}}\tilde{v}^{a\bar{b}} v_{,i\bar{l}r\bar{b}} v_{,\bar{j}k\bar{s}a},$$

where the covariant differentiation is with respect to $(u_0)_{i\bar{j}}$ and $\tilde{v}^{i\bar{j}}$ represents the inverse of the time dependent metric $\tilde{v}_{i\bar{j}}(x,t) = (u_0)_{i\bar{j}}(x) + v_{i\bar{j}}(x,t)$. By the previous section, this norm is equivalent to that using $(u_0)_{i\bar{j}}$.

Lemma 4.8. *There are constants* $K_3, K_4 > 0$ *depending only on* K_0*,* K_1*,* K_2 *such that* $|Q_3|_{C^0} < K_3$ *and* $|Q_4|_{C^0} < K_4$ *for all* $t \in [0,T]$*.*

Proof. The above estimates are known in the special case that v is a solution to (3.1) and have appeared in several places in various equivalent forms. We describe some of these briefly. Calabi first estimated $|Q_3|_{C^0}$ for the elliptic Monge–Ampère equation on a compact manifold. This estimate was later used by Aubin [3] and by Yau [24] in proving the Calabi conjecture. Calabi's estimate was applied directly by Cao [6] to (3.1) and later by Shi [19] to (1.1). In [19], Shi goes further to estimate an appropriate second derivative of the solution to (1.1) and observes that this is equivalent to estimating the curvature tensor of the evolving metric. An equivalent estimate can be found in [9] where $|Q_4|_{C^0}$ is estimated for (3.1).

In our case that v is a solution to (2.3), we point out that it is straightforward to adapt the arguments of the authors cited above to our case. □

Notice that while the estimates in Lemma 4.8 follow rather painlessly from the corresponding estimates for (3.1), such is not the case for our laplacian estimate in Lemma 4.5. The difference is that, in Lemma 4.8, we have already estimated all derivatives of lower order and second derivatives of the form $v_{i\bar{j}}$, while in the case of Lemma 4.5 do not have *a priori* gradient estimates. This does not cause a problem in (3.1) while in our case it does.

Note that the *a priori* estimates obtained so far imply that for any $t \in [0, T]$, $n + \Delta_{u_0} v(\cdot, t) \in C^\alpha$ with uniform bounds. Thus, elliptic Schauder theory implies that $v(\cdot, t) \in C^{2+\alpha}$. Differentiating (2.3) yields $v_{i\bar{j}} \in C^{\alpha, \frac{\alpha}{2}}$ and $v \in C^{2+\alpha, 1+\frac{\alpha}{2}}$ with uniform bounds.

Lemma 4.9. *Let v be a solution to* (2.3) *and let $C^{k+\alpha, \frac{k}{2}+\frac{\alpha}{2}}$ be the Hölder spaces on $\mathbb{C}^n$ relative to the metric $\tilde{v}_{i\bar{j}}$. Then, for every k, v is bounded in $C^{k+\alpha, \frac{k}{2}+\frac{\alpha}{2}}$ independent of t.*

Proof. We prove the respective result for Hölder spaces with respect to the background metric. The corresponding results in these Hölder spaces imply the claimed estimates. Consider an arbitrary coordinate neighborhood V_β with coordinates (z^i) as in Definition 2.4. Differentiating (2.3) with respect to z^i in these coordinates and rearranging terms gives

$$\begin{aligned} \frac{d}{dt} v_i =& \tilde{v}^{r\bar{s}} v_{,r\bar{s}i} - \tfrac{1}{2}\left(\hat{z}^I v_I\right)_i \\ =& \tilde{v}^{r\bar{s}} v_{,ir\bar{s}} - \tfrac{1}{2}\hat{z}^I v_{,iI} + \left(\tilde{v}^{r\bar{s}} R_{i\bar{s}r\bar{d}} \tilde{u}_0^{e\bar{d}} v_e - \tfrac{1}{2}\hat{z}^I_{,i} v_I\right), \end{aligned} \tag{4.5}$$

where $\hat{z}^i$ are the local components of the global vector field z^i. For covariant differentiation and the curvature tensor, we use the background metric $(u_0)_{i\bar{j}}$. We view (4.5) as a parabolic equation for $v_i(x, t)$ on the coordinate domain $V_\beta \times [0, \infty)$ with the third term on the right-hand side considered as a single inhomogeneous term. In what follows, all bounds stated will be independent of β and t. It is readily seen that our estimates from above provide us with a $C^{\alpha, \frac{\alpha}{2}}$ bound for the coefficients and terms of (4.5). We may then apply standard parabolic Schauder estimates to obtain a $C^{2+\alpha, 1+\frac{\alpha}{2}}$ bound for $v_i(x, t)$ in an interior domain of V_β. A standard bootstrapping argument [6] combined with the fact that the metric $\tilde{v}_{i\bar{j}}$ has bounded geometry of order ∞ then allows us to obtain a $C^{k+\alpha, \frac{k}{2}+\frac{\alpha}{2}}$ bound on $v(x, t)$ in V_β for all k. The lemma now follows readily from Definition 2.5. □

Corollary 4.10. *The solution v is smooth and exists for all time. Moreover, the metric $\tilde{v}_{i\bar{j}}(x, t) = (u_0)_{i\bar{j}}(x) + v_{i\bar{j}}(x, t)$ remains equivalent to $(u_0)_{i\bar{j}}(x)$ uniformly over all t and the curvature of $\tilde{v}_{i\bar{j}}(x, t)$ remains bounded on $\mathbb{C}^n$ independent of t.*

Proof. By our *a priori* estimates, it is straightforward to see that the curvature of the metric $\tilde{v}_{i\bar{j}}(x,t)$ stays uniformly bounded on $[0,T]$. Corollary 4.7 implies that the metrics stay uniformly equivalent. Thus, to prove the corollary, it suffices to prove the assertion of long time existence. Moreover, long time existence for v follows from long time existence for (1.2) with initial condition $\tilde{v}(x,0) = u_0(x) + v(x,0)$. Begin by assuming that T is the maximal time up to which we have a smooth solution. Choosing a time $T' < T$ arbitrarily close to T and applying Lemma 3.1 to the metric $\tilde{v}_{i\bar{j}}(x,T')$, we may extend $\tilde{v}$ past T as a solution to (1.2) thus arriving at a contradiction and thus proving the corollary. □

5. Barrier Construction.

Before we can construct a barrier, we have to determine the precise asymptotic behavior of our soliton. According to [5], we may assume that the function $\varphi(s) = \hat{u}'(s)$ fulfilling

$$\varphi^{n-1}\varphi' e^{\varphi} = e^{ns} \tag{5.1}$$

and

$$\varphi(s) \to 0 \quad \text{for } s \to -\infty, \tag{5.2}$$

where $s = \log |z|^2$, gives rise to our soliton. The second condition is required to obtain a smooth solution at the origin. We derive in Appendix B the following expansions for φ and its derivatives at infinity

$$\begin{aligned} \varphi =& ns + o(s), \\ \varphi' =& n + o(1), \\ \varphi'' =& \frac{n-1}{s^2} + o\left(\frac{1}{s^2}\right), \end{aligned}$$

and

$$\varphi''' = -2\frac{n-1}{s^3} + o\left(\frac{1}{s^3}\right).$$

In the following, we construct barriers in the case $n \geq 2$. Now, we assume that our perturbation $u(x,0)$ of the initial value is such that

$$|u(x,0)| \leq K \cdot \min\left\{1, s^{-\alpha}\right\}, \quad \text{where } s = 2\log|x|, \quad 0 < \alpha < 1.$$

For our barrier, we make the ansatz

$$\varphi_b(s) = \varphi(s) \mp K s^{-1-\alpha} \alpha (2R)^\alpha \psi\left(\frac{s}{R}\right) \tag{5.3}$$

with φ as above, that corresponds to the barrier

$$\hat{b}(s) = \hat{u}(s) \pm \int_s^\infty K \sigma^{-1-\alpha} \alpha (2R)^\alpha \psi\left(\frac{\sigma}{R}\right) d\sigma.$$

Here, ψ is a smooth monotone function such that

$$\psi(s) = \begin{cases} 0 & \text{if } s \le 1, \\ 1 & \text{if } s \ge 2. \end{cases}$$

Assume from now on that $R \ge \frac{1}{2}$. It is straightforward to check that $\hat{b}(s)$ lies above/below our perturbed initial value.

To prove that for $R \gg 1$ fixed sufficiently large

$$\varphi_b > 0, \quad \varphi_b' > 0, \quad \varphi_b - \varphi_b' > 0, \quad \varphi_b'^2 - \varphi_b \varphi_b'' > 0, \quad \varphi_b''^2 - \varphi_b' \varphi_b''' > 0 \tag{5.4}$$

is again a technical calculation, we refer to Appendix C.

Note that it is essentially the integrability condition for φ_b and not (5.4) that determines the possible exponents in the decay condition.

For $n = 1$, our method does not seem to work. In this case, $\varphi(s)$ is even explicitly known to be $\log(1 + e^s)$, but

$$\varphi''^2 - \varphi' \varphi''' = e^{-s} + O\left(e^{-2s}\right)$$

seems to exclude such a barrier construction. For results concerning long time behavior of solutions to Ricci flow in the corresponding real dimension 2, we refer to [14, 23].

6. Convergence to Zero.

In this section, we prove Theorem 1.2. We use the radially symmetric decaying barriers constructed in Section 5 to enclose our initial perturbation from above and from below. By smoothly evolving our barriers and perturbed initial value using (2.3) for all time, the maximum principle of [10] implies that our perturbation will converge to zero provided such is true of our barriers. In particular, the perturbed soliton converges back to the original soliton as $t \to \infty$. We will only show that the upper barrier converges back to the original soliton. Studying the behavior during Kähler–Ricci flow is simpler for the barriers as they are rotationally symmetric and decaying in $|z|$.

Lemma 6.1. *Let b be the upper (lower) barrier constructed in Section* 5. *Then,* (2.3) *with initial condition b has a long time smooth solution, which we also denote by b, which converges to zero as $t \to \infty$ in the C^0 norm.*

The proof is divided into several steps. We sketch the proof for the case of the upper barrier and note that the case of the lower barrier is similar. Part of the argument is a modification of the convergence proof in [18]. We first show that the condition that b initially decays monotonely in $|z|$ is preserved for all time, so we get especially $b(0, t) \geq b(x, t)$ for all $(x, t) \in \mathbb{C}^n \times [0, \infty)$. We do this in Lemma 6.2. The strong maximum principle then guarantees that $\sup_{\mathbb{C}^n} b$ is strictly decreasing in t. In fact, we claim that b must converge to a constant. This can be seen as follows. In view of our *a priori* estimates, we can find for every sequence $t_n \to \infty$ a subsequence, again denoted by t_n, such that the maps

$$\mathbb{C}^n \times [-t_n, \infty) \ni (x, t) \mapsto b(x, t + t_n)$$

converge locally uniformly in any C^k-norm to a smooth function $b^\infty(x, t)$ satisfying the evolution equation (2.3) everywhere in $\mathbb{C}^n \times \mathbb{R}$. Moreover, since the oscillation of b decreases strictly in time by the strong maximum principle [1], it must converge to some non-negative constant. In other words, the limit solution $b^\infty(x, t)$ has non-negative oscillation which is constant in time. But it is easy to see that the rotational symmetry and decay condition on $b(x,t)$ also holds for $b^\infty(x,t)$ and thus by the strong maximum principle, the oscillation of $b^\infty(x,t)$ cannot be a positive constant. Thus, $b^\infty(x,t)$ is constant in space. The monotonicity of $b(0, t)$ shows that this constant is independent of the chosen subsequence and hence, b actually converges to a constant. In Corollary 6.4, we show that during the evolution the L^p-norm, for some $p \geq 2$, of b is dominated by its value at $t = 0$. We compute the L^p-norm with respect to an evolving volume form which stays uniformly equivalent to the volume form for the initial soliton metric, thus the integral of any positive constant over $\mathbb{C}^n$ with respect to this volume form for fixed t is infinite. By Remark C.4, the L^p-norm is finite for $t = 0$ provided $p > \frac{n+1}{\alpha}$ with α as in Section 5. So, b has to converge uniformly to zero on compact subsets of $\mathbb{C}^n$ as $t \to \infty$. Note, that the monotonicity in $|z|$ is preserved during the evolution and when we extract subsequences. Moreover, $b(0,t)$ is decreasing in t. So, $b(0, t)$ has to converge to zero and it follows that the perturbed soliton converges back to the original soliton in $\mathbb{C}^n$.

Lemma 6.2. *Let b be the upper barrier constructed in Section* 5*. Then, b stays rotationally symmetric and the property that b decays in $|z|$ is preserved during the evolution of b by* (2.3)*.*

It is quite evident that this lemma is true. Thus, we defer its proof to Appendix A.

Lemma 6.3. *Let u_0 be a C-potential and b a barrier as constructed in Section* 5*. Then, there exists a metric $a_{i\bar{j}}$, uniformly equivalent to $(u_0)_{i\bar{j}}$ and $(u_0)_{i\bar{j}} + b_{i\bar{j}}$, such that for b evolving according to* (2.3)*, we have*

$$\int\limits_{\mathbb{C}^n} |b(t)|^p \det(a_{i\bar{j}}(t)) \equiv I_p(t) \leq I_p(0)$$

for $p \geq 2$ and $t \geq 0$.

Proof. Interpolating between the two determinants in (2.3) and using upper indices to denote inverses, we get

$$\begin{aligned}\frac{d}{dt}b =& \log\det((u_0)_{i\bar{j}} + b_{i\bar{j}}) - \log\det((u_0)_{i\bar{j}}) + \tfrac{1}{2}z^I b_I \\ =& \int\limits_0^1 ((u_0)_{\cdot\cdot} + \tau b_{\cdot\cdot})^{i\bar{j}} d\tau b_{i\bar{j}} + \tfrac{1}{2}z^I b_I \\ \equiv& a^{i\bar{j}} b_{i\bar{j}} + \tfrac{1}{2}z^I b_I.\end{aligned}$$

Now, we define $(a_{i\bar{j}})$ to be the inverse of $\left(a^{i\bar{j}}\right)$. By definition, $(a_{i\bar{j}})$ is uniformly equivalent to $(u_0)_{i\bar{j}}$ and $(u_0 + b)_{i\bar{j}}$ as these two metrics stay uniformly equivalent during the evolution. For showing the definiteness of time derivatives of certain metrics, it will be convenient to make a substitution so that we almost come back to the original evolution equation (1.2). Set $\underline{b}(x,\, t) := b\left(e^{-\frac{1}{2}t}x,\, t\right)$, $\underline{u}_0(x, t) := u_0\left(e^{-\frac{1}{2}t}x\right)$. This implies that $(\underline{u}_0 + \underline{b})(x,\, t) = B(x,\, t) + \frac{1}{2}nt^2$, where $B = B(u_0 + b)$ is as in (1.2). As the metric $B_{i\bar{j}}$ has positive holomorphic bisectional curvature for $t = 0$ (Appendix C), this is preserved during the evolution [19], so the Ricci curvature also stays positive definite. From (1.1), we obtain that

$$\frac{d}{dt}(\underline{u}_0 + \underline{b})_{i\bar{j}} \leq 0 \text{ and similarly } \frac{d}{dt}(\underline{u}_0)_{i\bar{j}} \leq 0 \tag{6.1}$$

in the sense of matrices. The second inequality follows by noting that $(\underline{u}_0)_{i\bar{j}} = (U_0)_{i\bar{j}}$ and thus also corresponds to a solution to (3.2) with positive holomorphic bisectional curvature. The chain rule and the transformation formula for integrals imply that

$$I_p(t) = \int\limits_{\mathbb{C}^n} |\underline{b}|^p \det(\underline{a}_{i\bar{\jmath}})$$

as $\det(a_{i\bar{\jmath}}) = e^{-nt} \det(\underline{a}_{i\bar{\jmath}})$ and the volume elements differ by a factor e^{nt}. Here, $(\underline{a}_{i\bar{\jmath}})$ is the inverse of

$$\int\limits_0^1 ((\underline{u}_0)_{\cdot\cdot} + \tau \underline{b}_{\cdot\cdot})^{i\bar{\jmath}}\, d\tau.$$

Note that

$$(\underline{u}_0)_{i\bar{\jmath}} + \tau \underline{b}_{i\bar{\jmath}} = \tau((\underline{u}_0)_{i\bar{\jmath}} + \underline{b}_{i\bar{\jmath}}) + (1-\tau)(\underline{u}_0)_{i\bar{\jmath}}$$

and we get from (6.1)

$$\frac{d}{dt}\left((\underline{u}_0)_{i\bar{\jmath}} + \tau \underline{b}_{i\bar{\jmath}}\right) \le 0.$$

As $(\underline{a}_{i\bar{\jmath}})$ is obtained by taking the inverse of this matrix, integrating, and taking the inverse once more, the definiteness for the time derivative is inverted twice, so $\frac{d}{dt}\underline{a}_{i\bar{\jmath}} \le 0$. Finally, $(\underline{a}_{i\bar{\jmath}})$ is positive definite, so it follows that

$$\frac{d}{dt}\det(\underline{a}_{i\bar{\jmath}}) \le 0. \tag{6.2}$$

It is not obvious, whether $I_p(t)$ is differentiable with respect to t or not. Therefore, we define for radii $R > 0$ and balls B_R with respect to the flat metric of $\mathbb{C}^n$

$$I_{p,R}(t) := \int\limits_{B_R} |\underline{b}|^p \det(\underline{a}_{i\bar{\jmath}}) dV,$$

where dV denotes the Euclidean volume element. In order to compute $\frac{d}{dt}I_{p,R}(t)$, we have to compute the evolution equation for $\underline{b}$,

$$\frac{d}{dt}\underline{b} = \log \frac{\det((\underline{u}_0)_{i\bar{\jmath}} + \underline{b}_{i\bar{\jmath}})}{\det((\underline{u}_0)_{i\bar{\jmath}})} = \underline{a}^{i\bar{\jmath}} \underline{b}_{i\bar{\jmath}}. \tag{6.3}$$

Using (6.2) and (6.3)

$$\begin{aligned} \frac{d}{dt}I_{p,R}(t) &= \int\limits_{B_R} p|\underline{b}|^{p-2}\underline{b}\left(\frac{d}{dt}\underline{b}\right)\det(\underline{a}_{i\bar{\jmath}})dV + \int\limits_{B_R} |\underline{b}|^p \frac{d}{dt}\det(a_{i\bar{\jmath}})dV \\ &\le \int\limits_{B_R} p|\underline{b}|^{p-2}\underline{b}\det(\underline{a}_{k\bar{l}})\underline{a}^{i\bar{\jmath}}\underline{b}_{i\bar{\jmath}}dV. \end{aligned}$$

To estimate further, we denote by g the real metric corresponding to $(\underline{a}_{i\bar{j}})$, see e.g. [19], and obtain in real coordinates

$$\frac{d}{dt} I_{p,R}(t) \le \int\limits_{B_R} p|\underline{b}|^{p-2}\underline{b}\Delta_g\underline{b}\sqrt{\det(g)}\,dV \equiv \int\limits_{B_R} p|\underline{b}|^{p-2}\underline{b}\Delta_g\underline{b}\,d\mu_g.$$

We apply the divergence theorem and use ν to denote the exterior unit normal to B_R with respect to the metric g which coincides with $\frac{x}{|x|}$ up to a positive factor

$$\frac{d}{dt} I_{p,R}(t) \le -\int\limits_{B_R} p(p-1)|\underline{b}|^{p-2}\langle\nabla\underline{b},\nabla\underline{b}\rangle_g\,d\mu_g + \int\limits_{\partial B_R} p|\underline{b}|^{p-2}\underline{b}\langle\nabla\underline{b},\nu\rangle_g d\,\mathcal{H}^{2n-1}_g.$$

Here, we used suggestive invariant notation. We apply Lemma 6.2 to see that the boundary integral is non-positive and get $I_{p,R}(t_1) \ge I_{p,R}(t_2)$ for $0 \le t_1 \le t_2$. Finally, we let $R \to \infty$ and obtain the claimed inequality. □

Corollary 6.4. *Let u_0 be a C-soliton and $b : \mathbb{C}^n \to \mathbb{R}$ the barrier constructed in Section* 5. *Assume that $p \ge 2$ is chosen such that the L^p-norm*

$$\|b\|_{L^p} := \int\limits_{\mathbb{C}^n} |b|^p \det((u_0)_{i\bar{j}} + b_{i\bar{j}})$$

is finite for $t = 0$. Then, the L^p-norm of b stays uniformly bounded when b evolves by Kähler–Ricci flow (2.3)

$$\|b(t)\|_{L^p} \le c \cdot \|b(0)\|_{L^p},$$

where the constant depends only on the uniform equivalence of the metrics $(u_0)_{i\bar{j}}$ and $(u_0)_{i\bar{j}} + b_{i\bar{j}}$ that is guaranteed during the evolution.

Proof of Lemma 6.1 *and Theorem* 1.2. Lemmata 6.2 and 6.4 together with the arguments at the beginning of the section complete the proof of Lemma 6.1 and thus of Theorem 1.2. □

Appendix A. Preserving Monotonicity.

Proof of Lemma 6.2. It is clear that the rotational symmetry is preserved during the evolution.

If b is not a monotone decaying function of $|z|$ for all $t > 0$, we choose $0 \le T < \infty$ maximal such that b is monotone decaying in $|z|$ for $t \in [0, T]$. Note that b is clearly monotone on a relatively closed subset in time. Our lemma follows if we can show that b stays monotone for a while after T. To simplify notation, we note, that applying (the independently proven) Lemma 6.3 to the time interval $[0, T]$, where b is monotone, yields that $\lim_{|z|\to\infty} b(|z|, T) = 0$. A similar argument works if we do not use this fact, we just have to take into account the possibly different $\inf b(\cdot, T)$.

First, we consider b on $\mathbb{C}^n \setminus B_R(0)$ for $R \gg 1$. The radius R depends only on the fact, that certain coefficients in the ordinary differential equation are not too far from the corresponding values in the asymptotic expansion for the soliton. So, the value of R depends only on $b(\cdot, 0)$ and our initial soliton as the initial soliton and the perturbed soliton stay uniformly equivalent during the evolution. We have $b(R, T) > 0$ as otherwise the strong maximum principle would imply $b(\cdot, T) \equiv 0$, so $b(\cdot, t) \equiv 0$ for $t > T$, contradicting the maximality of T. Due to the uniformly bounded geometry during the evolution, there exists $T^* > T$ such that $b(R, t) - \frac{1}{2} b(R, T) \ge \frac{1}{c} > 0$ for $t \in [T, T^*]$.

Note that both u_0 and $\tilde{b} = u_0 + b$ solve (2.1). We consider u_0 and b as functions of $s = \log |z|^2$ and t and use $\overline{u}_0$ and $\overline{b}$ to indicate that. Equation (2.1) and calculations similar to those in Section 2.2 imply that

$$\frac{d}{dt}\overline{u}_0 = \log \overline{u}_0'' + (n-1) \log \overline{u}_0' - ns + \overline{u}_0'$$

and

$$\frac{d}{dt}\left(\overline{u}_0 + \overline{b}\right) = \log\left(\overline{u}_0'' + \overline{b}''\right) + (n-1)\log\left(\overline{u}_0' + \overline{b}'\right) - ns + \overline{u}_0' + \overline{b}'.$$

Considering the difference of these two evolution equations gives

$$\frac{d}{dt}\overline{b} = \int\limits_0^1 \frac{1}{\overline{u}_0'' + \tau \overline{b}''} d\tau \cdot \overline{b}'' + (n-1) \int\limits_0^1 \frac{1}{\overline{u}_0' + \tau \overline{b}'} d\tau \cdot \overline{b}' + \overline{b}'.$$

As $\overline{u}_0' = \varphi$ in the notation of Appendix B, we see that $\overline{b}$ fulfills a parabolic equation of the form

$$\frac{d}{dt}\overline{b} = \alpha \overline{b}'' + \beta \overline{b}',$$

where α, β, $\alpha^{-1} \in L^\infty((\log R^2, \infty))$ for $R \gg 1$ fixed appropriately. As $b(R, t) - \frac{1}{2}b(R, T) \ge \frac{1}{c} > 0$ for $T \le t \le T^*$, we can extend α, β, and $\overline{b}$ from $[\log R^2, \infty] \times [T, T^*]$ to $\mathbb{R} \times [T, T^*]$ as in the case with boundary in [2] and apply the result of this paper to see that for $h \in \left(0, \frac{1}{2}b(R, T)\right)$, $\#\{r \ge R : b(r, t) = h\} = 1$ for fixed $t \in (T, T^*]$. This implies monotonicity for $r \ge R$.

It remains to prove that monotonicity is preserved for $b > \frac{1}{2}b(R, T)$. Similarly, as above, we can fix a radius $R_* > R$ and $T_* > T$ such that $b(R_*, t) < \frac{1}{2}b(R, T)$ for $T \le t \le T_*$. Fix $\varepsilon > 0$ and assume that for $t_0 \in [T, T_*]$, there exist $0 \le r_1 < r_2 < R_*$ such that $b(r_2, t_0) \ge b(r_1, t_0) + \varepsilon$ and t_0 is chosen minimal with this property. $b(r, t_0)$ tends to zero as $r \to \infty$. Choose $r_3 > r_2$ minimal such that $b(r_3, t_0) = \frac{1}{2}(b(r_1, t_0) + b(r_2, t_0))$ and $r_0 < r_1$ maximal such that $b(r_0, t_0) = \frac{1}{2}(b(r_1, t_0) + b(r_2, t_0))$ (if such an r_0 exists). Set $\Omega := B_{r_3} \setminus \overline{B}_{r_0}$ if r_0 with this property exists, otherwise $\Omega := B_{r_3}$. From our assumptions, we get that

$$\operatorname{osc}(b, t, \Omega) := \sup_{x \in \Omega} b(x, t) - \inf_{x \in \Omega} b(x, t)$$

is strictly smaller than ε for $T \le t < t_0$ and equals ε for $t = t_0$. Note that for t close to t_0, b is close to $\frac{1}{2}(b(r_1, t_0) + b(r_2, t_0))$ on $\partial\Omega$. So, $b|_{\partial\Omega}$ does not "contribute" to the oscillation for t close to t_0 and we get a contradiction to the strong maximum principle as a positive oscillation has to be strictly decreasing in time (Huisken, see e.g. [1]).

As ε was arbitrary, we see that monotonicity is preserved in $B_{R_*}(0)$ for $T \le t \le T_*$, so monotonicity is preserved everywhere for $T \le t \le \min\{T^*, T_*\}$ and our lemma follows. □

Appendix B. Asymptotic Soliton Behavior.

Lemma B.1. *A solution* $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi = \varphi(s)$, *fulfilling*

$$\varphi^{n-1}\varphi' e^{\varphi} = e^{ns} \tag{B.1}$$

and $\varphi \to 0$ for $s \to -\infty$ has the following asymptotic behavior at infinity

$$
\begin{aligned}
\varphi =& ns - \log\left(n^n s^{n-1}\right) + (n-1)\frac{\log\left(n^n s^{n-1}\right)}{ns} + (n-1)\frac{1}{ns} \\
&+ \frac{1}{2}(n-1)\frac{\log^2\left(n^n s^{n-1}\right)}{n^2 s^2} - (n-1)(n-2)\frac{\log\left(n^n s^{n-1}\right)}{n^2 s^2} \\
&- \frac{1}{2}(n-1)(3n-5)\frac{1}{n^2 s^2} + \frac{1}{3}(n-1)\frac{\log^3\left(n^n s^{n-1}\right)}{(ns)^3} \\
&- \frac{1}{2}(n-1)(3n-5)\frac{\log^2\left(n^n s^{n-1}\right)}{(ns)^3} \\
&+ (n-1)\left(n^2 - 6n + 7\right)\frac{\log\left(n^n s^{n-1}\right)}{(ns)^3} \\
&+ \frac{1}{6}(n-1)\left(11n^2 - 46n + 47\right)\frac{1}{(ns)^3} + o\left(\frac{1}{s^3}\right),
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
\varphi' =& n - \frac{n-1}{s} - (n-1)\frac{\log\left(n^n s^{n-1}\right)}{ns^2} + (n-1)(n-2)\frac{1}{ns^2} \\
&- (n-1)\frac{\log^2\left(n^n s^{n-1}\right)}{n^2 s^3} + (n-1)(3n-5)\frac{\log\left(n^n s^{n-1}\right)}{n^2 s^3} \\
&- (n-1)\left(n^2 - 6n + 7\right)\frac{1}{n^2 s^3} + o\left(\frac{1}{s^3}\right),
\end{aligned}
\tag{B.3}
$$

$$
\varphi'' = \frac{n-1}{s^2} + 2(n-1)\frac{\log\left(n^n s^{n-1}\right)}{ns^3} - (n-1)(3n-5)\frac{1}{ns^3} + o\left(\frac{1}{s^3}\right),
\tag{B.4}
$$

and

$$
\varphi''' = -2\frac{n-1}{s^3} + o\left(\frac{1}{s^3}\right).
\tag{B.5}
$$

We wish to emphasize that for the application we have in mind, we do not need the high precision of (B.2) explicitly. But as we are not only aiming for the asymptotic expansion for φ, but also for φ', φ'' and φ''', we have to compute the expansion for φ with high precision, as we have to use (B.1) and derivatives of this equation to determine derivatives of φ iteratively. Obviously, derivatives of the expansion of a function do not necessarily have to coincide with expansions of the derivatives. In our situation, however, these two operations commute. This is essentially due to the fact that φ satisfies (B.1).

Proof. We start as in [5]. Separation of variables, integration by parts and induction give

$$\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{n!}{k!}\varphi^k e^{\varphi}=e^{ns}+(-1)^{n-1}n!, \tag{B.6}$$

where the constant on the right-hand side is chosen such that $\varphi(s)\to 0$ for $s\to-\infty$. From this formula, Cao deduces that

$$\varphi(s)=ns+o(s)\quad\text{and}\quad\varphi'(s)=n+o(1)\quad\text{for } s\to\infty.$$

To get the asymptotic behavior of φ in (B.2), we can directly plug an appropriate ansatz for φ in (B.6) and obtain an expression for the next correction. This results in carrying out long computations with increasing precision.

To verify that the expansion (B.2) is correct, it is convenient to rewrite (B.6) as

$$1+(-1)^{n-1}n!e^{-ns}=\left(\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{n!}{k!}\frac{\varphi^k}{n(ns)^{n-1}}\right)\left(n(ns)^{n-1}e^{\varphi-ns}\right).$$

We note that (B.2) implies

$$\begin{aligned}
e^{\varphi-ns}n(ns)^{n-1}={}&1+(n-1)\frac{\log\left(n^n s^{n-1}\right)}{ns}+\frac{n-1}{ns}\\
&+\frac{1}{2}n(n-1)\frac{\log^2\left(n^n s^{n-1}\right)}{n^2s^2}+(n-1)\frac{\log\left(n^n s^{n-1}\right)}{n^2s^2}\\
&-(n-1)(n-2)\frac{1}{n^2s^2}+\frac{1}{6}(n-1)n(n+1)\frac{\log^3\left(n^n s^{n-1}\right)}{n^3s^3}\\
&-\frac{1}{2}(n-1)\left(n^2-2n-1\right)\frac{\log^2\left(n^n s^{n-1}\right)}{n^3s^3}\\
&-(n-1)\left(n^2-3\right)\frac{\log\left(n^n s^{n-1}\right)}{n^3s^3}\\
&+\frac{1}{2}(n-1)\left(n^2-8n+11\right)\frac{1}{n^3s^3}+o\left(\frac{1}{s^3}\right)
\end{aligned}$$

and

$$
\begin{aligned}
\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{n!}{k!}\frac{\varphi^k}{n(ns)^{n-1}} &= 1-(n-1)\frac{\log\left(n^n s^{n-1}\right)}{ns}-\frac{n-1}{ns}\\
&+\frac{1}{2}(n-1)(n-2)\frac{\log^2\left(n^n s^{n-1}\right)}{n^2s^2}+(n-1)(2n-3)\frac{\log\left(n^n s^{n-1}\right)}{n^2s^2}\\
&+(n-1)(2n-3)\frac{1}{n^2s^2}-\frac{1}{6}(n-1)(n-2)(n-3)\frac{\log^3\left(n^n s^{n-1}\right)}{n^3s^3}\\
&-\frac{1}{2}(n-1)\left(3n^2-12n+11\right)\frac{\log^2\left(n^n s^{n-1}\right)}{n^3s^3}\\
&-2(n-1)(n-2)(2n-3)\frac{\log\left(n^n s^{n-1}\right)}{n^3s^3}\\
&-\frac{1}{2}(n-1)\left(7n^2-24n+21\right)\frac{1}{n^3s^3}+o\left(\frac{1}{s^3}\right).
\end{aligned}
\tag{B.7}
$$

Moreover, it is not too complicated to see that additional terms do not improve the approximation unless they belong to the class $o\left(s^{-3}\right)$. Thus, (B.2) follows.

Note that the right-hand side of (B.7) can also be used for the expansion of $\frac{\exp(ns-\varphi)}{n(ns)^{n-1}}$, because

$$
\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{n!}{k!}\frac{\varphi^k}{n(ns)^{n-1}}=\frac{\exp(ns-\varphi)}{n(ns)^{n-1}}+o\left(\frac{1}{s^3}\right).
$$

To determine the behavior of φ' at infinity, we note that direct calculations give

$$
\begin{aligned}
\left(\frac{ns}{\varphi}\right)^{n-1} =&1+(n-1)\frac{\log\left(n^n s^{n-1}\right)}{ns}+\frac{1}{2}(n-1)n\frac{\log^2\left(n^n s^{n-1}\right)}{n^2s^2}\\
&-(n-1)^2\frac{\log\left(n^n s^{n-1}\right)}{n^2s^2}-(n-1)^2\frac{1}{n^2s^2}\\
&+\frac{1}{6}(n-1)n(n+1)\frac{\log^3\left(n^n s^{n-1}\right)}{n^3s^3}\\
&-\frac{1}{2}(n-1)^2(2n+1)\frac{\log^2\left(n^n s^{n-1}\right)}{n^3s^3}\\
&-2(n-1)^2\frac{\log\left(n^n s^{n-1}\right)}{n^3s^3}+\frac{1}{2}(n-1)^2(3n-5)\frac{1}{n^3s^3}+o\left(\frac{1}{s^3}\right).
\end{aligned}
$$

Combining this with (B.1), (B.7), and the remark following (B.7) gives (B.3).

To obtain (B.4) and (B.5), we make use of the Taylor expansion of $\frac{ns}{\varphi}$. We get

$$\begin{aligned}\frac{1}{\varphi} =& \frac{1}{ns} + \frac{\log\left(n^n s^{n-1}\right)}{n^2 s^2} + \frac{\log^2\left(n^n s^{n-1}\right)}{n^3 s^3} - (n-1)\frac{\log\left(n^n s^{n-1}\right)}{n^3 s^3} \\ &- (n-1)\frac{1}{n^3 s^3} + o\left(\frac{1}{s^3}\right), \\ \varphi'' =& \varphi'\left(n - \varphi' - (n-1)\frac{\varphi'}{\varphi}\right), \\ \frac{\varphi'}{\varphi} =& \frac{1}{s} + \frac{\log\left(n^n s^{n-1}\right)}{ns^2} - (n-1)\frac{1}{ns^2} + \frac{\log^2\left(n^n s^{n-1}\right)}{n^2 s^3} \\ &- 3(n-1)\frac{\log\left(n^n s^{n-1}\right)}{n^2 s^3} + (n-1)(n-3)\frac{1}{n^2 s^3} + o\left(\frac{1}{s^3}\right), \\ \varphi''' =& \varphi''\left(n - 2\varphi' - 2(n-1)\frac{\varphi'}{\varphi}\right) + (n-1)\left(\frac{\varphi'}{\varphi}\right)^2 \varphi',\end{aligned}$$

and deduce directly (B.4) and (B.5). □

Appendix C. Positive Holomorphic Bisectional Curvature.

Lemma C.1. For the function φ_b introduced in (5.3), we have

$$\varphi_b > 0, \tag{C.1}$$

$$\varphi_b' > 0, \tag{C.2}$$

$$\varphi_b - \varphi_b' > 0, \tag{C.3}$$

$$\left(\varphi_b'\right)^2 - \varphi_b\varphi_b'' > 0, \tag{C.4}$$

$$\left(\varphi_b''\right)^2 - \varphi_b'\varphi_b''' > 0 \tag{C.5}$$

for $R \gg 1$ sufficiently large.

Remark C.2. Before we give a proof of Lemma C.1, we wish to note that it implies that $(u_0)_{i\bar{\jmath}} + b_{i\bar{\jmath}}$ has positive holomorphic bisectional curvature. This follows from the calculations in [5]. Cao gives a proof of this lemma for a C-soliton, so it suffices to proof it in regions where we have changed φ.

Note that the proof of Lemma C.1 shows also that the metric of the barrier is uniformly equivalent to the soliton metric.

Proof. We differentiate the definition of φ_b, use Lemma B.1, and get

$$\begin{aligned}
\varphi_b(s) =& ns + o(s) \mp K s^{-1-\alpha}\alpha(2R)^\alpha \psi\left(\frac{s}{R}\right),\\
\varphi_b'(s) =& n + o(1) \pm K(1+\alpha)s^{-2-\alpha}\alpha(2R)^\alpha\psi\left(\frac{s}{R}\right)\\
&\mp K s^{-1-\alpha}\alpha(2R)^\alpha \frac{1}{R}\psi'\left(\frac{s}{R}\right),\\
\varphi_b''(s) =& \frac{n-1}{s^2} + o\left(\frac{1}{s^2}\right) \mp K(1+\alpha)(2+\alpha)s^{-3-\alpha}\alpha(2R)^\alpha\psi\left(\frac{s}{R}\right)\\
&\pm 2K(1+\alpha)s^{-2-\alpha}\alpha(2R)^\alpha\frac{1}{R}\psi'\left(\frac{s}{R}\right)\\
&\mp K s^{-1-\alpha}\alpha(2R)^\alpha\frac{1}{R^2}\psi''\left(\frac{s}{R}\right),\\
\varphi_b'''(s) =& -2\frac{n-1}{s^3} + o\left(\frac{1}{s^3}\right)\\
&\pm K(1+\alpha)(2+\alpha)(3+\alpha)s^{-4-\alpha}\alpha(2R)^\alpha\psi\left(\frac{s}{R}\right)\\
&\mp 3K(1+\alpha)(2+\alpha)s^{-3-\alpha}\alpha(2R)^\alpha\frac{1}{R}\psi'\left(\frac{s}{R}\right)\\
&\pm 3K(1+\alpha)s^{-2-\alpha}\alpha(2R)^\alpha\frac{1}{R^2}\psi''\left(\frac{s}{R}\right)\\
&\mp K s^{-1-\alpha}\alpha(2R)^\alpha\frac{1}{R^3}\psi'''\left(\frac{s}{R}\right).
\end{aligned}$$

To get (C.1), we study $s^{-1-\alpha}R^\alpha\psi$ in detail. When we choose R sufficiently large, $|s^{-1-\alpha}R^\alpha|$ becomes arbitrarily small for $s \ge R$. For $s \le R$, however, $\psi\left(\frac{s}{R}\right)$ vanishes. Thus, (C.1) follows for $s \ge R$ when R is sufficiently large and is true for $s < R$ by the calculations in [5].

Equations (C.2), (C.3), and (C.4) are proved similarly. Note, however, that the term $s^{-1-\alpha}R^{\alpha-2}\psi''$ is estimated by choosing R large, as $s^{-1-\alpha}$ decays slower as the "leading" term $\frac{n-1}{s^2}$ as a function of s. This works as ψ'' is zero outside $R \le s \le 2R$. The same arguments can also be applied to ψ'''. Thus, for φ_b, φ_b', φ_b'', and φ_b''', the additional terms with a factor K can all be absorbed in the original error terms for $R \gg 1$ fixed sufficiently large. We wish to stress, that the sign of φ_b''', as $s \to \infty$, is important to get (C.5). For this reason, we had to do all the approximations in Appendix B up to such a high precision. □

Remark C.3. The expression for the Riemannian curvature tensor for a radially symmetric Kähler potential in [5] and the expansions of φ and φ_b at infinity imply $\|Rm\| \le c$ for the Kähler metrics corresponding to φ and

φ_b, respectively. Moreover, the vector fields $\left(z^i\right)$ and $(z^{\bar{j}})$ have finite length with respect to these metrics.

Remark C.4. We want to check, that the L^p-norm I_p considered in Corollary 6.4 and Lemma 6.3 is finite. Use that the metric $a_{i\bar{j}}$ is uniformly equivalent to the metric $(u_0)_{i\bar{j}}$. As in Section 2.2, we compute $\det((u_0)_{i\bar{j}})$. Lemma B.1 implies that this quantity behaves like $s^{n-1}r^{-2n}$, $r = |z| = e^{\frac{1}{2}s}$, near infinity. The definition of $\hat{b}$ implies that at infinity, $\hat{b}$ decays like $s^{-\alpha}$. We introduce polar coordinates and get that $I_p(0)$ is finite, if $p \geq 2$ is choosen so large that

$$\int\limits_e^\infty (\log r)^{n-1-\alpha p}\tfrac{1}{r}dr < \infty. \tag{C.6}$$

Choose p such that $\alpha p > n+1$, with $1 > \alpha > 0$ as in (5.3). Introducing a new variable for $\log r$, we see, that the integral in (C.6) is finite as $\int_1^\infty \rho^{-2}d\rho < \infty$.

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