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# Isoperimetric Estimate for the Ricci Flow on $S^2 \times S^1$

XIAODONG CAO

In this paper, we study the dilation limit of $S^2 \times S^1$ with a warped product metric under the Ricci flow. We first prove that the isoperimetric ratio on the base manifold $S^2$ has a lower bound, which excludes the $\Sigma^2 \times \mathbb{R}$ as the dilation limit. We also prove a monotonicity result under a certain condition.

## 1. Introduction.

Suppose, we have a solution to the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

on a compact Riemannian 3-manifold $N$ with Riemannian metric $g(t)$, and suppose $R$ becomes unbounded in some finite time $T_0$. In the paper [2], Hamilton proves that under the Ricci flow, the dilation limit converges to $S^3$, $S^2 \times \mathbb{R}$, $\Sigma^2 \times \mathbb{R}$ or their quotients, where the $\Sigma^2$ is the cigar soliton on the surface. Hamilton also conjectured that $\Sigma^2 \times \mathbb{R}$ cannot occur as dilation limit.

If the 3-manifold $N$ is $S^2 \times S^1$ topologically, and the metric is a warped product metric. It is well known that this condition is preserved by the Ricci flow. Here, the base manifold $S^2$ is also denoted as $M$.

The metric can be written as:

$$g = \pi^*(g_{S^2}) + \pi^*(f^2)d\theta^2$$

where $\pi$ is the standard projection from $S^2 \times S^1$ to $S^2$, $g_{S^2}$ is the corresponds metric on the sphere and $f$ is a positive function on the sphere $S^2$.

The Ricci curvature is

$$\pi^* \begin{pmatrix} \tilde{R_{ij}} - \frac{f_{,ij}}{f} & 0 \\ 0 & -f\Delta f \end{pmatrix}$$

where the $\tilde{R}_{ij}$ and $\Delta$ denote the Ricci curvature and the Laplacian on the base manifold $(S^2, g_{S^2})$ respectively.

Now, the Ricci flow equation can be written as

$$\frac{\partial}{\partial t}(g_{S^2})_{ij} = -2\tilde{R}_{ij} + 2\frac{f_{,ij}}{f} \tag{1.1}$$

$$\frac{\partial}{\partial t} f = \Delta f \tag{1.2}$$

The rest of this paper is organized as follows. In Section 2, we analyze the geometry on $S^2 \times S^1$, and state the isoperimetric ratio estimate and the technical lemma. In Section 3, we prove the isoperimetric estimate. In Section 4, we prove the technical lemma. In the last section, we prove a monotonicity result which shows that the isoperimetric ratio is strictly increasing.

## 2. Isoperimetric Estimate.

In this section, we shall establish an isoperimetric estimate on the base manifold $M$.

First, we shall compute the evolution equation for $|\frac{\nabla f}{f}|^2$.

**Lemma 2.1.**

$$\frac{\partial}{\partial t}\left|\frac{\nabla f}{f}\right|^2 = \Delta\left|\frac{\nabla f}{f}\right|^2 - 2\left|\frac{\nabla^2 f}{f}\right|^2 + 3\nabla\left(\left|\frac{\nabla f}{f}\right|^2\right)\cdot\frac{\nabla f}{f} \tag{2.1}$$

**Corollary 2.2.** There exists a constant $C_1$, which only depends on the initial metric and f at time 0, such that under the Ricci flow, we have

$$\left|\frac{\nabla f}{f}\right|^2 \le C_1^2$$

Now, we define the isoperimetric ratio on $S^2$ (also see [3]).

$$C_H(\gamma) = l(\gamma)^2\frac{A}{A_1 A_2}$$

for all smooth closed curves $\gamma$ and

$$C_H(S^2, g) = \inf_{\gamma} C_H(\gamma)$$

where the infimum is taken over all smooth closed curves $\gamma$, which divides $M$ into two open surfaces $M_1$ and $M_2$ with $\partial M_1 = \partial M_2 = \gamma$ and areas $A = Area(S^2)$, $A_1 = Area(M_1)$ and $A_2 = Area(M_2)$. This definition is equivalent to:

$$C_I(\gamma) = \frac{l(\gamma)^2}{\min\{A_1, A_2\}}$$

and

$$C_I(S^2, g) = \inf_{\gamma} C_I(\gamma).$$

Because

$$C_I \leq C_H \leq 2C_I,$$

we know that under Ricci flow, the dilation limit is a product of a surface with $\mathbb{R}$. By using this isoperimetric estimate, we shall prove that this surface cannot be $\Sigma^2$. Our claim is

**Theorem 2.3.** Under the Ricci flow, there exists an $\eta > 0$, such that

$$C_H(S^2, g)(t) > \eta > 0$$

for all $t < T_0$, where $T_0$ is the maximum existence time for the solution of the Ricci flow.

**Remark 2.4.** 1) The existence and smoothness of the optimal curve $\gamma$ such that $C_H(\gamma) = C_H(S^2, g)$ are proved in [3] by Hamilton. This does not depend on the flow.

2) Since $C_H$ is a dilation invariant, the estimate holds on any dilation solution, but the cigar soliton has isoperimetric constant 0. So, we have the following corollary.

**Corollary 2.5.** Under the Ricci flow, $\Sigma^2 \times \mathbb{R}$ can not occur as a limit of dilation for $S^2 \times S^1$ with warped product metric.

## 3. Proof of Theorem.

Let $g(t)$ be a solution to the Ricci flow, given some time $t_0$. Let $\gamma_0$ be any smooth embedded closed curve, which divides $M$ into two open surfaces $M_1$ and $M_2$ with $\partial M_1 = \partial M_2 = \gamma_0$. Let $\gamma_r$ denote the parallel curve of signed distance $r$ from $\gamma_0$ (with respect to the metric $g(t_0)$), where the signed distance is positive for the curves in $M_2$ and negative for the curves in $M_1$, as shown in Figure 1. For $r$ which is sufficiently small, $\gamma_r$ is a smooth embedded closed curve which separates $M$ into two open surface $M_{1,r}$ and $M_{2,r}$ (see [3] or [1]). We can consider the following as the functions of $r$ and $t$.

1) $l = l(\gamma_r, g(t))$

2) $A_1 = Area(M_{1,r}, g(t))$

3) $A_2 = Area(M_{2,r}, g(t))$

4) $C_H = C_H(\gamma_r, g(t))$

Our computation shows that:

$$\frac{\partial A_1}{\partial r} = l$$

$$\frac{\partial A_2}{\partial r} = -l$$

$$\frac{\partial l}{\partial r} = \int_{\gamma_r} k\, ds$$

where $k$ is the geodesic curvature of $\gamma_r$ with respect to the metric $g(t)$.

$$\frac{\partial^2 A_1}{\partial r^2} = \int_{\gamma_r} kds$$

$$\frac{\partial^2 A_2}{\partial r^2} = -\int_{\gamma_r} kds$$

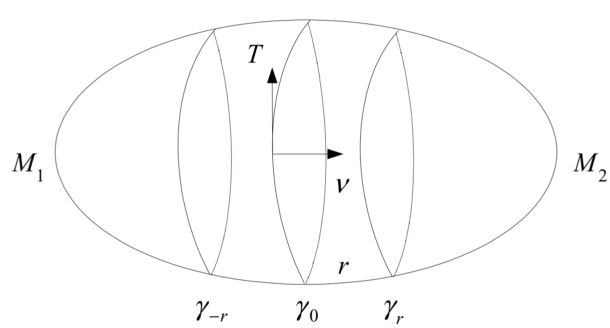


Figure 1: Isoperimetric ratio

By Gauss–Bonnet formula, we get

$$2\pi = \int_{M_{1,r}} K dA + \int_{\gamma_r} k ds$$

Differentiating the above with respect to $r$, we have

$$\int_{\gamma_r} K ds + \frac{\partial}{\partial r} \int_{\gamma_r} k ds = 0$$

and then

$$\frac{\partial^2 l}{\partial r^2} = -\int_{\gamma_r} K\, ds.$$

For time $t$, we can show

(3.1)

$$\frac{d}{dt} A = -2 \int_M K dA + \int_M \frac{\Delta f}{f} dA = -8\pi + \int_M \frac{\Delta f}{f} dA = -8\pi + \int_M \frac{|\nabla f|^2}{f^2} dA$$

Since

$$\frac{|\nabla f|^2}{f^2} \leq C_1^2$$

we have

$$\frac{d}{dt} A \leq C_1^2 A$$

and it follows,

$$A(t) \leq A(0) e^{C_1^2 t},$$

Therefore, we have the following:

**Corollary 3.1.** $A(t)$ is bounded above at any finite time $t$.

We also have

$$\frac{\partial}{\partial t} A_1 = -2 \int_{M_1} K dA + \int_{M_1} \frac{\Delta f}{f} dA \tag{3.2}$$

$$\frac{\partial}{\partial t} A_2 = -2 \int_{M_2} K dA + \int_{M_2} \frac{\Delta f}{f} dA \tag{3.3}$$

$$\frac{\partial}{\partial t} l = -\int_{\gamma_r} K ds + \int_{\gamma_r} \frac{|Tf|^2}{f^2} ds \tag{3.4}$$

where $T$ is the unit tangent vector along $\gamma_r$. Since

$$C_H = l^2 \frac{A}{A_1 A_2} \tag{3.5}$$

so,

$$\ln C_H = 2\ln l + \ln A - \ln A_1 - \ln A_2 \tag{3.6}$$

Differentiating twice with respect to $r$, we get

$$\begin{aligned}
&\frac{\partial^2}{\partial r^2} \ln C_H \\
&\quad = \frac{2}{l}\frac{\partial^2 l}{\partial r^2} - \frac{2}{l^2}\left(\frac{\partial l}{\partial r}\right)^2 - \frac{1}{A_1}\left(\frac{\partial^2 A_1}{\partial r^2}\right) + \frac{1}{A_1^2}\left(\frac{\partial A_1}{\partial r}\right)^2 \\
&\qquad - \frac{1}{A_2}\left(\frac{\partial^2 A_2}{\partial r^2}\right) + \frac{1}{A_2^2}\left(\frac{\partial A_2}{\partial r}\right)^2 \\
&\quad = -\frac{2}{l}\int_{\gamma_r} K ds - \frac{2}{l^2}\left(\int_{\gamma_r} kds\right)^2 - \frac{1}{A_1}\int_{\gamma_r} kds \\
&\qquad + \frac{l^2}{A_1^2} + \frac{1}{A_2}\int_{\gamma_r} kds + \frac{l^2}{A_2^2} \\
&\quad = -\frac{2}{l}\int_{\gamma_r} K ds - \frac{2}{l^2}\left(\int_{\gamma_r} kds\right)^2 + \left(\frac{1}{A_2} - \frac{1}{A_1}\right)\int_{\gamma_r} kds \\
&\qquad + l^2\left(\frac{1}{A_1^2} + \frac{1}{A_2^2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\ln C_H =& \frac{2}{l}\frac{\partial l}{\partial t} + \frac{1}{A}\left(\frac{dA}{dt}\right) - \frac{1}{A_1}\frac{\partial A_1}{\partial t} - \frac{1}{A_2}\frac{\partial A_2}{\partial t} \\
=& \frac{2}{l}\frac{\partial l}{\partial t} - \frac{A_2}{A_1 A}\frac{\partial A_1}{\partial t} - \frac{A_1}{A_2 A}\frac{\partial A_2}{\partial t} \\
=& -\frac{2}{l}\int_{\gamma_r} K ds + \frac{2}{l}\int_{\gamma_r} \frac{|Tf|^2}{f^2} ds \\
& - \frac{A_2}{A_1 A}\left(2\int_{\gamma_r} kds - 4\pi + \int_{M_1} \frac{\Delta f}{f} dA\right) \\
& - \frac{A_1}{A_2 A}\left(-2\int_{\gamma_r} kds - 4\pi + \int_{M_2} \frac{\Delta f}{f} dA\right)
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\frac{\partial}{\partial t}\ln C_H = &\frac{\partial^2}{\partial r^2}\ln C_H + 4\pi\left(\frac{A_2}{A_1 A}+\frac{A_1}{A_2 A}\right)+\frac{2}{l^2}\left(\int_{\gamma_r} k ds\right)^2 \\
&-\left(\frac{1}{A_2}-\frac{1}{A_1}\right)\int_{\gamma_r} k ds - l^2\left(\frac{1}{A_1^2}+\frac{1}{A_2^2}\right) \\
&+\frac{2}{l}\int_{\gamma_r}\frac{|Tf|^2}{f^2}ds - 2\left(\frac{A_2}{A_1 A}-\frac{A_1}{A_2 A}\right)\int_{\gamma_r} k ds - \frac{A_2}{A_1 A}\int_{M_1}\frac{\Delta f}{f}dA \\
&-\frac{A_1}{A_2 A}\int_{M_2}\frac{\Delta f}{f}dA \\
=&\frac{\partial^2}{\partial r^2}\ln C_H + 4\pi\left(\frac{A_2}{A_1 A}+\frac{A_1}{A_2 A}\right)+\frac{2}{l^2}\left(\int_{\gamma_r} k ds\right)^2 + \left(\frac{1}{A_2}-\frac{1}{A_1}\right)\int_{\gamma_r} k ds \\
&- l^2\left(\frac{1}{A_1^2}+\frac{1}{A_2^2}\right)+\frac{2}{l}\int_{\gamma_r}\frac{|Tf|^2}{f^2}ds - \frac{A_2}{A_1 A}\int_{M_1}\frac{|\nabla f|^2}{f^2}dA \\
&-\frac{A_1}{A_2 A}\int_{M_2}\frac{|\nabla f|^2}{f^2}dA - \frac{A_2}{A_1 A}\int_{\gamma_r}\frac{\nabla f}{f}\cdot\nu ds - \frac{A_1}{A_2 A}\int_{\gamma_r}\frac{\nabla f}{f}\cdot(-\nu)ds \\
=&\frac{\partial^2}{\partial r^2}\ln C_H + (4\pi - C_H)\left(\frac{A_2}{A_1 A}+\frac{A_1}{A_2 A}\right) \\
&+\frac{2}{l^2}\int_{\gamma_r} k ds\left[\int_{\gamma_r} k ds + \frac{l^2}{2}\left(\frac{1}{A_2}-\frac{1}{A_1}\right)\right] \\
&+\frac{2}{l}\int_{\gamma_r}\frac{|Tf|^2}{f^2}ds - \frac{A_2}{A_1 A}\int_{M_1}\frac{|\nabla f|^2}{f^2}dA - \frac{A_1}{A_2 A}\int_{M_2}\frac{|\nabla f|^2}{f^2}dA \\
&-\frac{A_2}{A_1 A}\int_{\gamma_r}\frac{\nabla f}{f}\cdot\nu ds - \frac{A_1}{A_2 A}\int_{\gamma_r}\frac{\nabla f}{f}\cdot(-\nu)ds
\end{aligned}$$

We can simplify the above as

(3.7)
$$\begin{aligned}
\frac{\partial}{\partial t}\ln C_H =&\frac{\partial^2}{\partial r^2}\ln C_H + (4\pi - C_H)\left(\frac{A_2}{A_1 A}+\frac{A_1}{A_2 A}\right) \\
&+\frac{1}{l}\left(\int_{\gamma_r} k ds\right)\left(\frac{\partial}{\partial r}\ln C_H\right) \\
&+\frac{2}{l}\int_{\gamma_r}\frac{|Tf|^2}{f^2}ds - \frac{A_2}{A_1 A}\int_{M_1}\frac{|\nabla f|^2}{f^2}dA - \frac{A_1}{A_2 A}\int_{M_2}\frac{|\nabla f|^2}{f^2}dA \\
&-\frac{A_2}{A_1 A}\int_{\gamma_r}\frac{\nabla f}{f}\cdot\nu ds - \frac{A_1}{A_2 A}\int_{\gamma_r}\frac{\nabla f}{f}\cdot(-\nu)ds
\end{aligned}$$

where $\nu$ is the unit normal vector of $\gamma_r$ towards $M_2$. We used the following identities in the above:

$$C_H = l^2 \frac{A}{A_1 A_2}$$

$$C_H \left( \frac{A_2}{A_1 A} + \frac{A_1}{A_2 A} \right) = l^2 \left( \frac{1}{A_1^2} + \frac{1}{A_2^2} \right)$$

$$\frac{\partial}{\partial r} \ln C_H = \frac{2}{l} \int_{\gamma_r} k ds - l \left( \frac{1}{A_1} - \frac{1}{A_2} \right)$$

and

$$\frac{A_2}{A_1 A} - \frac{A_1}{A_2 A} = \frac{A_2^2 - A_1^2}{A_1 A_2 A} = \frac{1}{A_1} - \frac{1}{A_2}$$

because

$$A_1 + A_2 = A$$

We shall prove that $C_H(M, g)$ has a positive lower bound. It suffices to show that when $C_H$ attains its minimum, the right hand side of Eq. (3.7) has a fixed lower bound. Without loss of generality, we assume that $A_2 \geq A_1$, then

(3.8)
$$\left| \frac{A_2}{A_1 A} \int_{\gamma_r} \frac{\nabla f}{f} \cdot \nu ds + \frac{A_1}{A_2 A} \int_{\gamma_r} \frac{\nabla f}{f} \cdot (-\nu) ds \right| = \left( \frac{A_2}{A_1 A} - \frac{A_1}{A_2 A} \right) \left| \int_{\gamma_r} \frac{\nabla f}{f} \cdot \nu ds \right|$$

Since $|\frac{\nabla f}{f}| \leq C_1$, $\forall t$, we have

$$\begin{aligned} \left( \frac{A_2}{A_1 A} - \frac{A_1}{A_2 A} \right) \left| \int_{\gamma_r} \frac{\nabla f}{f} \cdot \nu ds \right| &\leq \left( \frac{A_2}{A_1 A} - \frac{A_1}{A_2 A} \right) C_1 l \\ &= \left( \frac{1}{A_1} - \frac{1}{A_2} \right) C_1 l < \frac{C_1 l}{A_1} \end{aligned} \tag{3.9}$$

and

$$\frac{A_2}{A_1 A} \int_{M_1} \frac{|\nabla f|^2}{f^2} dA + \frac{A_1}{A_2 A} \int_{M_2} \frac{|\nabla f|^2}{f^2} dA \leq C_1^2 \left( \frac{A_2}{A_1 A} A_1 + \frac{A_1}{A_2 A} A_2 \right) = C_1^2$$

We then have the following lemma,

**Lemma 3.2.**

$$\begin{aligned} \frac{\partial}{\partial t} \ln C_H \geq & \frac{\partial^2}{\partial r^2} \ln C_H + (4\pi - C_H) \left( \frac{A_2}{A_1 A} + \frac{A_1}{A_2 A} \right) \\ & + \frac{1}{l} \left( \int_{\gamma_r} k ds \right) \left( \frac{\partial}{\partial r} \ln C_H \right) + \frac{2}{l} \int_{\gamma_r} \frac{|Tf|^2}{f^2} ds - C_1^2 - \frac{C_1 l}{A_1} \end{aligned} \tag{3.10}$$

When $C_H(\gamma)$ reaches its minimum, i.e.,

$$C_H(\gamma) = C_H(M, g) \geq \frac{l^2}{\min(A_1, A_2)}$$

it suffices to show that $\frac{C_1 l}{A_1}$ is bounded by $\delta \frac{A_2}{A_1 A}$ for some $\delta > 0$.

For any fixed $\delta > 0$ (for example, we can take $\delta = 2\pi$), we consider two cases:

(I) If $l \geq \frac{\delta}{2C_1}$, then

$$C_H(\gamma) = \frac{l^2 A}{A_1 A_2} \geq \frac{l^2}{A} \geq \frac{\delta^2 e^{-C_1^2 T_0}}{4C_1^2 A(0)} > 0$$

i.e, we already have a positive lower bound.

(II) If $l \leq \frac{\delta}{2C_1}$, then

$$\delta \frac{A_2}{A_1 A} - \frac{C_1 l}{A_1} \geq (\frac{\delta}{2} - C_1 l)\frac{1}{A_1} > 0$$

hence,

$$\begin{aligned} \frac{\partial}{\partial t} \ln C_H \geq & \frac{\partial^2}{\partial r^2} \ln C_H + (4\pi - \delta - C_H)\left(\frac{A_2}{A_1 A} + \frac{A_1}{A_2 A}\right) \\ & + \frac{1}{l}\left(\int_{\gamma_r} k ds\right)\left(\frac{\partial}{\partial r}\ln C_H\right) + \frac{2}{l}\int_{\gamma_r} \frac{|Tf|^2}{f^2} ds - C_1^2 \end{aligned} \tag{3.11}$$

So, for any finite time $t < T_0$, $C_H$ has a positive lower bound.

Now, we have proved that there exists $\eta > 0$, such that

$$C_H > \eta > 0$$

$\forall t \in [0, T_0)$.

Since $C_H > 0$ on a compact manifold is a dilation invariant, we have

$$\lim_{t \to T_0} C_H > \eta$$

on any dilation. This is also true on the limit, and the dilation limit cannot be $\Sigma^2 \times \mathbb{R}^1$. This finishes the proof of Theorem 2.3.

## 4. Proof of Lemma.

*Proof.* The left-hand side of Eq. (2.1) is:

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{|\nabla f|^2}{f^2} =& \frac{\tilde{R}|\nabla f|^2 - 2\frac{f_{,ij}}{f}f_{,i}f_{,j}}{f^2} - \frac{2|\nabla f|^2 f f_t}{f^4} + \frac{2g^{ij}_{S^2}f_{t,i}f_{,j}}{f^2} \\
=& \frac{\tilde{R}|\nabla f|^2 - 2\frac{f_{,ij}}{f}f_{,i}f_{,j} + 2g^{ij}_{S^2}(\Delta f)_{,i}f_{,j}}{f^2} - \frac{2|\nabla f|^2\Delta f}{f^3}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Delta\left(\frac{|\nabla f|^2}{f^2}\right) =& \nabla\left(\frac{\nabla(|\nabla f|^2)f^2 - |\nabla f|^2\cdot 2f\nabla f}{f^4}\right) \\
=& \nabla\frac{\nabla(|\nabla f|^2)}{f^2} - \frac{2|\nabla f|^2\cdot\nabla f}{f^3} \\
=& \frac{\Delta(|\nabla f|^2)f^2 - \nabla(|\nabla f|^2)\cdot 2f\nabla f}{f^4} \\
&- \frac{2\nabla(|\nabla f|^2)\cdot(\nabla f)f^3 + 2|\nabla f|^2(\Delta f)f^3 - 2|\nabla f|^2\nabla f\cdot 3f^2\nabla f}{f^6} \\
=& \frac{\Delta|\nabla f|^2}{f^2} - \frac{4\nabla(|\nabla f|^2)\cdot\nabla f}{f^3} - \frac{2|\nabla f|^2\Delta f}{f^3} + \frac{6|\nabla f|^4}{f^4}
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\frac{|\nabla f|^2}{f^2} - \Delta\left(\frac{|\nabla f|^2}{f^2}\right) \\
&= -\frac{\Delta|\nabla f|^2}{f^2} + \frac{4\nabla(|\nabla f|^2)\cdot\nabla f}{f^3} - \frac{6|\nabla f|^4}{f^4} \\
&\quad + \frac{\tilde{R}|\nabla f|^2}{f^2} - \frac{2f_{,ij}f_{,i}f_{,j}}{f^3} + \frac{2f_{,jji}f_{,i}}{f^2}
\end{aligned}
$$

Since

$$\Delta|\nabla f|^2 = 2f^2_{,ij} + 2(\Delta f)_i f_{,i} + 2\tilde{R_{ij}}f_{,i}f_{j},$$

we can show

$$
\begin{aligned}
&\frac{\partial}{\partial t}\frac{|\nabla f|^2}{f^2} - \Delta\left(\frac{|\nabla f|^2}{f^2}\right) \\
&= -\frac{2f_{,ij}^2 + 2\tilde{R_{ij}} f_{,i} f_{,j}}{f^2} + \frac{4\nabla(|\nabla f|^2)\cdot\nabla f}{f^3} \\
&\quad - \frac{6|\nabla f|^4}{f^4} + \frac{\tilde{R}|\nabla f|^2}{f^2} - \frac{2f_{,ij}f_{,i}f_{,j}}{f^3} \\
&= -\frac{2|\nabla^2 f|^2}{f^2} + \frac{4\nabla(|\nabla f|^2)\cdot\nabla f}{f^3} - \frac{6|\nabla f|^4}{f^4} - \frac{2f_{,ij}f_{,i}f_{,j}}{f^3} \\
&= -\frac{2|\nabla^2 f|^2}{f^2} + 4\nabla(\frac{|\nabla f|^2}{f^2})\cdot\frac{\nabla f}{f} + \frac{2|\nabla f|^4}{f^4} - \frac{2f_{,ij}f_{,i}f_{,j}}{f^3}
\end{aligned}
$$

where we used

$$\nabla\left(\frac{|\nabla f|^2}{f^2}\right) = \frac{\nabla(|\nabla f|^2)}{f^2} - \frac{|\nabla f|^2\cdot 2f\nabla f}{f^3}$$

□

The Corollary 2.2 follows from Lemma 2.1 and maximum principle.

## 5. Monotonicity Result.

In the above, we exclude $\Sigma^2 \times R^1$ as a dilation limit, but we still need to prove the base manifold converges to the round sphere $S^2$. We actually have a partial monotonicity result about isoperimetric ratio by using that $C_H$ is bounded from below and proceed our proof more carefully. In fact, we need to assume that when $\gamma$ attains the minimum of $C_H(\gamma)$, $l(\gamma, t) \to 0$ as $t \to T_0$. Our claim is:

**Theorem 5.1.** Under Ricci flow, if $l(\gamma, t) \to 0$ as $t \to T_0$ for the $\gamma$ satisfies $C_H(\gamma) = C_H(S^2)$, then the isoperimetric ratio will strictly increase unless $C_H = 4\pi$.

In order to prove this, we need to show that whenever $C_H < 4\pi$, the right-hand side of Eq. (3.7) is non-negative.

*Proof.* Since $l \to 0$ and

$$\eta < C_H \le \frac{l^2}{A_1} \le 8\pi,$$

so as $t \to T_0$, $A_1 \to 0$ (we still assume $A_1 \le A_2$).

For any fixed small $\delta > 0$, if $t$ is close to $T_0$ enough, we shall have

$$l < \frac{\delta}{4C_1}$$

and

$$A_1 < \frac{\delta}{4C_1^2}$$

Hence,

$$\frac{\delta}{2}\left(\frac{A_2}{A_1 A} + \frac{A_1}{A_2 A}\right) - \frac{C_1 l}{A_1} > \frac{1}{A_1}\left(\frac{\delta}{4} - C_1 l\right) > 0$$

and

$$\frac{\delta}{2}\left(\frac{A_2}{A_1 A} + \frac{A_1}{A_2 A}\right) - C_1^2 \ge \frac{\delta}{4A_1} - C_1^2 > 0,$$

then

$$\begin{aligned}(5.1)\qquad \frac{\partial}{\partial t}\ln C_H \ge & \frac{\partial^2}{\partial r^2}\ln C_H + (4\pi - \delta - C_H)(\frac{A_2}{A_1 A} \\ & + \frac{A_1}{A_2 A}) + \frac{1}{l}(\int_{\gamma_r} k ds)(\frac{\partial}{\partial r}\ln C_H) + \frac{2}{l}\int_{\gamma_r}\frac{|Tf|^2}{f^2}ds\end{aligned}$$

Therefore, whenever $C_H < 4\pi$, $C_H$ will increase. □

**Remark 5.2.** The more general result to exclude $\Sigma^2 \times \mathbb{R}$ has been proved by Perelman in [4] recently via an entropy estimate. In that paper, he confirms Hamilton's conjecture. Our work was done during 2001.

## 6. Acknowledgement.

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Department of Mathematics
Columbia University
New York,NY 10027
*E-mail address*: cao@math.columbia.edu

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