

Patterson–Sullivan measures and quasi-conformal deformations

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In this paper we relate the ergodic action of a Kleinian group on the space of line elements to the conformal action of the group on the sphere at infinity. In particular, we show that for a pair of geometrically isomorphic convex co-compact Kleinian groups, the ratio of the length of the Patterson–Sullivan measure on line element space to the length of its push-forward is bounded below by the ratio of the Hausdorff dimensions of the limit sets. Our primary techniques come from ergodic theory and Patterson–Sullivan theory.

1. Basics and Statement of Results.

Let $Isom_+(\mathbf{H}^n)$ $n \geq 2$ be the space of orientation-preserving isometries of \mathbf{H}^n . As is well known, this space of isometries can be given the topology induced by uniform convergence on compact sets. A *Kleinian group* Γ is a discrete subgroup of $Isom_+(\mathbf{H}^n)$. As such, Γ acts discontinuously on \mathbf{H}^n , and because we make a standing assumption that the action is torsion-free, the quotient manifold $N = \mathbf{H}^n/\Gamma$ is a complete Riemannian manifold of constant curvature -1 .

A Kleinian group Γ also acts as a discrete subgroup of conformal automorphisms of the sphere at infinity \mathbf{S}_∞^{n-1} ; this action partitions \mathbf{S}_∞^{n-1} into two disjoint sets. The *regular set* Ω_Γ is the largest open set in \mathbf{S}_∞^{n-1} on which Γ acts properly discontinuously, and the *limit set* L_Γ is its complement. In the case that L_Γ contains more than 2 points, the limit set is characterized as being the smallest closed Γ -invariant subset of \mathbf{S}_∞^{n-1} . Define the *convex hull* $CH(L_\Gamma)$ of the limit set L_Γ to be the smallest convex subset of \mathbf{H}^n , so that all geodesics with both limit points in L_Γ are contained in $CH(L_\Gamma)$. We can take the quotient of $CH(L_\Gamma)$ by Γ (denoted by $C(\Gamma)$); this is the *convex core*. It is the smallest convex submanifold of $N = \mathbf{H}^n/\Gamma$, so that

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the inclusion map is a homotopy equivalence. A Kleinian group is *convex co-compact* if its associated convex core is compact, and it is *geometrically finite* if the volume of the unit neighborhood of the convex core is finite (see Bowditch [4]). The content of this paper deals almost exclusively with convex co-compact Kleinian groups.

One can view the property of Γ being geometrically finite as being a restriction on how the limit set is formed from an orbit $\Gamma(0)$, $0 \in \mathbf{H}^n$. The *conical limit set* CL_Γ is the set of points $x \in L_\Gamma$ so that for each such point, there exists a Euclidean cone C_x based at x in \mathbf{H}^n , and an infinite sub-orbit $\{\gamma_j(0)\} \subset \Gamma(0)$, so that $\gamma_j(0) \in C_x \neq \emptyset$ and $\lim_{j \rightarrow \infty} \gamma_j(0) = x$. Given the isometric action of Γ on \mathbf{H}^n , it is an observation that Γ being convex co-compact implies that L_Γ is purely conical, i.e. that $L_\Gamma = CL_\Gamma$. We will not consider Kleinian groups containing parabolic elements, and thus, we do not detail the dynamical restrictions on the approximation of parabolic fixed points imposed by the assumption of geometric finiteness. We also note that this paper deals exclusively with Kleinian groups having the property that $\text{vol } \mathbf{H}^n/\Gamma$ is unbounded (the so-called *co-infinite Kleinian groups*), so as to avoid situations where the Mostow Rigidity Theorem makes our results automatic. For the basics in the theory of Kleinian groups, we refer the reader to Maskit [9].

Let Γ be a geometrically finite Kleinian group, and suppose there exists a quasi-conformal homeomorphism of \mathbf{S}_∞^{n-1} to itself, so that conjugation by this map results in another Kleinian group $\hat{\Gamma}$. The Kleinian group $\hat{\Gamma}$ is said to be *quasi-conformally conjugate* to Γ ; note that the conjugacy induces a type-preserving isomorphism between the groups. We call such an isomorphism a *geometric isomorphism*. Quasi-conformal deformations extend equivariantly to quasi-isometries of \mathbf{H}^n , see [15], [12], and [8].

We now give a brief description of the parts of the Patterson–Sullivan theory needed to state the Main Theorem. For further background in the basics of Patterson–Sullivan theory, see Nicholls [10]. Let $x \in \mathbf{H}^n$ and give \mathbf{S}_∞^{n-1} the spherical metric obtained by taking the Poincaré model for \mathbf{H}^n with x as the origin. Define $\mu_{x,s}$, the *s-dimensional Hausdorff measure* on \mathbf{S}_∞^{n-1} , as follows. If E is a Borel set in \mathbf{S}_∞^{n-1} and $\epsilon > 0$, then let

$$\mu_{x,s}^\epsilon(E) = \inf \left\{ \sum_{j=1}^{\infty} c_j^s : E \subset \bigcup B(x_j, c_j); c_j \leq \epsilon \right\}$$

where $B(x_j, c_j)$ denotes an open ball centered at $x_j \in \mathbf{S}_\infty^{n-1}$ of radius c_j in the spherical metric on \mathbf{S}_∞^{n-1} . As $\mu_{x,s}^\epsilon$ is clearly non-decreasing as ϵ decreases,

we can take the (possibly infinite) limit

$$\mu_{x,s}(E) = \lim_{\epsilon \rightarrow 0} \mu_{x,s}^\epsilon(E).$$

Given $x, y \in \mathbf{H}^n$, the measures μ_x, μ_y satisfy,

$$\frac{d\mu_{x,s}}{d\mu_{y,s}}(\xi) = \left(\frac{P(x, \xi)}{P(y, \xi)} \right)^s$$

where $P(x, \xi) = \frac{1-|x|^2}{|x-\xi|^2}$ is the Poisson kernel (see Section 1 of [13]). Therefore, it follows that

$$e^{-sd(x,y)}\mu_{x,s}(E) \leq \mu_{y,s}(E) \leq e^{sd(x,y)}\mu_{x,s}(E)$$

where $d(x, y)$ is the hyperbolic distance from x to y . Therefore, we say a set E has s -dimensional Hausdorff measure zero if $\mu_{x,s}(E) = 0$ for any choice of x . Furthermore, the Hausdorff dimension $D(E)$ of a set $E \subset \mathbf{S}_\infty^{n-1}$ is well-defined by

$$D(E) = \inf\{s : \mu_{x,s}(E) = 0\} = \sup\{s : \mu_{x,s}(E) = \infty\}.$$

Again fix $s \in \mathbf{R}^+ \cup \{0\}$, and let $x, y \in \mathbf{H}^n$ be two fixed (chosen) points. In all that follows, we will denote the hyperbolic metric on \mathbf{H}^n by d . We define the Poincaré series of a Kleinian group Γ to be

$$(1.1) \quad g_s(x, y) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma y)}$$

It is an easy calculation to show, for every discontinuous group acting isometrically on \mathbf{H}^n , that if $s > n - 1$, then $g_s(x, y) < \infty$ for all $(x, y) \in \mathbf{H}^n \times \mathbf{H}^n$. Let

$$(1.2) \quad \delta(\Gamma) = \inf\{s : g_s(x, y) < \infty\};$$

then $\delta(\Gamma)$ is called the *exponent of convergence* of the Poincaré series. We say a group is of *convergence type* if its Poincaré series converges at the critical exponent and of *divergence type* otherwise. It is a result of Sullivan [14] and Tukia [16] that if Γ is a geometrically finite Kleinian group having non-empty regular set, then $\delta(\Gamma) < n - 1$.

Following the work of Patterson and Sullivan, a measure can be constructed on \mathbf{S}_∞^{n-1} which is supported on L_Γ – one that is descriptive of the geometric and conformal actions of Γ . For $x, y \in \mathbf{H}^n$ and $s > \delta(\Gamma)$, we

consider the measure $\sigma_{x,s}$ to have Dirac point mass of weight $\frac{e^{-sd(x,\gamma y)}}{g_s(y,y)}$ at each point γy . A *Patterson–Sullivan measure* σ_x is constructed by taking the limit as $s \rightarrow \delta^+(\Gamma)$ of these measures in the weak* topology. If Γ is assumed to be geometrically finite, as will always be the case in the context of this paper, then this construction of Patterson and Sullivan yields a unique measure class (independent of the choice of x and y .) Amongst its many properties, perhaps the most basic is that the Patterson–Sullivan measure is a finite and positive measure which is supported on the limit set (being of divergence type guarantees that the support of the Patterson–Sullivan measure is supported only on the limit set).

Patterson–Sullivan theory provides a remarkable description of the conformal action of a geometrically finite Kleinian group in terms of its Patterson–Sullivan measure. One facet of this description, restricted to the setting of interest to us, is given below.

Theorem 1.1 (Patterson [11], Sullivan [14] and Tukia [16]). Let Γ be a non-elementary, geometrically finite Kleinian group acting on \mathbf{H}^n . Then, $\delta(\Gamma) = D(L_\Gamma)$ and each Patterson–Sullivan measure is supported on the limit set L_Γ .

Further, if Γ is convex co-compact, then each Patterson–Sullivan measure σ_x is non-atomic, and is equal to the Hausdorff measure $\mu_{x,\delta(\Gamma)}$ (up to scaling) of dimension $\delta(\Gamma)$ restricted to the limit set L_Γ .

Remark 1.2. One can find a complete presentation of these results, from first principles, in chapters 3 and 4 of Nicholls [10]. For instance, the most important of the results listed above, that the Patterson–Sullivan measure is a constant multiple of the Hausdorff measure, is Theorem 4.6.4 of [10].

We will relate the distortion of a convex co-compact hyperbolic structure by a quasi-isometry via the distortion of a canonical measure built from the Patterson–Sullivan measure acting on a portion of the unit tangent bundle. We now describe this canonical measure. The space of oriented geodesics $G(\mathbf{H}^n)$ in \mathbf{H}^n can be identified with their endpoints in \mathbf{S}_∞^{n-1} . Therefore, $G(\mathbf{H}^n) \cong (\mathbf{S}_\infty^{n-1} \times \mathbf{S}_\infty^{n-1}) - \text{diagonal}$. A measure on $G(\mathbf{H}^n)$ which is supported on $(L_\Gamma \times L_\Gamma - \text{diagonal})$ and that is invariant under Γ is called a *geodesic current* (see [3]). One such geodesic current is the *Patterson–Sullivan geodesic current* m , which is given differentially by

$$(1.3) \quad dm(a, b) = \frac{d\sigma_0(a)d\sigma_0(b)}{|b - a|^{2\delta(\Gamma)}}.$$

Here, σ_0 is the Patterson–Sullivan measure based at the origin in the Poincaré model of \mathbf{H}^n , and $|\cdot|$ is the chordal distance on \mathbf{S}_∞^{n-1} calculated with respect to the standard metric on \mathbf{R}^n . We again consider the Poincaré model for hyperbolic space, and let $\Omega(n) = \mathbf{H}^n \times \mathbf{S}_\infty^{n-1}$. This space is the space of *line elements* of \mathbf{H}^n , with a pair (x, ξ) representing a directed geodesic through x and in the direction of ξ . We write ν_- and ν_+ for the endpoints of this geodesic (with $+$ and $-$ specified by the orientation.) Let z be the Euclidean midpoint of this geodesic, and let s be the signed hyperbolic distance of z to x . Then, we have a natural correspondence between $\Omega(n)$ and $G(\mathbf{H}^n) \times \mathbf{R}$ given by $(x, \xi) \rightarrow (\nu_-, \nu_+, s)$. The *geodesic flow* is a family of diffeomorphisms $g_t : \Omega(n) \rightarrow \Omega(n)$ defined by $g_t(\nu_-, \nu_+, s) \rightarrow (\nu_-, \nu_+, s + t)$. There is of course a natural extension of the action of $Isom_+(\mathbf{H}^n)$ on \mathbf{H}^n to an action on $\Omega(n)$. Let Γ be a Kleinian group, then the geodesic flow on $\Omega(n)$ descends to the *geodesic flow* on the *quotient line element space* $\Omega(n)/\Gamma$ by virtue of the fact that it commutes with any element of $\gamma \in \Gamma$.

We will need more structure in order to define a quantity that measures the distortion of the Patterson–Sullivan geodesic current under deformation. Using the above notation, it is clear that there is a fiber map

$$\pi : \Omega(n) \rightarrow G(\mathbf{H}^n)$$

defined by $\pi(\nu_-, \nu_+, s) = (\nu_-, \nu_+)$, with the fiber over the geodesic (ν_-, ν_+) being all line elements in $\Omega(n)$ belonging to the geodesic. Using the fiber map π (see Lemma 4.2), we can lift any geodesic current μ to a *line element measure* μ^* on $\Omega(n)$ by taking the product of μ with the hyperbolic length measure λ , i.e.

$$(1.4) \quad d\mu^* = d\mu \, d\lambda.$$

Clearly, μ^* is also invariant under the action of Γ on $\Omega(n)$, and so naturally descends to a measure on $\Omega(n)/\Gamma$. In particular, the Patterson–Sullivan geodesic current m lifts to a measure on $\Omega(n)$, which we denote by $M = m^*$. (This discussion is partially reprised in Section 4.) By abuse of notation, we will also refer to the measure M generated by dm^* as the *Patterson–Sullivan measure*. We define the *length of a geodesic current* μ by $L(\mu) = \mu^*(\Omega(n)/\Gamma)$. We compile the crucial facts in the following theorem, as used in this paper, concerning this Patterson–Sullivan measure:

Theorem 1.3 (See Theorem 3 in [14]). Let Γ be a convex co-compact Kleinian group. Then, geodesic flow acts ergodically on $\Omega(n)$ with respect to the measure M , induced from the Patterson–Sullivan geodesic current

m . Furthermore, the length of the Patterson–Sullivan geodesic current m is finite, i.e. $M(\Omega(n)/\Gamma) < \infty$.

Remark 1.4. A detailed discussion of both the Patterson–Sullivan geodesic current and the measure, in the context of dynamical systems, is also provided in [13], Section 4.

Let Γ_1, Γ_2 be quasi-conformally conjugate Kleinian groups. It is clear that a quasi-conformal deformation preserves the regular set and the limit set. Thus, we have a homeomorphism $f : L_{\Gamma_1} \rightarrow L_{\Gamma_2}$ of the limit sets induced by the quasi-conformal deformation, with inverse g . Let m_1, m_2 be the corresponding Patterson–Sullivan geodesic currents for Γ_1, Γ_2 respectively. We can push forward m_1, m_2 by the maps f, g to define *the distortion ratios* $\mathcal{R}_{12}, \mathcal{R}_{21}$ as follows:

$$\mathcal{R}_{12} = \frac{L_2(f_*m_1)}{L_1(m_1)} \quad \mathcal{R}_{21} = \frac{L_1(g_*m_2)}{L_2(m_2)},$$

where L_1, L_2 are the length functions on the spaces of geodesic currents $\mathcal{C}(\Gamma_1), \mathcal{C}(\Gamma_2)$ respectively.

Theorem 1.5 (Main). Suppose $\Gamma_{i=1,2}$ are geometrically isomorphic, convex co-compact, co-infinite Kleinian groups. Let Γ_1 and Γ_2 have critical exponents $\delta(\Gamma_1)$ and $\delta(\Gamma_2)$ respectively. Then,

$$\frac{1}{\mathcal{R}_{12}} \leq \frac{\delta(\Gamma_2)}{\delta(\Gamma_1)} \leq \mathcal{R}_{21}.$$

As an immediate corollary of our main result, we observe that the distortion ratio detects geometrically when a quasi-conformal deformation increases Hausdorff dimension.

Corollary 1.6. Let $\Gamma_{i=1,2}$ be as in the Main Theorem. If $\mathcal{R}_{12} < 1$, then $\delta(\Gamma_1) < \delta(\Gamma_2)$.

Remark 1.7. In the case where Γ_1, Γ_2 are quasi-conformally conjugate Fuchsian groups, the Main Theorem implies $\mathcal{R}_{12} \geq 1$. Bonahon [3], proved this inequality using the language of Liouville geodesic currents and then used it to realize the Weil–Petersson metric on Teichmüller space. (See [17] for a comprehensive discussion of the Weil–Petersson metric on Teichmüller space.) This paper is part of a larger project to investigate whether a metric on the deformation space of a convex co-compact Kleinian group can be

formed from the distortion ratio. Indeed, in [6], we have been able to show that there is a natural symmetric bilinear two form on quasi-Fuchsian space which extends the Weil-Petersson metric on Teichmüller space. We also note that the Main Theorem is trivially true for convex co-compact groups uniformizing finite volume hyperbolic (i.e. closed) manifolds. By the Mostow Rigidity Theorem, the only deformations of such groups are isometric deformations. Thus, the resulting ratios are identically equal to one. It is also well known that the limit set of any such group is the whole sphere \mathbf{S}_∞^{n-1} .

Plan of Paper: In the next two sections, we continue to develop the technical definitions and results we need to both define the length distortion function and to characterize it in terms of the associated Poincaré series. Section 4 culminates in a proof of the Main Theorem (Theorem 4.14); here, we also prove the necessary geometric and analytic relationships between the associated spaces of geodesic currents defined by the two Kleinian groups. Ergodic theory, in the guise of the Birkhoff Ergodic Theorem and the ergodicity of geodesic flow, is central to the arguments in this section. We now wish to comment briefly on the connection between this paper and the contents of our previous work. In [5], we restrict ourselves to convex co-compact quasi-Fuchsian hyperbolic manifolds; in this setting, there is a natural geometric isomorphism between the Fuchsian holonomy of the hyperbolic structure on a boundary component of the convex core and the hyperbolic action of the quasi-Fuchsian group. By expanding our repertoire of techniques, we enlarge our focus from the first paper to the consideration of any two geometrically isomorphic convex co-compact Kleinian groups.

2. Poincaré series and limit sets.

The purpose of this section is to prove a technical result concerning the relationship between the exponent of convergence of the Poincaré series and the Hausdorff dimension of the conical limit set. Though the result and proof techniques are standard, we include a proof for completeness. Let (\mathbf{H}^n, d) be the ball model of hyperbolic space, and let K be a countable collection of Möbius transformations acting discontinuously on \mathbf{H}^n . An equivalent formulation of the usual Poincaré series has a more Euclidean presentation. Fix the origin $0 \in \mathbf{H}^n$ and let $s \in \mathbf{R}^+ \cup \{0\}$, define the *Euclidean s -Poincaré series* to be

$$\Sigma(s, 0) = \sum_{k \in K} (1 - |k(0)|)^s;$$

here, $|k(0)|$ is the Euclidean distance of the point $k(0)$ to the origin. (As with the familiar form of the Poincaré series, the convergence properties of $\Sigma(s, x)$ does not depend on the choice of $x \in \mathbf{H}^n$; see Section 1.6 of [10] for an introduction to the Euclidean s -Poincaré series.) Note, via the hyperbolic distance formula for a point to the origin in the ball model, we can observe that

$$\frac{(1 - |k(0)|)^s}{2^s} \leq e^{-sd(0, k(0))} \leq (1 - |k(0)|)^s.$$

Define

$$\delta_{\text{euc}}(K) = \inf\{s : \Sigma(s, 0) < \infty\};$$

it is thus an observation, using the inequality above, that $\delta(K) = \delta_{\text{euc}}(K)$. We thus denote the exponent of convergence by δ regardless of the choice of a hyperbolic or Euclidean presentation of the Poincaré series.

Let CL_K be the collection of conical limit points of K ; we assume that K is such that $CL_K \neq \emptyset$ to avoid the trivial case. We will show:

Lemma 2.1. Let K be a countable collection of Möbius transformations acting discontinuously on \mathbf{H}^n . Then, $D(CL_K) \leq \delta(K)$; in particular, if $s > \delta(K)$, then the s -dimensional Hausdorff measure of CL_K is zero.

Proof. Fix $M > 0$. For $k \in K - \{\text{id}\}$, denote by $B_k(M)$ the Euclidean ball in $\mathbf{S}_{\infty}^{n-1}$ with radius $M(1 - |k(0)|)$ and center $\frac{k(0)}{|k(0)|}$. Define

$$E_M = \{x \in \mathbf{S}_{\infty}^{n-1} : x \in B_k(M) \text{ for infinitely many } k \in K\}.$$

It is well-known that

$$CL_K = \bigcup_{M=1}^{\infty} E_M.$$

The proof proceeds by a covering argument that estimates, from above, the Hausdorff dimension of each E_M . Let $r > 0$ be an arbitrarily chosen radial length. We have, for each fixed M , that

$$\#\{k \in K : \text{diam}(B_k(M)) \geq r\} = \#\left\{k \in K : |k(0)| \leq 1 - \frac{r}{2M}\right\};$$

because K acts discontinuously on \mathbf{H}^n , we observe that $\{B_k(M) : \text{diam}(B_k(M)) < r\}$ is a cover of E_M . Denote by $K(r)$ the subset of K such that $\text{diam}(B_k(M)) < r$.

Let $\delta = \delta(K)$, and fix an arbitrary $\epsilon > 0$. Recalling our definition of the Hausdorff measure, let $\mu_{\delta+\epsilon}^r = \mu_{0,\delta+\epsilon}^r$. Thus,

$$\mu_{\delta+\epsilon}^r(E_M) = \inf \left\{ \sum_j (\text{diam } U_j)^{\delta+\epsilon} : E_M \subset \bigcup_j U_j, \text{diam } U_j \leq r \right\}.$$

Thus,

$$\mu_{\delta+\epsilon}^r(E_M) \leq \sum_{k \in K(r)} (\text{diam } B_k(M))^{\delta+\epsilon} \leq \sum_{k \in K} (\text{diam } B_k(M))^{\delta+\epsilon}.$$

Since $\sum_{k \in K} (\text{diam } B_k(M))^{\delta+\epsilon} = (2M)^{\delta+\epsilon} \sum_{k \in K} (1 - |k(0)|)^{\delta+\epsilon} < \infty$ via the definition of E_M and the definition of δ , we have that

$$\mu_{\delta+\epsilon}^r(E_M) < \infty$$

independent of r . Since $\epsilon > 0$ was chosen arbitrarily, this shows that $D(E_M) \leq \delta(K)$ for any choice of M . The conclusion follows by noting that because $E_M \subset E_{M+1}$ and $CL(K) = \bigcup_{M=1}^\infty E_M$, we have that $D(CL_K) \leq \limsup_{M \rightarrow \infty} D(E_M) \leq \delta(K)$. \square

Remark 2.2. This is a standard argument, see [2] for instance. We remark that a much stronger fact is proven in [2], using in part the above argument: *If Γ is a non-elementary Kleinian group, then the exponent of convergence of the group is the Hausdorff dimension of the conical limit set.*

3. Length Distortion Facts.

In all that follows, we will be keeping track of the distortion, under quasi-isometry, of two convex co-compact hyperbolic structures. Suppose Γ is a Kleinian group acting on \mathbf{H}^n ; define $a(\gamma) = e^{(-d(x,\gamma x))}$, where d is again the hyperbolic metric on the ball model of \mathbf{H}^n and $x \in \mathbf{H}^n$ is a fixed choice of basepoint. These are the terms of the 1-Poincaré series $g_1(x, x)$ of Γ .

Let $i : \Gamma_1 \rightarrow \Gamma_2$ be an isomorphism of Kleinian groups Γ_i , $i = 1, 2$, with inverse $j : \Gamma_2 \rightarrow \Gamma_1$. We define $R_{12} : \Gamma_1 - \{id\} \rightarrow \mathbf{R}$ by

$$(3.1) \quad R_{12}(\gamma) = \frac{d(x_2, i(\gamma)x_2)}{d(x_1, \gamma x_1)},$$

where $x_1, x_2 \in \mathbf{H}^n$ are fixed base points for the Poincaré series of Γ_1, Γ_2 respectively. Similarly, we define $R_{21} : \Gamma_2 - \{id\} \rightarrow \mathbf{R}$ by

$$R_{21}(\gamma) = \frac{d(x_1, j(\gamma)x_1)}{d(x_2, \gamma x_2)}.$$

Therefore, if a_1, a_2 are the terms of the 1-Poincaré series of Γ_1, Γ_2 respectively, we have that

$$[a_1(\gamma)]^{R_{12}(\gamma)} = a_2(i(\gamma)) \quad \text{and} \quad [a_2(\gamma)]^{R_{21}(\gamma)} = a_1(j(\gamma)).$$

Let $\delta(\Gamma_1), \delta(\Gamma_2)$ be the critical exponents of Γ_1, Γ_2 , and fix $\epsilon > 0$. We define the set $\Gamma_{12}^\epsilon \subset \Gamma_1$ by

$$\Gamma_{12}^\epsilon = \left\{ \gamma \in \Gamma_1 : R_{12}(\gamma) \leq \frac{\delta(\Gamma_1) - \epsilon}{\delta(\Gamma_2)} \right\}.$$

Proposition 3.1. Let Γ_1, Γ_2 be isomorphic Kleinian groups, and fix $\epsilon > 0$. Then,

$$\sum_{\gamma \in \Gamma_{12}^\epsilon} [a_1(\gamma)]^s < \infty \text{ for } s > \delta(\Gamma_1) - \epsilon.$$

Proof. If $k > \delta(\Gamma_2)$, then by definition

$$\sum_{\gamma \in \Gamma_2} [a_2(\gamma)]^k < \infty.$$

As $a_2(\gamma) = [a_1(j(\gamma))]^{1/R_{21}(\gamma)}$, we observe that

$$\sum_{\gamma \in \Gamma_2} [a_1(j(\gamma))]^{k/R_{21}(\gamma)} < \infty.$$

Letting $\gamma' = j(\gamma)$, we can change the sum to being over Γ_1 to get

$$\sum_{\gamma' \in \Gamma_1} [a_1(\gamma')]^{kR_{12}(\gamma')} < \infty.$$

Because $\Gamma_{12}^\epsilon \subseteq \Gamma_1$, we immediately see that

$$\sum_{\gamma' \in \Gamma_{12}^\epsilon} [a_1(\gamma')]^{kR_{12}(\gamma')} \leq \sum_{\gamma' \in \Gamma_1} [a_1(\gamma')]^{kR_{12}(\gamma')} < \infty.$$

Fix $\epsilon > 0$. If $\gamma' \in \Gamma_{12}^\epsilon$, then by definition $R_{12}(\gamma') \leq \frac{\delta(\Gamma_1) - \epsilon}{\delta(\Gamma_2)}$, and therefore,

$$kR_{12}(\gamma') \leq \frac{k(\delta(\Gamma_1) - \epsilon)}{\delta(\Gamma_2)}$$

for any $k > 0$. Thus, if $s > \delta(\Gamma_1) - \epsilon$, then

$$\sum_{\gamma \in \Gamma_{12}^\epsilon} [a_1(\gamma)]^s < \infty.$$

□

By applying Lemma 2.1, we immediately obtain:

Proposition 3.2. Let $K = \Gamma_{12}^\epsilon$ then $D(CL_K) \leq \delta(\Gamma_1) - \epsilon$.

Define

$$CL_{12} = \bigcup_{n \in \mathbf{Z}^+} CL_{\Gamma_{12}^{\frac{1}{n}}}.$$

Via Proposition 3.1, and Lemma 2.1, we have that $CL(\Gamma_{12}^{\frac{1}{n}})$ has $\delta(\Gamma_1)$ -Hausdorff measure zero for all $n \in \mathbf{Z}^+$. Thus, we have

Corollary 3.3. The set CL_{12} is of $\delta(\Gamma_1)$ -Hausdorff measure zero.

For the remainder of this section, we assume that Γ_1 is both convex co-compact and is geometrically isomorphic to a (necessarily) convex co-compact Kleinian group Γ_2 . By Theorem 1.1, the Patterson–Sullivan measure σ_x associated to Γ_1 , is equal to (up to scaling) the $\delta(\Gamma_1)$ -dimensional Hausdorff measure $\mu_{x, \delta(\Gamma_1)}$. Therefore, as CL_{12} is of $\delta(\Gamma_1)$ -Hausdorff dimension zero, CL_{12} is of σ_x measure zero (Corollary 3.3.) As Γ_1 is convex co-compact, every point $\nu \in L_{\Gamma_1}$ is a conical limit point [1]. We define the *length distortion function* $R_{12} : \mathbf{H}^n \times \mathbf{H}^n \times \mathbf{S}_\infty^{n-1} \rightarrow \mathbf{R}$ by

$$\tilde{R}_{12}(x_1, x_2, \nu) = \inf \left\{ \liminf_{i \rightarrow \infty} R_{12}(\gamma_i) \mid \gamma_i x_1 \rightarrow \nu \text{ conically} \right\},$$

where x_1 and x_2 are the chosen base points in the definition of $R_{12}(\gamma)$ (see equation 3.1.) We can of course define an analogous function \tilde{R}_{21} for Γ_2 . In the next lemma, we will see that the function \tilde{R}_{12} is independent of the choice of base points x_1, x_2 and bounded below by the ratio of the exponents of convergence.

Lemma 3.4. Let Γ_1, Γ_2 be convex co-compact and quasi-conformally conjugate and let σ_x be the Patterson–Sullivan measure of Γ_1 based at x . Then, for any choice of $x_1, x_2, y_1, y_2 \in \mathbf{H}^n$,

$$\tilde{R}_{12}(x_1, x_2, \nu) = \tilde{R}_{12}(y_1, y_2, \nu).$$

Defining $\tilde{R}_{12} : \mathbf{S}_\infty^{n-1} \rightarrow \mathbf{R}$, by $\tilde{R}_{12}(\nu) = \tilde{R}_{12}(x_1, x_2, \nu)$, then

$$\tilde{R}_{12}(\nu) \geq \frac{\delta(\Gamma_1)}{\delta(\Gamma_2)} \quad \text{a.e. with respect to } (\sigma_x).$$

Proof. Suppose that (y_1, y_2) is another pair of base points, with $d(x_1, y_1) = d_1$ and $d(x_2, y_2) = d_2$. By the triangle inequality, we observe that

$$d(x_1, \gamma x_1) - 2d_1 \leq d(y_1, \gamma y_1) \leq d(x_1, \gamma x_1) + 2d_1,$$

and

$$d(x_2, i(\gamma)x_2) - 2d_2 \leq d(y_2, i(\gamma)y_2) \leq d(x_2, i(\gamma)x_2) + 2d_2.$$

Since $d(\gamma x_1, \gamma y_1) = d(x_1, y_1) = d_1$ for all $\gamma \in \Gamma_1$, if $\{\gamma_i\}$ is a sequence in Γ_1 , then $\gamma_i x_1$ tends to $\nu \in \mathbf{S}_\infty^{n-1}$ conically if and only if $\gamma_i y_1$ tends to $\nu \in \mathbf{S}_\infty^{n-1}$ conically. Also, as $N_1 = \mathbf{H}^n/\Gamma_1$ and $N_2 = \mathbf{H}^n/\Gamma_2$ are quasi-isometric, then $d(x_1, \gamma_i x_1)$ and $d(x_2, i(\gamma_i)x_2)$ both tend to infinity as $\gamma_i x_1$ tends to $\nu \in \mathbf{S}_\infty^{n-1}$ conically. Therefore, by the above inequalities, we have that

$$\liminf_{i \rightarrow \infty} \frac{d(x_2, i(\gamma_i)x_2)}{d(x_1, \gamma_i x_1)} = \liminf_{i \rightarrow \infty} \frac{d(y_2, i(\gamma_i)y_2)}{d(y_1, \gamma_i y_1)}.$$

Thus, taking the infimum over all such sequences gives $\tilde{R}_{12}(x_1, x_2, \nu) = \tilde{R}_{12}(y_1, y_2, \nu)$. Thus, we can define $\tilde{R}_{12}(\nu) = \tilde{R}_{12}(x_1, x_2, \nu)$. Now, let $\tilde{R}_{12}(\nu) = k < \delta(\Gamma_1)/\delta(\Gamma_2)$. Then, there exists a sequence of group elements $\{\gamma_i \in \Gamma_1\}$, such that $\gamma_i x_1 \xrightarrow{\text{conically}} \nu$ with $k \leq \lim_{i \rightarrow \infty} R_{12}(\gamma_i) < \delta(\Gamma_1)/\delta(\Gamma_2)$. Therefore, $\nu \in CL_{12}$, which has σ_x measure zero (see Corollary 3.3.) Therefore,

$$\tilde{R}_{12}(\nu) \geq \frac{\delta(\Gamma_1)}{\delta(\Gamma_2)} \quad \text{a.e. with respect to } (\sigma_x).$$

□

Remark 3.5. We now extend \tilde{R}_{12} to the line element space $\Omega(n)$. Let $p \in \Omega(n)$ and $\nu^+ \in \mathbf{S}_\infty^{n-1}$ be the endpoint of the associated ray. Then, we

define $\tilde{R}_{12}(p) = \tilde{R}_{12}(\nu^+)$; this function is a geodesic flow invariant function. By the definition of the measure M_1 of Γ_1 on $\Omega(n)$ in terms of σ_0 , any set $A = \mathbf{H}^n \times O \subset \Omega(n)$ with O having σ_0 measure zero, has M_1 measure zero. Therefore, it follows that

$$\tilde{R}_{12}(p) \geq \frac{\delta(\Gamma_1)}{\delta(\Gamma_2)} \quad \text{a.e. with respect to } (M_1).$$

4. Geodesic Currents.

Geodesic currents arise in the study of the collection of homotopy classes of closed geodesics. To illustrate the naturalness of geodesic currents, we start with the simplest possible construction of such an object. If $N = \mathbf{H}^n/\Gamma$ is a hyperbolic n -manifold, then consider a non-trivial and non-peripheral homotopy class $[\alpha]$ of closed curves so that the geodesic representative of α is a closed primitive oriented geodesic in N . Let $\gamma \in \Gamma$ be a representative of α , and let $\text{fix}(\gamma)$ be its fixed point pair. Orbit $\text{fix}(\gamma)$ under the action of Γ to produce a Γ -invariant discrete subset A of $G(\mathbf{H}^n) = \mathbf{S}_\infty^{n-1} \times \mathbf{S}_\infty^{n-1} - \text{diagonal}$. Place the Dirac measure on the orbit A . In this way, we identify this homotopy class of closed curves on \mathbf{H}^n/Γ with a Γ -invariant measure on $G(\mathbf{H}^n)$; this is the simplest example of a geodesic current. We call a line element measure that is so associated to a closed geodesic a *simple line element measure*. It is clear that the length of a simple line element measure is simply the length of the associated closed geodesic.

Let $C_c(X)$ be the space of real-valued continuous functions with compact support in the topological space X . The measure μ acts on $\phi \in C_c(X)$ by $\mu(\phi) = \int \phi d\mu$. Recall the definition of the weak* topology on a space of measures \mathcal{C} defined on X : we say that $\mu_i \rightarrow \mu$ in the weak* topology on \mathcal{C} if and only if for every $\phi \in C_c(X)$, then $\mu_i(\phi) \rightarrow \mu(\phi)$.

Definition 4.1. A *geodesic current* is a positive locally finite Borel measure on $G(\mathbf{H}^n)$ that is both invariant under the action of Γ and supported on the set of geodesics with endpoints belonging to L_Γ . The set $\mathcal{C}(\Gamma)$ of geodesic currents of Γ is endowed with the weak* topology.

We provide a detailed description of the line element space $\Omega(n)$ in the introduction. Let $M(\Omega(n))$ be the space of measures on $\Omega(n)$ endowed with the weak* topology. If μ is a geodesic current, recall that the *line element measure* $\mu^* \in M(\Omega(n))$ (see equation 1.4) is locally the product measure on $G(\mathbf{H}^n) \times \mathbf{R}$ given by $\mu \times \lambda$, where λ is the Lebesgue measure on \mathbf{R} . We

thus, have a well-defined map

$$\pi^* : \mathcal{C}(\Gamma) \rightarrow M(\Omega(n))$$

given by $\pi^*(\mu) = \mu^*$. Let $\mathcal{M}(\Gamma)$ be the image of $\mathcal{C}(\Gamma)$ under π^* ; we endow it with the subspace topology from $M(\Omega(n))$. A *flow box* in $\Omega(n) = G(\mathbf{H}^n) \times \mathbf{R}$ is a closed set $B = D \times [a, b]$, where D is a closed ball in $G(\mathbf{H}^n)$ and $[a, b]$ is a closed interval in \mathbf{R} . Therefore, the product $x \times [a, b] \subset B$ projects to a geodesic segment in \mathbf{H}^n of length $b - a$. Furthermore, by definition of the product measure, we have $\mu^*(B) = \mu(D) \cdot (b - a)$. Thus, $\mathcal{M}(\Gamma)$ consists of Γ -invariant measures on $\Omega(n)$ having the following two properties: first, each line element in the support of a line element measure defines a hyperbolic geodesic with both endpoints in L_Γ , and secondly, each measure in $\mathcal{M}(\Gamma)$ is locally the product of the hyperbolic length measure along geodesics with a measure on the space of geodesics. We will show in the next lemma that the function π^* is a homeomorphism between $\mathcal{C}(\Gamma)$ and $\mathcal{M}(\Gamma)$. Note also that each measure $\mu^* \in \mathcal{M}(\Gamma)$ (necessarily Γ -invariant) descends to a measure on the quotient space $\Omega(n)/\Gamma$ which, by abuse of notation, we label by μ^* as well. We recall that the *length of a geodesic current* μ is $L(\mu) = \mu^*(\Omega(n)/\Gamma)$. It is an easy observation, in the case of a geodesic current induced by a closed geodesic, that the length of such a geodesic current is the length of the geodesic as one would expect.

Recall the fiber bundle map $\pi : \Omega(n) \rightarrow G(\mathbf{H}^n)$ defined in the introduction. The following lemma equates the space of geodesic currents with the space of line element measures.

Lemma 4.2. Let Γ be a Kleinian group. Then, the map $\pi^* : \mathcal{C}(\Gamma) \rightarrow \mathcal{M}(\Gamma)$ is a homeomorphism.

Proof. We observe that π^* is onto by the definition of $\mathcal{M}(\Gamma)$. We will show in the following order continuity, injectivity, and the existence of a continuous inverse.

Let $\mu_j \rightarrow \mu$ in $\mathcal{C}(\Gamma)$, and let $\phi : \Omega(n) \rightarrow \mathbf{R}$ be a continuous function with compact support $K \subset \Omega(n)$. We choose a partition of unity $\{\rho_i\}_{i=1}^n$ for a neighborhood of K with $\text{support}(\rho_i) \subset B_i$, where each B_i is a flowbox. Then,

$$\mu_j^*(\phi) = \int_{\Omega(n)} \phi \, d\mu_j^* = \sum_{i=1}^n \int_{B_i} \rho_i \phi \, d\mu_j^*.$$

As $B_i \cong D_i \times [0, 1]$, we break the integral into a double integral

$$\int_{\Omega(n)} \phi \, d\mu_j^* = \sum_{i=1}^n \int_{D_i} \left(\int_{x \times [0,1]} \rho_i \phi \, d\lambda \right) d\mu_j.$$

We define $\widehat{\phi} : G(\mathbf{H}^n) \rightarrow \mathbf{R}$ by

$$\widehat{\phi}_i(x) = \int_{x \times [0,1]} \rho_i \phi \, d\lambda.$$

Thus,

$$\mu_j^*(\phi) = \sum_{i=1}^n \mu_j(\widehat{\phi}_i).$$

Note that $\widehat{\phi}_i \in C_c(G(\mathbf{H}^n))$. As $\mu_j \rightarrow \mu$, then $\mu_j(\widehat{\phi}_i) \rightarrow \mu(\widehat{\phi}_i)$ and by the above sum, we have $\mu_j^* \rightarrow \mu^*$. Thus, π^* is continuous. It remains to show that π^* is injective and has a continuous inverse. Let $\phi \in C_c(G(\mathbf{H}^n))$ be a continuous function with compact support. Also let $f \in C_c(\mathbf{R})$ be a bump function on the real line with total integral one. Then, we define $\widehat{\phi}(g, x) = \phi(g)f(x)$. Thus, by definition of the product measure, we have that $\mu^*(\widehat{\phi}) = \mu(\phi)(\int f dx) = \mu(\phi)$. To show that π^* is injective, we let μ_i be two geodesic currents so that $\mu_1 \neq \mu_2$. Thus, there exists a $\phi \in C_c(G(\mathbf{H}^n))$ such that $\mu_1(\phi) \neq \mu_2(\phi)$. As $\mu_i^*(\widehat{\phi}) = \mu(\phi)$, we therefore have that $\mu_1^*(\widehat{\phi}) \neq \mu_2^*(\widehat{\phi})$. Thus, μ^* is injective. Now, let $\mu_i^* \rightarrow \mu^*$, and let $\phi \in C_c(G(\mathbf{H}^n))$. Thus, $\mu_i^*(\widehat{\phi}) \rightarrow \mu^*(\widehat{\phi})$. As $\mu^*(\widehat{\phi}) = \mu(\phi)$, we have that $\mu_i(\phi) \rightarrow \mu(\phi)$, thus by the definition of the weak* topology, we observe that $\mu_i \rightarrow \mu$, and therefore, π^* has a continuous inverse. \square

Using Lemma 4.2, we can show that the length function is a continuous map.

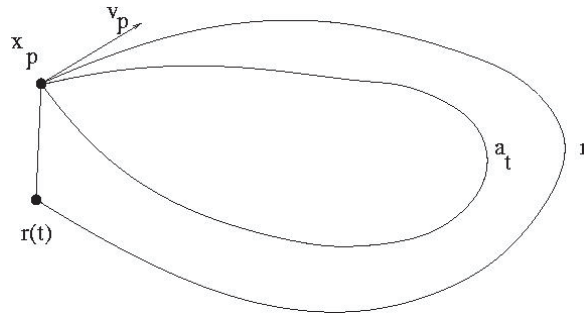
Proposition 4.3. Let Γ be convex co-compact, then the length function $L : \mathcal{C}(\Gamma) \rightarrow \mathbf{R}$ is continuous.

Proof. As the map π^* is continuous, we need only show that the map $\mu^* \rightarrow \mu^*(\Omega(n)/\Gamma)$ is continuous. Recall, given the definition of a geodesic current and from the assumption that Γ is convex co-compact, that the subset of $\Omega(n)/\Gamma$ on which any line element measure has support is compact. The result follows immediately from the definition of weak* convergence. \square

Recall the definition (equation 1.3) of the Patterson–Sullivan geodesic current m , and of the induced Patterson–Sullivan line element measure $M = m^*$. We now give a construction of a family of simple line element measures that converge to the measure M .

We describe the natural metric D on $\Omega(n)$ given by parallel translation. Let d be the hyperbolic distance function on \mathbf{H}^n and d_x be the spherical metric on the unit tangent space at $x \in \mathbf{H}^n$. Then, for $p, q \in \Omega(n)$, with basepoints $x, y \in \mathbf{H}^n$, we define $D(p, q) = d(x, y) + d_y(g_y(p), q)$ where $g_y(p)$ is the line element obtained by parallel translating p to y . In particular, we note that $d(x, y) \leq D(p, q)$. Also, D naturally descends to a metric on the quotient $\Omega(n)/\Gamma$.

Let $p \in \Omega(n)/\Gamma$, with base point x_p and in the direction of the tangent vector v_p . We define the *ray in the direction of p* to be the geodesic parameterized by length given by $r : [0, \infty) \rightarrow \mathbf{H}^n/\Gamma$ so that $r(0) = x_p$ and $r'(0) = v_p$. We fix an embedded ball neighborhood B of x_p . If, after a distance t , the geodesic ray r returns to B , we obtain a closed curve r_t in \mathbf{H}^n/Γ by joining the point $r(t)$ to $x_p = r(0)$. We let $a_t : [0, l_t] \rightarrow \mathbf{H}^n/\Gamma$ be the geodesic (parameterized by arclength) of length l_t that is homotopic to r_t with $a_t(0) = a_t(l_t) = x_p$. Then, a_t is a closed curve through x_p that is geodesic except at x_p .



We define $R(s) = (r(s), r'(s))$ and $A_t(s) = (a_t(s), a_t'(s))$, and thus, obtain parameterized curves $R : [0, \infty) \rightarrow \Omega(n)/\Gamma$ and $A_t : [0, \infty) \rightarrow \Omega(n)/\Gamma$. As a_t is a closed curve, the distance $D(A_t(0), A_t(l_t))$ is just the angle of a_t at x_p . The lemma below says that if the ray r returns to a small neighborhood of the initial line element in the D metric on $\Omega(n)/\Gamma$, then $D(A_t(0), A_t(l_t))$ is small. The proof is an elementary exercise in hyperbolic geometry that we omit for the sake of brevity.

Lemma 4.4. Given $\epsilon > 0$, there exist constants $\delta > 0$ and $K > 0$,

each depending on ϵ , such that if $t > K$ and $D(R(0), R(t)) < \delta$, then $D(A_t(0), A_t(t)) < \epsilon$.

The next lemma shows that for sufficiently large t , if the angle of a_t is small, then a_t is homotopic to a closed geodesic by a bounded homotopy. Recall the definition of a bounded homotopy: let X be a Riemannian manifold, then a homotopy $H : [0, 1] \times [0, 1] \rightarrow X$ has *bounded height* if there exists a K such that for each $s \in [0, 1]$ the curve $p_s(t) = H(s, t)$ have length less than or equal to K . The minimum value of K is the *height* of the homotopy, a homotopy for which a height can be found is called a *bounded homotopy*.

Lemma 4.5. Assume that Γ is a purely loxodromic Kleinian group. Given any $\epsilon < \pi$, the following is true: There exists a constant K so that if $\alpha : [0, t] \rightarrow \mathbf{H}^n/\Gamma$ is a geodesic parameterized by arc length with $t > K$, and if

1. $\alpha(0) = \alpha(t) = x$,
2. $d_x(\alpha'(0), \alpha'(t)) < \epsilon$,

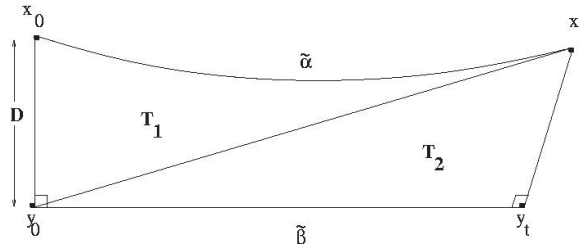
then α is homotopic to a closed geodesic by a bounded homotopy of height less than or equal to $\frac{K}{2}$.

Proof. We will show that $K = 2 \cosh^{-1}(\frac{\pi}{\pi-\epsilon})$ works.

Let $\pi : \mathbf{H}^n \rightarrow \mathbf{H}^n/\Gamma$ be the covering map, and let $x_0 \in \mathbf{H}^n$ belong to $\pi^{-1}(x)$. Let $\tilde{\alpha} : [0, t] \rightarrow \mathbf{H}^n$ be the unique lift of α starting at x_0 . Denote by $x_t \in \pi^{-1}(x)$ the point $x_t = \tilde{\alpha}(t)$. Let θ be the angle between the tangent vectors $\alpha'(0)$ and $\alpha'(t)$ at x . We let $\gamma \in \Gamma$ be the (loxodromic) element such that $x_t = \gamma(x_0)$, and let $\tilde{\beta}$ be the axis of γ . We denote by $\langle \gamma \rangle$ the cyclic group generated by γ . We orthogonally project (x_0, x_t) to the points $(y_0, y_t) \in \tilde{\beta}$, and denote by D the distance $d(x_0, y_0) = d(x_t, y_t)$. Thus, $\gamma(y_0) = y_t$ and therefore, the geodesic segment $[y_0, y_t] \subset \tilde{\beta}$ projects to a closed geodesic β in $\mathbf{H}^n/\langle \gamma \rangle$. The convexity of the length function in hyperbolic space implies that the geodesic arc segment $[x_0, x_t] \subset \mathcal{N}_D([y_0, y_t])$ (here, the symbol \mathcal{N}_D refers to the radius D neighborhood). Therefore, by taking orthogonal projection, we obtain a homotopy of the geodesic arc $[x_0, x_t]$ to $[y_0, y_t]$ by a homotopy of height at most D . Descending to the quotient space \mathbf{H}^n/Γ , we obtain a homotopy of α to β of height at most D .

Our goal is to show that D is less than or equal to $\frac{K}{2}$, and to do so, we use the simplicial surface techniques of Thurston. We now describe

a simplicial annulus A with boundary components α and β . We form a simplicial quadrilateral Q in \mathbf{H}^n with vertices x_0, x_t, y_0, y_t which is the union of two triangles T_1, T_2 . The triangle T_1 has sides $x_0y_0, x_0x_t, x_t y_0$ and T_2 has sides $y_0y_t, y_tx_t, x_t y_0$. Let $Q = T_1 \cup T_2$. Then, Q is a quadrilateral and Q projects to an annulus A in $\mathbf{H}^n / \langle \gamma \rangle$ with boundary components α and β . At the vertex $x \in A$, the exterior angle is θ . Therefore, the interior angle θ_x in A at x satisfies $\theta_x \geq \pi - \theta > \pi - \epsilon$.



If $D > t/2$, we let S be the set of points within a distance $t/2$ of x in A . As $D > t/2$, then S is an embedded hyperbolic sector of angle θ_x and radius $t/2$. Thus,

$$\text{Area}(A) \geq \text{Area}(S) \geq \frac{\theta_x}{2\pi}(4\pi \sinh^2(t/4)) = 2\theta_x \sinh^2(t/4).$$

As the interior angles of Q at y_0 and y_t are at least $\pi/2$, we have $\text{Area}(A) \leq \pi - \theta_x$. Therefore, $2\theta_x \sinh^2(t/4) \leq \pi - \theta_x$ giving $\theta_x(1 + 2 \sinh^2(t/4)) \leq \pi$. As $\cosh(2x) = 1 + 2 \sinh^2(x)$, we get

$$\cosh(t/2) \leq \frac{\pi}{\theta_x} \leq \frac{\pi}{\pi - \epsilon}.$$

Thus, $t \leq K$, which gives us a contradiction. Therefore, $D \leq t/2$. We now let S be the set of points within a distance D of x in A . Then, S is an embedded hyperbolic sector of angle θ_x and radius D . Thus, as above, we have

$$D \leq \cosh^{-1} \left(\frac{\pi}{\pi - \epsilon} \right) = \frac{K}{2}$$

and therefore, α is homotopic to β via a homotopy of height at most $K/2$. \square

Remark 4.6. In fact, the above proof shows more: the height of the homotopy tends to zero as the angle of deflection tends to zero.

A direct corollary of the above facts is:

Corollary 4.7. There exist constants δ, K_1 , and K_2 , all greater than zero, so that if $t > K_1$ and $D(R(0), R(t)) \leq \delta$, then a_t is homotopic to a closed geodesic $\alpha_{(t,p)}$ via a homotopy with height bounded by K_2 .

Proof. Fix $\epsilon \in (0, \pi)$. By Lemma 4.4, there are constants δ and k_1 such that if $D(R(0), R(t)) \leq \delta$ and $t > k_1$, then a_t has angle at most ϵ at x_p . Let l_t be the length of a_t ; then, applying Lemma 4.5 insures that there exists a constant k_2 (depending only on ϵ) such that for $l_t > k_2$, the geodesic curve a_t is boundedly homotopic to its geodesic representative $\alpha_{(t,p)}$. We let $K_2 = k_2/2$. To finish the proof, we need to choose a constant K_1 such that if $t > K_1$, then both $t > k_1$ and $l_t > k_2$. By the triangle inequality, we have that $|l_t - t| \leq \delta$, thus the quantity $K_1 = \max\{k_1, k_2 + \delta\}$ works. \square

We now use the geodesics $\alpha_{(t_i,p)}$ to approximate the Patterson–Sullivan geodesic current m .

Theorem 4.8. Let Γ be convex co-compact, and denote by M the Patterson–Sullivan measure on $\Omega(n)/\Gamma$. Then, for almost every $p \in \Omega(n)$ with respect to M , there exists a sequence t_i such that

$$\lim_{t_i \rightarrow \infty} \frac{\alpha_{(t_i,p)}^*}{L(\alpha_{(t_i,p)}^*)} = \frac{M}{M(\Omega(n)/\Gamma)}$$

Proof. Recall the statement of Theorem 1.3, in particular, that the Patterson–Sullivan line element measure M assigns finite mass to $\Omega(n)/\Gamma$ and that the geodesic flow on $\Omega(n)/\Gamma$ is ergodic with respect to the measure M (see [14].) Thus, by the Birkhoff Ergodic Theorem, given $\phi \in C_c(\Omega(n)/\Gamma)$, there exists a set $A_\phi \subset \Omega(n)/\Gamma$ of full M -measure such that for all $p \in A_\phi$

$$(4.1) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \phi(g_s(p)) ds = \frac{M(\phi)}{M(\Omega(n)/\Gamma)}.$$

To prove that there is a set A of full M -measure in $\Omega(n)/\Gamma$ for which the above holds for every continuous function with compact support, we choose a countable dense subset $\{\phi_i\}$ of the space of continuous functions with compact support on $\Omega(n)/\Gamma$ (with the $\|\cdot\|_\infty$ normed topology). Therefore, as the countable union of measure zero sets is of measure zero, we have that

there is a set $A \subset \Omega(n)/\Gamma$ of full M -measure, such that for all $p \in A$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \phi_i(g_s(p)) ds = \frac{M(\phi_i)}{M(\Omega(n)/\Gamma)}.$$

Let ϕ be any continuous function with compact support in $\Omega(n)/\Gamma$, then there exists a subsequence $\{\phi_j\} \subset \{\phi_i\}$ such that $\phi_j \rightarrow \phi$ in the $\|\cdot\|_\infty$ norm. An elementary bit of analysis shows that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \phi(g_s(p)) ds = \frac{M(\phi)}{M(\Omega(n)/\Gamma)}$$

for all $p \in A$. Note that as M is supported on those line elements associated to the convex core of \mathbf{H}^n/Γ , we can assume that the points of A are line elements of this type. Define the ray $R : [0, \infty) \rightarrow \Omega(n)$ by $R(t) = g_t(p)$ for $p \in A$.

As the geodesic flow is ergodic with respect to M , the geodesic flow is recurrent for almost every $p \in A$. We fix a flow box about $R(0)$ so that the diameter of the flow box is smaller than the minimum value of the injectivity radius function on \mathbf{H}^n/Γ . Using the recurrence property of the geodesic flow, we can form a sequence of homotopically non-trivial curves by “closing up” the ray $R(s)$ each time it returns to the flow box (recurrence insures that there is an infinite sequence of distances s for which return of the ray to the flow box is achieved.) From Corollary 4.7, there exist constants δ , K_1 and K_2 such that if $t > K_1$ and $D(R(0), R(t)) \leq \delta$, then a_t is homotopic via a bounded homotopy of height K_2 to the closed geodesic $\alpha_{(t,p)}$ in its homotopy class, (we can without loss of generality assume that δ is less than the diameter of the flow box.) By the recurrence property, there exists a sequence of distances $\{t_i\}$ from $R(0)$, with $t_i \rightarrow \infty$, such that $t_i > K_1$ and $D(R(0), R(t_i)) \leq \delta$ for all index i . We therefore obtain a sequence $\{a_i = a_{t_i}\}$ of geodesics, along with the associated sequence $\{\alpha_i = \alpha_{(t_i,p)}\}$ of closed geodesics (here, of course, a_{t_i} is homotopic to $\alpha_{(t_i,p)}$). We let L_i be the length of α_i , and we let $A_i : [0, L_i] \rightarrow \Omega(n)/\Gamma$ be the closed curve given by taking line elements along the geodesic α_i . We consider the covering map $p : \Omega(n) \rightarrow \Omega(n)/\Gamma$, and let $\tilde{\phi}$ be the pullback of $\phi \in C_c(\Omega(n)/\Gamma)$ given by $\tilde{\phi} = \phi \circ p$. Similarly, choose $\tilde{R} : [0, \infty) \rightarrow \Omega(n)$ to be a lift of R and let $\tilde{A}_i : [0, L_i] \rightarrow \Omega(n)$ be the lift of A_i corresponding to the choice of \tilde{R} via the homotopy between a_i and α_i . Lifting the integral with respect to the covering map p , we get both

$$\frac{1}{t} \int_0^t \phi(R(s)) ds = \frac{1}{t} \int_0^t \tilde{\phi}(\tilde{R}(s)) ds$$

and

$$\frac{\alpha_i^*}{L_i}(\phi) = \frac{1}{L_i} \int_0^{L_i} \tilde{\phi}(\tilde{A}_i(s)) ds$$

(recall that α_i^* is the line element measure induced by the simple geodesic current defined by α_i .) We will now show that

$$\lim_{i \rightarrow \infty} \frac{1}{t_i} \int_0^{t_i} \tilde{\phi}(\tilde{R}(s)) ds = \lim_{i \rightarrow \infty} \frac{1}{L_i} \int_0^{L_i} \tilde{\phi}(\tilde{A}_i(s)) ds.$$

We start with the elementary observation that if f is a bounded function, K a non-negative constant, and for each $t \in \mathbf{R}$, let c_t, d_t be a pair of real numbers so that $|c_t|$ and $|d_t|$ are both less than or equal to K for all $t \in \mathbf{R}$, then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_{c_t}^{t+d_t} f(s) ds.$$

Under the fiber bundle map $\pi : \Omega(n) \rightarrow \mathbf{H}^n$, the curve \tilde{A}_i is mapped to a geodesic arc $\tilde{\alpha}_i : [0, L_i] \rightarrow \mathbf{H}^n$ and the ray \tilde{R} is mapped to a geodesic ray that we label by $\tilde{r} : [0, \infty) \rightarrow \mathbf{H}^n$. We orthogonally project \tilde{r} onto the axis $\tilde{\alpha}_i$. By the description of the homotopy from a_i to α_i , it follows that the point $x = \tilde{r}(0)$ projects to the point $x'_i = \tilde{\alpha}_i(0)$. We let $y_i = \tilde{r}(t_i)$, and let the projection of this point to $\tilde{\alpha}_i$ be denoted by y'_i . For each s , let $f(s)$ be the distance from x'_i on $\tilde{\alpha}_i$ to the projection of the point $\tilde{r}(s)$. Therefore, the point $\tilde{r}(s)$ is projected to the point $\tilde{\alpha}_i(f(s))$, and as orthogonal projection is distance decreasing, then f is a continuous distance decreasing homeomorphism onto its image. As y_i is within a distance δ of a point that projects to $\tilde{\alpha}_i(L_i)$, we have that $|f(t_i) - L_i| \leq \delta$. Using the fact that a_i is homotopic to α_i via a homotopy whose height is bounded by K_2 then by the triangle inequality, we have that $d(x, x'_i) \leq C$ and $d(y_i, y'_i) \leq C$, where $C = K_2 + \delta$. In particular, we have that $|L - t_i| \leq 2C$, and thus, we have asymptotically two relatively long geodesic arcs having endpoints within a bounded distance from each other.

It follows from elementary hyperbolic geometry (see Theorem 2.4.6 in [7]) that, given any $\delta_1 > 0$, there exists a $k_1 > 0$ so that $d(\tilde{r}(s), \tilde{\alpha}_i(f(s))) \leq \delta_1$ for $k_1 \leq s \leq s - k_1$. It follows similarly that the tangent vectors are close, and we get that there exists a constant $k_2 > 0$ so that $D(\tilde{R}(s), \tilde{A}_i(f(s))) \leq \delta_1$ for $k_2 \leq s \leq t - k_2$.

The function f is obviously differentiable, and it has the following properties. As f is distance decreasing, then $f(s) \leq s$ and $f'(s) \leq 1$. Letting $D(s) = D(\tilde{R}(s), \tilde{A}_i(f(s)))$, then we observe that as $D(s)$ tends to zero the

function $f'(s)$ tends uniformly to one. Therefore, there is a monotonically increasing function g satisfying $g(x) \rightarrow 1$ as $x \rightarrow 0$ such that

$$\frac{1}{g(D(s))} \leq f'(s) \leq 1.$$

Fix $\epsilon > 0$. As ϕ has compact support in $\Omega(n)/\Gamma$, it is uniformly continuous. Therefore, $\tilde{\phi}$ is uniformly continuous, and thus, there exists a $\delta_1 > 0$ so that $|\tilde{\phi}(p) - \tilde{\phi}(q)| \leq \epsilon$ whenever $D(p, q) \leq \delta_1$, where p and q are in the support of $\tilde{\phi}$. We can, without loss of generality, further assume that $\delta_1 \leq \epsilon$. Compiling the information above, we thus have that

$$\left| \tilde{\phi}(\tilde{R}(s)) - \tilde{\phi}(\tilde{A}_i(f(s))) \right| \leq \epsilon \text{ for } k_2 \leq s \leq t_i - k_2,$$

and thus,

$$(4.2) \quad \left| \int_{k_2}^{t_i - k_2} \tilde{\phi}(\tilde{R}(s)) ds - \int_{k_2}^{t_i - k_2} \tilde{\phi}(\tilde{A}_i(f(s))) ds \right| \leq \epsilon t_i.$$

Letting $u = f(s)$, and as $D(\tilde{R}(s), \tilde{A}_i(f(s))) \leq \delta_1$ and using the monotonicity of g , we obtain for $k_2 \leq s \leq t - k_2$ that

$$\frac{1}{g(\delta_1)} \leq \frac{du}{ds} \leq 1.$$

Thus:

$$(4.3) \quad \begin{aligned} \int_{f(k_2)}^{f(t_i - k_2)} \tilde{\phi}(\tilde{A}_i(u)) du &\leq \int_{k_2}^{t_i - k_2} \tilde{\phi}(\tilde{A}_i(f(s))) ds \\ &\leq g(\delta_1) \int_{f(k_2)}^{f(t_i - k_2)} \tilde{\phi}(\tilde{A}_i(u)) du. \end{aligned}$$

Combining inequalities 4.2 and 4.3, we have that

$$\begin{aligned} &\left| \int_{k_2}^{t_i - k_2} \tilde{\phi}(\tilde{R}(s)) ds - \int_{f(k_2)}^{f(t_i - k_2)} \tilde{\phi}(\tilde{A}_i(s)) ds \right| \\ &\leq \epsilon t_i + (g(\delta_1) - 1) \left| \int_{f(k_2)}^{f(t_i - k_2)} \tilde{\phi}(\tilde{A}_i(s)) ds \right|. \end{aligned}$$

As $\phi \in C_c(\Omega(n)/\Gamma)$ has compact support, there exists a constant k_3 so that $|\tilde{\phi}(p)| \leq k_3$ for all $p \in \Omega(n)$. Also, as f is distance non-increasing, i.e. $0 \leq f(x) \leq x$, then

$$\left| \int_{k_2}^{t_i - k_2} \tilde{\phi}(\tilde{R}(s)) ds - \int_{f(k_2)}^{f(t_i - k_2)} \tilde{\phi}(\tilde{A}_i(s)) ds \right| \leq \epsilon t_i + (g(\delta_1) - 1)t_i k_3.$$

Observe that $0 \leq f(k_2) \leq k_2$; also, by the triangle inequality, $|f(t_i - k_2) - L_i| \leq |f(t_i - k_2) - f(t_i)| + |f(t_i) - L_i| \leq k_2 + \delta$. Thus, there are constants c_i and d_i such that $\max\{|c_i|, |d_i|\} \leq k_2 + \delta$, and so

$$\left| \int_{k_2}^{t_i - k_2} \tilde{\phi}(\tilde{R}(s)) ds - \int_{c_i}^{L_i - d_i} \tilde{\phi}(\tilde{A}_i(s)) ds \right| \leq \epsilon t_i + (g(\delta_1) - 1)t_i k_3.$$

As $|L_i - t_i| \leq 2C$, it follows that $\lim_{i \rightarrow \infty} \frac{L_i}{t_i} = 1$, and therefore,

$$\lim_{i \rightarrow \infty} \left| \frac{1}{t_i} \int_0^{t_i} \tilde{\phi}(\tilde{R}(s)) ds - \frac{1}{L_i} \int_0^{L_i} \tilde{\phi}(\tilde{A}_i(s)) ds \right| \leq \epsilon + (g(\delta_1) - 1)k_3.$$

As ϵ is arbitrary, we can let $\epsilon \rightarrow 0$, and thus, as $\delta_1 \leq \epsilon$, we have $\delta_1 \rightarrow 0$. As $\delta_1 \rightarrow 0$, we have $g(\delta_1) \rightarrow 1$, and thus:

$$\lim_{i \rightarrow \infty} \frac{1}{t_i} \int_0^{t_i} \tilde{\phi}(\tilde{R}(s)) ds = \lim_{i \rightarrow \infty} \frac{1}{L_i} \int_0^{L_i} \tilde{\phi}(\tilde{A}_i(s)) ds,$$

and therefore,

$$\lim_{i \rightarrow \infty} \frac{1}{L_i} \int_0^{L_i} \phi(A_i(s)) ds = \frac{M(\phi)}{M(\Omega(n)/\Gamma)}.$$

Thus,

$$\lim_{i \rightarrow \infty} \frac{\alpha_i^*}{L_i} = \frac{M}{M(\Omega(n)/\Gamma)}.$$

Therefore, for almost every $p \in \Omega(n)$ with respect to M , there exist a sequence t_i such that

$$\lim_{i \rightarrow \infty} \frac{\alpha_{(t_i, p)}^*}{L(t_i, p)} = \frac{M}{M(\Omega(n)/\Gamma)}.$$

□

We can immediately reformulate the theorem above in terms of geodesic currents using Lemma 4.2.

Corollary 4.9. Let Γ be a convex co-compact group. Then, for almost every $p \in \Omega(n)$ with respect to M , there exists a sequence $t_i \rightarrow \infty$ such that the resulting sequence $\{\alpha_{(t_i, p)}\}$ of simple geodesic currents satisfies

$$\lim_{i \rightarrow \infty} \frac{\alpha_{(t_i, p)}}{L(\alpha_{(t_i, p)})} = \frac{m}{M(\Omega(n)/\Gamma)}.$$

Recall that Γ_1 and Γ_2 are two quasi-conformally conjugate convex co-compact Kleinian groups. Define the *push-forward* $f_* : \mathcal{C}(\Gamma_1) \rightarrow \mathcal{C}(\Gamma_2)$ of $\mu \in \mathcal{C}(\Gamma_1)$ by the induced homeomorphism $f : L_{\Gamma_1} \rightarrow L_{\Gamma_2}$ by $(f_*\mu)(\phi) = \mu(\phi \circ f)$; $\phi \in C_c(G(\mathbf{H}^n))$. The map f extends to a quasi-isometric (i.e. biLipschitz) map $\bar{f} : \mathbf{H}^n \rightarrow \mathbf{H}^n$ conjugating Γ_1 to Γ_2 (see [8] and [12]). Observe that if α is a geodesic current corresponding to a closed geodesic γ , then $f_*\alpha$ is the geodesic current corresponding to the unique closed geodesic in the homotopy class of the closed curve $\bar{f}(\gamma)$.

Lemma 4.10. The map $f_* : \mathcal{C}(\Gamma_1) \rightarrow \mathcal{C}(\Gamma_2)$ is a homeomorphism.

Proof. We first observe that because f_* is functorial, we have that f_* is one to one and onto. Let $\mu_i \rightarrow \mu$ in $\mathcal{C}(\Gamma_1)$ and $\phi : G(\mathbf{H}^n) \rightarrow \mathbf{R}$ be a continuous function with compact support K . Then, $f_*\mu(\phi) = \mu(\phi \circ f)$. The support of $\phi \circ f$ is $f^{-1}(K)$ which is compact as f is a homeomorphism. As $\mu_i \rightarrow \mu$ and $\phi \circ f$ is continuous with compact support, we have that $\mu_i(\phi \circ f) \rightarrow \mu(\phi \circ f)$ and therefore, $f_*\mu_i \rightarrow f_*\mu$. Therefore, f_* is continuous and similarly g_* is continuous, where $g = f^{-1}$. Composing f_*, g_* , we see that f_* has g_* as its inverse and is therefore, a homeomorphism. \square

We define the *distortion of a current* $\mu \in \mathcal{C}(\Gamma_1)$ by

$$\bar{R}_{12}(\mu) = \frac{L_2(f_*\mu)}{L_1(\mu)},$$

where L_1, L_2 are the length functions on $\mathcal{C}(\Gamma_1), \mathcal{C}(\Gamma_2)$ respectively and f is a quasi-conformal deformation inducing an isomorphism taking a convex co-compact Kleinian group Γ_1 to Γ_2 . The distortion $\bar{R}_{21}(\mu)$ of a current $\mu \in \mathcal{C}(\Gamma_2)$ is defined analogously.

In terms of the induced line element measures, the distortion is given by

$$\bar{R}_{12}(\mu) = \frac{(f_*\mu)^*(\Omega(n)/\Gamma_2)}{\mu^*(\Omega(n)/\Gamma_1)}.$$

Because geodesic flow acting on $\Omega(n)$ is ergodic with respect to M_1 for convex co-compact Kleinian groups, and since $\bar{R}_{12}(p)$ is a measurable, geodesic flow invariant function (Remark 3.5), we know that \bar{R}_{12} is constant on $\Omega(n)$ almost everywhere (M_1). The following lemma uses the compactness of the convex core to relate this constant to the distortion of the Patterson–Sullivan geodesic current m_1 .

Lemma 4.11. Let Γ_i , $i = 1, 2$ be convex co-compact, quasi-conformally conjugate Kleinian groups. Then,

$$\tilde{R}_{12}(p) \leq \bar{R}_{12}(m_1) \quad \text{a.e. } (M_1).$$

Proof. By Corollary 4.9, there is a set of full measure $A \subset \Omega(n)$ with respect to M_1 , such that for each $p \in A$, there exists a sequence t_i such that the Patterson–Sullivan geodesic current can be approximated using $\alpha_{(t_i,p)}$. By definition, we have, if $p \in \Omega(n)$ with endpoint ν^+ , that

$$\tilde{R}_{12}(p) = \inf \liminf_{\gamma_{x_1} \xrightarrow{\text{conical}} \nu^+} R_{12}(\gamma) = \inf \liminf_{\gamma_{x_1} \xrightarrow{\text{conical}} \nu^+} \frac{d(x_2, i(\gamma)x_2)}{d(x_1, \gamma x_1)},$$

where x_1, x_2 are any choice of basepoints for the 1-Poincaré series of Γ_1 and Γ_2 respectively. For any $\gamma \in \Gamma_1 - \{\text{id}\}$, we join the point x_1 to the point $\gamma(x_1)$. In the manifold \mathbf{H}^n/Γ_1 , this curve projects to a homotopically non-trivial closed curve a_γ ; we let α_γ be the unique closed geodesic (Γ_1 is convex co-compact) in its homotopy class. Let p have basepoint x , and choose $x_1 = x$. Hence, by the bounded homotopy argument in Lemma 4.5, there exists a constant C and a sequence $\{t_i\}$ tending to infinity so that the geodesic curves a_{t_i} are homotopic to the closed geodesics $\alpha_{\gamma_{t_i}}$ by a homotopy of bounded height C . Taking the corresponding sequence $\{\gamma_i = \gamma_{t_i}\}$ of elements in Γ_1 , we see that for each index i , the axis of γ_i is within a C -neighborhood of the geodesic joining x_1 to $\gamma_i(x_1)$. Thus, we have, by the triangle inequality, that

$$|L_1(\alpha_{\gamma_i}) - d(x_1, \gamma_i(x_1))| \leq 2C,$$

where L_1 is the length function on the geodesic current space $\mathcal{C}(\Gamma_1)$. Let $\bar{f} : \mathbf{H}^n \rightarrow \mathbf{H}^n$ be a quasi-isometric (i.e. biLipschitz) extension of f conjugating Γ_1 to Γ_2 . By the definition of the map $f_* : \mathcal{C}(\Gamma_1) \rightarrow \mathcal{C}(\Gamma_2)$, if α_{γ_i} is induced by a closed geodesic in \mathbf{H}^n/Γ_1 , then the geodesic current $f_*(\alpha_{\gamma_i})$ is induced by a closed geodesic in \mathbf{H}^n/Γ_2 homotopic to the image under \bar{f} of the closed geodesic inducing α_{γ_i} .

We let $x_2 = \bar{f}(x_1)$, then $\bar{f}(\gamma(x_1)) = i(\gamma)(\bar{f}(x_1)) = i(\gamma)(x_2)$. Therefore, the geodesic segment joining x_2 to $i(\gamma_i)(x_2)$ projects under the covering map to a curve that is homotopic to $f_*(\alpha_{\gamma_i})$ (here, we regard $f_*(\alpha_{\gamma_i})$ as the geodesic that induces the simple geodesic current), and because \bar{f} is biLipschitz, there is a constant C_1 so that the resulting homotopy is a bounded homotopy of height C_1 . Thus, the geodesic joining x_2 and $i(\gamma_i)(x_2)$ is in the

C_1 -neighborhood of the axis of $i(\gamma_i)$, and we have by another application of the triangle inequality that

$$|L_2(f_*(\alpha_{\gamma_i})) - d(x_2, i(\gamma_i(x_2)))| \leq 2C_1.$$

(Here, as before, L_2 represents the length function on the space of geodesic currents $\mathcal{C}(\Gamma_2)$.)

Using the definition of \tilde{R} , and the above two inequalities, we have

$$\tilde{R}_{12}(p) \leq \liminf_{i \rightarrow \infty} \frac{d(x_2, i(\gamma_i(x_2)))}{d(x_1, \gamma_i(x_1))} = \liminf_{i \rightarrow \infty} \frac{L_2(f_*(\alpha_{\gamma_i}))}{L_1(\alpha_{\gamma_i})}.$$

By Corollary 4.9, we have that

$$\lim_{i \rightarrow \infty} \frac{\alpha_{\gamma_i}}{L_1(\alpha_{\gamma_i})} = \frac{m_1}{M_1(\Omega(n)/\Gamma_1)}.$$

(Note that here m_1 is the Patterson–Sullivan geodesic current for Γ_1 , and M_1 is the Patterson–Sullivan line element measure for Γ_1 .) Therefore, by the continuity of f_* (see Lemma 4.10), we have that

$$\lim_{i \rightarrow \infty} \frac{f_*(\alpha_{\gamma_i})}{L_1(\alpha_{\gamma_i})} = \frac{f_*(m_1)}{M_1(\Omega(n)/\Gamma_1)}.$$

By the continuity of the map $\alpha \rightarrow \alpha^*$ (see Lemma 4.2), we have that

$$\lim_{i \rightarrow \infty} \frac{[f_*(\alpha_{\gamma_i})]^*}{L_1(\alpha_{\gamma_i})} = \frac{[f_*(m_1)]^*}{M_1(\Omega(n)/\Gamma_1)}.$$

Because Γ is convex co-compact, there exists a compact subset of $\Omega(n)/\Gamma$ that contains the support of every line element measure. Therefore, by the definition of the weak* topology, we have that

$$\lim_{i \rightarrow \infty} \frac{[f_*(\alpha_{\gamma_i})]^*(\Omega(n)/\Gamma_2)}{L_1(\alpha_{\gamma_i})} = \frac{[f_*(m_1)]^*(\Omega(n)/\Gamma_2)}{M_1(\Omega(n)/\Gamma_1)}.$$

Recall that, by definition, the length function L_2 acts on $\mathcal{C}(\Gamma_2)$ by $L_2(\alpha) = \alpha^*(\Omega(n)/\Gamma_2)$. Thus, via Proposition 4.3, we have

$$\bar{R}_{12}(m_1) = \lim_{i \rightarrow \infty} \frac{L_2(f_*(\alpha_{\gamma_i}))}{L_1(\alpha_{\gamma_i})}.$$

□

An immediate corollary of Theorem 4.8 is:

Corollary 4.12. If N_1 and N_2 are isometric, then $\overline{R}_{12}(m_1) = \overline{R}_{21}(m_2) = 1$.

We conjecture:

Conjecture 4.13. If $N_{i=1,2}$ are convex co-compact quasi-isometric infinite volume hyperbolic manifolds with the property that $\overline{R}_{12}(m_1) = \overline{R}_{21}(m_2) = 1$, then the N_i are isometric.

The Main Theorem is now an immediate consequence of Lemma 4.11 and Theorem 4.8.

Theorem 4.14 (Main Theorem). Let Γ_i ; $i = 1, 2$ be convex co-compact, quasi-conformally conjugate Kleinian groups. Then,

$$\mathcal{R}_{12} = \overline{R}_{12}(m_1) \geq \frac{\delta(\Gamma_1)}{\delta(\Gamma_2)}.$$

Proof. By the above lemma we have that $\tilde{R}_{12}(p) \leq \overline{R}_{12}(m_1)$ a.e. (M_1). Also Lemma 3.4 shows that $\tilde{R}_{12}(p) \geq \delta(\Gamma_1)/\delta(\Gamma_2)$ a.e. (M_1). Therefore,

$$\mathcal{R}_{12} = \overline{R}_{12}(m_1) \geq \frac{\delta(\Gamma_1)}{\delta(\Gamma_2)}$$

almost everywhere with respect to M_1 , as desired. We can relate \mathcal{R}_{12} and \mathcal{R}_{21} through the following inequality: switching the roles of Γ_1, Γ_2 , we obtain

$$\frac{1}{\mathcal{R}_{12}} \leq \frac{\delta(\Gamma_2)}{\delta(\Gamma_1)} \leq \mathcal{R}_{21}.$$

□

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