COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 13, Number 2, 461-478, 2005

Characterization for Balls by Potential Function of Kähler-Einstein Metrics for Domains in $\mathbb{C}^n$

Song-Ying Li

1. Introduction and Main results.

Let D be a bounded domain in $\mathbb{C}^n$. For $u \in C^2(D)$, we denote $H(u)(z) = [\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}]_{n\times n}$ the complex hessian matrix of u at z. A plurisubharmonic function $U(z)$ on D is called a potential function for the Kähler-Einstein metric if U satisfies the Monge-Ampère equation:

$$\det H(U) = e^{(n+1)U} \quad \text{in } D; \quad \text{and} \quad U = \infty \quad \text{on } \partial D. \tag{1.1}$$

Equivalently the function

$$\rho(z) = -e^{-U(z)}, \quad z \in D \tag{1.2}$$

is called the potential function for the Fefferman's metric, and ρ satisfies the Fefferman equation:

$$J(\rho) = -\det \begin{bmatrix} \rho & \overline{\partial}\rho \\ (\overline{\partial}\rho)^* & H(\rho) \end{bmatrix} = 1, \quad z \in D, \quad \rho(z) = 0 \ \text{ on } \ \partial D, \tag{1.3}$$

where $\overline{\partial}\rho$ denotes the row vector with entries $\frac{\partial\rho}{\partial\bar{z}_1}, \cdots, \frac{\partial\rho}{\partial\bar{z}_n}$. When D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$, a formal positive solution of (1.3) was given by C. Fefferman in [7]. The existence of a positive solution was proved by Cheng and Yau [5]. Moreover, they also proved that $\rho \in C^{n+3/2}(\overline{D})$. Lee and Melrose [21] gave an asymptotic expansion for ρ, which implies that $\rho \in C^{n+2-\epsilon}(\overline{D})$. When D is a smoothly bounded weakly pseudoconvex domain in $\mathbb{C}^n$, it was proved by Cheng and Yau [5] that there is a complete Kähler–Einstein metric. The same result on the existence of a complete Kähler-Einstein metric was obtained later by Mok and Yau in [27] without an assumption on the smoothness of the boundary ∂D.

Several very interesting and fundamental theorems on the characterization of the unit ball in $\mathbb{C}^n$ have bee discovered before. For example, B.

Wong's characterization theorem for the unit ball by using non-compact automorphism group in [29] or [17]. A celebrated theorem of Stoll in [28] and Burns in [3] on a characterization theorem of the ball, by using the degenerate complex Monge-Ampère equation, can be stated as follows:

Theorem 1.1. *Let M be a complex manifold of dimension n. If there is a smooth strictly plurisubharmoinc function $u : M \to [0, 1)$ so that it is onto and satisfies*

$$\det H(\log u)(z) = 0, \quad z \in \{z \in M : u(z) > 0\}, \tag{1.4}$$

then there is a biholomorphic map $\phi : M \to B_n$.

For convenience, we let

$$\tilde{U}(z) = U(z) - c_0 \quad \text{with } c_0 = \min\{U(z) : z \in D\}. \tag{1.5}$$

Then U satisfies (1.1) if and only if $\tilde{U}$ satisfies

$$\det H(U) = g(z)e^{(n+1)U} \ \text{ in } D, \quad U = \infty \ \text{ on } \ \partial D, \quad \min\{U(z) : z \in D\} = 0 \tag{1.6}$$

with $g(z) \equiv e^{(n+1)c_0}$. In other words, $\tilde{\rho}(z) = \rho(z)e^{c_0} = -e^{-(U-c_0)} = -e^{-\tilde{U}}$ satisfies

$$J(\rho) = g(z) \text{ in } \ D, \quad \rho = 0 \ \text{ on } \ \partial D \quad \min\{\rho(z) : z \in D\} = -1 \tag{1.7}$$

with $g(z) \equiv e^{(n+1)c_0}$.

One observes that if $\phi : D \to B_n$ is a biholomorphic map with $\phi(z_0) = 0$, then

$$U(z) = -\log(1 - |\phi(z)|^2) \tag{1.8}$$

satisfies (1.6) with

$$g(z) = |\det \phi'(z)|^2 > 0 \tag{1.9}$$

and $\log g(z)$ being pluriharmonic in D. Moreover, if $\rho(z) = -e^{-U(z)}$ then

$$\det H(\log(1 + \rho(z))) = \det H(\log |\phi|^2) = 0, \quad z \in D \setminus \{z_0\}. \tag{1.10}$$

Conversely, we shall prove the following main theorem of this paper.

Theorem 1.2. *Let D be a smoothly bounded weakly pseudoconvex domain in $\mathbb{C}^n$. Let U be pluri-subharmonic in D such that (1.6) holds for some positive function g in D with $\log g(z)$ being bounded and pluriharmonic in D. Then*

(a) If $\rho(z) = -e^{-U(z)}$ satisfies that

$$\liminf_{z\to\partial D} \det H\Big(\log\Big(1+\rho(z)\Big)\Big) \geq 0, \tag{1.11}$$

then D is biholomorphically equivalent to the unit ball in $\mathbb{C}^n$.

(b) In addition to the assumption (1.11), if $g(z) \equiv c$ for some positive constant c, then there is a biholomorphic map $\phi : D \to B_n$ so that $\det \phi'(z) = \sqrt{c}$ on D.

The paper is organized as follows: In Section 2, we will provide several fundamental results, which are mainly stated in Theorem 2.1. As a corollary of Theorem 2.1, we prove Part (a) of Theorem 1.2. In Section 3, we first prove a theorem building up a relationship between subharmonicity of $\log(1-e^{-U})$ in D and the biholomorphic mapping from D to B_n with constant Jacobian. As an application, we will prove Part (b) of Theorem 1.2.

2. The Proof of Part (a) Theorem 1.2.

In this section, we will prove several preliminary results related to the Kähler-Einstein metric. We will also give a few characterizations of the unit ball using some quantities associated to the Kähler–Einstein metric. Mainly, we will prove the following theorem.

Theorem 2.1. *Let D be a bounded weakly pseudoconvex domain in $\mathbb{C}^n$. Let U be a strictly plurisubharmonic solution of the Monge-Ampère equation:*

$$\det H(U) = g(z)e^{(n+1)U} \text{ in } D;\ u = \infty \text{ on } \partial D \text{ and } \min\{U(z) : z \in D\} = 0, \tag{2.1}$$

where $g(z) > 0$ and $\log g(z)$ is pluriharmonic in D. Let $\rho(z) = -e^{-U(z)}$. Then the following three statements hold.

(a) If $\log(1+\rho(z))$ is plurisubharmonic in D then D is biholomorphically equivalent to the unit ball B_n in $\mathbb{C}^n$.

(b) If $\lim_{z\to w} e^{U(z)}(U^{i\bar{j}}U_iU_{\bar{j}} + e^{-U(z)} - 1) = 0$ for all $w \in \partial D$ then D is biholomorphically equivalent to B_n.

(c) If $\liminf_{z\to\xi}\Big(g(z)^{-1}\det(H(\rho)(z)\Big) \geq 1$ *for all* $\xi \in \partial D$*, then* D *is biholomorphically equivalent to* B_n*.*

Notice that without the condition $m = \min\{U(z) : z \in D\} = 0$, a theorem of Cheng and Yau in [5] shows that (2.1) has a strictly plurisubhamonic solution $U(z) \in C^\infty(D)$ so that, the Kähler-Einstein metric $U_{i\bar{j}}dz_i d\bar{z}_j$ is complete on D. Therefore $U - m$ is a strictly plurisubharmonic solution of (2.1) if we replace $g(z)$ by $g(z)e^{(n+1)m}$.

Let

$$U_i = \frac{\partial U}{\partial z_i}, \quad U_{\bar{j}} = \frac{\partial U}{\partial \bar{z}_j} \quad \text{and} \quad U_{i\bar{j}} = \frac{\partial^2 U}{\partial z_i \partial \bar{z}_j}. \tag{2.2}$$

and let $H(U) = [U_{i\bar{j}}(z)]_{n\times n}$ be positive definite on D. We use the notation $[U^{i\bar{j}}]_{n\times n} = (H(U)^{-1})^t$ and

$$|\partial U|^2(z) = U^{i\bar{j}}U_iU_{\bar{j}}(z). \tag{2.3}$$

Let

$$\rho(z) = -e^{-U(z)}, \quad v(z) = 1 + \rho(z), \quad u(z) = \log v(z). \tag{2.4}$$

We will prove the following lemma.

Lemma 2.2. *Let* $U(z)$ *be strictly pluri-subharmonic and satisfy (2.1). Then*

$$\det H(u)(z) = \frac{g(z)}{v(z)^{n+1}}e^{U(z)}\Big(1 - e^{-U(z)} - |\partial U|^2\Big) \tag{2.5}$$

and

$$\det H(v)(z) = g(z)e^{U(z)}(1 - |\partial U|^2). \tag{2.6}$$

Proof. Since

$$\partial_i u(z) = \frac{1}{v(z)}\partial_i v(z) = \frac{e^{-U(z)}U_i(z)}{v(z)}$$

and

$$\begin{aligned}
\partial_{i\bar{j}}u(z) &= \partial_{\bar{j}}\Big[\frac{e^{-U(z)}U_i(z)}{v(z)}\Big] \\
&= \frac{e^{-U(z)}}{v(z)}[-U_iU_{\bar{j}} + U_{i\bar{j}} - \frac{e^{-U}U_iU_{\bar{j}}}{v}] \\
&= \frac{e^{-U(z)}}{v(z)}[U_{i\bar{j}} - \frac{e^{-U} + v(z)}{v}U_iU_{\bar{j}}] \\
&= \frac{e^{-U(z)}}{v(z)}[U_{i\bar{j}} - \frac{1}{v}U_iU_{\bar{j}}]
\end{aligned}$$

Let ∂U denote the row vector with entries $\partial_1 U, \cdots, \partial_n U$. Then

$$\begin{aligned}
\det H(u)(z) &= \frac{e^{-nU(z)}}{v(z)^n} \det\Big(H(U)(z) - \frac{1}{v(z)}(\partial U)^*(\partial U)\Big) \\
&= \frac{e^{-nU(z)}}{v(z)^n} \det\Big(H(U)(z)\Big)\Big(1 - \frac{1}{v(z)}(\partial U)H(U)^{-1}(\partial U)^*\Big) \\
&= \frac{e^{-nU(z)}}{v(z)^n} g(z)e^{(n+1)U}\Big(1 - \frac{1}{v(z)}(\partial U)H(U)^{-1}(\partial U)^*\Big) \\
&= g(z)\frac{e^{U(z)}}{v(z)^n}\Big(1 - \frac{1}{v(z)}|\partial U|^2\Big) \\
&= g(z)\frac{e^{U(z)}}{v(z)^{n+1}}(1 - e^{-U(z)} - |\partial U|^2),
\end{aligned}$$

and

$$\begin{aligned}
\det(v_{i\bar{j}})(z) &= \det(e^{-U(z)}[U_{i\bar{j}} - U_i U_{\bar{j}}]) \\
&= e^{-nU(z)} \det(U_{i\bar{j}})(1 - |\partial U(z)|^2) \\
&= g(z)e^{U(z)}\,(1 - |\partial U(z)|^2).
\end{aligned}$$

Therefore, (2.5) and (2.6) hold, and the proof of the lemma is complete. ▯

Corollary 2.3. *Let $U(z)$ be a strictly plurisubharmonic solution of (2.1). Let $u(z)$ and $v(z)$ be defined by (2.4) in term of U. Then the following statements hold.*

(a) $u(z)$ is plurisubharmonic in D if and only if $1 - e^{-U(z)} - |\partial U|^2(z) \geq 0$ on D.

(b) If $u(z)$ is plurisubharmonic in D then $v(z)$ is strictly plurisubharmonic in D.

Proof. We first prove Part (a). Since U is strictly plurisubharmonic in D and

$$u_{i\bar{j}}(z) = \frac{e^{-U(z)}}{v(z)}(U_{i\bar{j}} - \frac{1}{v(z)}U_i U_{\bar{j}}),$$

one can easily see that $u(z)$ is plurisubharmonic in D if and only if $\det(u_{i\bar{j}}) \geq 0$ on D. The Part (a) follows from (2.5).

Next we prove Part (b). Since u is plurisubharmonic in D then $|\partial U|^2 + e^{-U(z)} \leq 1$ by Part (a). Thus $v(z) = e^{u(z)}$ is plurisubharmonic in D. Moreover,

$$\det(v_{i\bar{j}}) = g(z)e^U(1 - |\partial U|^2) \geq g(z)e^U e^{-U(z)} = g(z) > 0.$$

Therefore, v is strictly plurisubharmonic in D; and the proof of the corollary is complete. $\square$

For simplicity, we shall use the notation:

$$T(z) = U^{i\bar{j}} U_i U_{\bar{j}} + e^{-U(z)} = |\partial U|^2 + e^{-U(z)}. \tag{2.7}$$

In order to prove Theorem 2.1, We will need the following theorem.

Theorem 2.4. *Let D be a bounded weakly pseudoconvex domain in $\mathbb{C}^n$. Let U be a strictly plurisubharmonic solution of (2.1). Then*

(a) If $T(z) \le 1$ on D then $T(z) \equiv 1$ on D.

(b) If $\limsup_{z\to\xi} e^{U(z)}(T(z)-1)=0$ for any $\xi \in \partial D$ then $T(z)\equiv 1$ on D.

Proof. Let

$$\mathcal{L} = U^{k\bar{\ell}} \partial_{k\bar{\ell}} + 2\mathrm{Re}\,(U^{k\bar{\ell}} U_k \partial_{\bar{\ell}}). \tag{2.8}$$

Then if $T(z) \le 1$ in D then we shall prove

$$\mathcal{L}T(z) \ge 0 \quad \text{on} \quad D. \tag{2.9}$$

For any $w \in D$, we shall calculate $\mathcal{L}T$ at w. Let $z = \phi(\xi)$ be a biholomorphic map between a neighborhood of 0 and a neighborhood of w. Then we define

$$\tilde{U}(\xi) = U(\phi(\xi)). \tag{2.10}$$

It is easy to choose such a map ϕ so that $\phi(0) = w$ and, in the new coordinates,

$$\tilde{U}_{i\bar{j}k}(0) = 0, \quad 1 \le i, j, k \le n. \tag{2.11}$$

By the definition of $T(z)$, if we let

$$V(z) = U^{i\bar{j}} U_i U_{\bar{j}} \tag{2.12}$$

and

$$\tilde{V}(\xi) = \tilde{U}^{i\bar{j}} \tilde{U}_i \tilde{U}_{\bar{j}}(\xi), \quad \tilde{T}(\xi) = \tilde{V}(\xi) + e^{-\tilde{U}(\xi)}, \tag{2.13}$$

then

$$\tilde{T}(\xi) = T(\phi(\xi)) = \tilde{V}(\xi) + e^{-\tilde{U}(\xi)} \tag{2.14}$$

and

$$\det\left(\tilde{U}_{i\bar{j}}\right)(\xi) = |f(\xi)|^2 g(\phi(\xi)) e^{(n+1)\tilde{U}(\xi)} = F(\xi) e^{(n+1)\tilde{U}(\xi)} \tag{2.15}$$

where $f(\xi) = \det(\phi'(\xi))$ and $F(\xi) = |f(\xi)|^2 g(\phi(\xi))$. Thus
(2.16)

$$\tilde{V}_k(\xi) = \partial_k[\tilde{U}^{i\bar{j}}]\tilde{U}_i\tilde{U}_{\bar{j}} + \tilde{U}^{i\bar{j}}\tilde{U}_{ik}\tilde{U}_{\bar{j}} + \tilde{U}^{i\bar{j}}\tilde{U}_i\tilde{U}_{k\bar{j}} = \partial_k[\tilde{U}^{i\bar{j}}]\tilde{U}_i\tilde{U}_{\bar{j}} + \tilde{U}^{i\bar{j}}\tilde{U}_{ik}\tilde{U}_{\bar{j}} + \tilde{U}_k.$$

Since $\tilde{U}_{i\bar{j}k}(0) = 0$ and

$$\frac{\partial \tilde{U}^{i\bar{j}}}{\partial \xi_k} = \frac{\partial \tilde{U}^{i\bar{j}}}{\partial \tilde{U}_{p\bar{q}}} \frac{\partial \tilde{U}_{p\bar{q}}}{\partial \xi_k} = -\tilde{U}^{p\bar{j}}\tilde{U}^{i\bar{q}}\tilde{U}_{p\bar{q}k}, \tag{2.17}$$

we have

$$\frac{\partial \tilde{U}^{i\bar{j}}}{\partial \xi_k}(0) = \frac{\partial \tilde{U}^{i\bar{j}}}{\partial \bar{\xi}_k}(0) = 0. \tag{2.18}$$

and

$$\frac{\partial^2 \tilde{U}^{i\bar{j}}}{\partial \xi_k \partial \bar{\xi}_\ell}(0) = -\sum_{p,q=1}^{n} \tilde{U}^{p\bar{j}}\tilde{U}^{i\bar{q}}(0)\tilde{U}_{p\bar{q}k\bar{\ell}}(0). \tag{2.19}$$

Therefore, at $\xi = 0$, we have

$$\begin{aligned} \tilde{V}_{k\bar{\ell}}(0) &= \partial_{k\bar{\ell}}[\tilde{U}^{i\bar{j}}]\tilde{U}_i\tilde{U}_{\bar{j}} + \partial_k[\tilde{U}^{i\bar{j}}]\tilde{U}_{i\bar{\ell}}\tilde{U}_{\bar{j}} + \partial_k[\tilde{U}^{i\bar{j}}]\tilde{U}_i\tilde{U}_{\bar{j}\bar{\ell}} \\ &\quad + \partial_{\bar{\ell}}[\tilde{U}^{i\bar{j}}]\tilde{U}_{ik}\tilde{U}_{\bar{j}} + \tilde{U}^{i\bar{j}}\tilde{U}_{ik\bar{\ell}}\tilde{U}_{\bar{j}} + \tilde{U}^{i\bar{j}}\tilde{U}_{ik}\tilde{U}_{\bar{j}\bar{\ell}} + \tilde{U}_{k\bar{\ell}} \\ &= \partial_{k\bar{\ell}}[\tilde{U}^{i\bar{j}}]\tilde{U}_i\tilde{U}_{\bar{j}} + \tilde{U}^{i\bar{j}}\tilde{U}_{ik}\tilde{U}_{\bar{j}\bar{\ell}} + \tilde{U}_{k\bar{\ell}} \\ &= -\sum_{p,q=1}^{n} \tilde{U}^{p\bar{j}}\tilde{U}^{i\bar{q}}\tilde{U}_{p\bar{q}k\bar{\ell}}\tilde{U}_i\tilde{U}_{\bar{j}} + \tilde{U}^{i\bar{j}}\tilde{U}_{ik}\tilde{U}_{\bar{j}\bar{\ell}} + \tilde{U}_{k\bar{\ell}}. \end{aligned} \tag{2.20}$$

Since f is holomorphic, $f(0) \neq 0$ and $\log g(\phi(\xi))$ is pluriharmonic, we have $\partial_{p\bar{q}} \log F = \partial_{p\bar{q}}(\log|f|^2 + \log g(\phi(\xi))) = 0$ near $\xi = 0$. By (2.11) and (2.15), we have

$$\sum_{k,\ell=1}^{n} \tilde{U}^{k\bar{\ell}}\partial_{k\bar{\ell}}\tilde{U}_{p\bar{q}}(0) = \partial_{p\bar{q}}[(n+1)\tilde{U} + \log F](0) = (n+1)\tilde{U}_{p\bar{q}}(0). \tag{2.21}$$

Therefore, at $\xi = 0$ we have

$$\tilde{U}^{k\bar{\ell}}\tilde{V}_{k\bar{\ell}}(0) = -(n+1)\tilde{U}^{p\bar{j}}\tilde{U}^{i\bar{q}}\tilde{U}_{p\bar{q}}(0)\tilde{U}_i\tilde{U}_{\bar{j}} + \tilde{U}^{k\bar{\ell}}\tilde{U}^{i\bar{j}}\tilde{U}_{ik}\tilde{U}_{\bar{j}\bar{\ell}}(0) + n \tag{2.22}$$

$$= \quad -(n+1)\tilde U^{i\bar j}(0)\tilde U_i\tilde U_{\bar j} + \tilde U^{k\bar\ell}\tilde U^{i\bar j}\tilde U_{ik}\tilde U_{\bar j\bar\ell}(0) + n$$
$$= \quad -(n+1)\tilde V(0) + \tilde U^{k\bar\ell}\tilde U^{i\bar j}\tilde U_{ik}\tilde U_{\bar j\bar\ell}(0) + n$$

and

$$\tilde U^{k\bar\ell}\partial_{k\bar\ell}(e^{-\tilde U(\xi)}) = e^{-\tilde U(\xi)}(-\tilde U^{k\bar\ell}\tilde U_{k\bar\ell} + \tilde U^{k\bar\ell}\tilde U_k\tilde U_{\bar\ell}) = e^{-\tilde U(\xi)}(-n+\tilde V(\xi)). \tag{2.23}$$

Therefore, by combining (2.22) and (2.23)

$$\begin{aligned} &\tilde U^{k\bar\ell}\partial_{k\bar\ell}\tilde T(0) \\ =\ & -(n+1)\tilde V(0) + n + e^{-\tilde U(0)}[-n+\tilde V(0)] + \tilde U^{i\bar j}\tilde U^{k\bar\ell}\tilde U_{ik}\tilde U_{\bar j\bar\ell} \\ =\ & n - n\tilde V(0) - ne^{-\tilde U(0)} - \tilde V(0) + \tilde V(0)e^{-\tilde U(0)} + \tilde U^{i\bar j}\tilde U^{k\bar\ell}\tilde U_{ik}\tilde U_{\bar j\bar\ell} \\ =\ & n[1-\tilde T(0)] - \tilde V(0)[1-e^{-\tilde U(0)}] + \tilde U^{i\bar j}\tilde U^{k\bar\ell}\tilde U_{ik}\tilde U_{\bar j\bar\ell}. \end{aligned} \tag{2.24}$$

Now we let

$$\tilde{\mathcal L} = \tilde U^{k\bar\ell}\frac{\partial^2}{\partial\xi_k\partial\bar\xi_\ell} + 2\mathrm{Re}\,(\tilde U^{k\bar\ell}\tilde U_k\frac{\partial}{\partial\bar\xi_\ell}). \tag{2.25}$$

Then

$$(\mathcal L h)(\phi(\xi)) = \tilde{\mathcal L}\tilde h(\xi) \tag{2.26}$$

and $\tilde h(\xi) = h(\phi(\xi))$ for any $h \in C^2(D)$.

Since $\tilde U_{i\bar j k}(0) = 0$, we have $\partial_i\tilde U^{k\bar\ell}(0) = 0$ and $\partial_{\bar j}\tilde U^{k\bar\ell}(0) = 0$. Thus

$$\begin{aligned} \partial_i\tilde T(0) &= \partial_i(\tilde U^{k\bar\ell}\tilde U_k\tilde U_{\bar\ell} + e^{-\tilde U})(0) \\ &= 0 + \tilde U^{k\bar\ell}\tilde U_{ik}\tilde U_{\bar\ell}(0) + \tilde U^{k\bar\ell}\tilde U_k\tilde U_{i\bar\ell}(0) - e^{-\tilde U(0)}\tilde U_i(0) \\ &= \tilde U^{k\bar\ell}\tilde U_{ik}\tilde U_{\bar\ell}(0) + \tilde U_i(1-e^{-\tilde U(0)}) \end{aligned} \tag{2.27}$$

and

$$\partial_{\bar j}\tilde T(0) = \tilde U^{k\bar\ell}\tilde U_k\tilde U_{\bar j\bar\ell}(0) + \tilde U_{\bar j}(0)(1-e^{-\tilde U(0)}). \tag{2.28}$$

Thus

$$\tilde U^{i\bar j}\partial_i\tilde T(0)\tilde U_{\bar j}(0) = \tilde U^{i\bar j}\tilde U^{k\bar\ell}\tilde U_{\bar j}\tilde U_{ik}\tilde U_{\bar\ell}(0) + \tilde V(0)(1-e^{-\tilde U(0)}), \tag{2.29}$$

and

$$\tilde U^{i\bar j}\partial_{\bar j}\tilde T(0)\tilde U_i(0) = \tilde U^{i\bar j}\tilde U^{k\bar\ell}\tilde U_i\tilde U_k\tilde U_{\bar j\bar\ell}(0) + \tilde V(0)(1-e^{-\tilde U(0)}). \tag{2.30}$$

Therefore

$$\begin{aligned}
\mathrm{Re}\,(\tilde U^{k\overline{\ell}}\tilde U_k\partial_{\overline{\ell}})\tilde T(0) &= \mathrm{Re}\,(\tilde U^{k\overline{\ell}}\tilde U_k\tilde U^{i\overline{j}}\tilde U_i\tilde U_{\overline{j}\overline{\ell}}) + \tilde V(0)(1-e^{-\tilde U(0)})\\
&\geq -\Big(\sum_{i,j,k,\ell=1}^{n}\tilde U^{i\overline{j}}\tilde U^{k\overline{\ell}}\tilde U_{ik}\tilde U_{\overline{j}\overline{\ell}}\Big)^{1/2}(\sum_{i,j=1}^{n}\tilde U^{i\overline{j}}\tilde U_i\tilde U_{\overline{j}})\\
&\quad +\tilde V(0)(1-e^{-\tilde U(0)})\\
&= -\Big(\sum_{i,j,k,\ell=1}^{n}\tilde U^{i\overline{j}}\tilde U^{k\overline{\ell}}\tilde U_{ik}\tilde U_{\overline{j}\overline{\ell}}\Big)^{1/2}\tilde V + \tilde V(0)(1-e^{-\tilde U(0)}).
\end{aligned}$$

Since $\tilde T(0) = \tilde V(0) + e^{-\tilde U(0)} \leq 1$, we have $\tilde V(0)(1-e^{-\tilde U(0)}) \geq \tilde V(0)^2$. Thus

$$\begin{aligned}
\mathcal{L}T(w) &= \tilde{\mathcal{L}}\tilde T(0)\\
&\geq n[1-\tilde T(0)] - 2\Big(\sum_{i,j,k,\ell=1}^{n}\tilde U^{i\overline{j}}\tilde U^{k\overline{\ell}}\tilde U_{ik}\tilde U_{\overline{j}\overline{\ell}}\Big)^{1/2}\tilde V(0)\\
&\quad +\tilde V(0)(1-e^{-\tilde U(0)}) + \tilde U^{i\overline{j}}\tilde U^{k\overline{\ell}}\tilde U_{ik}\tilde U_{\overline{j}\overline{\ell}}(0)\\
&\geq n[1-\tilde T(0)] - 2\Big(\sum_{i,j,k,\ell=1}^{n}\tilde U^{i\overline{j}}\tilde U^{k\overline{\ell}}\tilde U_{ik}\tilde U_{\overline{j}\overline{\ell}}\Big)^{1/2}\tilde V(0) + \tilde V(0)^2\\
&\quad +\tilde U^{i\overline{j}}\tilde U^{k\overline{\ell}}\tilde U_{ik}\tilde U_{\overline{j}\overline{\ell}}(0)\\
&= n[1-\tilde T(0)] + [\tilde V(0) - \Big(\sum_{i,j,k,\ell=1}^{n}\tilde U^{i\overline{j}}\tilde U^{k\overline{\ell}}\tilde U_{ik}\tilde U_{\overline{j}\overline{\ell}}\Big)^{1/2}]^2\\
&\geq 0.
\end{aligned}$$

Therefore, since $w \in D$ is arbitrary, $\mathcal{L}T(z) \geq 0$ on D. Let $z_0 \in D$ so that $U(z_0) = \min\{U(z) : z \in D\} = 0$. Then $T(z_0) = 1$. Since $T(z) \leq 1$ and $T(z_0) = 1$. By the maximum principle, we have $T(z) \equiv T(z_0) = 1$, and the proof of Part (a) is complete.

Next we prove Part (b). Using (2.24), (2.29) and (2.30), we have

$$\begin{aligned}
&e^{-U(w)}U^{i\overline{j}}\partial_{i\overline{j}}\Big[e^{U(z)}(T(z)-1)\Big](w)\\
&\quad= e^{-\tilde U(0)}\tilde U^{i\overline{j}}\partial_{i\overline{j}}\Big[e^{\tilde U(\xi)}(\tilde T(\xi)-1)\Big](0)\\
&\quad= \tilde U^{i\overline{j}}(\tilde U_{i\overline{j}} + \tilde U_i\tilde U_{\overline{j}})(\tilde T(0)-1) + \tilde U^{i\overline{j}}(\partial_i\tilde T(0)\tilde U_{\overline{j}} + \tilde U_i\partial_{\overline{j}}\tilde T(0)) + \tilde U^{i\overline{j}}\partial_{i\overline{j}}\tilde T(0)\\
&\quad= (n+\tilde V(0))(\tilde T(0)-1) + \tilde U^{i\overline{j}}(\partial_i\tilde T(0)\tilde U_{\overline{j}} + \tilde U_i\partial_{\overline{j}}\tilde T(0)) + \tilde U^{i\overline{j}}\partial_{i\overline{j}}\tilde T(0)
\end{aligned}$$

$$
\begin{aligned}
&= (n+\tilde V(0))(\tilde T(0)-1)+\tilde U^{i\bar j}\tilde U^{k\bar\ell}(\tilde U_{\bar j}\tilde U_{ik}\tilde U_{\bar\ell}(0)+\tilde U_i\tilde U_k\tilde U_{\overline{j\ell}}(0))\\
&\quad +2\tilde V(0)(1-e^{-\tilde U(0)})+n[1-\tilde T(0)]-\tilde V(0)[1-e^{-\tilde U(0)}]+\tilde U^{i\bar j}\tilde U^{k\bar\ell}\tilde U_{ik}\tilde U_{\overline{j\ell}}\\
&= \tilde V(0)(\tilde T(0)-1)+\tilde U^{i\bar j}\tilde U^{k\bar\ell}(\tilde U_{\bar j}\tilde U_{ik}\tilde U_{\bar\ell}(0)+\tilde U_i\tilde U_k\tilde U_{\overline{j\ell}}(0))\\
&\quad +\tilde V(0)[1-\tilde T(0)]+\tilde V(0)^2+\tilde U^{i\bar j}\tilde U^{k\bar\ell}\tilde U_{ik}\tilde U_{\overline{j\ell}}\\
&= \tilde U^{i\bar j}\tilde U^{k\bar\ell}(\tilde U_{\bar j}\tilde U_{ik}\tilde U_{\bar\ell}(0)+\tilde U_i\tilde U_k\tilde U_{\overline{j\ell}}(0))+\tilde V(0)^2+\tilde U^{i\bar j}\tilde U^{k\bar\ell}\tilde U_{ik}\tilde U_{\overline{j\ell}}\\
&\geq 0.
\end{aligned}
$$

Therefore $e^{U(z)}(T(z)-1)$ attains its maximum over $\overline{D}$ at some point on ∂D. By the assumption: $\limsup_{z\to\xi} e^{U(z)}(T(z)-1)=0$ for all $\xi\in\partial D$, we have that $e^{U(z)}(T(z)-1)\leq 0$ on D. This implies that $T(z)\leq 1$ on D. By Part (a), we have that $T(z)\equiv 1$ on D, and the proof of Part (b) is complete. Therefore, we have completed the proof of Theorem 2.4. ▯

From the proof of Theorem 2.4, one has the following corollary.

Corollary 2.5. *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let U be a plurisubharmonic solution of (2.1) with $g\in C^2(\overline{D})$ being positive and $\log g$ being pluriharmonic in D. Let $\rho(z)=-e^{-U(z)}$. Then the function*

$$\frac{1}{g(z)}\det H(\rho)(z)$$

attains its minimum over $\overline{D}$ at some point in ∂D.

Proof. By the results in [5], we have $\rho\in C^2(\overline{D})$. By (2.6) and definition of $v(z)$, one can easily see that

$$\frac{1}{g(z)}\det H(-e^{-U})(z)=e^{U(z)}(1-|\partial U|^2)=e^{U(z)}(1-T(z))+1,\quad z\in D$$

The proof of Part (b) of Theorem 2.4 shows that $e^{U(z)}(T(z)-1)$ attains its maximum over $\overline{D}$ at some point in ∂D. This implies that the statement of Corollary 2.5 holds. ▯

Now we are ready to prove Theorem 2.1.

The proof of Theorem 2.1.

Proof of Part (a) of Theorem 2.1: Since $u(z)=\log(1-e^{-U(z)})$ is plurisubharmonic, we have by (2.5) that $1-T(z)\geq 0$ on D. By Part (a) of Theorem 2.4, we have $T(z)\equiv 1$. By (2.5) again, we have $\det H(u)(z)=0$ on $\{z\in D: u(z)>-\infty\}$. By Corollary 2.3, $v(z)=e^{u(z)}$ is strictly plurisubharmonic in D which maps D onto $[0,1)$. Theorem 1.1 (of Burns and Stoll) implies that D is biholomorphic to the unit ball in $\mathbb{C}^n$. This proves Part (a).

Proof of Part (b) of Theorem 2.1: The assumption of Part (b) of Theorem 2.1 is the same as the assumption of Part (b) of Theorem 2.4. Hence, Part (b) of Theorem 2.4 implies that $T(z) \equiv 1$ on D. Therefore, The proofs of Part (b) of Theorem 2.1 follows from Corollary 2.3 and Theorem 1.1 (of Burns and Stoll).

Proof of Part (c) of Theorem 2.1: By (2.6), we have

$$\det H(v)(z) = g(z)e^{U(z)}(1 - V(z)) = g(z)e^{U(z)}(1 - T(z)) + g(z).$$

Thus

$$\det H(-e^{-U})(z) - g(z) = g(z)e^{U(z)}(1 - T(z)).$$

By the assumption (c) of Theorem 2.1, we have

$$\begin{aligned}
\liminf_{z\to\partial D} e^{U(z)}(1 - T(z)) &= \liminf_{z\to\partial D} \Big(\frac{\det H(-e^{-U})(z)}{g(z)} - 1\Big) \\
&\geq 1 - 1 = 0.
\end{aligned}$$

Since $e^{U(z)}(1 - T(z))$ attains its minimum over $\overline{D}$ at some point on ∂D, we have $e^{U(z)}(1 - T(z)) \geq 0$ on D and so $T(z) \leq 1$. By Theorem 2.4 (a), we have $T(z) \equiv 1$. Corollary 2.3 implies that the function

$$v(z) = 1 - e^{-U(z)}, \quad z \in D$$

is strictly plurisubharmonic in D, and $v : D \to [0, 1)$ is onto. Moreover, $\det H(\log v(z)) = 0$ on $\{z \in D : v(z) > 0\}$. Applying Theorem 1.1 (of Burns and Stoll), we have that D is biholomorphically equivalent to the unit ball in $\mathbb{C}^n$. $\square$

3. The proof of Part (b) of Theorem 1.2.

In order to prove Part (b) of Theorem 1.2, we first consider biholomorphic mapping between D and B_n.

Let $\phi : D \to B_n$ be a biholomorphic mapping. Let

$$U(z) = -\log(1 - |\phi(z)|^2) + \frac{1}{n+1}\log|\det\phi'(z)|^2 - c, \tag{3.1}$$

with

$$c = \min\Big\{ -\log(1-|\phi|^2) + \frac{1}{n+1}\log|\phi'(z)|^2 : z \in D\Big\}. \tag{3.2}$$

Since $U(z)$ is plurisubharmonic, there is point $z_0 \in D$ so that $U(z_0) = 0$. Without loss of generality (since B_n is a symmetric domain), we may assume that

$$\phi(z_0) = 0. \tag{3.3}$$

Then U is a strictly plurisubharmonic solution of (2.1) with $g(z) = e^{(n+1)c}$. In particular, $\min\{U(z) : z \in D\} = 0$.

Let $\psi : B_n \to D$ be the inverse mapping of $\phi(z)$. Then

$$\det \phi'(\psi(w)) = \frac{1}{\det \psi'(w)}, \quad w \in B_n \tag{3.4}$$

and

$$\tilde{U}(w) = U(\psi(w)) = -\log(1-|w|^2) - \frac{1}{n+1}\log|\det\psi'(w)|^2 - c. \tag{3.5}$$

We shall prove the following theorem.

Theorem 3.1. *Let $\phi : D \to B_n$ be a biholomorphic mapping so that $\phi(z_0) = 0$. Let $U(z)$ be defined by (3.1) and (3.2). If $u(z) = \log(1-e^{-U(z)})$ is plurisubharmonic near ∂D, then $\det\phi'(z) =$ constant. In particular,*

(i) If $n = 1$ then D is a disk;

(ii) If D is circular with respect to a pint $w \in D$, then ϕ is a linear map: $\phi(z) = A(z-w)$, where A is an $n\times n$ non-singular scalar matrix.

Proof. Let $\psi(w) = \phi^{-1}(w) : B_n \to D$, and let

$$\tilde{U}(w) = U(\psi(w)) = -\log(1-|w|^2) - \log h(w), \ \ h(w) = |(\det\psi'(w))^{1/(n+1)}|^2 e^c. \tag{3.6}$$

Then $h(0) = 1$. Let

$$\tilde{v}(w) = 1-(1-|w|^2)h(w), \quad \text{and} \ \ \tilde{u}(w) = \log\tilde{v}(w). \tag{3.7}$$

Then by (2.5)

$$\det H(\tilde{u})(w) = \frac{h(w)^{n+1}}{\tilde{v}(w)^{n+1}} e^{\tilde{U}}(1-(1-|w|^2)h(w) - |\partial\tilde{U}|^2) \tag{3.8}$$

where

$$|\partial\tilde{U}|^2 = \tilde{U}^{i\bar{j}}\tilde{U}_i\tilde{U}_{\bar{j}}, \quad \partial_i\tilde{U} = \frac{\overline{w}_i}{1-|w|^2} - \partial_i \log h(w)$$

and

$$\tilde{U}_{i\bar{j}} = \frac{\delta_{ij}}{1-|w|^2} + \frac{\overline{w}_i w_j}{(1-|w|^2)^2}, \quad \tilde{U}^{i\bar{j}} = (1-|w|^2)(\delta_{ij} - w_i\overline{w}_j).$$

Let

$$X_j = \frac{\partial}{\partial w_j} - \overline{w}_j R, \quad R = \sum_{k=1}^{n} w_j \frac{\partial}{\partial w_j}.$$

Then

$$\begin{aligned}
|\partial\tilde{U}|^2 &= (1-|w|^2)(\delta_{ij}-w_i\overline{w}_j)\Big(\frac{\overline{w}_i}{1-|w|^2}-\partial_i \log h(w)\Big)\Big(\frac{w_j}{1-|w|^2}-\partial_{\bar{j}} \log h(w)\Big)\\
&= |w|^2 - (\delta_{ij} - w_i\overline{w}_j)[\overline{w}_i\partial_{\bar{j}} \log h(w) + w_j\partial_i \log h(w)]\\
&\quad + (1-|w|^2)(\delta_{ij} - w_i\overline{w}_j)\partial_i \log h(w)\, \partial_{\bar{j}} \log h(w)\\
&= |w|^2 - (1-|w|^2)(R+\overline{R}) \log h(w)\\
&\quad + (1-|w|^2)(\delta_{ij} - w_i\overline{w}_j)\partial_i \log h(w)\, \partial_{\bar{j}} \log h(w).
\end{aligned}$$

Thus

$$1-|\partial\tilde{U}|^2 = (1-|w|^2)[1+(R+\overline{R}) \log h(w) - (\delta_{ij}-w_i\overline{w}_j)\partial_i \log h(w)\partial_{\bar{j}} \log h(w)].$$

Therefore,

$$\begin{aligned}
&h(w)e^{\tilde{U}}(1-|\partial\tilde{U}|^2 - e^{-\tilde{U}})\\
&= [1+(R+\overline{R}) \log h(w) - (\delta_{ij} - w_i\overline{w}_j)\partial_i \log h(w)\partial_{\bar{j}} \log h(w) - h(w)]\\
&= [1+(R+\overline{R}) \log h(w) - \sum_{j=1}^{n} |X_j \log h|^2 - (1-|w|^2)|R\log h(w)|^2 - h(w)].
\end{aligned}$$

Since $\log(1-e^{-U(z)})$ is plurisubharmonic in D, so is $\log(1-e^{-\tilde{U}})$ in B_n. By (3.8) we have

$$\frac{h(w)^{n+1}}{\tilde{v}(w)^{n+1}} e^{\tilde{U}(w)}(1-|\partial\tilde{U}(w)|^2 - e^{-\tilde{U}(z)}) \geq 0, \quad w \in B_n$$

Since $h>0$ and $v>0$ (if $w \neq 0$), we have

$$h(w)e^{\tilde{U}(w)}(1-|\partial\tilde{U}(w)|^2 - e^{-\tilde{U}(z)}) \geq 0, \quad w \in B_n.$$

Moreover, since $h(w)$ is positive subharmonic and $\log h(w)$ is pluriharmonic, we have

$$\begin{aligned}
0 &\le \frac{1}{\sigma(\partial B_n)}\int_{\partial B_n} h(w)e^{\tilde U}(1-|\partial\tilde U|^2-e^{-\tilde U})d\sigma(w)\\
&= \frac{1}{\sigma(\partial B_n)}\int_{\partial B_n}[1+(R+\overline{R})\log h(w)-\sum_{j=1}^n|X_j\log h|^2-h(w)]d\sigma(w)\\
&= 1-\frac{1}{\sigma(\partial B_n)}\int_{\partial B_n}[\sum_{j=1}^n|X_j\log h|^2+h(w)]d\sigma(w)\\
&\le 1-h(0)-\frac{1}{\sigma(\partial B_n)}\int_{\partial B_n}\sum_{j=1}^n|X_j\log h|^2d\sigma(w)\\
&= -\frac{1}{\sigma(\partial B_n)}\int_{\partial B_n}\sum_{j=1}^n|X_j\log h|^2d\sigma(w)\\
&\le 0
\end{aligned}$$

This implies that

$$\sum_{j=1}^n|X_j\log h|\equiv 0,\qquad \text{on}\ \ \partial B_n. \tag{3.9}$$

and

$$\int_{\partial B_n} h(w)d\sigma(w)=h(0)\sigma(\partial B_n)=\sigma(\partial B_n) \tag{3.10}$$

Since $\log h(z)$ is real-valued, (3.9) implies that $\log h(z)$ is CR. It must be a constant on ∂B_n. Since $\log h(w)$ is pluriharmonic, by the Maximum and Minimum Principles, we have $h(w)\equiv h(0)=1$ on B_n. This implies that $|\det\psi'(w)|^2=e^{-(n+1)c}$. Since $\det\psi'(w)$ is holomorphic, we have

$$\det\psi'(w)\equiv e^{i\theta}e^{-(n+1)c/2} \tag{3.11}$$

for some $\theta\in[0,2\pi)$.

Parts (i) and (ii) follow directly from the Cartan's theorem (see [17]). Therefore, the proof of Theorem 3.1 is complete. ∎

As a corollary of Theorem 2.1 and Theorem 3.1, we have

Corollary 3.2. *Let D be a bounded weakly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Let $U(z)$ be a plurisubharmonic solution of (2.1) in D with $g(z)=e^{(n+1)c}$ for some constant c so that $\exp(-U)\in C^2(\overline{D})$ and*

$$\log(1-\exp(-U(z)))$$

is plurisubharmonic near ∂D. Then there is a biholomorphic map $\phi : D \to B_n$ so that $\det \phi'(z) = e^{i\theta} e^{(n+1)c/2}$ for some $\theta \in [0, 2\pi)$. In particular, when $n = 1$ or D is a circular domain with respect to $w \in D$, then D must be the unit ball after a linear transformation.

Proof. By Theorem 2.1, we have that D is biholomorphic to the unit ball B_n in $\mathbb{C}^n$. Let $\phi : D \to B_n$ be a biholomorphic map. Let

$$U^0(z) = -\log(1 - |\phi|^2) + \frac{1}{n+1} \log |\det \phi'(z)|^2 - c, \quad z \in D.$$

It is easy to show that

$$\det H(U^0)(z) = e^{(n+1)c} e^{(n+1)U^0}, \quad \min\{U^0(z) : z \in D\} = 0$$

Since $e^{-U} \in C^2(\overline{D})$, there exist two constants $0 < c < C < \infty$ such that

$$cH(U^0) \le H(U) \le CH(U^0), \quad z \in D.$$

By the uniqueness theorem in [5], we have that $U(z) = U^0(z)$. Thus $\min\{U^0(z) : z \in D\} = 0$. Applying Theorem 3.1, we have that $\det(\phi') \equiv e^{i\theta} e^{(n+1)c/2}$ for some $\theta \in [0, 2\pi)$. Moreover, if D is circular with respect to $w \in D$, Cartan's Theorem implies that $\phi(z) = A(z - w)$ with A being a scalar $n \times n$ matric. Therefore, the proof of the corollary is complete. ▯

Now we are ready to prove Part (b) of Theorem 1.2.

Proof. By Part (a) of Theorem 1.2, we have that D is biholomorphic to the unit ball in $\mathbb{C}^n$. Corollary 3.2 implies that there is a biholomorphic map $\phi : D \to B_n$ so that $\det \phi'(z) \equiv$constant in D. This completes the proof of Part (b) of Theorem 1.2. ▯

Finally in the section, we connect the domains, which are biholomorphically equivalent to the unit ball with constant Jacobian maps, to the Bergman kernel function $K(z, w)$.

Proposition 3.3. *Let D be a bounded domain in $\mathbb{C}^n$ which is biholomorphically equivalent to a ball in $\mathbb{C}^n$. Then there is $a \in D$ such that $K(z, a)$ is constant function of z if and only if there is a biholomorphic map $\phi : D \to B_n$ with $\det \phi'(z) =$constant.*

Proof. If $\phi : D \to B_n$ is a biholomorphic map with $\det \phi'(z)$=constant= c, then

$$K_D(z, w) = (1 - \langle \phi(z), \phi(w) \rangle)^{-n-1} |c|^2$$

Then
$$K_D(z, \phi^{-1}(0)) \equiv |c|^2, \quad z \in D.$$

On the other hands, if $K_D(z, a) \equiv |c|^2$ for $z \in D$, since D is biholomorphically equivalent to the unit ball, there is a biholomorphic map $\psi : D \to B_n$ so that $\psi(a) = 0$. Thus

$$K_D(z, a) = (1 - \langle \psi(z), \psi(a) \rangle)^{-n-1} \det \psi'(z) \det \psi'(a) = \det \psi'(z) \overline{\det \psi'(a)}$$

Thus
$$\det \psi'(z) = \frac{|c|^2}{\overline{\det \psi'(a)}}, \quad z \in D.$$

Therefore, the proof of the proposition is complete. ∎

References.

[1] M. S. Baouendi, P. Ebenfelt, L. P. Rothschild, Real submanifolds in complex space and their mappings. Princeton Mathematical Series, 47. Princeton University Press, Princeton, NJ, 1999.

[2] E. Bedford, Survey of Pluri-Potential Theory, *Several Complex Variables: Proceedings of the Mittag-Leffler Institute, 1987-1988*, John E. Fornaess, ed., Princeton University Press, Princeton, 1993, 11–65.

[3] D. Burns, Curvatures of Monge-Ampère foliations and parabolic manifolds, *Ann. Math.* 115 (1982), 349–373.

[4] L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations, II: complex Monge-Ampère, and uniformly elliptic equations, *Comm. on Pure and Appl. Math.* (1985), 209–252.

[5] S. Y. Cheng and S. T. Yau, On the existence of a complex Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation, *Comm. Pure Appl. Math.* 33 (1980) 507–544.

[6] S. S. Chern and S. Ji, On the Riemann mapping theorem, *Ann. of Math.*, 144(1996), 421–439.

[7] C. Fefferman, The Bergman kernel and biholomorphic mapping of pseudoconvex domains, *Invent. Math.* 65(1974), 1–65.

[8] C. Fefferman, Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains, *Ann. of Math.* 103(1976), 395–416.

[9] S. Fu, B. Wong, On strictly pseudoconvex domains with Khler-Einstein Bergman metrics, *Math. Res. Lett.* 4 (1997), no. 5, 697–703.

[10] B. Guan, The Dirichlet problem for complex Monge-Ampère equations and Regularity of the Green's function, *Comm. in Analysis and Geometry*, 6(1998), 687–703.

[11] B. Guan and P. Guan, Convex Hypersurfaces of Prescribed Curvature, *Ann. of Math.*, 156(2002), 655–674.

[12] P. Guan, Extremal Function associated to Intrinsin Norm, *Ann. of Math.*, 156(2002), 197–211.

[13] P. Guan, N. Trudinger and X-J. Wang, Bounary regularity for degenerate Monge-Ampère equations, *Acta Math.*, 182(1999), 87–104.

[14] Xiaojun Huang, On the rigidity of proper holomorphic maps between balls, *JDG*, 51(1999), 13–33.

[15] Xiaojun Huang, On some problems in several complex variables and CR geometry. First International Congress of Chinese Mathematicians (Beijing, 1998), 383–396,

[16] Xiaojun Huang and Shanyu Ji, Mapping $\mathbf{B}^n$ into $\mathbf{B}^{2n-1}$, *Invent. Math.* 145 (2001), no. 2, 219–250.

[17] S. G. Krantz, Function Theory of Several Complex Variables, 2nd Edition, Wadsworth & Brooks/Cole, Mathematics Series, 1992.

[18] S. G. Krantz aand S.-Y. Li, On the Existence of Smooth Plurisubharmonic Solutions for Certain Degenerate Monge-Ampère Equations, *Complex variables*, 41(2000), 207–219.

[19] John M. Lee, Pseudo-Einstein structures on CR manifolds, *Amer. J. Math.* 110 (1988), no. 1, 157–178.

[20] John M. Lee, The Fefferman metric and pseudo-Hermitian invariants, *Trans. Amer. Math. Soc.* 296 (1986), no. 1, 411–429.

[21] J. Lee and Melrose, Boundary behavior of the complex Monge-Ampère equation, *Acta Math.* (1982), 159–192.

[22] L. Lempert, Solving the degenerate Monge-Ampère equation with one concentrated singularity, *Math. Ann.* 263 (1983), 515-532.

[23] S.-Y. Li, On the existence and regularity of Dirichlet problem for complex Monge-Ampère equations on weakly pseudoconvex domains, *Calculus of Variations & PDEs*, 20 (2004), 119–132.

[24] S.-Y. Li, The Neumann Problem for Complex Monge-Ampere equation, *Indiana University J. of Mathematics*, 44(1995), 1099–1122.

[25] S.-Y. Li, On the Dirichlet Problem for Symmetric Function Equations of Eigenvalues of Complex Hessian, *Asian J. Math.*, 8(2004), 87–106.

[26] S-Y. Li and L. Ni, On holomorphicity of proper harmonic maps between balls with Bergman metrics, *Math. Ann.*, 316(2000), 333–354.

[27] N. Mok and S. T. Yau, Completeness of the Khler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions. *The mathematical heritage of Henri Poincar* Part 1 (Bloomington, Ind., 1980), 41–59, Proc. Sympos. Pure Math., 39, Amer. Math. Soc., Providence, RI, 1983.

[28] Wilhelm Stoll, The characterization of strictly parabolic manifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 7 (1980), no. 1, 87–154.

[29] B. Wong, Characterizations of the balls in $\mathbb{C}^n$ by its automorphism group, *Invent. Math.* 41(1977), 253–257.

[30] S. T. Yau, On the Ricci curvature of a compact Kähler maniford and the complex Monge-Ampère equation, *Comm. Pure Appl. Math.* 31 (1978), 339–411.

School of Mathematics and Computer Science,
Fujian Normal University, Fujian, China

Current Address:

Department of Mathematics, University of California,
Irvine, CA 92697-3875, USA
sli@math.uci.edu

Received April 13, 2004.