On the Volume of a Hyperbolic and Spherical Tetrahedron

JUN MURAKAMI AND MASAKAZU YANO

A new formula for the volume of a hyperbolic and spherical tetrahedron is obtained from the quantum $6j$-symbol. This formula is of symmetric form with respect to the symmetry of the tetrahedron.

Introduction.

A formula for the volume of a generic hyperbolic tetrahedron is given in [1]. In this paper, we give another formula, which is symmetric with respect to the permutation of the vertices of a tetrahedron. Our formula comes from the quantum $6j$-symbol [5]. The actual formulation of the quantum $6j$-symbol is given in Section 4. Shortly, the quantum $6j$-symbol $\begin{Bmatrix} i & j & k \\ \ell & m & n \end{Bmatrix}$ is a number defined for six spins i, j, k, l, m, n assigned to the edges of a tetrahedron as in Figure 1 . The spins correspond to representations of the quantum enveloping algebra $\mathcal{U}_q(sl_2)$, and this number is defined as a certain amplitude of a sequence of coupling and decoupling of representations corresponding to the tetrahedron.
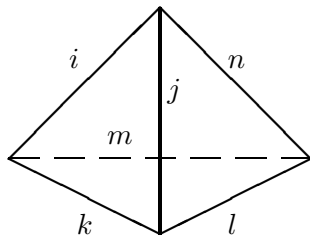

Figure 1: The six parameters i, j, k, l, m, n.

The relation between the volume of a hyperbolic tetrahedron and the quantum $6j$-symbol is expected by the following observations.

(1) R. Kashaev conjectured in [2] that the hyperbolic volume of the complement of a hyperbolic knot is equal to certain limit of the invariants defined by quantum R-matrices he constructed from quantum dilogarithm functions. These invariants are turned out in [7] to be specializations of colored Jones invariants. Moreover, in [6], it is observed that the volume of a hyperbolic manifold given by the Dehn surgeries along the figure-eight knot is obtained by applying Kashaev's method of computation to the Witten-Reshetikhin-Turaev invariants.

On the other hand, Turaev and Viro constructed a 3-manifold invariant in [11] by using a simplicial decomposition. It is defined by assigning the quantum $6j$-symbol to each tetrahedron of the decomposition. In [9], it is shown that this invariant is almost equivalent to the Reshetikhin-Turaev invariant, which seem to have some relation to the hyperbolic volume. Hence there may be some relation between the quantum $6j$-symbol and the volume of a hyperbolic tetrahedron.

(2) A relation between the volume of a Euclidean tetrahedron and certain asymptotics of the classical $6j$-symbols is conjectured by Ponzano and Regge in 1968, and proved by [10] in 1999. The classical $6j$-symbol is defined similarly as the quantum $6j$-symbol from the representations of the Lie algebra sl_2. This formula is quite surprising because the volume of the Euclidean tetrahedron, which is a basic quantity of geometry, is dominated by numbers coming from algebraic settings. Generalizing this relation to a hyperbolic and a spherical tetrahedron may reveal some fundamental relation between geometry and algebra we haven't noticed yet.

Encouraged by the above speculations, we started to apply Kashaev's method of computation to the quantum $6j$-symbol, and then we get the following formula.

Let T be a hyperbolic tetrahedron whose dihedral angles are A, B, C, D, E, F. Assume that A, B and C are the angles at the three edges having a common vertex, and D, E and F are the angles at the opposite position of A, B and C respectively as in Figure 2. Let $a = \exp \sqrt{-1}\,A$, $b = \exp \sqrt{-1}\,B$,

$\cdots$, $f = \exp \sqrt{-1}\, F$ and let $U(z, T)$ be the function

$$U(z,T) = \frac{1}{2}\ (\mathrm{Li}_2(z) + \mathrm{Li}_2(z\,a\,b\,d\,e) + \mathrm{Li}_2(z\,a\,c\,d\,f) + \mathrm{Li}_2(z\,b\,c\,e\,f)$$
$$- \mathrm{Li}_2(-z\,a\,b\,c) - \mathrm{Li}_2(-z\,a\,e\,f) - \mathrm{Li}_2(-z\,b\,d\,f) - \mathrm{Li}_2(-z\,c\,d\,e)), \quad (0.1)$$

where $\mathrm{Li}_2(x)$ is the dilogarithm function defined by the analytic continuation of the following integral.
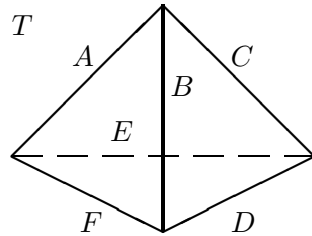

Figure 2: The six dihedral angles A, B, C, D, E, F of T.

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt \quad \text{for a positive real number } x. \qquad (0.2)$$

Let z_1, z_2 be the two non-trivial solutions of the equation

$$\frac{d}{dz}U(z,T) = \frac{\pi\sqrt{-1}}{z}\,k \quad (k \in \mathbf{Z}). \qquad (0.3)$$

Let

$$\tilde{\Delta}(a,b,c) = -\frac{1}{4}\ \Big(\mathrm{Li}_2(-a\,b\,c^{-1}) + \mathrm{Li}_2(-a\,b^{-1}\,c) + \mathrm{Li}_2(-a^{-1}\,b\,c)$$
$$+ \mathrm{Li}_2(-a^{-1}\,b^{-1}\,c^{-1}) + (\log a)^2 + (\log b)^2 + (\log c)^2\Big), \quad (0.4)$$

$$\Delta(T) = \tilde{\Delta}(a,b,c) + \tilde{\Delta}(a,e,f) + \tilde{\Delta}(b,d,f) + \tilde{\Delta}(c,d,e)$$
$$+ \frac{1}{2}\ (\log a\,\log d + \log b\,\log e + \log c\,\log f), \qquad (0.5)$$

and

$$V_1(T) = U(z_1,T) + \Delta(T),$$
$$V_2(T) = U(z_2,T) + \Delta(T), \qquad (0.6)$$
$$V(T) = \frac{U(z_1,T) - U(z_2,T)}{2}.$$

Let $\mathrm{Vol}(T)$ denote the hyperbolic volume of T. Then we get the following.

Theorem 1. *The volume $\mathrm{Vol}(T)$ of a hyperbolic tetrahedron T is given by the following.*

$$\mathrm{Vol}(T) = \mathrm{Im}\, V(T). \qquad\qquad (0.7)$$

In the following theorems, the solutions z_1 and z_2 in the definitions of the functions V, V_1 and V_2 are chosen adequately.

Theorem 2. *The volume $\mathrm{Vol}(T)$ of a hyperbolic tetrahedron T is given by the following.*

$$\mathrm{Vol}(T) = \mathrm{Im}\, V_1(T) = -\,\mathrm{Im}\, V_2(T). \qquad\qquad (0.8)$$

Theorem 3. *By taking an appropriate branch of $U(z,T)$, we have*

$$\mathrm{Re}\, V(T) = 0.$$

and

$$\mathrm{Vol}(T) = -\sqrt{-1}\; V(T). \qquad\qquad (0.9)$$

Theorem 4. *Let T be a tetrahedron T in S^3 with the constant curvature 1. Then the volume $\mathrm{Vol}(T)$ is given by*

$$\mathrm{Vol}(T) = V(T). \qquad\qquad (0.10)$$

1. Some property of the formula.

1.1. Quadratic equation for z_1 and z_2.

The equation $dU/dz = \pi k \sqrt{-1}/z$ is investigated. dU/dz is computed as follows:

$$\frac{dU(z,T)}{dz} =$$

$$-\frac{1}{2z} \left(\log(1-z) + \log(1-abde\,z) + \log(1-acdf\,z) + \log(1-bcef\,z) - \right.$$

$$\left. \log(1+abc\,z) - \log(1+aef\,z) - \log(1+bdf\,z) - \log(1+cde\,z) \right),$$

$$(k \in \mathbf{Z}). \quad (1.1)$$

Hence $dU/dz = \pi k \sqrt{-1}/z$ is equivalent to the following equation:

$$\log(1-z) + \log(1-abde\,z) + \log(1-acdf\,z) + \log(1-bcef\,z) =$$

$$\log(1+abc\,z) + \log(1+aef\,z) + \log(1+bdf\,z) + \log(1+cde\,z)$$

$$+ 2\pi\sqrt{-1}\,k. \quad (1.2)$$

Any solution of the above equation must be a solution of the following equation.

$$(1-z)(1-abde\,z)(1-acdf\,z)(1-bcef\,z)$$

$$- (1+abc\,z)(1+aef\,z)(1+bdf\,z)(1+cde\,z) = 0. \quad (1.3)$$

The constant term is equal to 0. We put

$$h(z) = -\frac{1}{z} \left((1-z)(1-abde\,z)(1-acdf\,z)(1-bcef\,z) - \right.$$

$$\left. (1+abc\,z)(1+aef\,z)(1+bdf\,z)(1+cde\,z) \right) \quad (1.4)$$

Then the equation

$$h(z) = 0 \qquad\qquad (1.5)$$

is a quadratic equation. Let α, β, γ be the coefficient of $h(z)$ of degrees 0, 1, 2 respectively. Then

$$\alpha = 1 + abde + acdf + bcef + abc + aef + bdf + cde,$$

$$\beta = -abcdef \left((a - \frac{1}{a})(d - \frac{1}{d}) + (b - \frac{1}{b})(e - \frac{1}{e}) + (c - \frac{1}{c})(f - \frac{1}{f}) \right),$$

$$\gamma = abcdef\,(abcdef + ad + be + cf + abf + ace + bcd + def).$$

Note that

$$\gamma = (a\,b\,c\,d\,e\,f)^2\,\overline{\alpha},$$

$$\frac{\beta}{a\,b\,c\,d\,e\,f} = \text{a real number,} \qquad (1.6)$$

since the absolute values of a, b, $\cdots$, f are all equal to 1, and so $\dfrac{\beta^2}{\alpha\,\gamma}$ is a non-negative real number.

Lemma. *Let A, B, C, D, E, F be the dihedral angles of a hyperbolic tetrahedron as in Figure 2. Let z_1, z_2 be the solutions of the equation $h(z) = 0$. Then*

$$|z_1| = |z_2| = 1.$$

Proof. Let $\mathrm{Gram}(T)$ be the Gram matrix of T defined by

$$\mathrm{Gram}(T) = \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}. \qquad (1.7)$$

Since T is a tetrahedron realized in a hyperbolic space,

$$\det \mathrm{Gram}(T) < 0. \qquad (1.8)$$

Putting $a = \exp\sqrt{-1}\,A$, $b = \exp\sqrt{-1}\,B$ and so on,

$$\mathrm{Gram}(T) =$$
$$\begin{pmatrix} 1 & -(a+a^{-1})/2 & -(b+b^{-1})/2 & -(f+f^{-1})/2 \\ -(a+a^{-1})/2 & 1 & -(c+c^{-1})/2 & -(e+e^{-1})/2 \\ -(b+b^{-1})/2 & -(c+c^{-1})/2 & 1 & -(d+d^{-1})/2 \\ -(f+f^{-1})/2 & -(e+e^{-1})/2 & -(d+d^{-1})/2 & 1 \end{pmatrix}. \qquad (1.9)$$

Let D be the discriminant of the equation $h(z) = 0$, i.e. $D = \beta^2 - 4\,\alpha\,\gamma$. Let β_1 be the real number defined by

$$\beta_1 = \frac{\beta}{a\,b\,c\,d\,e\,f}.$$

Then

$$D = (a\,b\,c\,d\,e\,f)^2\,\left(\beta_1{}^2 - 4\,|\alpha|^2\right)$$

An actual computation shows that

$$\frac{D}{16\,(a\,b\,c\,d\,e\,f)^2} = \det \mathrm{Gram}(T). \tag{1.10}$$

This and (1.8) implies that

$$\beta_1^{\,2} - 4\,|\alpha|^2 < 0. \tag{1.11}$$

The solutions z_1, z_2 are given by

$$z_1, z_2 = \frac{-\beta \pm \sqrt{\beta^2 - 4\,\alpha\,\gamma}}{2\,\gamma} = a\,b\,c\,d\,e\,f\,\frac{-\beta_1 \pm \sqrt{\beta_1^{\,2} - 4\,|\alpha|^2}}{2\,\gamma},$$

and so (1.11) implies that

$$|z_1|^2 = |z_2|^2 = \left|\frac{\alpha}{\gamma}\right|^2.$$

Since

$$|\alpha/\gamma| = \left|\frac{1 + a\,b\,c + a\,b\,d\,e + a\,c\,d\,f + a\,e\,f + b\,c\,e\,f + b\,d\,f + c\,d\,e}{1 + \overline{a\,b\,c} + \overline{a\,b\,d\,e} + \overline{a\,c\,d\,f} + \overline{a\,e\,f} + \overline{b\,c\,e\,f} + \overline{b\,d\,f} + \overline{c\,d\,e}}\right| = 1,$$

we have $|z_1| = |z_2| = 1$.   q.e.d.

1.2. Lobachevsky function.

The Lobachevsky function $\Lambda(x)$ is defined for real x by the following integral.

$$\Lambda(x) = -\int_0^x \log |2 \sin t|\, dt. \tag{1.12}$$

Note that $\Lambda(x)$ is a periodic function with period π. It is known (see e.g. [4]) that

$$\mathrm{Im}\,\mathrm{Li}_2(\exp \sqrt{-1}x) = 2\,\Lambda(\frac{x}{2}). \tag{1.13}$$

From the remark at the last subsection, we can rewrite $V(T)$, $V_1(T)$ and $V_2(T)$ for a hyperbolic tetrahedron T as follows. Let $Z_1 = \arg z_1$, $Z_2 = \arg z_2$, and $W_1 = A+B+C-\pi$, $W_2 = A+E+F-\pi$, $W_3 = B+D+F-\pi$,

$W_4 = C + D + E - \pi$ be the halves of the solid angles at the four vertices. Let

$$U_\Lambda(T, Z) = \Lambda(\frac{Z}{2}) +$$
$$\Lambda(\frac{Z + W_1 + W_4}{2} - C) + \Lambda(\frac{Z + W_1 + W_3}{2} - B) + \Lambda(\frac{Z + W_1 + W_2}{2} - A)$$
$$- \Lambda(\frac{Z + W_1}{2}) - \Lambda(\frac{Z + W_2}{2}) - \Lambda(\frac{Z + W_3}{2}) - \Lambda(\frac{Z + W_4}{2}), \quad (1.14)$$

$$\tilde{\Delta}_\Lambda(W, A, B, C) = \frac{\Lambda(W) - \Lambda(W - A) - \Lambda(W - B) - \Lambda(W - C)}{2}, \quad (1.15)$$

$$\Delta_\Lambda(T) = \tilde{\Delta}_\Lambda(\frac{W_1}{2}, A, B, C) + \tilde{\Delta}_\Lambda(\frac{W_2}{2}, A, E, F) +$$
$$\tilde{\Delta}_\Lambda(\frac{W_3}{2}, B, D, F) + \tilde{\Delta}_\Lambda(\frac{W_4}{2}, C, D, E), \quad (1.16)$$

$$V_{1,\Lambda}(T) = U_\Lambda(T, Z_1) + \Delta_\Lambda(T), \quad (1.17)$$

$$V_{2,\Lambda}(T) = U_\Lambda(T, Z_2) + \Delta_\Lambda(T). \quad (1.18)$$

$$V_\Lambda(T) = \frac{U_\Lambda(T, Z_1) - U_\Lambda(T, Z_2)}{2} \quad (1.19)$$

Then all the absolute values of $V_\Lambda(T)$, $V_{1,\Lambda}(T)$ and $V_{2,\Lambda}(T)$ coincide with the volume $\text{Vol}(T)$.

2. Volume of a tetrahedron with some ideal vertices.

In this section, the case that some of the vertices of T is an ideal one, i.e. some of them are located at infinity.

2.1. Tetrahedron with one ideal vertex.

Let v be such vertex and assume that v is the end point of the edges corresponding to A, B, C. In this case, A, B and C satisfy

$$A + B + C = \pi \quad \text{(i.e. } W_1 = 0) \quad (2.1)$$

Let parameters a, b, c, d, e and f be as before. Then (2.1) implies

$$a\,b\,c = \exp\sqrt{-1}\,\pi = -1.$$

Therefore, one of the solution of (1.5), say z_1, is equal to 1. Obtain the volume of T by (1.17).

$$U_\Lambda(T, 0) =$$
$$\Lambda(\frac{W_2}{2} - A) + \Lambda(\frac{W_3}{2} - B) + \Lambda(\frac{W_4}{2} - C) - \Lambda(\frac{W_2}{2}) - \Lambda(\frac{W_3}{2}) - \Lambda(\frac{W_4}{2}). \tag{2.2}$$

Hence we get

$$V_{1,\Lambda}(T) = \frac{1}{2} \left(\Lambda(A) + \Lambda(B) + \Lambda(C) \right.$$
$$+ \Lambda(\frac{W_2}{2} - A) - \Lambda(\frac{W_2}{2} - E) - \Lambda(\frac{W_2}{2} - F) - \Lambda(\frac{W_2}{2})$$
$$+ \Lambda(\frac{W_3}{2} - B) - \Lambda(\frac{W_3}{2} - D) - \Lambda(\frac{W_3}{2} - F) - \Lambda(\frac{W_3}{2})$$
$$\left. + \Lambda(\frac{W_4}{2} - C) - \Lambda(\frac{W_4}{2} - D) - \Lambda(\frac{W_4}{2} - E) - \Lambda(\frac{W_4}{2}) \right). \tag{2.3}$$

2.2. Tetrahedron with two ideal vertices.

Now consider the case that there are two ideal vertices. Assume that $W_1 = W_2 = 0$. This case, we have

$$V_{1,\Lambda}(T) = \frac{1}{2} \left(\Lambda(B) + \Lambda(C) + \Lambda(E) + \Lambda(F) \right.$$
$$+ \Lambda(\frac{W_3}{2} - B) - \Lambda(\frac{W_3}{2} - D) - \Lambda(\frac{W_3}{2} - F) - \Lambda(\frac{W_3}{2})$$
$$\left. + \Lambda(\frac{W_4}{2} - C) - \Lambda(\frac{W_4}{2} - D) - \Lambda(\frac{W_4}{2} - E) - \Lambda(\frac{W_4}{2}) \right). \tag{2.4}$$

2.3. Tetrahedron with three ideal vertices.

Consider the case that there are three ideal vertices. Assume that $W_1 = W_2 = W_3 = 0$. This case, we have

$$V_{1,\Lambda}(T) = \frac{1}{2} \left(\Lambda(C) + \Lambda(D) + \Lambda(E) \right.$$
$$\left. - \Lambda(\frac{W_4}{2} - C) - \Lambda(\frac{W_4}{2} - D) - \Lambda(\frac{W_4}{2} - E) - \Lambda(\frac{W_4}{2}) \right). \tag{2.5}$$

This coinsides with the formula (43) in [12].

2.4. Tetrahedron with four ideal vertices.

This case, the tetrahedron T is an ideal tetrahedron and $W_1 = W_2 = W_3 = W_4 = 0$, $D = A$, $E = B$, $F = C$. Therefore

$$V_{1,\Lambda}(T) = \Lambda(A) + \Lambda(B) + \Lambda(C). \tag{2.6}$$

This coincides with the well known formula of the volume of an ideal tetrahedron.

Remark. Let T be a tetrahedra with at least one ideal vertex. Then the last formula (2.6) suggest that $V_{1,\Lambda}(T)$ with $z_1 = 1$ is positive and equal to the volume of T. For the general tetrahedron, let the solution z_1 of (0.3) be a deformation of $z_1 = 1$ of the above case. Then $\operatorname{Im} V_1(T)$, $\operatorname{Im} V(T)$, $V_\Lambda(T)$ and $V_{1,\Lambda}(T)$ are positive and equal to the volume of T.

3. Proofs.

3.1. The formula by Cho-Kim.

A formula of the volume of a generic hyperbolic tetrahedron is given by Cho-Kim [1]. Let A, B, C, D, E, F denote the dihedral angles of T as before, and let $(P_1, Q_1, R_1, S_1, T_1)$ and $(P_2, Q_2, R_2, S_2, T_2)$ be the solutions of the following system of equations with respect to the variables P, Q, R, S, T.

$$P + Q = B, \ \ R + S = E, \ \ Q + R + T = F + \pi, \ \ P + S + T = C + \pi, \tag{3.1}$$

$$\begin{vmatrix} 1 & -\cos D & -\cos P & \cos B & \cos C \\ -\cos D & 1 & \cos(R+T) & \cos F & \cos E \\ -\cos P & \cos(R+T) & 1 & -\cos Q & \cos(S+T) \\ \cos B & \cos F & -\cos Q & 1 & -\cos A \\ \cos C & \cos E & \cos(S+T) & -\cos A & 1 \end{vmatrix} = 0. \tag{3.2}$$

This system can be reduced to a quadratic equation and there are two solutions.

Theorem 5. (Cho-Kim [1]) *The twice of the volume of T is given as*

follows.

$$2\,\mathrm{Vol}(T) = \Lambda(P_1) - \Lambda(Q_1) + \Lambda(R_1) - \Lambda(S_1)$$
$$- \Lambda\left(\frac{B-C-A+\pi}{2} - Q_1\right) + \Lambda\left(\frac{D-B-F+\pi}{2} + Q_1\right)$$
$$+ \Lambda\left(\frac{E-C-D+\pi}{2} - R_1\right) - \Lambda\left(\frac{A-E-F+\pi}{2} + R_1\right)$$
$$- \Lambda(P_2) + \Lambda(Q_2) - \Lambda(R_2) + \Lambda(S_2)$$
$$+ \Lambda\left(\frac{B-C-A+\pi}{2} - Q_2\right) - \Lambda\left(\frac{D-B-F+\pi}{2} + Q_2\right)$$
$$- \Lambda\left(\frac{E-C-D+\pi}{2} - R_2\right) + \Lambda\left(\frac{A-E-F+\pi}{2} + R_2\right). \quad (3.3)$$

The solutions $(P_1, \cdots)$ and $(P_2, \cdots)$ of the solutions of (3.1) and (3.2) are chosen so that the value of (3.3) is positive.

3.2. Discriminant of the quadratic equation.

From (3.1), we have

$$Q = B - P, \quad R = (-B - C + E + F)/2 + P,$$
$$S = (B + C + E - F)/2 - P, \quad T = (-B + C - E + F)/2 + \pi. \quad (3.4)$$

Let $a = \exp\sqrt{-1}\,A$, $b = \exp\sqrt{-1}\,B$, $c = \exp\sqrt{-1}\,C$, $d = \exp\sqrt{-1}\,D$, $e = \exp\sqrt{-1}\,E$, $f = \exp\sqrt{-1}\,F$ and $p = \exp\sqrt{-1}\,P$. Then (3.2) is reformulated as follows.

$$\begin{vmatrix} 1 & -\frac{1}{2}(d+\frac{1}{d}) & -\frac{1}{2}(p+\frac{1}{p}) & \frac{1}{2}(b+\frac{1}{b}) & \frac{1}{2}(c+\frac{1}{c}) \\ -\frac{1}{2}(d+\frac{1}{d}) & 1 & -\frac{1}{2}(\frac{fp}{b}+\frac{b}{fp}) & \frac{1}{2}(f+\frac{1}{f}) & \frac{1}{2}(e+\frac{1}{e}) \\ -\frac{1}{2}(p+\frac{1}{p}) & -\frac{1}{2}(\frac{fp}{b}+\frac{b}{fp}) & 1 & -\frac{1}{2}(\frac{p}{b}+\frac{b}{p}) & -\frac{1}{2}(\frac{p}{c}+\frac{c}{p}) \\ \frac{1}{2}(b+\frac{1}{b}) & \frac{1}{2}(f+\frac{1}{f}) & -\frac{1}{2}(\frac{p}{b}+\frac{b}{p}) & 1 & -\frac{1}{2}(a+\frac{1}{a}) \\ \frac{1}{2}(c+\frac{1}{c}) & \frac{1}{2}(e+\frac{1}{e}) & -\frac{1}{2}(\frac{p}{c}+\frac{c}{p}) & -\frac{1}{2}(a+\frac{1}{a}) & 1 \end{vmatrix}$$
$$= 0. \quad (3.5)$$

Multiplying p^2, this equation becomes a quadratic equation with respect to p^2, and we denote this equation as

$$g(p^2) = 0 \quad (3.6)$$

with a quadratic polynomial $g(x)$. Let D_g be the discriminant of $g(x)$. Let

$$g_1(x) = \frac{16\,a^2\,b^4\,c^2\,d^2\,e\,f^2\,g(x)}{(a\,b+c)\,(b+a\,c)\,(b\,d+f)\,(b+d\,f)}.$$

Then $g_1(x)$ is also a polynomial. Let D_g be the discriminant of $g_1(x) = 0$. An actual computation shows that

$$D_{g_1} = 16\, a^2\, b^4\, c^2\, d^2\, e^2\, f^2\, \det \mathrm{Gram}(T).\tag{3.7}$$

Noting (1.10), the quadratic equations (3.6) and(1.5) have similar discriminants.

3.3. Proof of Theorem 1.

To prove Theorem 1, we first investigate the derivation of (3.3) with respect to A. Let

$$h(x,T) = \mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{x}{b^2}\right) - \mathrm{Li}_2\left(-\frac{x}{a\,b\,c}\right) - \mathrm{Li}_2\left(-\frac{x\,a}{b\,c}\right)$$
$$- \mathrm{Li}_2\left(-\frac{x\,f}{b\,d}\right) - \mathrm{Li}_2\left(-\frac{x\,d\,f}{b}\right) + \mathrm{Li}_2\left(\frac{x\,f}{b\,c\,e}\right) + \mathrm{Li}_2\left(\frac{x\,e\,f}{b\,c}\right).\tag{3.8}$$

Lemma *Let x_1, x_2 be the non-trivial two solutions of*

$$\frac{d\,h(x,T)}{dx} = \frac{2\,\pi\,\sqrt{-1}}{x}\,k, \qquad (k \in \mathbf{Z}).\tag{3.9}$$

Then,

$$\mathrm{Vol}(T) = \frac{\mathrm{Im}\,h(x_1,T) - \mathrm{Im}\,h(x_2,T)}{4}.\tag{3.10}$$

Here x_1 and x_2 are chosen so that the value of the above formula is positive.

Proof. The equation (3.9) is reduced to the following quadratic equation with respect to x.

$$\frac{1}{x}\left((1-x)(1-\frac{x}{b^2})(1-\frac{xf}{bce})(1-\frac{xef}{bc}) - \right.$$
$$\left. (1+\frac{x}{abc})(1+\frac{xa}{bc})(1+\frac{xf}{bd})(1+\frac{xdf}{b})\right) = 0.\tag{3.11}$$

An actual computation shows that this equation is a multiple of the equation (3.6) where x corresponds to p^2. By using the relation (1.13) and $\Lambda(-x) = -\Lambda(x)$, it is proved that a half of the right hand side of (3.10) is a half of the right hand side of (3.3). q.e.d.

Proof of Theorem 1. First, we show that the imaginary part of the derivations of $V(T) = (U(z_1, T) - U(z_2, T))/2$ and $(h(x_1, T) - h(x_2, T))/4$ with respect to every dihedral angle of T is equal. Since $V(T)$ is symmetric with respect to the six angles A, B, C, D, E, F, it is enough to show for one parameter, say A.

Since z_1, z_2 in (0.6) are solutions of (0.3) and $|z_1| = |z_2| = 1$,

$$\frac{\partial U(z_i, T)}{\partial A} = \frac{\partial U(z_i, T)}{\partial a} \frac{d\,a}{dA}$$

$$= -\frac{1}{2\,a} \left(\log(1 - z_i\,a\,b\,d\,e) + \log(1 - z_i\,a\,c\,d\,f) - \right.$$

$$\left. \log(1 + z_i\,a\,b\,c) - \log(1 + z_i\,a\,e\,f) \right) \sqrt{-1}\,a + \frac{\partial U(z, T)}{\partial z} \frac{\partial z_i}{\partial a} \frac{d\,a}{dA}$$

$$= -\frac{\sqrt{-1}}{2} \left(\log(1 - z_i\,a\,b\,d\,e) + \log(1 - z_i\,a\,c\,d\,f) - \log(1 + z_i\,a\,b\,c) - \right.$$

$$\left. \log(1 + z_i\,a\,e\,f) \right) + \frac{\pi\,\sqrt{-1}\,k_i}{z_i} \frac{\partial z_i}{\partial A} \quad \left(\frac{\pi\,\sqrt{-1}\,k_i}{z_i} = \left. \frac{d\,U(z, T)}{dz} \right|_{z = z_i} \right)$$

$$= -\frac{\sqrt{-1}}{2} \log \frac{(1 - z_i\,a\,b\,d\,e)\,(1 - z_i\,a\,c\,d\,f)}{(1 + z_i\,a\,b\,c)\,(1 + z_i\,a\,e\,f)} - \pi\,k_i\,\alpha_i,$$

$$\left(\alpha_i = \frac{\partial \arg z_i}{\partial A} \in \mathbf{R} \right)$$

$$\text{(3.12)}$$

for $i = 1$, 2. Similarly, since x_1 and x_2 in (3.10) are solutions of (3.9), we have

$$\frac{\partial h(x_i, T)}{\partial A} = \frac{\partial h(x_i, T)}{\partial a} \frac{d\,a}{dA} = \sqrt{-1} \log \frac{1 + \frac{a\,x_i}{b\,c}}{1 + \frac{x_i}{a\,b\,c}} - \pi\,k_i'\,\alpha_i',$$

$$\left(\frac{2\,\pi\,\sqrt{-1}\,k_i'}{x_i} = \left. \frac{d\,h(x, T)}{dx} \right|_{x = x_i}, \quad \alpha_i' = 2 \frac{\partial \arg x_i}{\partial A} \in \mathbf{R} \right) \quad \text{(3.13)}$$

for $i = 1$, 2. Then, an actual computation shows that

$$\frac{(1 + z_1\,a\,b\,c)\,(1 + z_1\,a\,e\,f)\,(1 - z_2\,a\,b\,d\,e)\,(1 - z_2\,a\,c\,d\,f)}{(1 - z_1\,a\,b\,d\,e)\,(1 - z_1\,a\,c\,d\,f)\,(1 + z_2\,a\,b\,c)\,(1 + z_2\,a\,e\,f)} =$$

$$\frac{(1 + \frac{a\,x_1}{b\,c})\,(1 + \frac{x_2}{a\,b\,c})}{(1 + \frac{x_1}{a\,b\,c})\,(1 + \frac{a\,x_2}{b\,c})}, \quad \text{(3.14)}$$

for a suitable choice of z_1, z_2 and x_1, x_2. This identity implies that

$$\frac{\partial V(T)}{\partial A} = \frac{1}{4} \frac{\partial\,(h(x_1, T) - h(x_2, T))}{\partial A} + \alpha, \quad \text{(3.15)}$$

for some real number α and so we get

$$\operatorname{Im} \frac{\partial V(T)}{\partial A} = \frac{\partial}{\partial A} \operatorname{Vol}(T). \tag{3.16}$$

This implies that the difference $\operatorname{Im} V(T) - \operatorname{Vol}(T)$ is a constant $\mathcal{C}$. On the other hand, the functions $\operatorname{Im} V(T)$ and $\operatorname{Vol}(T)$ are both 0 if the determinant of the Gram matrix is 0, and they are continuous with respect to the parameters A, B, $\cdots$, F corresponding to hyperbolic and degenerate tetrahedra. Hence the constant $\mathcal{C}$ should be 0 and we get (0.7). q.e.d.

3.4. Proof of Theorem 2.

To prove Theorem 2, we show the following identity.

$$\operatorname{Im} \left(U(z_1, T) + U(z_2, T) + 2\,\Delta(T) \right) = 0. \tag{3.17}$$

To do this, we first show that

$$\operatorname{Im} \frac{\partial}{\partial A} \left(U(z_1, T) + U(z_2, T) + 2\,\Delta(T) \right) = 0. \tag{3.18}$$

To show (3.18), we prove that

$$\operatorname{Im} \exp 2\,a\, \frac{\partial}{\partial A} \left(U(z_1, T) + U(z_2, T) + 2\,\Delta(T) \right) = 1. \tag{3.19}$$

Since z_1 and z_2 are solutions of (1.5) and so they satisfy

$$(1 - z_1\, x)(1 - z_2\, x) = x^2\, h(x^{-1})/\gamma, \tag{3.20}$$

for any x where $h(z)$ is the quadratic polynomial introduced in (1.4) and α is the coefficient of the degree two term of $h(z)$. From this relation, we have

$$(1 - z_1\,a\,b\,d\,e)(1 - z_2\,a\,b\,d\,e) =$$
$$\frac{(a\,b\,d\,e)^3\,(1 + c\,d^{-1}\,e^{-1})\,(1 + b^{-1}\,d^{-1}\,f)\,(1 + a^{-1}\,e^{-1}\,f)\,(1 + a^{-1}\,b^{-1}\,c)}{\gamma},$$

$$(1 - z_1\,a\,c\,d\,f)(1 - z_2\,a\,c\,d\,f) =$$
$$\frac{(a\,c\,d\,f)^3\,(1 + b\,d^{-1}\,f^{-1})\,(1 + c^{-1}\,d^{-1}\,e)\,(1 + a^{-1}\,b\,c^{-1})\,(1 + a^{-1}\,e\,f^{-1})}{\gamma},$$

$$(1 + z_1\,a\,b\,c)(1 + z_2\,a\,b\,c) =$$
$$\frac{(a\,b\,c)^3\,(1 + a^{-1}\,b^{-1}\,c^{-1})(1 + c^{-1}\,d\,e)(1 + b^{-1}\,d\,f)(1 + a^{-1}\,e\,f)}{\gamma},$$

$$(1 + z_1\,a\,e\,f)(1 + z_2\,a\,e\,f) =$$
$$\frac{(a\,e\,f)^3\,(1 + a^{-1}\,e^{-1}\,f^{-1})\,(1 + b\,d\,f^{-1})\,(1 + c\,d\,e^{-1})\,(1 + a^{-1}\,b\,c)}{\gamma}.$$

$$(3.21)$$

Using the above and (3.12), we get

$$\frac{\partial}{\partial A}\,(U(z_1, T) + U(z_2, T) + 2\,\Delta(T))$$

$$= \frac{\sqrt{-1}}{2}\,\log\left(\frac{(1 + z_1\,a\,b\,c)\,(1 + z_1\,a\,e\,f)\,(1 + z_2\,a\,b\,c)\,(1 + z_2\,a\,e\,f)}{(1 - z_1 abde)\,(1 - z_1 acdf)\,(1 - z_2 abde)\,(1 - z_2 acdf)}\times\right.$$

$$\frac{(1 + a\,b\,c^{-1})(1 + a\,b^{-1}\,c)}{(1 + a^{-1}bc)(1 + a^{-1}b^{-1}c^{-1})}\times\frac{(1 + a\,e\,f^{-1})(1 + a\,e^{-1}\,f)}{(1 + a^{-1}ef)(1 + a^{-1}e^{-1}f^{-1})}\times\left.\frac{d^2}{a^4}\right)$$

$$-\,\pi\,k_1\,\alpha_1 - \pi\,k_2\,\alpha_2$$

$$= \frac{\sqrt{-1}}{2}\,\log\left(\frac{(1 + a^{-1}\,b^{-1}\,c^{-1})\,(1 + c^{-1}\,d\,e)\,(1 + b^{-1}\,d\,f)\,(1 + a^{-1}\,e\,f)}{(1 + cd^{-1}e^{-1})(1 + b^{-1}d^{-1}f)(1 + a^{-1}e^{-1}f)(1 + a^{-1}b^{-1}c)}\times\right.$$

$$\frac{(1 + a^{-1}\,e^{-1}\,f^{-1})\,(1 + b\,d\,f^{-1})\,(1 + c\,d\,e^{-1})\,(1 + a^{-1}\,b\,c)}{(1 + b\,d^{-1}\,f^{-1})\,(1 + c^{-1}\,d^{-1}\,e)\,(1 + a^{-1}\,b\,c^{-1})\,(1 + a^{-1}\,e\,f^{-1})}\times$$

$$\frac{(1 + a\,b\,c^{-1})\,(1 + a\,b^{-1}\,c)}{(1 + a^{-1}\,b\,c)(1 + a^{-1}\,b^{-1}\,c^{-1})}\times\frac{(1 + a\,e\,f^{-1})(1 + a\,e^{-1}\,f)}{(1 + a^{-1}ef)(1 + a^{-1}e^{-1}f^{-1})}\times\left.\frac{d^2}{a^4}\right)$$

$$-\,\pi\,k_1\,\alpha_1 - \pi\,k_2\,\alpha_2$$

$$= \frac{\sqrt{-1}}{2}\,\log\left(\frac{1}{a^4\,d^4}\,(c^{-1}\,d\,e)\,(b^{-1}\,d\,f)\,(b\,d\,f^{-1})\,(c\,d\,e^{-1})\times\right.$$

$$(a\,b\,c^{-1})\,(a\,b^{-1}\,c)\,(a\,e\,f^{-1})\,(a\,e^{-1}\,f)\big)\big) - \pi\,k_1\,\alpha_1 - \pi\,k_2\,\alpha_2$$

$$= -\pi\,k_1\,\alpha_1 - \pi\,k_2\,\alpha_2.$$

$$(3.22)$$

Hence

$$\mathrm{Im}\,\frac{\partial}{\partial A}\,(U(z_1, T) + U(z_2, T) + 2\,\Delta(T)) = 0.$$

The derivation with respect to the other parameters B, C, D, E, F also vanish and so we get

$$\mathrm{Im}\,(U(z_1, T) + U(z_2, T) + 2\,\Delta(T)) = \mathcal{C}'. \qquad (3.23)$$

The function $\mathrm{Im}\,(U(z_1, T) + U(z_2, T) + 2\,\Delta(T))$ is continuous for hyperbolic tetrahedra including ideal tetrahedra, and the constant $\mathcal{C}'$ is 0 for ideal tetrahedra by (2.6), we get (0.8).   q.e.d.

3.5. Proof of Theorem 3.

Let T_t, $t \in [0, 1]$ be tetrahedra whose dihedral angles continuously depend on the parameter t. Assume that, if $t > 0$ then T_t is a hyperbolic tetrahedron, and T_0 is a Euclidean tetrahedron, i.e. $\mathrm{rank}\,\mathrm{Gram}(T_0) = 3$ and $\mathrm{Gram}(T_0)$ is positive semidefinite. Let W be a neighborhood of 0 in $[0, 1]$. Let $z_1(t)$ and z_2 be the equation

$$\frac{dU(z, T_{t_0})}{dz} = \frac{\pi\,\sqrt{-1}}{z}\,k, \qquad (k \in \mathbf{Z})$$

and let k_1 and k_2 satisfy

$$\left.\frac{dU}{dz}\right|_{z=z_i} = \frac{\pi\,\sqrt{-1}}{z_i}\,k_i.$$

We assume that W is small enough so that k_i is constant for all $t \in W$.

Now let $\tilde{V}(T)$ be the branch of $V(T)$ which is the analytic continuation of

$$\tilde{V}(T_t) =$$
$$\frac{\left\{U(z_1(t), T_t) - \pi k_1 \sqrt{-1}\log z_1(t)\right\} - \left\{U(z_2(t), T_t) - \pi k_2 \sqrt{-1}\log z_2(t)\right\}}{2}.$$
$$(3.24)$$

Note that $\mathrm{Im}\,V(T) = \mathrm{Im}\,\tilde{V}(T)$ since $|z_i| = 1$.

For $t \in W$, (3.12) implies that

$$\frac{\partial \tilde{V}(T_t)}{\partial A} =$$
$$-\frac{\sqrt{-1}}{4}\,\log\frac{(1 - z_1\,a\,b\,d\,e)\,(1 - z_1\,a\,c\,d\,f)\,(1 + z_2\,a\,b\,c)\,(1 + z_2\,a\,e\,f)}{(1 + z_1\,a\,b\,c)\,(1 + z_1\,a\,e\,f)\,(1 - z_2\,a\,b\,d\,e)\,(1 - z_2\,a\,c\,d\,f)}, \qquad (3.25)$$

where $\alpha_i = \dfrac{\partial \arg z_i}{\partial A} \in \mathbf{R}$ for $i = 1, 2$. Since $\arg(1 - z) = \dfrac{\arg z - \pi}{2}$ and $\arg(1 + z) = \dfrac{\arg z}{2}$,

$$\arg \frac{(1 - z_1\, a\, b\, d\, e)\,(1 - z_1\, a\, c\, d\, f)\,(1 + z_2\, a\, b\, c)\,(1 + z_2\, a\, e\, f)}{(1 + z_1\, a\, b\, c)\,(1 + z_1\, a\, e\, f)\,(1 - z_2\, a\, b\, d\, e)\,(1 - z_2\, a\, c\, d\, f)} = 0,$$

and so

$$\operatorname{Re} \frac{\partial \tilde{V}(T_t)}{\partial A} = 0.$$

Hence $\operatorname{Re} \dfrac{\partial \tilde{V}(T)}{\partial A} = 0$ for any hyperbolic tetrahedron.   q.e.d.

3.6. Proof of Theorem 4.

Theorem 4 comes from the following result in [12].

Theorem 6. (Vinberg [12]) There is an analytic function ϕ defined on some open set of $\mathbf{C}^6$ corresponding to the six dihedral angles of a tetrahedron such that

$$\phi(T) = \begin{cases} \operatorname{Vol}(T) & (T \text{ is a hyperbolic tetrahedron}), \\ 0 & (T \text{ is an Euclidean tetrahedron}), \\ \sqrt{-1}\,\operatorname{Vol}(T) & (T \text{ is a tetrahedron in } S^3). \end{cases}$$

4. Relation to the quantum $6j$-symbol.

In this section, we explain how we derive our volume formula from the quantum $6j$-symbol.

4.1. Quantum $6j$-symbol.

Let N be a integer with $N \geq 3$. Let $I = \{0, 1/2, 1, 3/2, 2, \cdots, (N - 3)/2, (N - 2)/2\}$. Let i, j, k, l, m, n be six elements of I corresponding to the edges of a tetrahedron as in Figure 1. For these parameters, the quantum $6j$-symbol $\begin{Bmatrix} i & j & k \\ \ell & m & n \end{Bmatrix}$ is given as follows.

Let $q = \exp 2\pi \sqrt{-1}/N$. For a non-negative integer n, let

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}},$$

and

$$[n]! = [n][n-1][n-2]\cdots[2][1].$$

Three elements (a, b, c) of I is called *admissible triple* if $|a - b| \le c \le a + b$ and $a + b + c$ is a integer less than $N - 1$. For i, j, k, l, m, n such that (i, j, k), (i, m, n), (j, l, n), and (k, l, m) are all admissible triples, let

$$\begin{Bmatrix} i & j & k \\ \ell & m & n \end{Bmatrix} = \Delta(i,j,k)\Delta(i,m,n)\Delta(j,l,n)\Delta(k,l,m)\times$$

$$\sum_s (-1)^s [s+1]! \times$$

$$\{[s-i-j-k]![s-i-m-n]![s-j-l-m]![s-k-l-m]! \times$$

$$[i+j+l+m-s]![i+k+l+n-s]![j+k+m+n-s]!\}^{-1}. \quad (4.1)$$

Here the sum $\sum\limits_s$ runs over all integers s satisfying

$$s \le \min\{i+j+l+m, i+k+l+n, j+k+m+n\},$$
$$s \ge \max\{i+j+k, i+m+n, J+l+m, k+l+m\},$$

and

$$\Delta(i,j,k) = \left(\frac{[i+j-k]![i-j+k]![-i+j+k]!}{[i+j+k+1]!} \right)^{\frac{1}{2}}.$$

The quantum $6j$-symbol defined the above is a symmetrized version with respect to the symmetry of the tetrahedron.

4.2. Large N limit.

Let x_N be a sequence of integers such that $\dfrac{2\pi}{N} x_N \sim \dfrac{x}{2}$ $(N \to \infty)$. Then, by (1.12) and (1.13), we have

$$[x_n]! \sim \exp\left(-\frac{N}{\pi}\Lambda\left(\frac{x}{2}\right) \right) = \exp(-\frac{N}{2\pi}\operatorname{Im}\operatorname{Li}_2(\exp\sqrt{-1}\,x))$$

$$= \exp(\frac{N}{2\pi}\operatorname{Im}\operatorname{Li}_2(\exp-\sqrt{-1}\,x)). \quad (4.2)$$

Let i_N, j_N, k_N, l_N, m_N and n_N be sequences of half integers such that $\begin{Bmatrix} i_N & j_N & k_N \\ l_N & m_N & n_N \end{Bmatrix}$ is defined and $\dfrac{i_N}{N} \sim x_i$, $\dfrac{j_N}{N} \sim x_j$, $\dfrac{k_N}{N} \sim x_k$, $\dfrac{l_N}{N} \sim x_l$, $\dfrac{m_N}{N} \sim x_m$, $\dfrac{n_N}{N} \sim x_n$. Let $a = \exp\sqrt{-1}\,(x_i - \pi)$, $b = \exp\sqrt{-1}\,(x_j - \pi)$,

$c = \exp\sqrt{-1}\,(x_k - \pi)$, $d = \exp\sqrt{-1}\,(x_l - \pi)$, $e = \exp\sqrt{-1}\,(x_m - \pi)$, and $f = \exp\sqrt{-1}\,(x_n - \pi)$, we have

$$\frac{\pi}{N}\log\left|\left\{\begin{matrix} i_N & j_N & k_N \\ l_N & m_N & n_N \end{matrix}\right\}\right| \sim$$

$$\frac{1}{2}\,\mathrm{Im}\,\{(L(a,b,c) + L(a,e,f) + L(b,d,f) + L(c,d,e)) +$$

$$\int_z (\mathrm{Li}_2(z) + \mathrm{Li}_2(zabde) + \mathrm{Li}_2(zacdf) + \mathrm{Li}_2(zbdef) -$$

$$\mathrm{Li}_2(-zabc) - \mathrm{Li}_2(-zaef) - \mathrm{Li}_2(-zbdf) - \mathrm{Li}_2(-zcde))\,dz\}\,,\quad (4.3)$$

where

$$L(a,b,c) = \frac{1}{2}\,\mathrm{Li}_2(-\frac{ab}{c}) + \mathrm{Li}_2(-\frac{bc}{a}) + \mathrm{Li}_2(-\frac{ca}{b}) - \mathrm{Li}_2(-abc).$$

The integral path of z correspond to the range of s in (4.1)

4.3. Saddle points.

From (4.3), we tried to find out the relation of the saddle point of the function

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(zabde) + \mathrm{Li}_2(zacdf) + \mathrm{Li}_2(zbdef) -$$

$$\mathrm{Li}_2(-zabc) - \mathrm{Li}_2(-zaef) - \mathrm{Li}_2(-zbdf) - \mathrm{Li}_2(-zcde),\quad (4.4)$$

with respect to the parameter z. The saddle point means the point that the derivation with respect z vanish. We put this formula (4.4), as $U(z,T)$ in (0.1), and, after various numerical experimentations, we get our formula.

5. Discussion.

5.1. Orientation and mirror image.

Let T be a tetrahedron and let T' be its mirror image. Then we have

$$V(T) = V(T').$$

The function V is defined as a complex function, but it is pure imaginary for a hyperbolic tetrahedron and real for a spherical tetrahedron. On the other hand, we defined another functions V_1 and V_2 for the volume. Let $\tilde{V}_1$ and $\tilde{V}_2$ be the branch of V_1 and V_2 obtained similarly as $\tilde{V}$. Now assume that

V_1 corresponds to the solution z_1 which is equal to 1 when T has an ideal vertex. Then

$$\tilde{V}_1(T) - \tilde{V}_2(T) = 2\,\tilde{V}(T) = 2\,\mathrm{Vol}(T). \tag{5.1}$$

If T is hyperbolic, then $\mathrm{Re}\,\tilde{V}(T) = 0$ by Theorem 3 and so

$$\mathrm{Re}\,\tilde{V}_1(T) = \mathrm{Re}\,\tilde{V}_2(T). \tag{5.2}$$

Hence, if we assign $\tilde{V}_1(T)$ as the complexification of the volume of T, it may be natural to assign $-\tilde{V}_2(T')$ for the mirror image T' instead of $\tilde{V}_1(T)$ since $\mathrm{Re}(-\tilde{V}_2(T')) = -\mathrm{Re}\,\tilde{V}_1(T)$.

There are four candidates for the complexification of the volume function of a tetrahedron T; $\tilde{V}_1(T)$, $-\tilde{V}_1(T)$, $\tilde{V}_2(T)$, $-\tilde{V}_2(T)$. If the volume of T is assumed to be positive, there are two candidates; $\tilde{V}_1(T)$ and $-\tilde{V}_2(T)$. The natural way of this choice may be determined by the sign of the vertex orientation of T. Here, vertex orientation means the order of four vertices of T.

We would like to give one more remark. The real part $\mathrm{Re}\,\tilde{V}_1(T)$ may correspond to the scissors congruence invariant of hyperbolic polyhedra other than the volume. To define $\tilde{V}_1$ and $\tilde{V}_2$, we fix their branches. They are chosen so that their imaginary parts correspond to the volume. However, there is still some ambiguity for the choice of branch and we can fix the real part only up to modulo π^2.

5.2. Actual asymptotics of the quantum $6j$-symbols.

Our formula is obtained by considering the asymptotics of the quantum $6j$-symbol. Unfortunately, the integral path corresponding to the sum does not pass the saddle point of the function (4.4). Actually, it is known that the quantum $6j$-symbol is of polynomial growth with respect to N and we cannot apply the saddle point method. However, our result gives a hidden relation between the quantum $6j$-symbol and the volume of a tetrahedron.

5.3. Regge's symmetry.

The quantum $6j$-symbol is invariant under Regge's symmetry. This implies the following. Let T be a tetrahedron whose dihedral angles are A, B, C, D, E, F as before. Choose a pair of dihedral angles of opposite sides, say A, D. Let $L = (B + C + E + F)/2$. Let T' be the tetrahedron whose dihedral angles are A, $L - B$, $L - C$, D, $L - E$, $L - F$. Since this operation induces

a permutation of the terms of $U(z, T)$ and $\Delta(T)$, we have

$$V_1(T) = V_1(T'), \quad V_2(T) = V_2(T'), \quad V(T) = V(T'). \tag{5.3}$$

Hence, if these functions actually equal to the volume, we have

$$\mathrm{Vol}(T) = \mathrm{Vol}(T'). \tag{5.4}$$

5.4. Higher dimensional case.

The area of a hyperbolic triangle is determined by the sum of the three angles. The angle is given as a argument of a complex number, i.e. the imaginary part of its logarithm. The volume of a hyperbolic tetrahedron is given by dilogarithm functions of some complex values relating its dihedral angles. So it may be natural to seek a formula of the volume of a higher-dimensional simplex given by polylogarithm functions of certain numbers related to the simplex. In [3] and in the papers cited in it, such formulas are actually given for some special cases.

References.

[1] Y. Cho and H. Kim, *On the volume formula for hyperbolic tetrahedra,* Discrete Comput. Geom. **22** (1999), 347–366.

[2] R. M. Kashaev, *The hyperbolic volume of knots from quantum dilogarithm,* Modern Phys. Lett. A **10** (1995), 1409–1418.

[3] R. Kellerhals, *On volumes of non-Euclidean polytopes,* T. Bisztriczky et al. (eds.), POLYTOPES: Abstract, Convex and Computational, 1994 Kluwer Academic Publishers, pp. 231–239.

[4] A. N. Kirillov, Dilogarithm Identities, Lectures in Mathemical Sciences, The University of Tokyo (1995).

[5] Kirillov, A. N. and Reshetikhin, N. Yu. *Representations of the algebra $U_q(\mathrm{sl}(2))$, q-orthogonal polynomials and invariants of links,* Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), 285–339, Adv. Ser. Math. Phys., 7, World Sci. Publishing, Teaneck, NJ, 1989.

[6] H. Murakami, *Optimistic calculations about the Witten–Reshetikhin–Turaev invariants of closed three-manifolds obtained from the figure-eight knot by integral Dehn surgeries,* Recent progress towards the volume conjecture (Kyoto, 2000). Sūrikaisekikenkyūsho Kōkyūroku No. 1172, (2000), 70–79. .

[7] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot,* Acta Math. **186** (2001), 85–104.

[8] N. Reshetikhin and V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups,* Invent. Math. **103** (1991), 547–597.

[9] J. Roberts, *Skein theory and Turaev-Viro invariants,* Topology **34** (1995), 771–787.

[10] J. Roberts, *Classical 6j-symbols and the tetrahedron,* Geometry and Topology **3** (1999), 21–66.

[11] V. G. Turaev and O. Ya. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols,* Topology **31** (1992), 865–902.

[12] E. B. Vinberg, *The volume of polyhedra on a sphere and in Lobachevski space,* Amer. Math. Soc. Tans. (2), **148** (1991), 15–27.

[13] Y. Yokota, *On the volume conjecture of hyperbolic knots,* preprint, math.QA/0009165.

JUN MURAKAMI
DEPARTMENT OF MATHEMATICAL SCIENCE
SCHOOL OF SCIENCE AND ENGINEERING
WASEDA UNIVERSITY
3-4-1 OKUBO SHINJUKU-KU TOKYO, 169-8555, JAPAN
murakami@waseda.jp

MASAKAZU YANO
DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
OSAKA UNIVERSITY
MACHIKANEYAMA-CHO 1-1, TOYONAKA, OSAKA 560-0043, JAPAN