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On Moving Ginzburg-Landau Vortices

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In this note, we establish a quantization property for the heat equation of Ginzburg-Landau functional in R^4 which models moving vortices of surface types. It asserts that if the energy is sufficiently small on a parabolic ball in $R^4 \times R_+$ then there is no vortice in the parabolic ball of $\frac{1}{2}$ radius. This extends a recent result of Lin-Rivière [LR3] in R^3.

1. Introduction.

For $n \geq 2$ and $\epsilon > 0$, the heat equation for the Ginzburg-Landau functional on R^n is

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} - \Delta u_\epsilon &= \frac{1}{\epsilon^2}(1 - |u_\epsilon|^2)u_\epsilon, \quad (x,t) \in R^n \times R_+, \\ u_\epsilon(x,0) &= g_\epsilon(x), \qquad\qquad\qquad x \in R^n. \end{aligned} \tag{1.1}$$

Here $g_\epsilon : R^n \to R^2$ are given smooth maps. Notice that (1.1) is the negative gradient flow for the Ginzburg-Landau functional:

$$E_\epsilon(v) = \int_{R^n} \frac{1}{2}|Dv|^2 + \frac{1}{4\epsilon^2}(1 - |v|^2)^2. \tag{1.2}$$

Asymptotic behaviors for minimizers of E_ϵ in dimension two was first studied by Bethuel-Brezis-Hélein [BBH] (see also Struwe [S1] and recent important works by Pacard-Rivière [PR] on steady solutions to (1.1)). Moreover, such static theories were developed by Rivière [R1] [R2] and Lin-Rivière [LR1] in higher dimensions in connection with codimension two area minimizing currents. In particular, a crucial quantization property for steady solutions to the equation (1.1) was proved by Lin-Rivière [LR2] for $n = 3$. The asympototics for the equation (1.1) in dimension two was initiated by Lin [L1][L2] and also by Jerrard-Soner [JS]. To study the limiting behavior for a sequence of u_ϵ which are either static or time-dependent solutions to (1.1), one encounters the main difficulty that u_ϵ may vanish on sets, called Ginzburg-Landau

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vortices, where the equation (1.1) degenerates and $\frac{u_\epsilon}{|u_\epsilon|}$ gives nontrivial topological obstruction. On the other hand, it is well-known that the existence of vortices requires the Ginzburg-Landau energy at least of the order $\log\frac{1}{\epsilon}$. Since g_ϵ is usually assumed to have $E_\epsilon(g_\epsilon) = O(\log(\frac{1}{\epsilon}))$, we have

$$E_\epsilon(u_\epsilon(\cdot,t)) \leq O(\log\frac{1}{\epsilon}). \tag{1.3}$$

From the analytic point of view, the size estimate for the bad set, $B_\epsilon = \{(x,t) \in R^n \times R_+ : |u_\epsilon|(x,t) \leq \frac{1}{2}\}$, plays a critical role for $W^{1,p}$ compactness for $p \in (1,2)$ (see [BBH] and [PR]). To obtain sharp size estimates of B_ϵ, one needs the so-called η-compactness property for u_ϵ which roughly says that if $E_\epsilon(u_\epsilon)$ is of order $\eta\log\frac{1}{\epsilon}$ for sufficiently small $\eta > 0$ then there is no interior bad points for u_ϵ. This has been established for (i) minimizers of E_ϵ by Rivière [R1] [R2] for $n = 3$ and by Lin-Rivière [LR1] for $n \geq 3$; and (ii) critical points of E_ϵ by Lin-Rivière [LR2] for $n = 3$. Moreover, such η-compactness property was also proved for solutions to the equation (1.1) by Lin-Rivière [LR3] in the case $n = 3$. It was believed that their result still holds for R^n with $n \geq 4$. In this note, we confirm such a belief in the case that $n = 4$. More precisely, we prove

Theorem A. *For $n = 4$ and $\epsilon > 0$, let $u_\epsilon : R^4 \times R_+ \to R^2$ be solutions to the equation (1.1) satisfying $|u_\epsilon| \leq 1$ and $|Du_\epsilon| \leq \frac{C_0}{\epsilon}$. Then there exist $\epsilon_0 > 0$ and $\eta > 0$ depending only on C_0 such that if, for $(x_0,t_0) \in R^4 \times R_+$, $0 < \rho < \sqrt{t_0}$, and $\epsilon \leq \epsilon_0$, it holds*

$$\frac{1}{\rho^4}\int_{t_0-\rho^2}^{t_0}\int_{R^4}(\frac{1}{2}|Du_\epsilon|^2 + \frac{(1-|u_\epsilon|^2)^2}{4\epsilon^2})e^{\frac{|x-x_0|^2}{4(t-t_0)}} \leq \eta\log\frac{\rho}{\epsilon} \tag{1.4}$$

then

$$|u_\epsilon|(x_0,t_0) \geq \frac{1}{2}.$$

We would like to remark that the idea developed by Lin-Riviere [LR2] [LR3] was to interpolate between the Lorentz spaces $L^{2,1}$ and $L^{2,\infty}$ on generic two dimensional slices which therefore work very well in R^3, but it seems unclear how to extend the idea of [LR2] to R^n with $n \geq 4$. On the other hand, there is the interpolation technique between L^1 and L^∞ developed by Bethuel-Brezis-Orlandi [BBO] for the static case in R^n for all $n \geq 3$, where they made very clever use of the energy monotonicity formula for static solutions to the equation (1.1). Our method starts with the observation that there exists an energy monotonicity inequality for all time slice $R^n \times \{t\}$ when $n = 4$, which enables us to adapt the main ideas from [BBO] and some of those ideas from [LR3]. Since one can always view solutions to the equation (1.1) in $R^3 \times R_+$ as solutions to the equation (1.1) in $R^4 \times R_+$ which are independent of the fourth spatial variable, we also gives a different proof of the main theorem of [LR3]. As an important consequence of the η-compactness theorem, it can be shown that the vortice is moving by its mean curvature in the generalized sense.

The paper is organized as follows. In §2, we derive the needed elliptic type energy inequality in $R^4 \times \{t\}$. In §3, we recall the parabolic type energy monotonicity inequalities by Struwe [S2] and Lin-Rivière [LR3] and extract a good time slice. In §4, we illustrate the main estimate by performing an intrinsic Hodge decomposition on good time slices and prove theorem A.

Added in Proof. After the paper has been accepted for publication, the author received a preprint by Bethuel-Orlandi-Smets [BOS] where theorem A is proved for any dimension $n \geq 5$ through a much more delicate method.

2. Euclidean monotonicity at time slice for $n = 4$.

This section is devoted to the slice monotonicity inequality (2.1) for $u_\epsilon : R^n \times R_+ \to R^2$ satisfying (1.1) for $n = 4$. For $n \geq 4$, $x \in R^n$, $r > 0$, and $t > 0$, we denote

$$E_\epsilon(x, r) := \int_{B_r(x)} (\frac{1}{2}|Du_\epsilon|^2 + \frac{n(1 - |u_\epsilon|^2)^2}{4(n-2)\epsilon^2})(y)\, dy.$$

Then we have

Lemma 2.1. *For $n \geq 4$ and $\epsilon > 0$, let $u_\epsilon : R^n \times R_+ \to R^2$ be a solution to (1.1). Then, for any $(x, t) \in R^n \times R_+$ and $r > 0$, we have*

$$\frac{d}{dr}(r^{2-n} E_\epsilon(x, r) + \frac{r^{3-n}}{3-n} \int_{B_r(x)} |\frac{\partial u_\epsilon}{\partial t}||\frac{\partial u_\epsilon}{\partial r}|)$$
$$\geq r^{2-n} \int_{\partial B_r(x)} |\frac{\partial u_\epsilon}{\partial r}|^2 + \frac{(1 - |u_\epsilon|^2)^2}{2(n-2)\epsilon^2} + \frac{r^{3-n}}{3-n} \int_{\partial B_r(x)} |\frac{\partial u_\epsilon}{\partial t}||\frac{\partial u_\epsilon}{\partial r}|. \tag{2.1}$$

Proof. We assume that $x = 0$ and denote u for u_ϵ. Multiplying (1.1) by

$x \cdot Du$, integrating over B_r, and using integration by parts, we obtain

$$\begin{aligned}
\int_{B_r} u_t x \cdot Du &= \int_{B_r} \Delta u x \cdot Du - \frac{1}{4\epsilon^2} x \cdot D(1-|u|^2)^2 \\
&= \int_{B_r} D \cdot (Dux \cdot Du) - Du \cdot D(x \cdot Du) - x \cdot D \frac{(1-|u|^2)^2}{4\epsilon^2} \\
&= r \int_{\partial B_r} |\frac{\partial u}{\partial r}|^2 - \int_{B_r} |Du|^2 \\
&- \int_{B_r} x \cdot D(\frac{1}{2}|Du|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2}) \\
&= r \int_{\partial B_r} (|\frac{\partial u}{\partial r}|^2 - \frac{1}{2}|Du|^2 - \frac{(1-|u|^2)^2}{4\epsilon^2}) \\
&\quad + (n-2) \int_{B_r} (\frac{1}{2}|Du|^2 + \frac{n(1-|u|^2)^2}{4(n-2)\epsilon^2}).
\end{aligned}$$

This yields

$$(n-2)E_\epsilon(0,r) = \int_{B_r} u_t x \cdot Du + r \int_{\partial B_r} (\frac{1}{2}|Du|^2 - |\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2}).$$

Therefore

$$\begin{aligned}
\frac{d}{dr}(r^{2-n}E_\epsilon(0,r)) &= (2-n)r^{1-n}E_\epsilon(0,r) + r^{2-n} \int_{\partial B_r} (\frac{1}{2}|Du|^2 + \frac{n(1-|u|^2)^2}{4(n-2)\epsilon^2}) \\
&= -r^{1-n} \int_{B_r} u_t x \cdot Du + r^{2-n} \int_{\partial B_r} (|\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{2(n-2)\epsilon^2}).
\end{aligned}$$

Observe that

$$\begin{aligned}
-r^{1-n} \int_{B_r} u_t x \cdot Du &\geq -r^{2-n} \int_{B_r} |u_t||\frac{\partial u}{\partial r}| \\
&= -\frac{d}{dr}(\frac{r^{3-n}}{3-n} \int_{B_r} |u_t||\frac{\partial u}{\partial r}|) + \frac{r^{3-n}}{3-n} \int_{\partial B_r} |u_t||\frac{\partial u}{\partial r}|.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{d}{dr}(r^{2-n}E_\epsilon(0,r) + \frac{r^{3-n}}{3-n} \int_{B_r} |u_t||\frac{\partial u}{\partial r}|) \\
&\geq r^{2-n} \int_{\partial B_r} (|\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{2(n-2)\epsilon^2}) + \frac{r^{3-n}}{3-n} \int_{\partial B_r} |\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}|.
\end{aligned}$$

This completes the proof of (2.1). ■

Now we have the following slice energy monotonicty inequality for $n = 4$.

Proposition 2.2. *For $\epsilon > 0$, let $u_\epsilon : R^4 \times R_+ \to R^2$ be a solution to (1.1). Then, for any $(x,t) \in R^4 \times R_+$ and $0 \le r \le R < \infty$, we have*

$$\begin{aligned} &r^{-2}E_\epsilon(x,r) + \int_r^R \frac{dr}{r^2}\int_{\partial B_r(x)} (\frac{1}{2}|\frac{\partial u_\epsilon}{\partial r}|^2 + (4\epsilon^2)^{-1}(1-|u_\epsilon|^2)^2 \\ &\le 2R^{-2}E_\epsilon(x,R) + 2\int_{B_R(x)} |\frac{\partial u_\epsilon}{\partial t}|^2. \end{aligned} \tag{2.2}$$

In particular,

$$\int_{B_R(x)} |y-x|^{-2}\frac{(1-|u_\epsilon(y)|^2)^2}{\epsilon^2} \le 8R^{-2}E_\epsilon(x,R) + 8\int_{B_R(x)} |\frac{\partial u_\epsilon}{\partial t}|^2. \tag{2.3}$$

Proof. Write u for u_ϵ. It is clear that (2.2), with r tending to zero, gives (2.3). Therefore, it suffices to prove (2.2). For $n = 4$, integrating (2.1) from r to R, we have

$$\begin{aligned} &R^{-2}E_\epsilon(x,R) \\ &\ge r^{-2}E_\epsilon(x,r) + R^{-1}\int_{B_R(x)} |\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}| - r^{-1}\int_{B_r(x)} |\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}| \\ &+ \int_r^R s^{-2}\int_{\partial B_s(x)} (|\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2}) - \int_r^R s^{-1}\int_{\partial B_s(x)} |\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}|. \end{aligned}$$

For $n = 4$, we have the following estimate:

$$\begin{aligned} r^{-1}\int_{B_r(x)} |\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}| &\le \frac{1}{2}r^{-2}\int_{B_r(x)} |\frac{\partial u}{\partial r}|^2 + \frac{1}{2}\int_{B_r(x)} |\frac{\partial u}{\partial t}|^2 \\ &\le \frac{1}{2}r^{-2}\int_{B_r(x)} |\frac{\partial u}{\partial r}|^2 + \frac{1}{2}\int_{B_R(x)} |\frac{\partial u}{\partial t}|^2. \end{aligned}$$

Applying the Young inequality, we also have, for $r \le s \le R$,

$$s^{-1}\int_{\partial B_s(x)} |\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}| \le \frac{1}{2}s^{-2}\int_{\partial B_s(x)} |\frac{\partial u}{\partial r}|^2 + \frac{1}{2}\int_{\partial B_s(x)} |\frac{\partial u}{\partial t}|^2$$

so that

$$\int_r^R s^{-1}\int_{\partial B_s(x)} |\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}| \le \frac{1}{2}\int_r^R s^{-2}\int_{\partial B_s(x)} |\frac{\partial u}{\partial r}|^2 + \int_{B_R(x)} |\frac{\partial u}{\partial t}|^2.$$

Putting these inequality together, we obtain

$$R^{-2}E_\epsilon(x,R) \geq \frac{1}{2}r^{-2}E_\epsilon(x,r) - \int_{B_R(x)} |\frac{\partial u}{\partial t}|^2$$
$$+ \int_r^R s^{-2} \int_{\partial B_s(x)} (\frac{1}{2}|\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2}).$$

This implies (2.2). ■

3. Parabolic monotonicity and extracting a good time.

In this section, we gather together two more parabolic energy monotonicty inequalities by Struwe [S2] and by Lin-Rivière [LR3]. The formula is valid for all $n \geq 2$.

Lemma 3.1 (Energy monotonicity). *Let $u_\epsilon : R^n \to R_+ \to R^2$ be solutions to (1.1) and $(x_0, t_0) \in R^n \times R_+$. Then, for any $0 < \rho \leq \sqrt{t_0}$, we have*

$$\frac{d}{d\rho}[\frac{1}{\rho^n}\int_{t_0-\rho^2}^{t_0}\int_{R^n}(\frac{1}{2}|Du_\epsilon|^2 + \frac{(1-|u_\epsilon|^2)^2}{4\epsilon^2})e^{\frac{|x-x_0|^2}{4(t-t_0)}}]$$
$$= \frac{1}{\rho^{n+1}}\int_{t_0-\rho^2}^{t_0}\int_{R^n}[\frac{1}{2(t_0-t)}|(x-x_0)\cdot Du_\epsilon + 2(t-t_0)\frac{\partial u_\epsilon}{\partial t}|^2$$
$$\frac{(1-|u_\epsilon|^2)^2}{2\epsilon^2}]e^{\frac{|x-x_0|^2}{4(t-t_0)}}. \tag{3.1}$$

Proof. It follows exactly the same lines of the proof of [LR3] Lemma 2.1. ■

The next identity indicates how the energy decays along the spatial infinity.

Lemma 3.2. *Under the same notations as Lemma 3.1. For any $t_0 > 0$ and $0 < \rho \leq \sqrt{t_0}$, the following holds:*

$$\int_{t_0-\rho^2}^{t_0}\int_{R^n}\{(1+\frac{|x|^2}{4(t_0-t)})(\frac{1}{2}|Du_\epsilon|^2 + \frac{(1-|u_\epsilon|^2)^2}{4\epsilon^2})$$
$$\frac{1}{4(t_0-t)}|x\cdot Du_\epsilon + 2(t-t_0)\frac{\partial u_\epsilon}{\partial t}|^2\}e^{\frac{|x|^2}{4(t-t_0)}}$$
$$\leq \rho^2\int_{R^n}\int_{R^n\times\{t_0-\rho^2\}}[\frac{1}{2}|Du_\epsilon|^2 + \frac{(1-|u_\epsilon|^2)^2}{4\epsilon^2}]e^{\frac{-|x|^2}{4\rho^2}}$$
$$+ \int_{t_0-\rho^2}^{t_0}\frac{x}{4(t_0-t)}\cdot Du_\epsilon\cdot[x\cdot Du_\epsilon + 2(t-t_0)\frac{\partial u_\epsilon}{\partial t}]e^{\frac{|x|^2}{4(t-t_0)}}. \tag{3.2}$$

Proof. It again follows from the same argument as that of [LR3] Lemma 2.2. ■

Now we describe the extraction of a good time slice as follows. We follow [LR3] §2.2 closely and the reader may refer to [LR3] for the detail. For simplicity, we assume that $(x_0, t_0) = (0, 0)$ and (1.1) holds in $R^4 \times R_-$. Assume that (1.4) holds for some $\rho > 0$. Then, by integrating (3.1) from ϵ to ρ and using the Fubini's theorem, there exists a $\rho_1 = \rho_\epsilon \in (\epsilon, \rho)$ such that

$$\frac{1}{\rho_1^4} \int_{-\rho_1^2}^0 \int_{R^4} j_\epsilon(u_\epsilon) e^{\frac{|x|^2}{4t}} \leq \eta. \tag{3.3}$$

Here

$$j_\epsilon(u_\epsilon) \equiv \frac{1}{2|t|} |x \cdot Du_\epsilon + 2t \frac{\partial u_\epsilon}{\partial t}|^2 + \frac{(1 - |u_\epsilon|^2)^2}{2\epsilon^2} \tag{3.4}$$

so that

$$\frac{1}{\rho_1^2} \inf_{\rho \in (\frac{\rho_1}{2}, \rho_1)} \int_{R^4 \times \{-\rho^2\}} j_\epsilon(u_\epsilon) e^{-\frac{|x|^2}{4\rho^2}} \leq 2\eta. \tag{3.5}$$

Denote

$$E = \frac{1}{\rho_1^4} \int_{-\rho_1^2}^0 \int_{R^4} e_\epsilon(u_\epsilon) e^{\frac{|x|^2}{4t}} \tag{3.6}$$

where

$$e_\epsilon(u_\epsilon) \equiv (\frac{1}{2} |Du_\epsilon|^2 + \frac{(1 - |u_\epsilon|^2)^2}{4\epsilon^2}).$$

Then (3.1) implies

$$\begin{aligned} E &\leq \inf_{\frac{\rho_1}{2} \leq \rho \leq \rho_1} \frac{1}{\rho^4} \int_{-\rho^2}^0 \int_{R^4} e_\epsilon(u_\epsilon) e^{\frac{|x|^2}{4t}} + \int_{\frac{\rho_1}{2}}^{\rho_1} \rho^{-5} \int_{R^4} j_\epsilon(u_\epsilon) e^{\frac{|x|^2}{4t}} \\ &\leq \inf_{\frac{\rho_1}{2} \leq \rho \leq \rho_1} \frac{1}{\rho^4} \int_{-\rho^2}^0 \int_{R^4} e_\epsilon(u_\epsilon) e^{\frac{|x|^2}{4t}} + \frac{4}{\rho_1^4} \int_{-\rho_1^2}^0 \int_{R^4} j_\epsilon(u_\epsilon) e^{\frac{|x|^2}{4t}} \\ &\leq \inf_{\frac{\rho_1}{2} \leq \rho \leq \rho_1} \frac{1}{\rho^4} \int_{R^4} e_\epsilon(u_\epsilon) e^{\frac{|x|^2}{4t}} + 4\eta. \end{aligned}$$

As in [LR3], we may assume

$$E >> C\eta \tag{3.7}$$

so that

$$\inf_{\frac{\rho_1}{2} \leq \rho \leq \rho_1} \frac{1}{\rho^4} \int_{R^4} e_\epsilon(u_\epsilon) e^{-\frac{|x|^2}{4\rho^2}} \leq E \leq 2 \inf_{\frac{\rho_1}{2} \leq \rho \leq \rho_1} \frac{1}{\rho^4} \int_{R^4} e_\epsilon(u_\epsilon) e^{-\frac{|x|^2}{4\rho^2}}. \tag{3.8}$$

Therefore, there exists a $\rho_0 \in [\frac{\rho_1}{2}, \rho_1]$ such that

$$\max\{\frac{1}{\rho_0^4}\int_{-\rho_0^2}^0\int_{R^4} j_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}}, \frac{1}{\rho_0^2}\int_{R^4\times\{-\rho_0^2\}} j_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}}\} \leq C\eta, \tag{3.9}$$

$$\frac{1}{\rho_0^4}\int_{-\rho_0^2}^0\int_{R^4} e_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}} \leq E \leq \frac{C}{\rho_0^4}\int_{-\rho_0^2}^0\int_{R^4} e_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}}, \tag{3.10}$$

$$\frac{1}{\rho_0^2}\int_{R^4\times\{-\rho_0^2\}} e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}} \leq E \leq \frac{C}{\rho_0^2}\int_{R^4\times\{-\rho_0^2\}} e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}}. \tag{3.11}$$

These inequalities, combined with Lemma 3.2, also yield

$$\frac{1}{\rho_0^2}\int_{R^4\times\{-\rho_0^2\}} \frac{|x|^2}{|t|}e_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}} \leq CE. \tag{3.12}$$

Observe that (3.9) and (3.11) also imply

$$\int_{R^4\times\{-\rho_0^2\}} |\frac{\partial u_\epsilon}{\partial t}|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \leq CE. \tag{3.13}$$

In particular, for any $\lambda >> 1$ to be chosen later, one has

$$\int_{B_{4\lambda\rho_0}\times\{-\rho_0^2\}} |\frac{\partial u_\epsilon}{\partial t}|^2 \leq Ce^{4\lambda^2}E. \tag{3.14}$$

Hence, applying the monotonicity inequality (2.3) for u_ϵ at $t = -\rho_0^2$, we obtain the following key inequality:

$$\int_{B_{2\lambda\rho_0}(x)\times\{-\rho_0^2\}} |y-x|^{-2}\frac{(1-|u_\epsilon|^2)^2}{\epsilon^2} \leq Ce^{4\lambda^2}E, \ \forall x \in B_{2\lambda\rho_0}. \tag{3.15}$$

On the other hand, (3.9) also yields

$$\frac{1}{\rho_0^2}\int_{B_{4\lambda\rho_0}\times\{-\rho_0^2\}} \frac{(1-|u_\epsilon|^2)^2}{\epsilon^2} \leq Ce^{4\lambda^2}\eta. \tag{3.16}$$

Notice that (3.12) implies that

$$\frac{1}{\rho_0^2}\int_{(R^4\setminus B_{\frac{\lambda\rho_0}{2}})\times\{-\rho_0^2\}} e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}} \leq \frac{C}{\lambda^2}E. \tag{3.17}$$

This, combined with suitable choice of $\lambda >> 1$ according to the Fubini's theorem, gives

$$\frac{1}{\rho_0}\int_{\partial B_{\lambda\rho_0}} e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}} \leq \frac{C}{\lambda^3}E, \tag{3.18}$$

$$\frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}\times\{-\rho_0^2\}} e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}} \geq \frac{E}{3}. \tag{3.19}$$

Together with the inequalities from (3.9) to (3.18), we can proceed estimating E by estimating the left hand side of (3.19) as in §4 below.

4. An intrinsic Hodge decomposition to estimate $u_\epsilon \times du_\epsilon$.

This section is devoted to the proof of theorem A. The main techinical part is to obtain L^2-estimate of $u_\epsilon \times du_\epsilon$ on $B_{\lambda\rho_0} \times \{-\rho_0^2\}$. To do it, we need an intrinsic Hodge decompostion of $u_\epsilon \times du_\epsilon$ at $t = -\rho_0^2$. For this purpose, we adopt ideas from both [BBO] and [LR3]. In this section, we work on $t = -\rho_0^2$ and denote u as u_ϵ.

First, define $H : B_{\lambda\rho_0} \to R^2$ by the auxillary Neumann problem:

$$\frac{\partial}{\partial x_i}(e^{-\frac{|x|^2}{4\rho_0^2}}\frac{\partial H}{\partial x_i}) = \frac{\partial}{\partial x_i}(e^{-\frac{|x|^2}{4\rho_0^2}} u \times \frac{\partial u}{\partial x_i}), \text{ in } B_{\lambda\rho_0}, \tag{4.1}$$

$$\frac{\partial H}{\partial r} = u \times \frac{\partial u}{\partial r}, \qquad \text{on } \partial B_{\lambda\rho_0}. \tag{4.2}$$

Observe that

$$\begin{aligned} |\frac{\partial}{\partial x_i}(e^{-\frac{|x|^2}{4\rho_0^2}} u \times \frac{\partial u}{\partial x_i})| &= e^{-\frac{|x|^2}{4\rho_0^2}}|\frac{(-2\rho_0^2\frac{\partial u}{\partial t} + x\cdot Du)}{2\rho_0^2} \times u| \\ &\leq e^{-\frac{|x|^2}{4\rho_0^2}}\frac{|-2\rho_0^2\frac{\partial u}{\partial t} + x\cdot Du|}{2\rho_0^2} \\ &\leq 2\rho_0^{-1}e^{-\frac{|x|^2}{4\rho_0^2}}(j_\epsilon(u_\epsilon))^{\frac{1}{2}} \end{aligned}$$

so that we can establish the following estimate for DH.

Lemma 4.1 *Under the same notations as above. There exists a $C_\lambda > 0$ such that*

$$\begin{aligned} \frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |DH|^2 e^{-\frac{|x|^2}{4\rho_0^2}} &\leq C_\lambda\rho_0^{-2}\int_{B_{\lambda\rho_0}} j_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}} \\ &\quad + \frac{C\lambda}{\rho_0}\int_{\partial B_{\lambda\rho_0}} |\frac{\partial u}{\partial r}|^2 e^{-\frac{|x|^2}{4\rho_0^2}}. \end{aligned} \tag{4.3}$$

In particular, we have

$$\frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |DH|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \le C_\lambda \eta + \frac{CE}{\lambda^2}. \tag{4.4}$$

Proof. First, notice that (4.4) is the consequence of (4.3) and the inequalities (3.9) and (3.18). Secondly, the proof of (4.3) can be obtained by copying lines of arguments of [LR3] Lemma 2.4. ∎

Observe that (4.1) and (4.2) can be rewritten as

$$\frac{\partial}{\partial x_i}(e^{-\frac{|x|^2}{4\rho_0^2}}(\frac{\partial H}{\partial x_i} - u \times \frac{\partial u}{\partial x_i})\mathcal{I}_{B_{\lambda\rho_0}}) = 0 \tag{4.5}$$

in the sense of distributions on R^4, here $\mathcal{I}_{B_{\lambda\rho_0}}$ denotes the characteristic function of the ball $B_{\lambda\rho_0}$.

Define $\delta \in C^\infty(R_+, R_+)$ by $\delta(r) = r^2$ for $0 \le r \le 2\lambda\rho_0$, $\delta(r) = (4\lambda\rho_0)^2$ for $r \ge 4\lambda\rho_0$, and $(2\lambda\rho_0)^2 \le \delta(r) \le (4\lambda\rho_0)^2$ for $r \in [2\lambda\rho_0, 4\lambda\rho_0]$. Let $g_{ij}(x) = e^{-\frac{\delta(|x|)}{4\rho_0^2}}\delta_{ij}$ be the new conformal metric on R^4, which is readily seen to be bilipschitzly equivalent to the standard metric on R^4. Denote d_g^* as the adjoint of d with respect to g and $\Delta_g \equiv d_g^* d + dd_g^*$ as the Laplace-Beltrami operator with respect to g. Notice that (4.5) is equivalent to

$$d_g^*((dH - u \times du)\mathcal{I}_{B_{\lambda\rho_0}}) = 0, \text{ in } R^4. \tag{4.6}$$

Therefore, by the classical Hodge decompostion theory (see, e.g., Iwaniec-Martin [IW]), there exists a 2-form $\alpha \in H_g^1(R^4, \Lambda^2(R^4))$ such that

$$d_g^*\alpha = (dH - u \times du)\mathcal{I}_{B_{\lambda\rho_0}}, \quad d\alpha = 0, \tag{4.7}$$

$$\|D\alpha\|_{L_g^2(R^4)} \le C(\|Du\|_{L_g^2(B_{\lambda\rho_0})} + \|DH\|_{L_g^2(B_{\lambda\rho_0})}). \tag{4.8}$$

Here H_g^1 (or L_g^1 resp.) denotes H^1 (or L^2 resp.) with respect to g. Notice that

$$\|Df\|^2_{L_g^2(R^4)} = \int_{R^4} |Df|^2(x) e^{-\frac{\delta(|x|)}{4\rho_0^2}}.$$

In order to estimate $D\alpha$ in L_g^2, we modify the approach of [BBO] as follows. Let $\beta \in (0, \frac{1}{2}$ be determined later, and $f : R_+ \to [1, \frac{1}{1-\beta}]$ be a smooth function such that $f(t) = \frac{1}{t}$ for $t \ge 1-\beta$, $f(t) = 1$ for $t \le 1 - 2\beta$, and $|f'| \le 4$. On R^4, define the function a such that $a(x) = f^2(|u|(x))$ on $B_{\lambda\rho_0}$

and $a(x) = 1$ elsewhere, so that $0 \le a - 1 \le 4\beta$ holds on R^4. Observe that $f^2(|u|^2)u \times du = f(|u|)u \times d(f(|u|)u)$. Therefore, (4.7) implies

$$\begin{aligned} d(ad_g^*\alpha) &= \mathcal{I}_{B_{\lambda\rho_0}} d(f(|u|)u) \times d(f(|u|)u) \\ &\quad + f(|u|)u \times du \wedge d|x|\sigma^g_{\partial B_{\lambda\rho_0}} - d(\mathcal{I}_{B_{\lambda\rho_0}} adH) \\ &= \omega_1 + \omega_2 + \omega_3 \end{aligned} \tag{4.9}$$

where $\sigma^g_{\partial B_{\lambda\rho_0}}$ denotes the surface measure of $\partial B_{\lambda\rho_0}$ with respect to the metric g. Observe that if $|u| \ge 1-\beta$ then $d(f(|u|)u) \times d(f(|u|)u) = d(\frac{u}{|u|}) \times d(\frac{u}{|u|}) = 0$, otherwise we have $1 \le \beta^{-2}(1-|u|^2)^2$ so that

$$|\omega_1|(x) \le C\epsilon^{-2} \le C\beta^{-2}\frac{(1-|u(x)|^2)^2}{\epsilon^2}, \forall x \in B_{\lambda\rho_0}. \tag{4.10}$$

Using the fact that $d\alpha = 0$, we get

$$\Delta_g\alpha = dd_g^*\alpha = d(ad_g^*\alpha) + d((1-a)d_g^*\alpha) = \omega_1+\omega_2+\omega_3+d((1-a)d_g^*\alpha) \tag{4.11}$$

Denote $G(x,y) = G(|x-y|)$ as the fundamental solution of Δ_g on R^4. Then it follows from the bilipschitz equivalence between g and the euclidean metric on R^4 that there exists a $C > 0$ such that

$$Ce^{-4\lambda^2}|x-y|^{-2} \le G(x,y) \le Ce^{4\lambda^2}|x-y|^{-2}, |D_yG(x,y)| \le Ce^{4\lambda^2}|x-y|^{-3}. \tag{4.12}$$

Let $\alpha_i = G * \omega_i$ for $1 \le i \le 3$. Then $\alpha_4 = \alpha - \sum_{i=1}^3 \alpha_i$ solves

$$\Delta_g\alpha_4 = d((1-a)d_g^*\alpha). \tag{4.13}$$

Direct calculations, using $|a-1| \le 4\beta$ and smallness of β, yield

$$\|D\alpha_4\|^2_{L^2_g(R^4)} \le C\beta\sum_{i=1}^3 \|D\alpha_i\|^2_{L^2_g(R^4)}. \tag{4.14}$$

The main difficulty comes from the estimate of $D\alpha_1$ which can be done as follows, due to the monotonicity inequality (3.15) and (3.16). Indeed, by the maximum principle, we have $\|\alpha_1\|_{L^\infty(R^4)} = \|\alpha_1\|_{L^\infty(B_{\lambda\rho_0})}$ and, by (4.10), (4.12), and (3.15),

$$\begin{aligned} \|\alpha_1\|_{L^\infty(B_{\lambda\rho_0})} &\le \sup_{x\in B_{\lambda\rho_0}} \int_{B_{\lambda\rho_0}} G(x-y)|\omega_1|(y) \\ &\le C_\lambda\beta^{-2}\sup_{x\in B_{\lambda\rho_0}} \int_{B_{\lambda\rho_0}} |x-y|^{-2}\frac{(1-|u(y)|^2)^2}{\epsilon^2} \\ &\le C_\lambda\beta^{-2}E. \end{aligned} \tag{4.15}$$

This, combined with (3.16), implies

$$\|D\alpha_1\|^2_{L^2_g(R^4)} \leq \|\omega_1\|_{L^1(R^4)}\|\alpha_1\|_{L^\infty(R^4)} \leq C_\lambda \beta^{-2}\rho_0^2\eta E. \tag{4.16}$$

For α_3, using integration by parts and (4.4), we have

$$\|D\alpha_3\|^2_{L^2_g(R^4)} \leq C\|DH\|^2_{L^2_g(B_{\lambda\rho_0})} \leq C_\lambda \eta\rho_0^2 + \frac{C\rho_0^2 E}{\lambda^2}. \tag{4.17}$$

For α_2, we can modify the Lemma A1 of appendix in [BBO] to conclude that

$$\|D\alpha_2\|^2_{L^2_g(R^4)} \leq C\lambda\rho_0\|Du\|^2_{L^2_g(\partial B_{\lambda\rho_0})} \tag{4.18}$$

this, combined with (3.18), gives

$$\|D\alpha_2\|^2_{L^2_g(R^4)} \leq \frac{C\rho_0^2}{\lambda^2}E. \tag{4.19}$$

Putting these estimates for α_i for $1 \leq i \leq 4$ and Lemma 4.1 together, we then obtain

$$\frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |u\times du|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \leq C_\lambda\eta + \frac{CE}{\lambda^2} + C_\lambda\beta^{-2}\eta E. \tag{4.20}$$

This, combined with the fact that $4|u|^2|du|^2 = 4|u\times du|^2 + |d|u|^2|^2$ and the following estimate (see (2.67) of [LR3] page 845)

$$\frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |D|u|^2|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \leq C\eta^{\frac{1}{4}}E + C\eta^{\frac{1}{2}} \tag{4.21}$$

implies

$$
\begin{aligned}
&\frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |Du|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \\
&= \frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} (1-|u|^2)|Du|^2 e^{-\frac{|x|^2}{4\rho_0^2}} + \frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |u|^2|Du|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \\
&\leq \frac{C}{\rho_0^2}\int_{B_{\lambda\rho_0}} \frac{(1-|u|^2)}{\epsilon}|Du| e^{-\frac{|x|^2}{4\rho_0^2}} \\
&\quad + \frac{4}{\rho_0^2}\int_{B_{\lambda\rho_0}} (|u\times du|^2 + |D|u|^2|^2) e^{-\frac{|x|^2}{4\rho_0^2}} \\
&\leq \frac{C_\lambda}{\rho^2}\int_{B_{\lambda\rho_0}} \frac{(1-|u|^2)^2}{\epsilon^2} e^{-\frac{|x|^2}{4\rho_0^2}} + (C_\lambda \eta + C\eta^{\frac{1}{2}}) \\
&\quad + (\lambda^{-1} + C\lambda^{-2} + C_\lambda \beta^{-2}\eta + C\eta^{\frac{1}{4}})E \\
&\leq (\lambda^{-1} + C\lambda^{-2} + C_\lambda \beta^{-2}\eta + C\eta^{\frac{1}{4}})E + (C_\lambda \eta + C\eta^{\frac{1}{2}})
\end{aligned} \tag{4.22}
$$

Therefore, for any given $\delta > 0$, we can first choose a sufficiently large $\lambda > 1$ and a sufficiently small β and then choose much smaller η so that

$$E \leq C\delta \tag{4.23}$$

so that, using the monotonocity inequality (3.1) again,

$$\frac{1}{\epsilon^6}\int_{-\epsilon^2}^{0}\int_{B_\epsilon(0)} \frac{(1-|u_\epsilon|^2)^2}{\epsilon^2} e^{\frac{|x|^2}{4t}} \leq \delta. \tag{4.24}$$

This, combined with the fact that $|Du_\epsilon| \leq C\epsilon^{-1}$, yields $|u_\epsilon(0,0)| \geq \frac{1}{2}$. Therefore, the proof of theorem A is complete. ■

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