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Regularity of Entire Solutions to the Complex Monge-Ampère Equation

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A regularity theorem for the solutions of the complex Monge-Ampère equation in $\mathbb{C}^n$ with the right hand side having finite mass is proved.

Introduction.

The aim of this paper is to study the regularity of those entire solutions to the complex Monge-Ampère equation which have logarithmic growth. Such solutions are obtained, after suitable normalization, when the right hand side of the equation has finite total mass. As usual, we denote this class of functions by $\mathcal{L}_+$.

$$\mathcal{L}_+ := \{\varphi \in PSH(\mathbb{C}^n) : \sup |\varphi(z) - \log(1 + |z|)| < const.\}.$$

It is known (see [T]) that if $u \in \mathcal{L}_+$ then $\int_{\mathbb{C}^n} (dd^c u)^n = (2\pi)^n$. So the following problem is well posed.

$$\begin{aligned} &u \in \mathcal{L}_+ \\ &(dd^c u)^n = f\, d\lambda \quad (d\lambda \text{ the Lebesgue measure }) \\ &\int f\, d\lambda = (2\pi)^n, \quad f \geq 0, \end{aligned} \tag{$*$}$$

and the function f is given. It has been investigated in [BT3], [CK], [KL] and discussed in [B]. The existence of continuous solutions to $(*)$ when $f \in L^p((1 + |z|^2)^{-n-1}\, d\lambda(z)),\ p > 1$, was shown in [KL]. It follows from a generalized version of the Yau theorem.

Theorem. *[Y] Let us consider a compact n-dimensional Kähler manifold M with the fundamental form ω and the Monge-Ampère equation on M:*

$$(\omega + dd^c\phi)^n = F\omega^n, \tag{$**$}$$

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where ϕ is the function we search such that $\omega + dd^c\phi$ is a non-negative (1,1) form. The given non-negative function $F \in L^1(M)$ is normalized by the condition

$$\int_M F\omega^n = \int_M \omega^n.$$

If $F > 0, F \in C^k(M), k \geq 3$, then there exists a solution to $(**)$ belonging to Hölder class $C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$.

In [KL] the existence part of this result has been extended to cover the case $F \geq 0,\ F \in L^p(M), p > 1$. Then one can apply this for $(M, \omega) = (\mathbb{P}^n, \omega_0)$, where the Fubini-Study form ω_0 restricted to $\mathbb{C}^n$ (with standard identification) equals to $\omega_0(z) = (1/2)dd^c \log(1+|z|^2)$ and the volume form is $\omega_0^n(z) = n!(1+|z|^2)^{-n-1}d\lambda$. Setting $F(z) = f(z)(n!)^{-1}(1+|z|^2)^{n+1}$ we thus obtain $u(z) = \varphi(z)+(1/2)\log(1+|z|^2)$ which is continuous, plurisubharmonic and solves $(*)$. The solution is unique up to an additive constant by [BT3].

Regularity of u when f is smooth follows from Yau's theorem provided $f(z)(1+|z|^2)^{n+1}$ can be extended to a smooth positive function on $\mathbb{P}^n$. In particular, since F is bounded and positive on $\mathbb{P}^n$:

$$c_0(1+|z|^2)^{-n-1} \leq f(z) \leq c_1(1+|z|^2)^{-n-1},$$

but obviously there are also other restrictions on the behavior of f and its derivatives at infinity forced by smoothness of F.

Below we shall prove a stronger result.

Theorem 1. *If f in $(*)$ is positive, C^∞ smooth and for some positive constants $K > 1, K_1$ satisfies the inequalities*

$$f(z) \leq K(1+|z|^2)^{-n-1},$$

and

$$-\Delta f^{1/n}(z) \leq K_1 f^{1/n}(z),$$

then $u \in C^\infty$.

If we drop the assumption that f is positive and take $f \geq 0$ instead then we get $u \in C^1(\mathbb{C}^n)$ (see Corollary to Theorem 2). Both inequalities in the statement are obviously satisfied by every smooth positive f on any bounded subset of $\mathbb{C}^n$, so they can be viewed as "boundary conditions" on the hyperplane $\mathbb{P}^n \setminus \mathbb{C}^n$.

Due to a result of Riebesehl and Schulz [RS] and classical elliptic theory, in order to prove Theorem 1 it is enough to ensure that the Laplacian of u is locally bounded.

For this we shall prove a priori estimates of Δu when u is smooth. Derivation of those estimates is different than in the existing literature on the regularity of the complex Monge-Ampère equation (comp. [BT1], [CKNS], [CY], [GT], [RS], [Y]). We seek local bounds for the operator $T_u(\epsilon) = const.\epsilon^{-2}(u_\epsilon - u)$ independent of ϵ, where u_ϵ is a regularizing family for u. This is done by suitable application of the comparison principle and an inequality relating Monge-Ampère measures of u and u_ϵ.

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1. Preliminaries.

Let $B(R)$ denote the ball in $\mathbb{C}^n$ centered at 0 of radius R. For any real function f we denote throughout the paper, by f_ϵ the convolution of f with the characteristic function of $B(\epsilon)$ multiplied by $\sigma_n := (\int_{B(\epsilon)} d\lambda)^{-1}$.

$$f_\epsilon(z) = \sigma_n \int_{B(\epsilon)} f(z+w)\, d\lambda(w).$$

We refer to [BT1] or [K] for basic properties of the operator $T_f(\epsilon) := 4(n+1)\epsilon^{-2}(f_\epsilon - f)$. In particular, if $f \in C^2(\mathbb{C}^n)$, then $T_f(\epsilon) \to \Delta f$ locally uniformly as $\epsilon \to 0$, and

$$-T_f(\epsilon)(z) \le \max_{B(z,\epsilon)} -\Delta f. \tag{1}$$

The following proposition follows from Theorem 5.7 in [BT1].

Proposition 1. *If* $(dd^c u_\epsilon)^n = f_{(\epsilon)}\, d\lambda$, *then* $f_{(\epsilon)} \ge [(f^{1/n})_\epsilon]^n$.

Let us denote by v the potential of Fubini-Study metric and by g the density of the Monge-Ampère mass of v. So, $v(z) := (1/2)\log(1+|z|^2)$, $(dd^c v)^n = g(z)\, d\lambda$, where $g(z) = n!(1+|z|^2)^{-n-1}$. Note that the Laplacian of v is bounded from above by $n/8$ since

$$\frac{\partial^2 v}{\partial z_j \partial \bar{z}_j} = [2(1+|z|^2)]^{-1}[1 - |z_j|^2(1+|z|^2)^{-1}] < 1/2.$$

We shall need this fact in the proof of the next proposition.

Proposition 2. *If* $u \in C^2, (dd^c u)^n = f\, d\lambda$, *and* $f \le Kg$ *then*

$$(dd^c u)^{n-1} \wedge dd^c v \ge c_1(K,n)(\Delta u)^{1/(n-1)} f d\lambda,$$

where $c_1(K,n) = (n^{n+1}K)^{-1/(n-1)}$.

Proof. We shall use the notation $u_{j\bar{k}} := \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}$. Since $(u_{j\bar{k}})$ is positive semidefinite one can choose a coordinate system centered at a given point z so that the matrix becomes diagonal. Set $N := \Delta u(z) = (1/4) \sum u_{j\bar{j}}(z)$ and suppose that $u_{1\bar{1}}(z) \geq u_{j\bar{j}}(z) > 0,\ j = 2, 3, ..., n$ (if $u_{j\bar{j}}(z) = 0$ then $f(z) = 0$ and there is nothing to prove). So $u_{1\bar{1}}(z) \geq 4Nn^{-1}$.

Observe that denoting by m the maximum of $\frac{v_{j\bar{j}}(z)}{u_{j\bar{j}}(z)}$ we have the following inequalities

$$m^{n-1} \geq \left(\frac{\prod_{j=1}^n v_{j\bar{j}}}{\prod_{j=1}^n u_{j\bar{j}}} \right) \frac{u_{1\bar{1}}}{v_{1\bar{1}}}(z) \geq \frac{g u_{1\bar{1}}}{f v_{1\bar{1}}}(z) \geq \frac{8Ng}{n^2 f}(z),$$

as $u_{1\bar{1}}(z) \geq 4Nn^{-1}$ and $v_{1\bar{1}} \leq 4\Delta v \leq n/2$ (see the remark preceding the statement). We thus get $m \geq [\frac{8Ng}{n^2 f}(z)]^{1/(n-1)} \geq [\frac{8N}{n^2 K}]^{1/(n-1)}$. Using this one arrives at the desired estimate

$$\begin{aligned}(dd^c u)^{n-1} \wedge dd^c v(z) &= (2i)^n (\sum u_{j\bar{j}}(z) dz_j \wedge d\bar{z}_j)^{n-1} \wedge (\sum v_{j\bar{j}}(z) dz_j \wedge d\bar{z}_j) \\ &= 4^n (n-1)! (\prod u_{j\bar{j}}(z)) \sum \frac{v_{j\bar{j}}}{u_{j\bar{j}}}(z) d\lambda(z) = (1/n) f(z) \sum \frac{v_{j\bar{j}}}{u_{j\bar{j}}}(z) d\lambda(z) \\ &\geq \frac{m}{n} f(z) d\lambda(z) \geq c_1(K, n) (\Delta u(z))^{1/(n-1)} f(z)\, d\lambda(z).\end{aligned}$$

The proof is completed.

Finally, let us mention some basic facts about the function g.

Proposition 3. *We have* $\int_{\mathbb{C}^n} g\, d\lambda = (2\pi)^n$ *and for any* $\delta > 0$ *there exists* $R > 0$ *such that*

$$\epsilon^{-2} \int_{\mathbb{C}^n \setminus B(R)} \{g - [(g^{1/n})_\epsilon]^n\}\, d\lambda < \delta, \quad \epsilon \in (0, 1).$$

Proof. The first part follows by computation. To get the convergence of the other integral set $\alpha := \frac{n+1}{n}$, and compute

$$\begin{aligned}\Delta g^{1/n}(z) &= (n!)^{1/n} \Delta (1 + |z|^2)^{-\alpha} \\ &= (4\alpha)(n!)^{1/n} (1 + |z|^2)^{-\alpha-2} [(2 - n + n^{-1})|z|^2 - n].\end{aligned}$$

Using (1) and the above formula one obtains

$$\epsilon^{-2} [g^{1/n}(z) - (g^{1/n})_\epsilon(z)] \leq const.(n)(1 + |z|^2)^{-\alpha-1}.$$

Hence

$$\begin{aligned}\epsilon^{-2} \{g - [(g^{1/n})_\epsilon]^n\}(z) &\leq \epsilon^{-2} [g^{1/n} - (g^{1/n})_\epsilon] n g^{(n-1)/n}(z) \\ &\leq const.(n) n (1 + |z|^2)^{-n-2}.\end{aligned}$$

The result follows by integration of the above inequality.

2. Proofs of main results.

The key element of our proof is the following a priori estimate of second order derivatives of the solution.

Lemma. *Suppose* $f \in C^3(\mathbb{C}^n), n > 1,\ \int f\, d\lambda = (2\pi)^n,\ 0 < f \leq Kg,$

$$-\Delta f^{1/n} \leq K_1 f^{1/n}, \tag{2}$$

and $f = ag$ *on* $\mathbb{C}^n \backslash B(R_0)$ *for some* $a, K_1 > 0, K > 1$. *Let* $u \in PSH \cap C^2(\mathbb{C}^n)$ *solve*

$$(dd^c u)^n = f\, d\lambda$$

and satisfy $\inf(2u - v) = 0$. *Then*

$$\Delta u \leq c(K, K_1, n) + 2u - v$$

on $B(R_0)$, *where for* $c(K, K_1, n)$ *one may take* $(3^n n)^{n+1} K[4^n + K_1]^n$.

Proof. By the theorem of Yau $u \in C^4(\mathbb{C}^n)$. The use of Yau's theorem is justified since from the hypothesis $f = ag$ on $\mathbb{C}^n \setminus B(R_0)$ it follows that F in Theorem from Introduction equals to a constant in a neighborhood of $H = \mathbb{P}^n \setminus \mathbb{C}^n$ and thereby $F \in C^3(\mathbb{P}^n)$. Fix $c(K, K_1, n)$ as in the statement and set $M = c(K, K_1, n) - 3$. Suppose Lemma were not true and

$$E := \{\Delta u > M + 3 + 2u - v\} \cap B(R_0) \neq \emptyset. \tag{3}$$

Fix $\delta > 0$ satisfying

$$2\delta < \int_E f\, d\lambda.$$

Use Proposition 3 to find $R > R_0$ with

$$3^n a \int_{\mathbb{C}^n \backslash B(R)} g\, d\lambda + \epsilon^{-2} a \int_{\mathbb{C}^n \backslash B(R)} \{g - [(g^{1/n})_\epsilon]^n\}\, d\lambda < \delta, \quad \epsilon \in (0, 1). \tag{4}$$

Fix $\epsilon \in (0, \min(1/(2K_1), 1/2))$ such that

$$\delta < \int_{E \cap \{f_{(\epsilon)} < 3^n f\}} f\, d\lambda, \tag{5}$$

$$||h_\epsilon - \Delta u||_{B(R)} < 1, \quad \text{where } h_\epsilon := 4(n+1)\epsilon^{-2}(u_\epsilon - u), \tag{6}$$

and

$$||\Delta u_\epsilon - \Delta u||_{B(R)} < 1, \tag{7}$$

(which is possible since $f_{(\epsilon)}$ converges a.e. to f when ϵ tends to 0; and then $h_\epsilon \to \Delta u$ and $\Delta u_\epsilon \to \Delta u$ uniformly on $B(R)$ as $\epsilon \to 0$, see [BT1], [K]). Furthermore one can assume, using (1) and (2) that

$$|f^{1/n} - (f^{1/n})_\epsilon| \leq (2n)^{-1} K_1 \epsilon^2 f^{1/n} \quad \text{on } B(R). \tag{8}$$

Set

$$E(\epsilon) := \{2u - v < h_\epsilon - (M+2)\}.$$

By (6) (see also (3)) we have

$$E \subset E(\epsilon). \tag{9}$$

Apply the comparison principle [BT2] to obtain

$$\int_{E(\epsilon)} [dd^c(\epsilon^{-2}u_\epsilon + v)]^n \leq \int_{E(\epsilon)} (\epsilon^{-2}+2)^n (dd^c u)^n.$$

Hence, using the positivity of the forms involved and the inequality $\epsilon < 1/2$ we have

$$\epsilon^{-2n} \int_{E(\epsilon)} (dd^c u_\epsilon)^n + n\epsilon^{-2(n-1)} \int_{E(\epsilon)} dd^c v \wedge (dd^c u_\epsilon)^{n-1} \\ \leq \epsilon^{-2n} \int_{E(\epsilon)} (dd^c u)^n + 3^n \epsilon^{-2(n-1)} \int_{E(\epsilon)} (dd^c u)^n.$$

Upon regrouping the terms and multiplying by $\epsilon^{2(n-1)}$ one gets

$$n \int_{E(\epsilon)} dd^c v \wedge (dd^c u_\epsilon)^{n-1} \leq 3^n \int_{E(\epsilon)} f\, d\lambda + \epsilon^{-2} \int_{E(\epsilon)} (f - f_{(\epsilon)})\, d\lambda,$$

where $f_{(\epsilon)}\, d\lambda = (dd^c u_\epsilon)^n$. The integrand on the right hand side of this inequality is obviously negative whenever $f_{(\epsilon)} \geq 3^n f > 0$, so the inequality remains valid if we integrate only over the set

$$E_1(\epsilon) := E(\epsilon) \cap \{f_{(\epsilon)} < 3^n f\}.$$

Next we apply Proposition 1 to the last term and obtain

$$n \int_{E_1(\epsilon)} dd^c v \wedge (dd^c u_\epsilon)^{n-1} < 3^n \int_{E_1(\epsilon)} f\, d\lambda + \epsilon^{-2} \int_{E_1(\epsilon)} \{f - [(f^{1/n})_\epsilon]^n\}\, d\lambda.$$

To estimate the right hand side use (4) , (8) and the inequality

$$\epsilon^{-2}\{f - [(f^{1/n})_\epsilon]^n\}(z) \leq \epsilon^{-2} |f^{1/n} - (f^{1/n})_\epsilon| n f^{(n-1)/n}(z).$$

Then

$$
\begin{aligned}
n\int_{E_1(\epsilon)} dd^c v\wedge (dd^c u_\epsilon)^{n-1} &< 3^n\int_{E_1(\epsilon)\cap B(R)} f\,d\lambda \\
&+\epsilon^{-2}\int_{E_1(\epsilon)\cap B(R)}\{f-[(f^{1/n})_\epsilon]^n\}\,d\lambda+\delta \\
&\le [3^n+K_1]\int_{E_1(\epsilon)\cap B(R)} f\,d\lambda+\delta.
\end{aligned}
$$

Finally, by (5) and (9)

$$
n\int_{E_1(\epsilon)} dd^c v\wedge (dd^c u_\epsilon)^{n-1} < [3^n+1+K_1]\int_{E_1(\epsilon)\cap B(R)} f\,d\lambda. \tag{10}
$$

From (6) , (7) and the condition $\inf(2u-v)=0$ it follows that

$$
E(\epsilon)\cap B(R)\subset\{\Delta u_\epsilon > M\}\cap B(R).
$$

So applying Proposition 2 (note that $f_{(\epsilon)} < 3^n Kg$ on $E_1(\epsilon)$) we get

$$
\int_{E_1(\epsilon)} dd^c v\wedge (dd^c u_\epsilon)^{n-1} \ge \left(\frac{M}{3^n n^{n+1}K}\right)^{1/(n-1)}\int_{E_1(\epsilon)\cap B(R)} (dd^c u_\epsilon)^n\,d\lambda.
$$

To give a lower bound for the right hand side use Proposition 1, (8) and the inequality $\epsilon<\min(1/2K_1,1/2)$. Then

$$
\begin{aligned}
&\int_{E_1(\epsilon)} dd^c v\wedge (dd^c u_\epsilon)^{n-1} \\
&\ge \left(\frac{M}{3^n n^{n+1}K}\right)^{1/(n-1)}\int_{E_1(\epsilon)\cap B(R)} [f^{1/n}(1-K_1\epsilon^2)]^n\,d\lambda \\
&\ge \left(\frac{M}{(3^n n)^{n+1}K}\right)^{1/(n-1)}\int_{E_1(\epsilon)\cap B(R)} f\,d\lambda.
\end{aligned} \tag{11}
$$

Combining (10) and (11) one obtains $M\le(3^n n)^{n+1}K(3^n+1+K_1)^{n-1}$ which contradicts the choice of M. Thus the lemma follows.

Theorem 2. *Suppose $f\in C^1(\mathbb{C}^n), n>1$, satisfies the following assumptions:* $\int f\,d\lambda=(2\pi)^n$, $0\le f\le Kg$,

$$
-\Delta f^{1/n}\le K_1 f^{1/n},
$$

for some $K > 1, K_1 > 0$. *Then normalizing the solution of* $(dd^c u)^n = f\, d\lambda$ *by* $\inf(2u - v) = 0$ *one obtains*

$$\Delta u \leq c_0(K, K_1, n) + 2u - v. \tag{12}$$

Proof. First we assume that $f \in C^3(\mathbb{C}^n)$. We shall approximate f by a sequence of functions f_j which fulfil the hypothesis of the lemma with the uniform constants K, K_1, and then we shall prove that u inherits the a priori estimate from solutions u_j corresponding to f_j.

Let us choose a sequence of smooth functions $\varphi_j : [0, \infty) \to [1/j, \infty)$ enjoying the following properties:

$$\begin{gathered} \varphi_j \text{ tend uniformly to identity,} \\ x < \varphi_j(x) < 2x, \text{ for } x > 1/(2j) \\ 0 \leq \varphi_j'(x) < 1, \\ 0 \leq \varphi_j''(x), \\ \varphi_j(x) = 1/j \text{ for } x \in [0, 1/(2j)]. \end{gathered} \tag{13}$$

For any positive integer define the numbers s_j, t_j putting

$$s_j = [(jK)^{n/(n+1)} - 1]^{1/2}, \quad t_j = [(2jK)^{n/(n+1)} - 1]^{1/2}.$$

Then

$$1/j = Kg^{1/n}(s_j, 0, 0, ..., 0) = 2Kg^{1/n}(t_j, 0, 0, ..., 0) \tag{14}$$

and $t_j - s_j > s_j/2$. Due to the last inequality one can also choose, for j big enough (say $j > j_0$), a function $\chi_j \in C_0^\infty(\mathbb{C}^n)$ such that $\chi_j(|\cdot|)$ is non-decreasing, $\chi_j = 0$ on $B(s_j)$, $\chi_j = 1$ on the complement of $B(t_j)$ and $|\nabla \chi_j(z)| < 1, |\Delta \chi_j(z)| < 1$.

Now we can define the sequence f_j setting:

$$f_j^{1/n} = (1 - \chi_j)\varphi_j(f^{1/n}) + 2K\chi_j g^{1/n}, \quad j > j_0.$$

Then by (13) and the hypothesis we have

$$f_j^{1/n} \leq 2(1 - \chi_j)f^{1/n} + 2K\chi_j g^{1/n} \leq 2Kg^{1/n}.$$

Still using the properties of φ_j one can estimate the Laplacian of $f_j^{1/n}$ on $B(s_j)$ as follows.

$$\begin{aligned} -\Delta f_j^{1/n} &= -\Delta \varphi_j(f^{1/n}) = -\varphi_j'(f^{1/n})\Delta f^{1/n} - \varphi_j''(f^{1/n})\langle \nabla f^{1/n}, \nabla f^{1/n} \rangle \\ &\leq K_1 f^{1/n} \leq K_1 \varphi_j(f^{1/n}) = K_1 f_j^{1/n} \end{aligned}$$

It is easy to compute that one can increase j_0 so that $|\nabla g^{1/n}(z)| < g^{1/n}$, and $|\Delta g^{1/n}(z)| < g^{1/n}$ outside $B(s_{j_0})$. It then follows, by the relevant properties of χ_j, the Cauchy-Schwarz inequality and (14) that for $j > j_0$ the following estimate holds on $B(t_j) \setminus B(s_j)$:

$$\begin{aligned}-\Delta f_j^{1/n} &= (1/j)\Delta\chi_j - 2Kg^{1/n}\Delta\chi_j - 2K\chi_j\Delta g^{1/n} - 4\langle\nabla g^{1/n}, \nabla\chi_j\rangle \\ &\leq 1/j + 8Kg^{1/n} \leq 9/j \leq 9f_j^{1/n}.\end{aligned}$$

Finally, on the complement of $B(t_j)$ we have

$$-\Delta f_j^{1/n} = -2K\Delta g^{1/n} \leq 2Kg^{1/n} = 2Kf_j^{1/n}.$$

Thus we have proved that

$$f_j^{1/n} \leq 2Kg^{1/n} \text{ and } -\Delta f_j^{1/n} \leq K_2 f_j^{1/n}. \tag{15}$$

Note that, by the construction, $f_j \geq f$. So, choosing $\alpha_j = (2\pi)^n(\int f_j\, d\lambda)^{-1}$ and denoting $\tilde{f}_j = \alpha_j f_j$ one concludes that $\tilde{f}_j$ also satisfies the estimates (15). Let u_j be the solutions of $(*)$ corresponding to $\tilde{f}_j$. Then applying Lemma we get

$$\Delta u_j \leq 2u_j - v + c(2K, K_2, n), \tag{16}$$

with the constant independent of j.

From [KL, Chapter 2, Secion 3] we know that $|u_j - v|$ is uniformly bounded by a constant which we denote by A_0. By Lemma 2.2.1 in the same paper $u_j \to u := (\limsup u_j)^*$ in $L^1_{loc}(\mathbb{C}^n)$ and u solves $(dd^c u)^n = f\, d\lambda$.

To deduce (12) from (16) fix a ball $B = B(x, r)$ and define A to be the supremum of $v(z) + 2A_0 + c(2K, K_2, n)$ over B. Then by (16) and the Jensen formula (see [BT1], [K])

$$T_{u_j}(\epsilon) \leq A$$

on $B(x, r - \epsilon)$. From the convergence $u_j \to u$ we get $(u_j)_\epsilon(z) \to u_\epsilon(z)$ as $j \to \infty$, and thus $T_u(\epsilon) \leq A$ on $B(x, r - \epsilon)$. Letting ϵ to 0 we conclude that $\Delta u \leq A$ on B. Since we can take arbitrarily small r it follows that setting

$$c_0(K, K_1, n) := 2A_0 + c(2K, K_2, n)$$

we get (12). Thus the theorem follows for $f \in C^3(\mathbb{C}^n)$. To get the general case fix a radially symmetric smoothing kernel $\omega \in C_0^\infty(\mathbb{C}^n)$ and put $\omega_j(z) := (1/j^{2n})\omega(z/j)$. Set $f_j^{1/n} = f^{1/n} * \omega_j$. Since

$$-\Delta f_j^{1/n} = -(\Delta f^{1/n}) * \omega_j \leq K_1 f^{1/n} * \omega_j = K_1 f_j^{1/n}$$

the regularizing sequence fulfills the hypothesis of Theorem 2 (with uniform constants) and so by repeating the argument from the preceding paragraphs we get the result.

Now, a classical reasoning (see e.g. [BT1], [K]) leads to the following corollary to the theorem above.

Corollary. *If f is as in Theorem 2 then the solution u has its second order derivatives in L^p_{loc} for any $p \in [1, \infty)$. In particular, by Sobolev's injection, u is of class C^1.*

End of proof of Theorem 1.

If Δu is locally bounded then from positivity of $dd^c u$ it follows that all mixed second order (complex) derivatives are locally bounded. So, having $f > 0$, one can apply [RS, Theorem 1] for the approximating sequence (so that the solutions are smooth) to conclude that also third order mixed derivatives of the functions u_j from formula (16) are uniformly bounded on compact sets. In particular, the sequence $\Delta(\partial u_j / \partial z_k)$ is locally uniformly bounded for any k. Then regularity theory of Poisson's equation (see [GT]) says that u belongs to $C^{2,\alpha}$ for $0 < \alpha < 1$, and further, the application of Schauder estimates gives $u \in C^{k+2,\alpha}$ when $f \in C^{k,\alpha}$. Thus the theorem follows.

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