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The Dual Kähler Cone of Compact Kähler Threefolds

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Introduction.

The Kähler cone and its dual play an important role in the study of compact Kähler manifolds. Therefore it seems natural to ask whether one can read off the (dual) Kähler cone, whether the underlying manifold is projective or not. A classical theorem of Kodaira says that a compact Kähler manifold whose Kähler cone has an interior rational point, is projective. Indeed, a multiple of such a point defines a Hodge metric on the manifold. But not only the Kähler cone itself, also its dual in $H^{n-1,n-1}(X)$ (1.7), where n is the dimension of the Kähler manifold X, is of interest. This is parallel to the projective theory, where both the ample cone and its dual cone $\overline{NE}$, the Mori cone of curves, play an important role. One of the basic questions underlying this paper asks whether projectivity of Kähler manifolds can be expressed in terms of the dual Kähler cone. This question was first posed and treated for surfaces by Huybrechts [Hu99]; an geometric proof for surfaces was given in [OP00]. To be precise, we consider the following

Problem 0.1. *Let X be a compact Kähler manifold of dimension n and suppose that the dual Kähler cone $\mathcal{K}^*(X) \subset H^{n-1,n-1}(X)$ contains a rational interior point. How algebraic is X, i.e. what can be said about the algebraic dimension $a(X)$ of X?*

Since this problems seems to be rather hard, we restrict ourselves to special rational points, namely those represented by *effective* curves. Here is a rather partial answer to the problem.

Theorem 0.2. *Let X be a smooth compact Kähler threefold, $C \subset X$ an effective curve such that $[C] \in \mathrm{Int}\mathcal{K}^*(X)$, the interior of the dual Kähler cone. Then $a(X) \geq 2$ unless X is simple non-Kummer.*

The proof relies on classification theory of compact Kähler threefolds; the term "simple" means that X does not admit a covering family of curves. It is

expected that such a simple threefold should be Kummer, i.e. bimeromorphic to a quotient of a torus by a finite group. This conjecture is however still far from being proven, therefore we need to make the exception in the theorem. It seems not impossible that Theorem (0.2) is actually sharp; in (3.2)-(3.4) we give a potential procedure how to construct a Kähler threefold with $a(X) = 2$ and admitting an irreducible curve which represents an interior rational point of the dual Kähler cone.
Given a curve $C \subset X$, we would like to decide whether its class $[C]$ is an interior point of $\mathcal{K}^*(X)$ or a boundary point. Obvious examples for boundary points are provided by curves contracted by maps $\phi : X \to Y$ to Kähler spaces. It seems therefore very reasonable to expect the following

Conjecture 0.3. *If the normal bundle $N_{C/X}$ is ample, then $[C] \in \mathrm{Int}\mathcal{K}^*(X)$.*

If some multiple of C moves in a family that covers X, then $[C] \in \mathrm{Int}\mathcal{K}^*(X)$ by (4.13). Moreover we prove (0.3) for complete intersections of hyperplane sections. Combining this with problem (0.1) we are lead to

Problem 0.4. *Let $C \subset X$ be a smooth curve in the compact Kähler manifold X with ample normal bundle. How projective is X?*

Our expectation - and we prove this in many cases - is that at least the algebraic dimension $a(X) \geq 2$; and similarly as in (0.2) there seem to be examples where $a(X) = 2$. Of course a similar question can be asked for submanifolds of larger dimension. We prove

Theorem 0.5. *Let $C \subset X$ be a smooth curve of genus g with ample normal bundle N_C in the compact Kähler manifold X. Then*

(1) If $g \leq 1$, then X is projective. The same holds for general g, if $N_C \otimes T_C$ is ample, where T_C denotes the tangent bundle of C.

(2) If $\kappa(X) \leq 0$, then X is projective or simple non-Kummer.

(3) The algebraic dimension $a(X) \neq 1$.

(4) If $\dim X = 3$ and C moves in a covering family, then X is projective.

Finally we investigate threefolds containing an elliptic curve C with ample normal bundle. In particular we prove that either X is rationally connected, so $\pi_1(X) = 0$ or $\pi_1(X) = \mathbb{Z}^2$, so that

$$\mathrm{Im}(\pi_1(C) \to \pi_1(X))$$

has always finite index. One might asks whether this holds in general for submanifolds with ample normal bundle.

1. Preliminaries.

Here we collect basic results and concepts which are essential for this paper. If X is an irreducible reduced compact complex space, we denote by $a(X)$ its algebraic dimension. For this notion, for the notion of an algebraic reduction and related stuff we refer e.g. to [Ue75],[GPR94]. We say that a meromorphic map $f : X \rightharpoonup Y$ is *almost holomorphic*, if it is proper and holomorphic on some open non-empty set $U \subset X$. This means that f has some compact fibers, i.e. theses fibers do not meet the set of indeterminacies.

Definition 1.1. Let X be a compact Kähler manifold. X is called *algebraically connected* if there exists a family of curves (C_t) such that C_t is irreducible for general t and such that every two very general points can be joined by a chain of C_t's.

Then we have Campana's theorem [Ca81]

Theorem 1.2. *Every algebraically connected compact Kähler manifold is projective.*

An immediate consequence of this theorem is

Corollary 1.3. *Every algebraic reduction of a threefold with* $a(X) = 2$ *is almost holomorphic.*

Definition 1.4. A compact Kähler manifold X is *simple*, if there is no proper positive-dimensional subvariety through a very general point of X.

Concerning the structure of simple compact Kähler threefolds one has the

Conjecture 1.5. *Every simple compact Kähler threefold is Kummer, i.e. bimeromorphic to* T/G, *with a torus* T *and a finite group* G *acting on* T.

For the relation to Mori theory in the Kähler case, see [Pe98].

We will need the following classification of compact Kähler threefolds due to Fujiki [Fu83].

Theorem 1.6. *Let* X *be a compact Kähler threefold with* $a(X) \leq 1$.
(1) If $a(X) = 0$ *but not simple, then* X *is uniruled and moreover there exists a holomorphic map* $f : X \to S$ *to a normal surface* S *with* $a(S) = 0$, *such that the general fiber is* $\mathbb{P}_1$.

(2) If $a(X) = 1$ and if a holomorphic model of the algebraic reduction admits a multi-section, then X is bimeromorphic to $(A \times F)/G$, where A is a torus or K3 with $a(A) = 0$, where F is a smooth curve and where G is finite group acting on A and on F and acting on the product diagonally. The map $X \rightharpoonup F/G$ is an algebraic reduction over the smooth curve F/G.
If no holomorphic model of the algebraic reduction admits a multi-section, then f is holomorphic and the general fiber is a torus or K3.

Notations 1.7. *Let X be a compact Kähler manifold of dimension n.*
(1) The Kähler cone $\mathcal{K}(X)$ is the subset of $H^{1,1} := H^{1,1}(X) \cap H^2(X, \mathbb{R})$ consisting of the Kähler classes of X. It is an open cone in $H^{1,1}$.
(2) The dual Kähler cone $\mathcal{K}^*(X)$ is the dual cone in $H^{n-1,n-1} := H^{n-1,n-1}(X) \cap H^{2n-2}(X, \mathbb{R})$ with respect to the natural pairing $H^{1,1}(X) \times H^{1,1}(X) \to \mathbb{R}$.

By $\overline{\mathcal{K}(X)}$ we will denote the closure of the Kähler cone in $H^{1,1}$ with respect to the usual topology. In contrast,the dual Kähler cone is closed by definition.
The following seems to be well-known, however we could not find an explicit reference, so we give a short proof. A general good reference for the theory of current is [Ha77].

Proposition 1.8. *Let X be a compact Kähler manifold. Then $\mathcal{K}^*(X)$ is the cone $\mathcal{P}(X)$ of classes of positive closed currents of bidimension $(1, 1)$.*

Proof. By Demailly-Paun [DP04], Corollary (0.3), $\mathcal{K}^*(X)$ is the closed cone generated by classes of currents $[Y] \wedge \omega^{p-1}$, where Y is an irreducible analytic set of dimension p and ω a Kähler form. Of course, $[Y]$ is the current given by integration over Y. Since all these currents $[Y] \wedge \omega^{p-1}$, are positive, $\mathcal{K}^*(X)$ is contained in the cone $\mathcal{P}(X)$. The other inclusion being clear, the assertion follows. Q.E.D.

2. Blow-ups and Galois covers.

We investigate the behaviour of interior points of the dual Kähler cone under blow-ups and special Galois covers. The results will be needed in sect. 3. In this section X always denotes a compact Kähler **threefold** unless otherwise stated.

Proposition 2.1. *Let $\pi : \hat{X} \to X$ be the blow-up along a submanifold Y. If $\mathcal{K}^*(X)$ contains a rational interior point, then so does $\mathcal{K}^*(\hat{X})$.*

Before giving the proof, we explain the idea in the simple case that the rational interior point, which is in general represented by a positive closed current, is represented by an effective curve. So let $C = \sum a_i C_i$ be an effective curve, $a_i > 0$ such that $[C] \in \mathrm{Int}\mathcal{K}^*(X)$. In case $C_i \neq Y$, we let $\hat{C}_i \subset \hat{X}$ be the strict transform of C_i; if $C_i = Y$, we let $\hat{C}_i$ be a section of $\pi|E \to Y$.
Then $\sum a_i[\hat{C}_i] + m[l]$ is an interior point of $\mathcal{K}^*(\hat{X})$ for a suitable rational m. Here l is a fiber of $\pi|E$ if $\dim Y = 1$ resp. a line in $E \simeq \mathbb{P}_2$ if $\dim Y = 0$.
In the general case however, the arguments get more involved since "strict transforms of currents" cannot be defined in general [Me96,p.52/53]. This surprising point was explained to us by J.P.Demailly.

Proof. Let $\alpha \in \mathrm{Int}\mathcal{K}^*(X) \cap H^4(X, \mathbb{Q})$ and represent α by a positive current T. Let $E = \pi^{-1}(Y)$ be the exceptional divisor and l either a line in E, if $\dim Y = 0$ or a ruling line, if $\dim Y = 1$. We fix a section $\hat{Y} \subset E$ in case E is ruled. If $E = \mathbb{P}_2$, then we set formally $\hat{Y} = 0$.
We consider the canonical decomposition

$$T = \chi_Y T + \chi_{X \setminus Y} T;$$

χ_Y denoting the characteristic function of Y. All currents occuring in this decomposition are closed. If $\dim Y = 0$, then $\chi_Y T = 0$; if $\dim Y = 1$, then by Siu's theorem [Si74], $\chi_Y T = a T_Y$, where $a \geq 0$ and T_Y is the current "integration over Y". In particular $d\chi_{X \setminus Y} T = d\chi_Y T = 0$. Let $T' := \chi_{X \setminus Y} T$ for simplicity. We will proceed in three steps.

1. $\pi^*([T']) \in \mathcal{K}^*(\hat{X})$ (possibly on the boundary);
2. $\hat{\alpha} := a[\hat{Y}] + b[l] + \pi^*([T']) \in \mathrm{Int}\mathcal{K}^*(\hat{X})$ for $b > 0$;
3. $\hat{\alpha}$ can be chosen rational.

Starting with the proof of (1), let $\hat{\omega}$ be a Kähler form on $\hat{X}$, and we need to show

$$\pi^*([T']) \cdot \hat{\omega} \geq 0. \qquad (*)$$

Let $S := \pi_*(\hat{\omega})$; then S is a positive closed current on $\hat{X}$ which is smooth outside Y. We have

$$[\hat{\omega}] = \pi^*[S] + \lambda[E],$$

with some negative number λ. Therefore

$$\pi^*[T'] \cdot \hat{\omega} = \pi^*[T'] \cdot \pi^*[S] = [T'] \cdot [S].$$

We show that

$$[T'] \cdot [S] \geq 0.$$

In fact, using Demailly's regularization theorem, we can write [S] as a weak limit of smooth positive closed forms Θ_ϵ in the same cohomology class as T' such

$$\Theta_\epsilon \geq -\lambda_\epsilon u - O(\epsilon)\eta,$$

where η is a positive $(1,1)-$form, u a suitable semi-positive $(1,1)-$form and (λ_ϵ) a decreasing family of non-negative smooth functions for $0 < \epsilon < 1$ converging pointwise to 0 on $X \setminus Y$ and to the Lelong number $\nu(S,x)$ for $x \in Y$. We conclude

$$[S] \cdot [T'] = [\Theta_\epsilon] \cdot [T'] = T'(\Theta_\epsilon) \geq -\int_X \lambda_\epsilon u \wedge T' - O(\epsilon).$$

Then the monotone convergence theorem gives the claim since $\chi_Y T' = 0$.
(2) By (1) we know already that $\hat{\alpha} \in \mathcal{K}^*(\hat{X})$. If it is not in the interior, then there exists $\hat{\beta} \in \partial\mathcal{K}(\hat{X})$ such that

$$\hat{\alpha} \cdot \hat{\beta} = 0.$$

Hence

$$0 = a[\hat{Y}] \cdot \hat{\beta} + b[l] \cdot \hat{\beta} + \pi^*([T']) \cdot \hat{\beta}. \qquad (a)$$

Since all summands are semi-positive, we conclude first

$$[\hat{Y}] \cdot \hat{\beta} = [l] \cdot \hat{\beta} = 0. \qquad (b)$$

Now (b) implies that $\hat{\beta} = \pi^*(\beta)$, with $\beta \in \overline{\mathcal{K}(X)}$, as we shall see in a moment (claim (c)). Since by (a) and (b) we have $[T'] \cdot \beta = 0$, and since $[T]$ is an interior point of $\mathcal{K}^*(X)$, we conclude

$$[Y] \cdot \beta > 0$$

(this already settles the case $\dim Y = 0$). But then $[\hat{Y}] \cdot \hat{\beta} > 0$ (represent β by a form), contradiction.
It remains to prove claim (c) in order to settle (2). It is clear by (b) that $\hat{\beta} = \pi^*(\beta)$, it only remains to show that $\beta \in \overline{\mathcal{K}(X)}$. Notice that $\beta \cdot Y = 0$.

Assuming the contrary, there exists $[T] \in \mathcal{K}^*(X)$, such that $\beta \cdot [T] < 0$. By virtue of the decomposition

$$T = aT_Y + \chi_{X \setminus Y} T$$

we may assume $\chi_Y T = 0$. Hence by (1)

$$\hat{\beta} \cdot \pi^*[T] = \beta \cdot [T] < 0,$$

with $\pi^*[T] \in \mathcal{K}^*(\hat{X})$, contradicting $\hat{\beta} \in \overline{\mathcal{K}(\hat{X})}$. So (c) and therefore (2) are proved.
(3) It remains to show that $\hat{\alpha}$ can be made rational by chosing b appropriately. This is however completely obvious, having in mind that $\pi^*[T]$ is already rational. Q.E.D.

In case T is the integration over an effective curve, the proof of (2.1) also shows - as already mentioned

Proposition 2.2. *Let $\pi : \hat{X} \to X$ be as in (2.1), $C \subset X$ an effective curve. If $[C] \in \mathrm{Int}\mathcal{K}^*(X)$, then there exists an effective curve $\hat{C} \subset \hat{X}$ with $\pi_*(\hat{C}) = C$ as cycle such that $[\hat{C}] \in \mathrm{Int}\mathcal{K}^*(\hat{X})$.*

The "converse" of (2.1) is also true:

Proposition 2.3. *Let $\pi : \hat{X} \to X$ be as in (2.1). If $\mathcal{K}^*(\hat{X})$ contains an interior rational point, so does $\mathcal{K}^*(X)$.*

Proof. Let $\alpha \in \mathrm{Int}\mathcal{K}^*(\hat{X}) \cap H^4(\hat{X}, \mathbb{Q})$ and represent it by a positive current $\hat{T}$. Let $T = \pi_*(\hat{T})$. Then we claim that

$$[T] \in \mathrm{Int}\mathcal{K}^*(X) \cap H^4(X, \mathbb{Q}).$$

It is clear that $[T]$ is rational. Notice that

$$\pi^* : H^q(X, \mathbb{R}) \to H^q(\hat{X}, \mathbb{R})$$

is injective, since π is surjective. Hence π^* maps $\overline{\mathcal{K}(X)}$ *into* $\overline{\mathcal{K}(\hat{X})}$, therefore

$$T(\beta) = \hat{T}(\pi^*(\beta)) > 0$$

for all $\beta \in \overline{\mathcal{K}(X)} \setminus \{0\}$, proving our claim. Q.E.D.

Again we have the following special case:

Proposition 2.4. *Let $\hat{C} \subset \hat{X}$ be an effective curve, $C = \pi_*(\hat{C})$. If $[\hat{C}] \in \mathrm{Int}\mathcal{K}^*(\hat{X})$, then $[C] \in \mathrm{Int}\mathcal{K}^*(X)$.*

The same reasoning as in (2.3) actually shows (in all dimensions)

Proposition 2.5. *Let $f : X \to Y$ be a surjective holomorphic map of compact Kähler manifolds of positive dimension, let T be a positive closed current of bidimension $(1,1)$ on X such that $[T] \in \mathrm{Int}\mathcal{K}^*(X)$. Then $[f_*(T)] \in \mathrm{Int}\mathcal{K}^*(Y)$.*

Putting things together (including (2.1)) and applying [OP00], we obtain

Proposition 2.6. *Let X be a compact Kähler threefold, S a smooth compact surface and $f : X \rightharpoonup S$ a dominant meromorphic map. Suppose that $\mathrm{Int}\mathcal{K}^*(X) \cap H^4(X, \mathbb{Q}) \neq \emptyset$. Then S is projective.*

Finally we need also to consider very special singular threefolds. The Kähler assumption on X we make is just that X has a Kähler desingularisation. We do not try to define the closure of the Kähler cone and its dual but just give the minimum amount of definition we need.

Definition 2.7. Let X be a normal compact complex space, $\dim X = 3$. Let $\pi : \hat{X} \to X$ be a desingularisation with $\hat{X}$ Kähler. Let $C \subset X$ be an effective curve. We say that *"C is in the interior of the dual Kähler cone of X" iff there exists an effective curve $\hat{C} \subset \hat{X}$ such that $\pi_*(\hat{C}) = C$ and such that $[\hat{C}] \in \mathrm{Int}\mathcal{K}^*(\hat{X})$.*

By (2.2) and (2.4) this definition does not depend on the choice of the Kähler desingularisation. The main point is now

Proposition 2.8. *Let X be a compact Kähler manifold, Y a normal compact complex space and $h : X \to Y$ a Galois covering with Galois group G, étale in codimension 1. Let $C = \sum a_i C_i$ be an effective curve in Y such that $C_i \not\subset \mathrm{Sing} Y$ for all i. Suppose that C is an interior point of the dual Kähler cone of Y. Then there exists a G-stable effective 1-cycle $C' \subset X$ with $h_*(C') = C$ set-theoretically such that $[C'] \in \mathrm{Int}\mathcal{K}^*(X)$.*

Proof. First of all notice that Y admits a Kähler desingularisation $\pi : \hat{Y} \to Y$. The existence of C' with $h(C') = C$ is clear; moreover we can make it G-stable by averaging. Hence we may assume $h_*(C') = C$ as cycles.

Suppose $[C'] \in \partial \mathcal{K}^*(X)$. Then we find some $\alpha \in \overline{\mathcal{K}(X)} \setminus \{0\}$ with $C' \cdot \alpha = 0$. Substituting possibly α by

$$\tilde{\alpha} = \sum_{g \in G} g^* \alpha \in \overline{\mathcal{K}(X)} \setminus \{0\},$$

we may assume from the beginning that α is $G-$invariant, and we still have $C' \cdot \alpha = 0$ since C' is G-stable. Thus α is of the form

$$\alpha = h^*(\beta).$$

We are going to show that this contradicts $[C]$ to be an interior point of $\mathcal{K}^*(Y)$. By definition and our assumption that $[C]$ is an interior point, we have an effective curve $\hat{C} \subset \hat{Y}$ which is an interior point of $\mathcal{K}^*(\hat{Y})$ and projects to C. On the other hand we claim

$$\pi^*(\beta) \in \overline{\mathcal{K}(\hat{Y})} \setminus \{0\}, \tag{1}$$

and

$$\hat{C} \cdot \pi^*(\beta) = 0. \tag{2}$$

This will give a contradiction.
To verify (1) we may assume that α is represented by a Kähler form ω (then take closure). By averaging, we can write

$$\omega = h^*(\omega'),$$

where ω' is a Kähler form on Y_{reg} but ω' extends to all of Y as explained now. Fix a point $y_0 \in Y$ and a small neighborhood U and consider $V = h^{-1}(U)$. Then on V we can write

$$\omega = i\partial\overline{\partial}\phi$$

with a strictly plurisubharmonic function ϕ. By substituting ϕ by $\sum_{g \in G} g^* \phi$ and observing $\partial\overline{\partial}(g^*\phi) = g^*\partial\overline{\partial}\phi$, we may assume ϕ to be G-invariant, hence we can write

$$\phi = h^*(\phi')$$

with a plurisubharmonic function ϕ' on U. Therefore

$$\omega' = i\partial\overline{\partial}\phi'$$

on U_{reg} with an extendable function ϕ'. From this description it immediately follows that $\pi^*(\omega')$ extends to a semipositive $(1,1)-$form on all of $\hat{Y}$, proving (1).

(2) We choose a sequence of smooth blow-ups $\rho : Z \to X$ inducing a map $\sigma : Z \to \hat{Y}$ such that

$$\pi \circ \sigma = h \circ \rho.$$

Choose an effective curve $\tilde{C} \subset Z$ with $\rho_*(\tilde{C}) = C'$. Then $h_* \rho_*(\tilde{C}) = C$, hence $\pi_* \sigma_*(\tilde{C}) = C$ and thus $\hat{C}$ and $\sigma_*(\tilde{C})$ differ only by cycles supported on the exceptional locus of π. Consequently

$$\hat{C} \cdot \pi^*(\beta) = \tilde{C} \cdot \sigma^* \pi^*(\beta) = \tilde{C} \cdot \rho^*(\alpha) = \hat{C} \cdot \alpha = 0,$$

proving (2). Q.E.D.

3. Interior points and algebraicity.

Here we investigate the influence of a curve which is an interior point of the dual Kähler cone on the algebraicity on the underlying Kähler manifold.

Theorem 3.1. *Let X be a compact Kähler threefold, $C \subset X$ an effective curve. If $[C] \in \mathrm{Int}\mathcal{K}^*(X)$, then $a(X) \geq 2$, unless X is simple non-Kummer.*

Proof. Suppose that $a(X) \leq 1$. We have to prove that X is simple non-Kummer.
(1) Case $a(X) = 0$.
If X is not at the same time simple and non-Kummer, then by (1.6), X is uniruled or Kummer. In the first case we have a meromorphic map $f : X \rightharpoonup S$ to a smooth surface S with $a(S) = 0$, contradicting (2.6). So X is Kummer, hence X is bimeromorphically equivalent to T/G with a torus T and a finite group G. By (2.2) we may assume that there is a holomorphic bimeromorphic map $f : X \to T/G$. Then $\dim f(C) = 1$. This is either seen by desingularising T/G, making f again holomorphic and applying (2.2) or by directly using the arguments of (2.8). Hence T contains a compact curve which may be assumed G-stable. Since $a(T) = 0$, T has the structure of an elliptic fiber bundle $h : T \to T'$ to a torus T' with $a(T') = 0$. Now G acts also on T' and we have a map $T/G \to T'/G$. In total we obtain a map $X \to T'/G$ contradicting (2.6).

(2) Case $a(X) = 1$.
Here we may assume that we have a *holomorphic* algebraic reduction $f : X \to B$ to the smooth curve B. By (2.5), $f(C) = B$, so f has a multisection. By (1.6), X is therefore bimeromorphic to $(A \times F)/G =: X'$, where

A is a torus or K3 with $a(A) = 0$, F is a smooth curve and G a finite group acting diagonally on $A \times F$. Thus we have a holomorphic map

$$(A \times F)/G \to A/G$$

and thus a meromorphic dominant map $X \rightharpoonup A/G$. This contradicts Theorem 2.5. It is also possible to avoid the use of 2.5 (and thus of [OP00]) and argue directly by 2.8. Q.E.D.

We are now addressing the question whether it is possible to have a compact Kähler threefold X with $a(X) = 2$ admitting a smooth curve C such that $[C] \in \mathrm{Int}\mathcal{K}^*(X)$. In this direction we prove

Proposition 3.2. *Let $f : X \to S$ be a surjective holomorphic map having connected fibers from the smooth compact Kähler 3-fold X to the smooth projective surface S. Let $C \subset S$ be an irreducible curve with $C^2 > 0$. Suppose that $X_C = f^{-1}(C)$ is Moishezon and irreducible. Let $B \subset X_C$ be a general irreducible curve. Then $[B] \in \mathrm{Int}\mathcal{K}^*(X)$.*

A general curve $B \subset X_C$ is understood to be the image of a general hyperplane section $\hat{B} \subset \hat{X}_C$.

Proof. Suppose $[B] \in \partial\mathcal{K}^*(X)$. Then there exists a non-zero class $\alpha \in \partial\mathcal{K}(X)$, such that

$$\alpha \cdot B = 0.$$

We claim

$$\alpha \cdot X_C = 0. \tag{1}$$

Let $i : X_C \to X$ denote the inclusion and let $\pi : \hat{X}_C \to X_C$ be a desingularisation; set $g = \pi \circ i$. Then

$$g^*(\alpha) \cdot g^*([B]) = 0.$$

Now we can write

$$g^*([B]) = [\hat{B}] + \sum a_i E_i$$

with $a_i \geq 0$ and E_i the exceptional curves of π. Hence we conclude that

$$g^*(\alpha) \cdot \hat{B} = 0.$$

Now $g^*(\alpha) \in \overline{\mathcal{K}(\hat{X}_C)}$, thus the ampleness of $\hat{B}$ gives $g^*(\alpha) = 0$, i.e. $\alpha \cdot X_C = 0$, proving (1).

Since $C^2 > 0$ and C is irreducible, some multiple mC is generated by global sections (by a theorem of Zariski, see [Ha70,p.65]). So mC carries a metric whose curvature form ω is semipositive and positive outside finitely many curves. By (1) we obtain

$$\alpha \cdot f^*(\omega) = 0,$$

so that

$$\alpha \cdot f^*(\omega) \cdot \eta = 0 \tag{2}$$

for all Kähler forms η on X. In order to exploit this, we represent α by a positive closed $(1,1)$-current T. Let S_0 be the maximal open subset of S over which f is a submersion and let $X_0 = f^{-1}(S_0)$. Then a small local calculation in X_0 shows that $f^*(\omega) \wedge \eta$ is a strictly positive $(2,2)$-form on X_0. By (2) we have

$$T(f^*(\omega) \wedge \eta) = 0,$$

hence $\mathrm{supp}(T) \subset X \setminus X_0$. Since $\dim(X \setminus X_0) = 2$, Siu's well-known theorem says that

$$T = \sum \lambda_i Z_i,$$

where the Z_i are irreducible components of $X \setminus X_0$, where $\lambda_i > 0$ and where the right hand side of course means integration over the cycle. We fix a positive integer m such that mC is generated by sections and therefore defines a map $g : S \to S'$ to a normal projective surface S'. Now let $C' \subset |mC|$ be a general member. Then (2) yields

$$\sum \lambda_i Z_i \cdot X_{C'} = 0.$$

Since the support of both divisors $\sum \lambda_i Z_i$ and $X_{C'}$ do not have common components, it follows that $Z_i \cap X_{C'} = \emptyset$ for all i. Now either $\dim f(Z_i) = 0$ or $f(Z_i) \cdot C' = 0$, whence $f(Z_i)$ has to be contracted by g. So

$$\dim(g \circ f)(Z_i) = 0$$

for all i. This is incompatible with $\alpha = \sum \lambda_i [Z_i] \in \overline{\mathcal{K}(X)}$, and thus producing a contradiction. In fact, we decompose $\sum \lambda_i Z_i$ into connected components and write with the appropriate multiplicities:

$$\sum \lambda_i Z_i = \sum \mu_j W_j.$$

So the W_j are pairwise disjoint and are supported in fibers of $g \circ f$ so that clearly all W_j are not nef (recall that the general fiber of $g \circ f$ has dimension 1). So $\alpha \notin \overline{\mathcal{K}(X)}$. Q.E.D.

With the same proof we can also deal with the case that X_C is irreducible:

Proposition 3.3. *If in 3.2 X_C is reducible, say $X_C = A_1 \cup \ldots \cup A_r$, then we take $B_i \subset A_i$ general and put $B = \sum B_i$. Again $[B] \in \mathrm{Int}\mathcal{K}^*(X)$.*

Corollary 3.4. *Let X be a smooth compact Kähler 3-fold with $a(X) = 2$ with holomorphic algebraic reduction $f : X \to S$ to a smooth projective surface S. Let*

$$\Delta = \overline{\{s \in S | X_s \text{ singular, not multiple elliptic }\}}.$$

Suppose Δ contains an irreducible component C with $C^2 > 0$. Then X carries a (possibly reducible) curve B such that $[B] \in \mathrm{Int}\mathcal{K}^*(X)$. If $f^{-1}(C)$ is irreducible, i.e. the general fiber over C is irreducible (rational), then B can be taken irreducible.

Proof. We only have to show that $X_C = f^{-1}(C)$ is Moishezon. This however is clear since X_C is covered by rational curves, the fibers of f over C. Q.E.D.

Remark 3.5. (1) It remains to explicitly construct a Kähler 3-fold X with $a(X) = 2$ (with holomorphic algebraic reduction; this can always be achieved by blowing up X) fulfilling the requirements of (3.4). It seems unimaginable that such a 3-fold should not exist; non-existence would simply mean that any component C of Δ (as in (3.4)) has $C^2 \leq 0$. On the other hand, an explicit construction is not obvious. Of course there are projective elliptic 3-folds with a curve $C \subset \Delta$ and $C^2 > 0$. So one might try to deform X in a non-algebraic Kähler 3-fold keeping the curve C.
(2) It is easy to construct elliptic Kähler 3-folds $f : X \to S$ of algebraic dimension 2 admitting a curve $C \subset \Delta$ such that $C^2 = 0$. In fact, let $f_1 : S_1 \to B$ be an elliptic Kähler surface with $a(S_1) = 1$ and $f_2 : S_2 \to B$ a projective surface. Suppose that f_1 has some non-elliptic fibers and that the singular loci of f_1 and f_2 in B are disjoint. Then

$$X = S_1 \times_B S_2$$

is a smooth Kähler 3-fold with $a(X) = 2$ and algebraic reduction $f : X \to S_2$. Now $\Delta_f = \Delta_{f_1} \times_B S_2$ so that Δ_f contains of fibers of f_2.
(3) Here is a possible way how to construct $g : X \to S$ as required in (3.4). The procedure is as follows: we start with a fibration $f : X \to C$ from the compact Kähler 3-fold X with $a(X) = 2$ to the smooth curve C. Suppose that $a(X_c) = 1$ for the general fiber X_c of f such that the algebraic reduction has non-elliptic singular fibers. Then we can form the relative algebraic reduction $X \rightharpoonup S \to C$. Suppose that S is smooth and $f : X \rightharpoonup S$ is actually

holomorphic. Then f is an elliptic fibration and S is projective. Moreover "in general" Δ should dominate C. This gives hope to find $C \subset \Delta$ with $C^2 > 0$. The difficulty is to find a starting fibration $g : X \to C$; the possible most natural choice would be require X_c to be a Kummer surface and that g is a submersion. Then the above construction will work and therefore we are reduced to the following

Problem 3.6. *Does there exist a compact Kähler threefold Z with a submersion h to a curve C such that the general fiber is a torus of algebraic dimension 1?*

It is not so difficult to construct a non-Kähler submersion $Z \to C$ such that the general fiber is a torus of algebraic dimension 1, but the Kähler property seems difficult to achieve.

Here is a mild restriction for Kähler 3-folds X containing a curve C which is an interior point of the dual Kähler cone.

Proposition 3.7. *Let X be a smooth compact Kähler 3-fold with algebraic reduction $f : X \rightharpoonup S$ to the smooth projective surface S. Suppose* $\mathrm{Int}\mathcal{K}(X)$ *contains a (possibly reducible) curve. Then $q(X) = q(S)$.*

Proof. Assume $q(X) > q(S)$. By considering a holomorphic model for f and appyling the Leary spectral sequence, we see immdiately that $q(X) = q(S) + 1$. Let $q = q(S)$. If $\alpha : X \to \alpha(X) \subset A_{q+1}$ and $\beta : S \to \beta(S) \subset B_q$ denote the Albanese maps, then we conclude that $\alpha(X)$ cannot be projective, otherwise we would have $q(S) = q(X)$. Let $\gamma : \alpha(X) \to \beta(Y)$ be the induced map. Since $\beta(Y)$ is projective, γ is not generically finite. It is important to notice that $\dim \gamma\alpha(C) = 1$ by (2.4).

We are thus left with the following cases:
(a)$\dim \alpha(X) = 2$. Then $\alpha(C)$ is a multi-section of γ, making $\alpha(X)$ projective.
(b) $\dim \alpha(X) = 3$. Then necessarily $\dim \beta(Y) = 2$ and now γ has a partial multi-section $\alpha(X)$. But γ is the reduction of the canonical projection $A_{q+1} \to B_q$ which (up to finite étale cover) is an elliptic fiber bundle without transverse curves; contradiction. Q.E.D.

4. Interior points and the normal bundle.

In this section we investigate curves in Kähler manifolds, their normal bundles and the connection with the dual Kähler cone. Here it is not always necessary to restrict to low dimensions.

Conjecture 4.1. *Let X be a compact Kähler manifold and $C \subset X$ a smooth curve with ample normal bundle. Then $[C] \in \mathrm{Int}\mathcal{K}^*(X)$.*

As a direct consequence of Theorem 3.1 in dimension three, we are lead to

Conjecture 4.2. *Let X be a compact Kähler manifold, $Y \subset X$ a positive-dimensional compact submanifold with ample normal bundle. Then $a(X) \geq \dim Y + 1$.*

In our feeling, (4.2) seems more accessible than (4.1). Notice that (4.2) holds for hypersurfaces.

In this section we verify (4.1) and (4.2) in special cases. First we notice that (4.1) and (4.2) hold in the surface case, even without Kähler assumption.

Proposition 4.3. *Let X be a smooth compact surface, $C \subset X$ an effective curve with ample normal bundle. Then X is projective and $[C] \in \mathrm{Int}\ \mathcal{K}^*(X)$.*

Proof. The first assertion is classical using Riemann-Roch and the fact that Moishezon surfaces are projective.
The second assertion is clear if C is an ample divisor on X. If C is not ample, C is big and nef, in particular $C^2 > 0$. Now assume that $[C]$ is not an interior point of $\mathcal{K}^*(X)$. Then we find $0 \neq \alpha \in \overline{\mathcal{K}(X)}$ such that

$$C \cdot \alpha = 0.$$

Hence the Hodge Index Theorem for $H^{1,1}$ yields $\alpha \equiv mC$ with a positive real number m. So $C \cdot \alpha = mC^2 = 0$, contradiction. Q.E.D.

Now we prove a special case of conjecture 4.1.

Theorem 4.4. *(1) Let $C \subset X$ be an effective curve in a smooth projective threefold X. Suppose that there is a smooth ample surface $S \subset X$ with $C \subset S$ such that $N_{C|S}$ is ample (e.g. C is a complete intersection $C = S \cap S'$ with ample S'). Then $[C] \in \mathrm{Int}\ \mathcal{K}^*(X)$.*
(2) Let C be a smooth complete intersection curve of hyperplane sections in the projective manifold X. Then $[C] \in \mathrm{Int}\ \mathcal{K}^*(X)$.

Proof. (1) The inclusion $i : S \to X$ yields a map $i^* : \overline{\mathcal{K}(X)} \to \overline{\mathcal{K}(S)}$. Since S is ample, i^* is injective. Take $\alpha \in \overline{\mathcal{K}(X)} \setminus \{0\}$ and represent it by a form ω. Then $\int_C \omega = \int_C i^*(\omega)$, and since $[i^*(\omega)] \in \overline{\mathcal{K}(S)}$, we conclude from (4.3) that $\alpha \cdot C = \int_C \omega > 0$.
(2) follows now by induction. Q.E.D.

Concerning Conjecture 4.2 we prove

Theorem 4.5. *Let $C \subset X$ be a smooth curve with ample normal bundle N_C in the compact Kähler manifold X. Assume moreover that $N_C \otimes T_C$ is ample. Then X is projective.*

Proof. We check the projectivity of X by showing

$$H^2(X, \mathcal{O}_X) = H^0(X, \Omega_X^2) = 0. \qquad (*)$$

This is shown by adopting the method of [PSS99, 2.1]: by power series expansion, one has the inequality

$$h^0(X, \Omega_X^2) \leq \sum_{k=0}^{\infty} h^0(C, S^k N_C^* \otimes \Omega_X^2 | C). \qquad (**)$$

Taking $\bigwedge^2$ of the sequence

$$0 \to N_C^* \to \Omega_X^1 | C \to \Omega_C^1 \to 0,$$

we obtain a sequence

$$0 \to \bigwedge^2 N_C^* \to \Omega_X^2 | C \to \Omega_C^1 \otimes N_C^* \to 0.$$

Tensoring with $S^k N_C^*$ for $k \geq 0$, taking cohomology and having in mind our ampleness assumptions, we obtain the vanishing

$$H^0(C, S^k N_C^* \otimes \Omega^1 \otimes N_C^*) = 0$$

for $k \geq 0$ from which (*) follows by virtue of (**). Q.E.D.

We remark that in most cases in Theorem 4.5 it is easily seen that $[C] \in \mathrm{Int}\mathcal{K}^*(X)$.

Corollary 4.6. *Let X be a compact Kähler manifold, $C \subset X$ a smooth curve with ample normal bundle. If $g(C) \leq 1$, then X is projective.*

Of course one can formulate more general versions of (4.5) e.g. for reduced locally complete intersection curves; we leave that to the reader. We will prove more on the structure of X (in case C is elliptic and $\dim X = 3$) in sect.5. Observe that (4.5) and (4.6) hold also in the following more general context: it is sufficient to assume that X is in class $\mathcal{C}$, i.e. bimeromorphic to a Kähler manifold. Then the conclusion is that X is Moishezon.

Theorem 4.7. *Let X be a compact Kähler threefold and $C \subset X$ a smooth curve. Suppose that the normal bundle N_C is ample.*
(1) If $\kappa(X) \leq 0$, then X is projective or X is simple non-Kummer.
(2) $a(X) \neq 1$.

Proof. (I) We first show the following general statement. Suppose $f : X \to Y$ is a surjective holomorphic map, then $\dim f(C) = 1$.
In fact, suppose $C \subset f^{-1}(y)$ for some $y \in Y$. Let F be the complex-analytic fiber over y. Now choose k maximal such that the $k-$th infinitesimal neighborhood C_k is still a subspace of F. Then we obtain an exact sequence of conormal sheaves

$$N^*_{F/X}|C_k \to N^*_{C_k/X} \to N^*_{C_k/F} \to 0.$$

Restricting to C and observing that

$$N^*_{F/X}|C \to N^*_{C_k/X}|C$$

does not vanish by the choice of k, the spannedness of $N^*_{F/X}$ provides a non-zero section of $N^*_{C_k/F}|C$. Thus $S^k N^*_{C/X}$ has a non-zero section, contradicting the ampleness of N_C.

(II) We next show that $a(X) \neq 1$. We first claim that every resolution of indeterminacies of the algebraic reduction $f : X \rightharpoonup B$ admits a multi-section. In fact, otherwise f itself is holomorphic and would contract C contradicting (I). Then by (1.6), X is bimeromorphically of the form $(A \times F)/G$, with $a(A) = 0$ and F a curve. So A is K3 or a torus. In case A is K3, we subsitute A by the blow-down of all $(-2)-$curves. So we may assume that A does not contain any curve at the expense that A may be singular. Therefore the induced map

$$h : X \rightharpoonup A/G$$

is actually holomorphic. In particular h contracts C which again contradicts (I).

(III) Now suppose $\kappa(X) = -\infty$ and that X is not both simple non-Kummer. Suppose moreover $a(X) = 0$. In that case we have by (1.6) a holomorphic map $f : X \to S$ to a normal surface S with $a(S) = 0$, the general fiber of f being a smooth rational curve. Arguing as in (II), we may assume that S does not contain any curve. So $f(C)$ is a point, contradicting (I). So $a(X) \neq 0$.
By (II) it therefore remains to exclude $a(X) = 2$. So suppose $a(X) = 2$ and let $f : X \rightharpoonup S$ be an algebraic reduction. By [CP00], X is uniruled, so we can form the rational quotient $h : X \rightharpoonup Z$ of a covering family of rational curves. The fibers of h being rationally connected, we conclude that $\dim Z > 1$, otherwise X would be projective. Therefore $\dim Z = 2$ and since $a(X) = 2$, we must have $a(Z) = 1$. Using the algebraic reduction $Z \to B$ we obtain a meromorphic map $g : X \rightharpoonup B$. The general fiber of g is bimeromorphically a $\mathbb{P}_1$-bundle over an elliptic curve, hence algebraic. Again a holomorphic model of g cannot have a multi-section by (1.2), hence g must be holomorphic and again provides a contradiction. Q.E.D.

(IV) We are left with $\kappa(X) = 0$ and suppose X is not both simple non-Kummer. By [CP00], Theorem 8.1, X has a birational model X' which admits a finite cover, étale in codimension 1, say $\tilde{X}$, such that $\tilde{X}$ is either a torus or a product of an elliptic curve with a K3 surface.
(a) $a(X) = 0$. Then X is bimeromorphically T/G with T a torus. If T has no curves, then we actually have a holomorphic map $h : X \to T/G$ and conclude by (I). If T admits a curve, we have an elliptic fiber bundle structure $g : T \to T'$ to a 2-dimensional torus T' with $a(T') = 0$ and every curve in T is a fiber of g. Then there is an induced map $T/G \to T'/G$, and T'/G has no curves, so that the meromorphic map $X \rightharpoonup T'/G$ is actually holomorphic. We conclude by (I).
(b) Since $a(X) \neq 1$ by (II), we are left with $a(X) = 2$.
(b.1) Suppose first $\tilde{X} = W \times E$ with W a K3-surface, $a(W) = 1$ and E elliptic. With $X' = \tilde{X}/G$, we note that the $G-$action on $\tilde{X}$ is diagonal, hence we have by composing with the algebraic reduction $h : W/G \to C$ a meromorphic map

$$g : X \rightharpoonup C$$

to a rational curve C. Since h has no multi-sections, any multi-section of a resolution of $X \to W/G$ must be mapped to a fiber of h. Hence g is almost holomorphic, hence holomorphic. Since $\dim g(C) = 0$, we conclude by (I).
(b.2) Suppose finally that $\tilde{X}$ is a torus. Then the algebraic reduction of $\tilde{X}$ provides an elliptic bundle structure $\tilde{X} \to \tilde{B}$ to an abelian surface without

multi-sections. We obtain a map $\tilde{X}/G \to \tilde{B}/G$ without multi-sections and $X \sim \tilde{X}/G$. So the induced map $X \rightharpoonup \tilde{B}/G$ must be holomorphic and will contract C, contradicting (I). Q.E.D.

Corollary 4.8. *Let X be a compact Kähler threefold, $C \subset X$ a smooth curve. Assume that the normal bundle N_C is ample. Then $a(X) \geq 2$ unless X is simple non-Kummer.*

Corollary 4.9. *Let C be a smooth curve in the compact Kähler threefold with ample normal bundle. If $K_X \cdot C < 0$, then X is projective unless X is simple non-Kummer with $\kappa(X) = -\infty$.*

Proof. It is sufficient to prove $\kappa(X) = -\infty$; then we apply (4.7). This follows essentially from [PSS99,2.1] and is parallel to the argument in (4.5). Namely, we claim that for all $t \in \mathbb{N}$ and all $k \in \mathbb{N}$ the following vanishing holds

$$H^0(C, S^k N_C^* \otimes K_X^t|C) = 0. \qquad (*)$$

This is clear since N_C is ample and $K_X|C$ is negative by adjunction. On the other hand power series expansion gives

$$h^0(X, K_X^t) \leq \sum_{i=0}^{\infty} h^0(C, S^k N_C^* \otimes K_X^t|C).$$

Thus (*) gives $\kappa(X) = -\infty$. Q.E.D.

For the cases $\kappa(X) = 1, 2$ we need a stronger assumption than just the existence of the family (C_t).

Theorem 4.10. *Let X be a compact Kähler threefold, $C \subset X$ a smooth curve with ample normal bundle. Suppose that C moves in a family (C_t) that* **covers** *X. Then X is projective.*

Proof. By (4.7) we may assume $a(X) \geq 2$. So we have a meromorphic elliptic fibration $f : X \rightharpoonup S$ to a projective surface S. Since $g(C_t) \geq 2$, we conclude $\dim f(C_t) = 1$ for general t and therefore the C_t make X algebraically connected, thus X is projective, contradiction. Q.E.D.

Example 4.11. In general curves in the interior of the dual Kähler cone of course are far from having ample normal bundle. For a trivial example take a threefold X with $b_2(X) = 1$ containing a $(-1,-1)-$curve, i.e. a smooth rational curve C with normal bundle $N_C = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Another interesting class of curves are the connecting curves:

Definition 4.12. A family (C_t) of $1-$cycles (say with general C_t reduced) is a connecting family if and only if any general points $x, y \in X$ can be joined by a chain of curves of type C_t.

By (1.2), any variety X carrying a connecting family of $1-$cycles is projective.

Theorem 4.13. *Let X be a compact Kähler manifold, (C_t) a connecting family, then $[C_t] \in \mathrm{Int}\mathcal{K}^*(X)$.*

For the proof, see [WK01].

Remark 4.14. Having in mind the results of sect.3, it is likely that there exists a compact Kähler 3-fold X with $a(X) = 2$ carrying a smooth or at least a locally complete intersection curve with ample normal bundle. The difficulty of course is that in 3.4 the surface X_C need not be projective.
Notice also the following. If $C \subset X$ is a smooth curve with ample normal bundle in a projective manifold, then $\mathcal{M}(\hat{C})$ is a finite-dimensional $\mathcal{M}(X)-$vector space, where $\hat{C}$ is the formal completion of X along C and $\mathcal{M}$ is the sheaf of meromorphic functions. One says that C is $G2$ in X, see [Ha70]. A counterexample as above with ample normal bundle would indicate that this property does no longer hold in the Kähler setting. In fact, it seems likely that the ampleness of the normal bundle forces the field of formal meromorphic functions along C should have transcendence degree 3 over the complex numbers.

For some new results concerning Conjecture 4.1 we also refer to [BM04].

5. Structure of projective threefolds containing a smooth elliptic curve with ample normal bundle.

Let X be a smooth projective threefold containing a smooth curve C with ample normal bundle N_C. If C is rational, then X is rationally connected [KoMiMo92]. In this section we consider the case that C is elliptic and we shall fix this situation unless otherwise stated. By [PSS99,2.1], see (4.5), we have $\kappa(X) = -\infty$, hence X is uniruled. Applying the minimal model program, we find a birational rational map $f : X \rightharpoonup X'$, given by a sequence of birational contractions and flips, and a contraction $g : X' \to Y$ with $\dim Y \leq 2$. Notice that in case $\dim Y = 2$, Y has only rational singularities and also that $\pi_1(X) = \pi_1(X') = \pi_1(Y)$ (see e.g. [Ko95]). We have the following structure theorem.

Theorem 5.1. *(1) The irregularity $q(X) \leq 1$.*
(2) If $q(X) = 0$ then X is rationally connected. In particular we always have $\pi_1(X) = 0$.
(3) If $q(X) = 1$, then C is an étale multi-section of the Albanese map $\alpha : X \to A$ and α factor over the Albanese $\beta : Y \to A$. The general fiber of α is a rational surface. Moreover either $\beta = \mathrm{id}$, or $\dim Y = 2$, $\kappa(Y) = -\infty$ and Y is a generic $\mathbb{P}_1-$bundle over A. In particular $\pi_1(X) = \mathbb{Z}^2$; more precisely

$$\alpha_* : \pi_1(X) \to \pi_1(A) = \mathbb{Z}^2$$

is an isomorphism and $\mathrm{Im}(\pi_1(C) \to \pi_1(X))$ has finite image.

Proof. (1) It is known in general that $\mathrm{Alb}(Z) \to \mathrm{Alb}(X)$ is surjective, if $Z \subset X$ is a submanifold with ample normal bundle in X (see e.g. [Ha70,p.116]). Applying this to $Z = C$, we get $q(X) \leq q(C) = 1$.
(2) By [KoMiMo92], it is sufficient to show that

$$H^0(X, S^t\Omega^2_X) = 0 \qquad (*)$$

for all $t \geq 1$ (actually $t = 2$ suffices). We verify this by the same method as in (4.5). In fact, (*) will follow from

$$H^0(C, S^k N^*_C \otimes S^t\Omega^2_X|C) = 0 \qquad (**)$$

for all $k \geq 0$ and all $t \geq 1$. In order to verify (*), we use the sequence

$$0 \to N^*_C \to \Omega^1_X|C \to \Omega^1_C = \mathcal{O}_C \to 0$$

and take $\bigwedge^2$ to obtain

$$0 \to \det N^*_C \to \Omega^2_X|C \to N^*_C \otimes \Omega^1_C \to 0.$$

Hence $\Omega^2_X|C$ is a negative vector bundle and thus (**) is clear.
(3) Suppose now $q(X) = 1$. Then C, having ample normal bundle, is not contracted by α and therefore it is an étale multi-section of α. The existence of β and the factorisation property are clear. If $\dim Y = 1$, then we must have $Y = A$ and $\pi_1(X) = \pi_1(A) = \mathbb{Z}^2$. So let $\dim Y = 2$. Notice that (**) from (2) was independent of $q(X) = 0$, so it holds also in our context. This implies $\kappa(Y) = -\infty$, whence our claim. Q.E.D.

Remark 5.2. Notice that the proof of (5.1) yields actually the following. Let X be a projective manifold, $C \subset X$ an elliptic curve with ample normal bundle. Then

$$H^0(X, S^t\Omega^k_X) = 0$$

for all $t \geq 1$ and all $k \geq 2$. In particular, X does not admit any rational map to a variety Y with $\dim Y \geq 2$ and $\kappa(Y) \geq 0$.

In this context we notice the following very interesting result of Napier-Ramachandran [NR98]:

Theorem 5.3. *(Napier-Ramachandran) Let X be a projective manifold, $Y \subset X$ a submanifold of positive dimension with ample normal bundle N_Y. Then*

$$\mathrm{Im}(\pi_1(Y) \to \pi_1(X))$$

has finite index.

Observe that the surjectivity $\mathrm{Alb}(Y) \to \mathrm{Alb}(X)$, used already in the proof of (5.1), means that

$$\mathrm{Im}(H_1(Y, \mathbb{Z}) \to H_1(X, \mathbb{Z}))$$

has finite index. This is the abelianized version of (5.3). If the Shafarevich conjecture holds ("the universal cover of a projective (compact Kähler) manifold is holomorphically convex"), then by Kollár, see e.g. [Ko95], the so-called Shafarevich map $sh : X \to Sh(X)$ exists which contracts exactly those subvarieties $Z \subset X$ for which $\mathrm{Im}(\pi_1(Z) \to \pi_1(X))$ has finite index. Suppose $\pi_1(X)$ is not finite, i.e.$Sh(X)$ is not a point - in that case (5.3) is anyway trivial. Since N_Y is ample, it follows that $\dim sh(Y) \neq 0$, hence $\mathrm{Im}(\pi_1(Y) \to \pi_1(X))$ is at least infinite. Of course we used the ampleness here only in a very weak form, namely to conclude that $\dim sh(Y) > 0$. Actually $\dim sh(Y) = \dim Y$.

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