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On the First Twisted Dirichlet Eigenvalue

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In this paper we prove an isoperimetric inequality for the twisted Dirichlet eigenvalue which was introduced by Barbosa and Bérard in the context of constant mean curvature surfaces. More precisely, we show that in the Euclidean case this eigenvalue is minimized by the union of two equal balls.

1. Introduction.

Let (Ω, g) be a Riemannian manifold with boundary and denote by T_0 the set of functions $f : \Omega \to \mathbb{R}$ with zero average in Ω and belonging to $H_0^1(\Omega)$, the usual Sobolev space which is the closure of the space of C^∞ functions with compact support in Ω, for the norm $\|u\| := \left(\int_\Omega |\nabla u|^2 + u^2\right)^{1/2}$. In the context of constant mean curvature immersions, Barbosa and Bérard [BB] were led to the problem of minimizing the Rayleigh quotient

$$\min_{u \in T_0,\ u \not\equiv 0} \frac{\displaystyle\int_\Omega |\nabla u|^2 + bu^2}{\displaystyle\int_\Omega u^2}, \tag{1}$$

where $b : \Omega \to \mathbb{R}$ is a continuous function. This combination of Dirichlet boundary conditions with zero average gives rise to an eigenvalue problem that is interesting in its own right, which Barbosa and Bérard called the twisted eigenvalue problem, and for which they presented some basic properties in [BB]. More specifically, we have that the eigenvalue problem in question is given by the Euler–Lagrange equations associated with the above minimization problem and is of the form

$$\begin{cases} -\Delta v + bv = \lambda_1^T v - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \Delta v\, dx & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

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Due to the presence of the average of the Laplacian, problems of this type are often referred to as nonlocal eigenvalue problems – see [F] for an overview of some nonlocal eigenvalue problems.

Among other results, Barbosa and Bérard proved that the spectra of the Dirichlet and the twisted problems are intertwined, and also a Courant type nodal domain result for the eigenfunctions of the twisted problem.

The purpose of the present paper is to continue the study of this eigenvalue problem in the case where the potential b vanishes and in the Euclidean context. For a bounded open set in $\mathbb{R}^n$, we denote by $\lambda_1^T(\Omega)$ the first twisted eigenvalue defined by (1). In particular, our main result is the following isoperimetric inequality of the Rayleigh–Faber–Krahn type

Theorem 1. *Let Ω be any bounded open set in $\mathbb{R}^n$. Then*

$$\lambda_1^T(\Omega) \geq \lambda_1^T(B_1 \cup B_2) \tag{3}$$

where B_1 and B_2 are two disjoint balls of volume $|\Omega|/2$.

Equality holds (for regular Ω) if and only if $\Omega = B_1 \cup B_2$

It is clear that the eigenvalue $\lambda_1^T(\Omega)$ does not change if we add or remove sets of zero capacity (for the capacity associated to the Sobolev space $H_0^1(\Omega)$). This is the reason why we need to consider regular domains (e.g. Lipschitzian) to investigate the equality case.

Since the first eigenfunction u_1^T changes sign in Ω, the result above is more related to the Krahn-Szegö Theorem for the second Dirichlet eigenvalue, which states that among open sets of given volume, this second eigenvalue is minimized by the union of two identical balls. We refer to [HO] for details and extensions about the Krahn-Szegö Theorem and to [H] for a survey of general similar results about the eigenvalues of the Laplacian operator.

In Section 2 we present some basic properties of the twisted problem relating the first twisted eigenvalue to various Dirichlet eigenvalues, together with some elementary bounds. In Section 3 we prove Theorem 1. This uses a result on the ratio of the first zero of consecutive Bessel functions which we state and prove in the Appendix. In the last section we present some remarks and open problems.

2. Basic properties.

We begin with a simple consequence of when a number λ is an eigenvalue of the twisted and the Dirichlet problems (see also Proposition 2.4 in [BB]).

Proposition 2.1. *A positive number λ is an eigenvalue for both the twisted and Dirichlet problems if and only if there exists an associated eigenfunction u for the Dirichlet problem such that*

$$\int_\Omega u(x)\,dx = 0.$$

Proof. If there is a Dirichlet eigenfunction u associated to λ which has zero average, then the Laplacian of u also has zro average and the result follows.

Assume now that there are eigenfunctions u for the Dirichlet problem and v for the twisted problem, both associated to the value λ. Multiplying each equation by the other eigenfunction, integrating over Ω and taking the difference yields

$$\left(\int_\Omega u(x)\,dx\right)\left(\int_\Omega \Delta v(x)\,dx\right) = 0.$$

From this we deduce that either u is an eigenfunction for the twisted problem, or v is an eigenfunction for the Dirichlet problem. In both cases there is an eigenfunction of the Dirichlet problem with zero average. □

In the case of the classical Dirichlet problem if u is an eigenfunction for a domain Ω, then it is also an eigenfunction for any of the nodal domains that it divides Ω into, with the same eigenvalue. For the twisted problem there is, of course, no analogue of this result. It is, however, possible to relate the first twisted eigenvalue to the first Dirichlet eigenvalue of its nodal domains.

Proposition 2.2. *Let $\lambda_1^T(\Omega)$ be the first eigenvalue of the twisted problem, and denote by v a corresponding eigenfunction. Then v has precisely two nodal domains and, denoting by Ω_+ and Ω_- the sets where v is positive and negative, respectively, we have that*

$$\min\left\{\lambda_1^D(\Omega_-), \lambda_1^D(\Omega_+)\right\} \leq \lambda_1^T(\Omega) \leq \max\left\{\lambda_1^D(\Omega_-), \lambda_1^D(\Omega_+)\right\},$$

where $\lambda_1^D(\Omega_-)$ and $\lambda_1^D(\Omega_+)$ denote the first Dirichlet eigenvalues of Ω_+ and Ω_-, respectively.

Proof. As was mentioned in [BB], a variation of Courant's nodal domain theorem applies also to the twisted problem, giving that any eigenfunction associated with the first eigenvalue has precisely two nodal domains. It remains to prove the other assertions.

To prove the first inequality, we use the function

$$u = \begin{cases} u_+, & x \in \Omega_+ \\ u_-, & x \in \Omega_- \end{cases}$$

in the variational formulation for $\lambda_1^T(\Omega)$, where u_+ and u_- denote first Dirichlet eigenfunctions for Ω_+ and Ω_-, respectively, and scaled in such a way that u has zero average in Ω. This gives that

$$\begin{aligned} \lambda_1^T(\Omega) &\leq \frac{\displaystyle\int_{\Omega_+} |\nabla u_+|^2\,dx + \int_{\Omega_-} |\nabla u_-|^2\,dx}{\displaystyle\int_{\Omega_+} u_+^2 dx + \int_{\Omega_-} u_-^2 dx} \\ &= \frac{\lambda_1^D(\Omega_+)\displaystyle\int_{\Omega_+} u_+^2 dx + \lambda_1^D(\Omega_-)\int_{\Omega_-} u_-^2 dx}{\displaystyle\int_{\Omega_+} u_+^2 dx + \int_{\Omega_-} u_-^2 dx}, \end{aligned} \tag{4}$$

from which the result follows.

On the other hand

$$\begin{aligned} \lambda_1^T(\Omega) &= \frac{\displaystyle\int_{\Omega_+} |\nabla v|^2\,dx + \int_{\Omega_-} |\nabla v|^2\,dx}{\displaystyle\int_{\Omega_+} v^2 dx + \int_{\Omega_-} v^2 dx} \\ &\geq \frac{\lambda_1^D(\Omega_+)\displaystyle\int_{\Omega_+} v^2 dx + \lambda_1^D(\Omega_-)\int_{\Omega_-} v^2 dx}{\displaystyle\int_{\Omega_+} v^2 dx + \int_{\Omega_-} v^2 dx}, \end{aligned} \tag{5}$$

proving the second inequality. □

Remark 2.1. It is, of course, possible to extend this result in the obvious way to higher eigenvalues.

Corollary 2.3.

$$\lambda_1^T(\Omega) = \lambda_1^D(\Omega_+) \textit{ if and only if } \lambda_1^D(\Omega_+) = \lambda_1^D(\Omega_-) = \lambda_2^D(\Omega),$$

where $\lambda_2^D(\Omega)$ *is the second Dirichlet eigenvalue of* Ω. *A similar statement holds if we replace* $\lambda_1^D(\Omega_+)$ *by* $\lambda_1^D(\Omega_-)$.

Proof. If $\lambda_1^D(\Omega_+) = \lambda_1^D(\Omega_-)$, then the previous proposition immediately implies that $\lambda_1^T(\Omega) = \lambda_1^D(\Omega_+)$.

Assume now that $\lambda_1^T(\Omega) = \lambda_1^D(\Omega_+)$. Replacing this in both (4) and (5) yields

$$\lambda_1^D(\Omega_-) \leq \lambda_1^T(\Omega) \leq \lambda_1^D(\Omega_-),$$

and so $\lambda_1^D(\Omega_-) = \lambda_1^D(\Omega_+)$. Denoting by v_+ and v_- the restrictions of a first eigenfunction of the twisted problem to Ω_+ and Ω_-, respectively, we now use

$$u = \begin{cases} u_+, & x \in \Omega_+ \\ cu_-, & x \in \Omega_- \end{cases}$$

in the Rayleigh quotient for the Dirichlet problem, where c is such that u is orthogonal to the first Dirichlet eigenspace. We then obtain that $\lambda_2^D(\Omega) \leq \lambda_1^T(\Omega)$. Since we have that $\lambda_2^D(\Omega) \geq \lambda_1^T(\Omega)$, the result follows. □

We shall now give some other bounds for λ_1^T in terms of the two first eigenvalues λ_1^D, λ_2^D and the corresponding eigenfunctions u_1, u_2 of the Dirichlet Laplacian on Ω. We already know that $\lambda_1^D < \lambda_1^T \leq \lambda_2^D$ and we are now going to give some more precise estimates.

The first is a very simple upper bound for the first twisted eigenvalue in the terms of the first Dirichlet eigenvalue.

Proposition 2.4. *There exists a constant α_n which depends only on n and for which*

$$\lambda_1^D(\Omega) \leq \lambda_1^T \leq \alpha_n \lambda_1^D(\Omega).$$

Proof. It is only necessary to prove the second inequality. This follows immediately from the inequality $\lambda_1^T \leq \lambda_2^D(\Omega)$ and then using the fact (see [AB1]) that

$$\frac{\lambda_2^D(\Omega)}{\lambda_1^D(\Omega)} \leq \frac{\lambda_2^D(B)}{\lambda_1^D(B)} = \left(\frac{j_{n/2,1}}{j_{n/2-1,1}}\right)^2,$$

where B denotes the unit ball in $\mathbb{R}^n$. □

Let us now assume that λ is a real (positive) number which is not in the spectrum of the Dirichlet Laplacian. Therefore, we can solve the equation

$$\begin{cases} -\Delta v = \lambda v + 1 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{6}$$

obtaining

$$v = (-\Delta - \lambda Id)^{-1}(1)\ .$$

Then, λ will be an eigenvalue for the twisted problem if and only if v satisfies $\int_\Omega v(x)\,dx = 0$. This relation $\int_\Omega v(x)\,dx = (1,v) = 0$ (where $(.,.)$ denotes the usual scalar product on $L^2(\Omega)$) can also be written

$$\left([-\Delta - \lambda Id]^{-1}(1), 1\right) = 0\,. \tag{7}$$

We now introduce the expansion of the constant function 1 in the Hilbert basis of the Dirichlet eigenfunctions:

$$1 = \sum_{n=1}^{+\infty} a_n u_n \quad \text{with } a_n = \int_\Omega u_n(x)\,dx,$$

where we assume the eigenfunctions u_n to be normalized with L^2 norm equal to one. Then, the eigenvalues of the operator $(-\Delta - \lambda Id)^{-1}$ being the numbers $(\lambda_n^D - \lambda)^{-1}$, equation (7) can be rewritten as

$$\sum_{n=1}^{+\infty} \frac{a_n^2}{\lambda_n^D - \lambda} = 0\,. \tag{8}$$

All the zeros of equation (8) are eigenvalues for the twisted problem, but in the case where $a_n = \int_\Omega u_n(x)\,dx$ vanishes for some n, we must add the corresponding λ_n^D as an eigenvalue.

We come back to our equation (8). If we denote by $\phi : (\lambda_1, \lambda_2) \to \mathbb{R}$ the function

$$\phi(x) := \sum_{n=1}^{+\infty} \frac{a_n^2}{\lambda_n^D - x},$$

it is clear that ϕ will be negative for $x < \lambda_1^T(\Omega)$ and positive for $x > \lambda_1^T(\Omega)$.

Theorem 2.5. *Let us denote by λ_1, λ_2 the two first eigenvalues of the Dirichlet Laplacian on Ω and by u_1, u_2 the corresponding (normalized) eigenfunctions. We also introduce $a_1 = \int_\Omega u_1(x)\,dx$, $a_2 = \int_\Omega u_2(x)\,dx$. Then, we have the following estimate for $\lambda_1^T(\Omega)$:*

$$\frac{\lambda_1 |\Omega| + \lambda_2\, a_1^2}{|\Omega| + a_1^2} \le \lambda_1^T(\Omega) \le \frac{\lambda_1\, a_2^2 + \lambda_2\, a_1^2}{a_2^2 + a_1^2}\,. \tag{9}$$

Proof. First, we set $x = \frac{\lambda_1 a_2^2 + \lambda_2 a_1^2}{a_2^2 + a_1^2}$ and plug it into the definition of ϕ giving

$$\phi(x) \geq \frac{a_1^2}{\lambda_1 - x} + \frac{a_2^2}{\lambda_2 - x} = \frac{a_1^2(a_1^2 + a_2^2)}{(\lambda_1 - \lambda_2)a_1^2} + \frac{a_2^2(a_1^2 + a_2^2)}{(\lambda_2 - \lambda_1)a_2^2} = 0.$$

The upper bound now follows thanks to the remark preceding the statement of Theorem 2.5.

In the same way, if we take now $x = \frac{\lambda_1|\Omega| + \lambda_2 a_1^2}{|\Omega| + a_1^2}$, we have

$$\sum_{n=2}^{+\infty} \frac{a_n^2}{\lambda_n - x} \leq \frac{1}{\lambda_2 - x} \sum_{n=2}^{+\infty} a_n^2 \leq \frac{\|1\|_{L^2}}{\lambda_2 - x} = \frac{|\Omega|}{\lambda_2 - x} .$$

Therefore,

$$\phi(x) \leq \frac{a_1^2}{\lambda_1 - x} + \frac{|\Omega|}{\lambda_2 - x} = 0$$

thanks to the definition of x. The result follows. □

Remark 2.2. We can obtain more precise bounds by taking one supplementary term. These bounds will involve λ_3 and u_3.

Remark 2.3. The upper bound is, in some sense, "best possible" since we have equality as soon as $a_2 = 0$ (see Proposition 2.1). This will happen for example when Ω is symmetric with respect to a hyperplane or when λ_2 is a multiple eigenvalue. This bound can also be obtained by making a linear combination of u_1 and u_2 with zero average, and then plugging it in the Rayleigh quotient.

From the first inequality in Theorem 2.5 it is possible to obtain a lower bound for the first twisted eigenvalue that depends only on the volume of the domain and its first two Dirichlet eigenvalues. To do this, we use the following inequality due to Kohler-Jobin, which is an extension to the $n-$dimensional case of an inequality of Payne and Rayner [KJ]:

$$a_1^2 \geq \frac{2\omega_n j_{n/2-1,1}^{n-2}}{\left(\lambda_1^D\right)^{n/2}} \int_\Omega u_1^2 dx, \tag{10}$$

where ω_n denotes the area of the unit sphere in $\mathbb{R}^n$. Using this in the first inequality in Theorem 2.5 yields

Corollary 2.6.

$$\lambda_1^T \geq \frac{1}{2}\lambda_1^D + \frac{\omega_n j_{n/2-1,1}^{n-2}}{|\Omega|\left(\lambda_1^D\right)^{n/2}}\lambda_2^D .$$

Since we have equality in (10) in the case of the ball (in dimension two this is known to be the only case [PR]), we might expect this bound not to be very good for *long* domains. Indeed, if one considers rectangles in the plane, the bound is larger than the first Dirichlet eigenvalue only up to a ratio of the larger to the smaller side which is approximately 1.678.

3. The isoperimetric inequality.

The goal of this section is to prove Theorem 1. The first part of the proof is similar to that of Ashbaugh and Benguria in [AB2] where they study the same question for the first eigenvalue of the clamped problem. Let us denote by u (one of) the first eigenfunction(s) for the twisted problem on Ω,

$$\Omega_+ = \{x \in \Omega, u(x) > 0\} \qquad \Omega_- = \{x \in \Omega, u(x) < 0\} .$$

Then,

$$\lambda_1^T(\Omega) = \frac{\displaystyle\int_{\Omega_+} |\nabla u|^2\, dx + \int_{\Omega_-} |\nabla u|^2\, dx}{\displaystyle\int_{\Omega_+} u^2\, dx + \int_{\Omega_-} u^2\, dx} .$$

We first prove:

Lemma 3.1. *Let us denote by B_+ (resp. B_-) the balls of same volume as Ω_+ (resp. Ω_-). Then,*

$$\lambda_1^T(\Omega) \geq \lambda_1^T(B_+ \cup B_-) .$$

Proof. Let us introduce u_+^* (resp. $-u_-^*$) the Schwarz decreasing rearrangement of $u \setminus \Omega_+$ (resp. $u \setminus \Omega_-$). The classical properties of the rearrangement provide:

$$\lambda_1^T(\Omega) \geq \frac{\displaystyle\int_{B_+} |\nabla u_+^*|^2\, dx + \int_{B_-} |\nabla u_-^*|^2\, dx}{\displaystyle\int_{B_+} {u_+^*}^2\, dx + \int_{B_-} {u_-^*}^2\, dx} \tag{11}$$

and

$$\int_{B_+} u_+^*\,dx - \int_{B_-} u_-^*\,dx = \int_{\Omega_+} u\,dx + \int_{\Omega_-} u\,dx = \int_{\Omega} u\,dx = 0\,. \tag{12}$$

In view of (11) and (12), we have the following inequality:

$$\lambda_1^T(\Omega) \geq \lambda^* := \inf_{\substack{(f,g)\in H_0^1(B_+)\times H_0^1(B_-)\\ \int_{B_+} f\,dx = \int_{B_-} g\,dx}} \frac{\int_{B_+} |\nabla f|^2\,dx + \int_{B_-} |\nabla g|^2\,dx}{\int_{B_+} f^2\,dx + \int_{B_-} g^2\,dx}\,. \tag{13}$$

Now, it is standard, using the classical method of calculus of variations, to prove that the infimum in the definition of λ^* is attained for a couple that we denote by (f_+, f_-). The Euler-Lagrange condition satisfied by (f_+, f_-), taking into account the constraint $\int_{B_-} g\,dx - \int_{B_+} f\,dx = 0$, can be written as

$$\left\{\begin{array}{l} \forall(\phi,\psi)\in H_0^1(B_+)\times H_0^1(B_-)\\ \int_{B_+} \nabla f_+.\nabla\phi\,dx + \int_{B_-} \nabla f_-.\nabla\psi\,dx - \lambda^*\left(\int_{B_+} f_+\phi\,dx + \int_{B_-} f_-\psi\,dx\right)\\ \quad = \mu_0\left(\int_{B_+} f_+^2\,dx + \int_{B_-} f_-^2\,dx\right)\left(\int_{B_-}\psi\,dx - \int_{B_+}\phi\,dx\right), \end{array}\right. \tag{14}$$

where μ_0 is a Lagrange multiplier. Due to the homogeneity of the problem, we can obviously assume that $\int_{B_+} f_+^2\,dx + \int_{B_-} f_-^2\,dx = 1$. Now, taking first $\psi = 0$, then $\phi = 0$ in (14) we see that f_+ and f_- solve:

$$\left\{\begin{array}{ll} -\Delta f_+ = \lambda^* f_+ - \mu_0 & \text{in } B_+\\ f_+ = 0 & \text{on } \partial B_+ \end{array}\right., \left\{\begin{array}{ll} -\Delta f_- = \lambda^* f_- + \mu_0 & \text{in } B_-\\ f_- = 0 & \text{on } \partial B_-\,. \end{array}\right. \tag{15}$$

Integrating the two equations and taking the difference yields

$$-\int_{B_+} \Delta f_+\,dx + \int_{B_-} \Delta f_-\,dx = -\mu_0(|\Omega^+| + |\Omega_-|) = -\mu_0|\Omega|\,.$$

Now, we introduce the open set $\tilde{\Omega} = B_+ \cup B_-$ and the function w defined on $\tilde{\Omega}$ by

$$w = \left\{\begin{array}{ll} f_+ & \text{in } B_+\\ -f_- & \text{in } B_-\,. \end{array}\right.$$

This function satisfies $-\int_{\tilde{\Omega}} \Delta w\, dx = -\mu_0|\Omega| = -\mu_0|\tilde{\Omega}|$ and then, replacing in (15):

$$\begin{cases} -\Delta w = \lambda^* w - \frac{1}{|\tilde{\Omega}|}\int_{\tilde{\Omega}} \Delta w\, dx & \text{in } \tilde{\Omega} \\ w = 0 & \text{on } \partial\tilde{\Omega} \end{cases} \tag{16}$$

this shows that λ^* is an eigenvalue of the twisted problem on $\tilde{\Omega}$ and therefore, $\lambda_1^T(\Omega) \geq \lambda^* \geq \lambda_1^T(\tilde{\Omega})$. □

To finish the proof of Theorem 1, it remains to prove that the union of two identical balls gives the lowest possible value of λ_1^T among union of balls. This is not as simple as in the purely Dirichlet case since the extra zero average condition on the eigenfunction couples the eigenfunction on the two balls making the eigenvalue equation more complicated.

Let us establish the equation allowing to compute the first twisted eigenvalue of the union Ω of two (disjoint) balls B_1 and B_2 in $\mathbb{R}^n$ of respective radii R_1 and R_2, with $R_1 \leq R_2$. Without loss of generality, we can assume that the volume of Ω is one which implies

$$R_1^n + R_2^n = 1\ . \tag{17}$$

There is a first possibility which consists in taking an eigenfunction which is zero on the smaller ball B_1 and which coincides on B_2 with the first eigenfunction u_2 of the larger ball. In this case, we would have

$$\lambda_1^T(B_1 \cup B_2) = \lambda_1^T(B_2)\ .$$

We will see later that this situation actually occurs for a large range of value of the ratio R_1/R_2! Following L. Barbosa and P. Bérard, see [BB], we see that, in such a case, we will have $\lambda_1^T(B_2) = \left(j_{\frac{n}{2},1}/R_2\right)^2$ where $j_{\frac{n}{2},1}$ is the first zero of the Bessel function $J_{\frac{n}{2}}(x)$.

We have now to look at the case where the eigenfunction, say u, does not vanish on any of the two balls. We write

$$u = \begin{cases} u_1 & \text{in } B_1 \\ u_2 & \text{in } B_2\ . \end{cases}$$

Since, as was mentioned in Proposition 2.2, Courant's Theorem about the number of nodal domains holds here, we see that u_1 is, for example, positive in B_1 while u_2 is negative in B_2. Moreover, the proof of Lemma 3.1 shows that we can restrict ourselves to the case where u_1 and u_2 are radially symmetric functions. Then, the ordinary differential equation that we have to solve (for $j = 1, 2$) is:

$$\begin{cases} \dfrac{d^2 u_j}{dr^2} + \dfrac{n-1}{r}\dfrac{d u_j}{dr} + \lambda^T u_j = c \\ \dfrac{d u_j}{dr}(0) = 0, \quad u_j(R_j) = 0 \end{cases} \tag{18}$$

where c is the constant unknown *a priori* which corresponds to the term $\int_\Omega \Delta u \, dx$ (we recall that we have chosen the volume of the union of the two balls to be one). Setting $\lambda^T(B_1 \cup B_2) = \omega^2$, the solution of (18) is known to be

$$u = \begin{cases} u_1 = \alpha_1 \left(r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\omega r) - R_1^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\omega R_1) \right) & \text{in } B_1 \\ u_2 = -\alpha_2 \left(r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\omega r) - R_2^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\omega R_2) \right) & \text{in } B_2 \end{cases} \tag{19}$$

Now, we express the coupling condition $\int_\Omega u(x)\, dx = 0$:

$$0 = \int_{B_1} u_1 \, dx + \int_{B_2} u_2 \, dx = \alpha_1 \left(\gamma_n \int_0^{R_1} J_{\frac{n}{2}-1}(\omega r) r^{\frac{n}{2}} \, dr - \delta_n R_1^{\frac{n}{2}+1} J_{\frac{n}{2}-1}(\omega R_1) \right)$$
$$-\alpha_2 \left(\gamma_n \int_0^{R_2} J_{\frac{n}{2}-1}(\omega r) r^{\frac{n}{2}} \, dr - \delta_n R_2^{\frac{n}{2}+1} J_{\frac{n}{2}-1}(\omega R_2) \right) .$$

where γ_n is the $(n-1)$-measure of the unit sphere in $\mathbb{R}^n$ and δ_n the n-measure of the unit ball. Using classical results for Bessel functions (see e.g. [W]), namely,

$$\int_0^R J_{\frac{n}{2}-1}(\omega r) r^{\frac{n}{2}} \, dr = \frac{1}{\omega} R^{\frac{n}{2}} J_{\frac{n}{2}}(\omega r) \quad \text{and} \quad 2\frac{\nu}{x} J_\nu(x) - J_{\nu-1}(x) = J_{\nu+1}(x),$$

together with $\gamma_n = n\delta_n$, we get

$$0 = \alpha_1 R_1^{\frac{n}{2}+1} J_{\frac{n}{2}+1}(\omega R_1) - \alpha_2 R_2^{\frac{n}{2}+1} J_{\frac{n}{2}+1}(\omega R_2).$$

Therefore, it is possible to take

$$\alpha_1 = R_2^{\frac{n}{2}+1} J_{\frac{n}{2}+1}(\omega R_2) \quad \text{and} \quad \alpha_2 = R_1^{\frac{n}{2}+1} J_{\frac{n}{2}+1}(\omega R_1) \tag{20}$$

in (19). It remains to express that we want the constant c in (18) to be the same for the two equations $j = 1$ and $j = 2$. Since

$$\Delta u_1 = -\alpha_1 \omega^2 r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\omega r)$$
$$\Delta u_2 = \alpha_2 \omega^2 r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\omega r)$$

we have

$$c = \Delta u_1 + \omega^2 u_1 = -\alpha_1 \omega^2 R_1^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\omega R_1)$$
$$c = \Delta u_2 + \omega^2 u_2 = \alpha_2 \omega^2 R_2^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\omega R_2)\ .$$

Comparing these two relations and taking into account (20) yields the following transcendental equation whose zeros give eigenvalues of the twisted problem for the union of two balls of radii R_1 and R_2:

$$R_2^{\frac{n}{2}+1} J_{\frac{n}{2}+1}(\omega R_2) R_1^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\omega R_1) + R_1^{\frac{n}{2}+1} J_{\frac{n}{2}+1}(\omega R_1) R_2^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\omega R_2) = 0 \tag{21}$$

which, unless $R_1 = R_2$ (see below) can also be written as

$$R_1^n \frac{J_{\frac{n}{2}+1}(\omega R_1)}{J_{\frac{n}{2}-1}(\omega R_1)} + R_2^n \frac{J_{\frac{n}{2}+1}(\omega R_2)}{J_{\frac{n}{2}-1}(\omega R_2)} = 0\ . \tag{22}$$

We will denote by $\omega(R_1, R_2)$ (or ω if no misunderstanding can occur) the first positive root of the equation (21) or (22). Its square is always an eigenvalue for the twisted problem on $B_1 \cup B_2$ but not necessarily the first one. Actually, numerical computations (and also asymptotic expansion for $R_1 \to 0$) show that there exist a constant c_n depending on the dimension n (we get e.g. $c_2 \approx 0.56714, c_3 \approx 0.64715$) such that if $R_1/R_2 < c_n$, then $j_{\frac{n}{2},1}/R_2 < \omega(R_1, R_2)$. That we always have $c_n < 1$ is actually a consequence of Corollary A.2 – see the comment just after Lemma 3.3. Summing up, the following situation holds:

Proposition 3.2. *There exists a constant c_n, depending on the dimension n, such that:*

- *if $R_1/R_2 < c_n$ then $\lambda_1^T(B_1 \cup B_2) = \left(j_{\frac{n}{2},1}/R_2\right)^2$*
- *if $R_1/R_2 \geq c_n$ then $\lambda_1^T(B_1 \cup B_2) = \omega(R_1, R_2)^2$.*

where $\omega(R_1, R_2)$ is the first positive zero of equation (21) or (22).

In the case $R_1 = R_2 (= 2^{-1/n})$, from the equation in its form (21), we see that

$$\omega(R_1, R_1) = \omega(2^{-1/n}, 2^{-1/n}) = j_{\frac{n}{2}-1,1}/2^{-1/n} = 2^{1/n} j_{\frac{n}{2}-1,1} \; .$$

We will denote this value by $\omega^* = 2^{1/n} j_{\frac{n}{2}-1,1}$ which will play an important role in the next analysis since we want to prove that $\lambda_1^T(B_1 \cup B_2) \geq \omega^{*2}$.

Let us denote by $\phi(x)$ the function which appears in (22):

$$\phi(x) := x^n \frac{J_{\frac{n}{2}+1}(x)}{J_{\frac{n}{2}-1}(x)} \; .$$

We easily get, thanks to the recurrence relations satisfied by Bessel functions, that

$$\phi'(x) = \frac{x^{n+1}}{n} \left[1 + \frac{J_{\frac{n}{2}+1}(x)}{J_{\frac{n}{2}-1}(x)}\right]^2 = \frac{x^{n+1}}{n} \left[1 + \frac{\phi(x)}{x^n}\right]^2 \; . \tag{23}$$

This shows, in particular, that ϕ is increasing on each interval where it is defined, that is, on intervals of the form

$$(0, j_{\frac{n}{2}-1,1}) \text{ and } (j_{\frac{n}{2}-1,k}, j_{\frac{n}{2}-1,k+1}), \quad k \geq 1.$$

Let us now introduce the function $\psi(\omega, R_1, R_2)$ for which we want to calculate the zeros:

$$\psi(\omega, R_1, R_2) := \phi(\omega R_1) + \phi(\omega R_2) \; .$$

It is defined when ω belongs to the intersection of all the intervals $(j_{\frac{n}{2}-1,k}/R_1, j_{\frac{n}{2}-1,k+1}/R_1)$ and $(j_{\frac{n}{2}-1,k}/R_2, j_{\frac{n}{2}-1,k+1}/R_2)$. The two first such intervals are

$$]0, j_{\frac{n}{2}-1,1}/R_2[\cup]j_{\frac{n}{2}-1,1}/R_2, \min(j_{\frac{n}{2}-1,1}/R_1, j_{\frac{n}{2}-1,2}/R_2)[.$$

On the first interval ψ is positive, while on the second it goes from $-\infty$ to $+\infty$. It implies the following first rough estimate:

$$j_{\frac{n}{2}-1,1}/R_2 < \omega(R_1, R_2) < \min(j_{\frac{n}{2}-1,1}/R_1, j_{\frac{n}{2}-1,2}/R_2) \; .$$

Moreover,

$$\psi(j_{\frac{n}{2}+1,1}/R_2, R_1, R_2) = (j_{\frac{n}{2}+1,1}/R_2)^n \frac{J_{\frac{n}{2}+1}(j_{\frac{n}{2}+1,1} R_1/R_2)}{J_{\frac{n}{2}-1}(j_{\frac{n}{2}+1,1} R_1/R_2)}$$

so, if we are in the case $j_{\frac{n}{2}+1,1} R_1/R_2 < j_{\frac{n}{2}-1,1}$ both numerator and denominator in the previous fraction will be positive which shows that

$\omega(R_1,R_2) < j_{\frac{n}{2}+1,1}/R_2$. Now, it is known that the zeros of $J_{\frac{n}{2}-1}$ and $J_{\frac{n}{2}+1}$ are intertwined, which means in particular $j_{\frac{n}{2}+1,1}/R_2 < j_{\frac{n}{2}-1,2}/R_2$. Therefore, we finally have

$$j_{\frac{n}{2}-1,1}/R_2 < \omega(R_1,R_2) < \min(j_{\frac{n}{2}-1,1}/R_1, j_{\frac{n}{2}+1,1}/R_2) \,. \tag{24}$$

Remark 3.1. From (24), it is clear that when the ratio R_1/R_2 tends to 1, $\omega(R_1,R_2) \to j_{\frac{n}{2}-1,1}/(2^{-1/n}) = \omega^*$.

Now, if we assume that $R_1 = R$ is fixed and we look at the function $\omega \mapsto \psi(\omega, R, (1-R^n)^{1/n})$, the previous analysis shows that this function is well defined and increasing on the interval

$$I = \left(\frac{j_{\frac{n}{2}-1,1}}{R_2}, \min\left\{\frac{j_{\frac{n}{2}-1,1}}{R_1}, \frac{j_{\frac{n}{2}+1,1}}{R_2}\right\}\right).$$

We can now introduce the function $G : (0, 2^{-1/n}) \to \mathbb{R}$ defined by:

$$G(r) := \psi(\omega^*, r, (1-r^n)^{1/n}) = \phi(\omega^* r) + \phi(\omega^*(1-r^n)^{1/n}) \,. \tag{25}$$

Let us remark that $\omega^* r = j_{\frac{n}{2}-1,1}2^{1/n} r < j_{\frac{n}{2}-1,1}$, so the expression $\phi(\omega^* r)$ is always well defined. For the expression $\phi(\omega^*(1-r^n)^{1/n})$, it will also be true if $\omega^* < j_{\frac{n}{2}-1,2}$. The chain of inequalities $2^{1/n} j_{\frac{n}{2}-1,1} < j_{\frac{n}{2},1} < j_{\frac{n}{2}-1,2}$ (the first inequality coming from Corollary A.2) shows that it is the case. Therefore, the function G is well defined on the interval $[0, 2^{-1/n})$.

Lemma 3.3. *If G takes only negative values on $(0, 2^{-1/n})$, then $\omega^* \leq \omega(R_1,R_2)$ for all R_1, R_2.*

Proof. Since $\omega \mapsto \psi(\omega, R, (1-R^n)^{1/n})$ is increasing, if $\psi(\omega^*, R, (1-R^n)^{1/n}) < 0$ for all $R \in (0, 2^{-1/n})$, it will remain negative for $\omega < \omega^*$, and then no zero of ψ (i.e. $\omega(R_1,R_2)$) can be in the range $(0, \omega^*)$. □

This Lemma implies Theorem 1 because of Proposition 3.2 and the inequality $\omega^* = j_{\frac{n}{2}-1,1}2^{1/n} < j_{\frac{n}{2},1} \leq j_{\frac{n}{2},1}/R_2$ for all $R_2 \leq 1$ which comes from Corollary A.2.

It remains to prove that G takes only negative values. For this, we compute its derivative.

$$G'(r) = \omega^* \phi'(\omega^* r) - \omega^* r^{n-1}(1-r^n)^{1/n-1}\phi'(\omega^*(1-r^n)^{1/n}) \,.$$

Using the expression of ϕ' given in (23), a straightforward computation gives:

$$G'(r)=\frac{\omega^{*n+2}r^{n-1}}{n}\left[r^2\left(1+\frac{\phi(\omega^* r)}{(\omega^* r)^n}\right)^2-(1-r^n)^{\frac{2}{n}}\left(1+\frac{\phi(\omega^*(1-r^n)^{1/n})}{\omega^{*n}(1-r^n)}\right)^2\right].$$

So, if we introduce the function

$$h(r):=r\left[1+\frac{\phi(\omega^* r)}{(\omega^* r)^n}\right]$$

we can write $G'(r)$ as

$$G'(r)=\frac{\omega^{*n+2}r^{n-1}}{n}\left[h^2(r)-h^2\left((1-r^n)^{\frac{1}{n}}\right)\right]\ .$$

Now, the relations satisfied by the Bessel functions show that

$$h(r)=r\left[1+\frac{J_{\frac{n}{2}+1}(\omega^* r)}{J_{\frac{n}{2}-1}(\omega^* r)}\right]=\frac{n}{\omega^*}\ \frac{J_{\frac{n}{2}}(\omega^* r)}{J_{\frac{n}{2}-1}(\omega^* r)}\ .$$

Since $\omega^* r\in]0,j_{\frac{n}{2}-1,1}[$, h is well defined and positive for $r\in]0,2^{-1/n}[$. Furthermore, using the Mittag-Leffler representation (see e.g. [W]), we get

$$h(r)=\frac{n}{\omega^*}\,2\omega^* r\sum_{m=1}^{+\infty}\frac{1}{(j_{\frac{n}{2}-1,m})^2-(\omega^* r)^2}$$

which shows that h is an increasing function (as a product of increasing positive functions). Therefore,

$$h^2(r)-h^2\left[(1-r^n)^{\frac{1}{n}}\right]<0$$

and $G'(r)<0$ for $r\in(0,2^{-1/n})$. Now,

$$G(0)=\phi(\omega^*)=\omega^{*n}\frac{J_{\frac{n}{2}+1}(\omega^*)}{J_{\frac{n}{2}-1}(\omega^*)}$$

is negative since $j_{\frac{n}{2}-1,1}<\omega^*<j_{\frac{n}{2}+1,1}<j_{\frac{n}{2}-1,2}$ according to Corollary A.2.

Finally, we investigate the equality case. According to the analysis of the equality case in the Polya inequality relating $\int_{\Omega^*}|\nabla u^*|^2$ and $\int_\Omega|\nabla u|^2$ (see e.g. [K]), we see from (11) that equality can occur only if Ω is already the union of two balls. Now, the proof of Lemma 3.3 shows that we have actually $\omega^*<\omega(R_1,R_2)$ if $R_1\neq R_2$.

This finishes the proof of Theorem 1.

4. Discussion and open problems.

Since the minimum is obtained for a set which is not connected, the obvious question that arises is what the infimum is if we restrict ourselves to connected sets. As in the case of the second Dirichlet eigenvalue referred to previously, this is not the good question, since the minimum can be approximated by a sequence of connected domains: take, for instance, two equal balls connected by a thin tube. By using in the Rayleigh quotient an eigenfunction corresponding to the first twisted eigenvalue problem for the two balls alone, we can easily see that by making the tube thinner and increasing the radius of the balls such that the total volume is kept fixed, we can approach the optimal eigenvalue as much as desired. Thus, and like in the Dirichlet problem, a more interesting situation is to consider the minimization over convex domains. Following the lines of [HO] where the case of the second Dirichlet eigenvalue is considered, we may actually prove the following properties:

- There exists a convex domain, say Ω^*, which minimizes $\lambda_1^T(\Omega)$ among convex sets of given volume.
- This domain Ω^* does not contain arc of circle (or pieces of sphere in dimension greater than 2) on its boundary. In particular, the optimal domain is not the stadium (convex hull of two identical tangent balls).
- The optimal domain is at least C^1.

To continue the study of the optimal convex domain, in particular its geometric properties, we need to know more about the nodal line of a convex domain. Here are some open problems related to that question.

Open problem 1: Prove that the nodal line of the first twisted eigenfunction of a plane convex domain Ω hits the boundary of Ω at exactly two points (see [M] for the Dirichlet case).

Open problem 2: Prove that the optimal plane convex domain has exactly two parallel segments on its boundary.

Open problem 3: Prove that the optimal plane convex domain has two axis of symmetry.

Appendix A. A result about the ratio of the first zero of two consecutive Bessel functions.

In the proof of Lemma 3.3 (and also at some other places in section 3), we used the inequality $j_{\frac{n}{2},1} > j_{\frac{n}{2}-1,1}2^{1/n}$. We will prove in this section that it is a consequence of a sharper inequality involving the ratio of the first zero of two consecutive Bessel functions:

Theorem A.1. *Let us denote by j_ν the first positive zero of the Bessel function $J_\nu(x)$, $\nu \geq 0$. Then, for any $\nu = \frac{n}{2}$ where n is an integer, the following estimate holds:*

$$3^{1/(\nu+1)} \geq \frac{j_{\nu+1}}{j_\nu} \geq \left(\frac{j_1}{j_0}\right)^{1/(\nu+1)} . \tag{26}$$

Let us point out that the estimate from below in inequality (26) is "the best possible" since equality obviously holds for $\nu = 0$. For the estimate from above, see comment in the Remark A.1.

Corollary A.2. *For every integer n, we have the inequality*

$$\frac{j_{\frac{n}{2}+1}}{j_{\frac{n}{2}-1}} > \frac{j_{\frac{n}{2}}}{j_{\frac{n}{2}-1}} > 2^{\frac{1}{n}} . \tag{27}$$

Proof. Indeed, the first inequality is trivial, while Theorem A.1 applied to $\nu = \frac{n}{2} - 1$ yields

$$\frac{j_{\frac{n}{2}}}{j_{\frac{n}{2}-1}} > \left(\frac{j_1}{j_0}\right)^{2/n}$$

and the result follows since $\left(\frac{j_1}{j_0}\right)^2 > 2$. □

Proof of the Theorem : The starting point is the following inequality which can be found, for example, in [QW]:

$$\nu - \frac{\alpha_1}{2^{1/3}}\,\nu^{1/3} < j_\nu < \nu - \frac{\alpha_1}{2^{1/3}}\,\nu^{1/3} + \frac{3}{20}\,\alpha_1^2\frac{2^{1/3}}{\nu^{1/3}} \tag{28}$$

where α_1 is the first negative zero of the Airy function $A_i(x)$, its numerical value being $\alpha_1 \simeq -2.3381$. From now on, we will assume $\nu \geq 2$, the case $\nu < 2$ will be considered at the end of the proof. We use the inequalities

$$1 + u/3 \geq (1+u)^{1/3} \geq 1 + u/4 \quad \text{(valid for } 0 \leq u \leq 1) , \tag{29}$$

$$(1+u)^{-1/3} \le 1 - u/4 \quad \text{(valid for } 0 \le u \le 1/2) \tag{30}$$

together with (28) to get on the one hand

$$j_{\nu+1} > \nu + 1 - \frac{\alpha_1}{2^{1/3}}\,(\nu+1)^{1/3} \ge \nu + 1 - \frac{\alpha_1}{2^{1/3}}\,\nu^{1/3}(1+\frac{1}{4\nu}) \tag{31}$$

and, on the other hand,

$$j_{\nu+1} < \nu + 1 - \frac{\alpha_1}{2^{1/3}}\,(\nu+1)^{1/3} + \frac{3}{20}\,\alpha_1^2\,2^{1/3}(\nu+1)^{-1/3}.$$

This implies

$$j_{\nu+1} < \nu + 1 - \frac{\alpha_1}{2^{1/3}}\,\nu^{1/3}(1+\frac{1}{3\nu}) + \frac{3}{20}\,\alpha_1^2\,2^{1/3}\nu^{-1/3}(1-\frac{1}{4\nu}). \tag{32}$$

Let us set $c = \log(j_1/j_0)$. We now use the inequality $e^u \le 1 + 1.3u$ valid for every number $u \le 1/2$ (we have $c/(\nu+1) \le c \le 1/2$) to get

$$\left(\frac{j_1}{j_0}\right)^{\frac{1}{\nu+1}} \le 1 + \frac{1.3\,c}{\nu+1} \le 1 + \frac{1.3\,c}{\nu}\ . \tag{33}$$

Using (28) for j_ν, (31) for $j_{\nu+1}$ and (33), we finally get:

$$\left(\frac{j_1}{j_0}\right)^{\frac{1}{\nu+1}} j_\nu - j_{\nu+1} < \left(1 + \frac{1.3\,c}{\nu}\right)\left(\nu - \frac{\alpha_1}{2^{1/3}}\,\nu^{1/3} + \frac{3}{20}\,2^{1/3}\alpha_1^2\nu^{-1/3}\right) - ...$$

$$\left(\nu + 1 - \frac{\alpha_1}{2^{1/3}}\,\nu^{1/3} - \frac{\alpha_1}{4.2^{1/3}}\,\nu^{-2/3}\right)\ .$$

which gives the following upper bound for $\left(\frac{j_1}{j_0}\right)^{\frac{1}{\nu+1}} j_\nu - j_{\nu+1}$:

$$1.3\,c - 1 + \frac{3}{20}\,2^{1/3}\alpha_1^2\nu^{-1/3} + \frac{\alpha_1}{2^{1/3}}\,(\frac{1}{4} - 1.3\,c)\nu^{-2/3} + \frac{3.9\,c}{20}\,2^{1/3}\,\alpha_1^2\nu^{-4/3}\ .$$

If we denote by $x = \nu^{1/3}$, the previous estimate leads to study the polynomial

$$P(x) := (1.3\,c - 1)x^4 + \frac{3}{20}\,2^{1/3}\alpha_1^2 x^3 + \frac{\alpha_1}{2^{1/3}}\,(\frac{1}{4} - 1.3\,c)x^2 + \frac{3.9\,c}{20}\,2^{1/3}\,\alpha_1^2\ .$$

Now, a straightforward calculation shows that $P(x) \le 0$ as soon as $x \ge 3.19226$, which yields that the lower bound in inequality (26) holds when $\nu \ge 3.19226^3 \simeq 32.53$.

For the upper bound, we proceed in the same way. We use $j_\nu > \nu - \frac{\alpha_1}{2^{1/3}}\,\nu^{1/3}$ and

$$3^{1/(\nu+1)} \geq 1 + \frac{\log 3}{\nu+1} \geq 1 + \frac{\log 3}{\nu} - \frac{\log 3}{\nu^2}$$

together with (32) to get

$$j_\nu 3^{1/(\nu+1)} - j_{\nu+1} \geq \log 3 - 1 - \frac{3}{20}\,2^{1/3}\alpha_1^2\nu^{-1/3} + \frac{\alpha_1}{2^{1/3}}\,(\frac{1}{3} - \log 3)\nu^{-2/3} + ...$$

$$- \log 3\,\nu^{-1} + \frac{3}{80}\,2^{1/3}\,\alpha_1^2\nu^{-4/3} + \frac{\alpha_1}{2^{1/3}}\,\log 3\nu^{-5/3}\,.$$

This leads us to consider the polynomial

$$\begin{aligned} Q(x) := (\log 3 - 1)x^5 - \frac{3}{20}\,2^{1/3}\alpha_1^2 x^4 + \frac{\alpha_1}{2^{1/3}}\,(\frac{1}{3} - \log 3)x^3 - \log 3\,x^2 \\ + \frac{3}{80}\,2^{1/3}\,\alpha_1^2 x + \frac{\alpha_1}{2^{1/3}}\,\log 3 \end{aligned}$$

which is non positive if $x > 9.0161$. It means that the upper bound in inequality (26) holds when $\nu \geq 9.0161^3 \simeq 732.2$.

It remains to check the inequalities in (26) for a finite number of values, namely, $\nu = \frac{n}{2}, n = 0, 1, \ldots, 1464$ for the left–hand inequality, and $n = 0, 1, \ldots, 66$ for the right–hand side inequality. This was done using Matlab.

Remark A.1. *The number 3 in the upper bound has been chosen to give numerical computations which were not too long. Actually, according to the asymptotic expansion given by (*28*), we could certainly choose, instead of 3, any number k greater than $e = 2.7128....$ But obviously, the closer k is to e, the more numerical computations we need to do to complete the proof.*

Remark A.2. *We were only motivated by the quotient $j_{\nu+1}/j_\nu$, so we have not considered the case of higher zeros. But, since the estimate (*28*) holds for any zero $j_{\nu,k}$ (we just have to replace α_1 by α_k, the k-th negative zero of the Airy function), it is clear that the proof we give can be adapted to look at any ratio of the type $j_{\nu+1,k}/j_{\nu,k}$.*

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