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Convergence of the J-flow on Kähler Surfaces

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Donaldson defined a parabolic flow of potentials on Kähler manifolds which arises from considering the action of a group of symplectomorphisms on the space of smooth maps between manifolds. One can define a moment map for this action, and then consider the gradient flow of the square of its norm. Chen discovered the same flow from a different viewpoint and called it the J-flow, since it corresponds to the gradient flow of his J-functional, which is related to Mabuchi's K-energy. In this paper, we show that in the case of Kähler surfaces with two Kähler forms satisfying a certain inequality, the J-flow converges to a zero of the moment map.

1. Introduction.

In [Do], Donaldson described how a number of geometric situations fit into a general framework of diffeomorphism groups and moment maps. In the Kähler setting, he used this framework to define a natural parabolic flow, as follows. Suppose that (M, ω) is a compact Kähler manifold of dimension n and let χ_0 be another Kähler form on M, in a different Kähler class. Consider the infinite-dimensional manifold $\mathcal{M}$ of diffeomorphisms $f : M \to M$, homotopic to the identity. $\mathcal{M}$ carries a natural symplectic form Ω defined by

$$\Omega_f(v, w) = \int_M \omega(v, w) \frac{\chi_0^n}{n!},$$

for sections v, w of $f^*(TM)$. The group $\mathcal{G}$ of exact χ_0-symplectomorphisms of M acts on $\mathcal{M}$ by composition on the right, preserving Ω. We can identify the Lie algebra of $\mathcal{G}$ with the space of functions on M of integral zero with respect to the volume form induced by χ_0. A moment map $\mu : \mathcal{M} \to \mathrm{Lie}(\mathcal{G})^*$ for the group action is given by

$$\mu(f) = \frac{f^*(\omega) \wedge \chi_0^{n-1}}{\chi_0^n} - \frac{\int_M \omega \wedge \chi_0^{n-1}}{\int_M \chi_0^n},$$

where we are using the L^2 inner product to identify $\mathrm{Lie}(\mathcal{G})$ with its dual. It is natural to look for solutions of

$$\mu(f) = 0 \qquad (\mathrm{mod}\ \mathcal{G}). \tag{1.1}$$

These points form the symplectic quotient. Under certain conditions, one would hope that the gradient flow f_t of the function $\|\mu\|^2$ on $\mathcal{M}$ would converge to give a solution of (1.1). The gradient flow can be rewritten as a flow of Kähler forms $(f_t^*)^{-1}(\chi_0)$ on M. This defines a parabolic flow on the space of Kähler potentials and is the object of study of this paper.

At around the same time, Chen [C1] independently discovered the same flow as the gradient flow of his J-functional. He later called it the J-flow [C2]. He showed in [C1] that the J-functional is related to the Mabuchi K-energy [Ma], which plays a key role in the study of Kähler geometry and stability in the sense of geometric invariant theory (see [Y2], [T2], [T3] and [PS] for example).

Explicitly, the J-flow is defined as follows. Let c be the constant given by

$$c = \frac{\int_M \omega \wedge \chi_0^{n-1}}{\int_M \chi_0^n},$$

and let $\mathcal{H}$ be the space of Kähler potentials

$$\mathcal{H} = \{\phi \in C^\infty(M) \mid \chi_\phi = \chi_0 + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi > 0\}.$$

The J-flow is the flow on $\mathcal{H}$ given by

$$\begin{aligned} \frac{\partial \phi_t}{\partial t} &= c - \frac{\omega \wedge \chi_{\phi_t}^{n-1}}{\chi_{\phi_t}^n}. \\ \phi_0 &= 0. \end{aligned} \tag{1.2}$$

A critical point of the J-flow gives a Kähler metric χ satisfying

$$\omega \wedge \chi^{n-1} = c\chi^n. \tag{1.3}$$

Donaldson [Do] asked whether one can find a solution to (1.3) in the class $[\chi_0]$ under certain assumptions. He noted that a necessary condition is that $[nc\chi_0 - \omega]$ be a Kähler class, and conjectured that this condition be sufficient. Chen [C1] confirmed this conjecture in the case $n = 2$, without using the J-flow, by observing that (1.3) reduces to a Monge-Ampère equation which can be solved by the well-known result of Yau [Y1]. The conjecture is still open for $n > 2$.

Chen [C1] shows that Donaldson's conjecture would imply a result on the lower bound of the Mabuchi K-energy for compact Kähler manifolds M with negative first Chern class. Namely, if $-\omega \in c_1(M)$ with $\omega > 0$, then for Kähler classes $[\chi_0]$ satisfying

$$nc[\chi_0] - [\omega] > 0,$$

the Mabuchi K-energy would have a lower bound in the class $[\chi_0]$.

Solutions of the J-flow exist for a short time by general theory, since the flow is parabolic. In [C2], Chen showed that the flow always exists for all time for any smooth initial data. He also showed that if the bisectional curvature of ω is non-negative then the J-flow converges to a critical metric.

In general, the behaviour of the flow is not known. In this paper, we deal with the case $n = 2$ with no curvature restrictions. Our main result is as follows.

Main Theorem *Suppose that (M, ω) has dimension $n = 2$ and that*

$$nc\chi_0 - \omega > 0.$$

Then the J-flow (1.2) converges in C^∞ to a smooth critical metric.

The outline of the paper is as follows. In section 2 we state some preliminary facts about the flow and introduce notation. In section 3, the maximum principle is used to derive an estimate on the second derivatives of ϕ in terms of ϕ itself. In section 4, a C^0 estimate for ϕ is given. The argument uses the second order estimate, a Moser iteration argument applied to the exponential of $-\phi$ and the result of Tian [T1] (see also [TY]) on the existence of constants $\alpha > 0$ and C such that

$$\int_M e^{-\alpha\phi} \frac{\chi_0^n}{n!} \leq C,$$

for all ϕ in $\mathcal{H}$ with $\sup_M \phi = 0$. In section 5, the proof of the main theorem is completed.

2. Preliminaries and notation.

From now on, assume that ω has been scaled so that $c = 1/n$. We will work in local coordinates, and write

$$\omega = \frac{\sqrt{-1}}{2} g_{i\overline{j}} dz^i \wedge dz^{\overline{j}}, \qquad \chi_0 = \frac{\sqrt{-1}}{2} \chi_{0\, i\overline{j}} dz^i \wedge dz^{\overline{j}},$$

and

$$\chi = \frac{\sqrt{-1}}{2} \chi_{i\overline{j}} dz^i \wedge dz^{\overline{j}} = \frac{\sqrt{-1}}{2} (\chi_{0\, i\overline{j}} + \partial_i \partial_{\overline{j}} \phi) dz^i \wedge dz^{\overline{j}},$$

where $\chi = \chi_\phi$ (suppressing the t-subscript.) The operators Λ_ω and Λ_χ act on $(1,1)$ forms $\alpha = \frac{\sqrt{-1}}{2} \alpha_{i\overline{j}} dz^i \wedge dz^{\overline{j}}$ by

$$\Lambda_\omega \alpha = g^{i\overline{j}} \alpha_{i\overline{j}}, \qquad \text{and} \qquad \Lambda_\chi \alpha = \chi^{i\overline{j}} \alpha_{i\overline{j}}.$$

The J-flow (1.2) can be written

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{1}{n}(1 - \Lambda_{\chi}\omega) \\ \phi|_{t=0} &= 0. \end{aligned} \tag{2.1}$$

Differentiating with respect to t gives

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) = \tilde{\triangle}\left(\frac{\partial \phi}{\partial t}\right), \tag{2.2}$$

where the operator $\tilde{\triangle}$ acts on functions f by

$$\tilde{\triangle} f = \frac{1}{n}\chi^{k\bar{j}}\chi^{i\bar{l}}g_{i\bar{j}}\partial_k\partial_{\bar{l}} f.$$

For convenience, write

$$h^{k\bar{l}} = \chi^{k\bar{j}}\chi^{i\bar{l}}g_{i\bar{j}}.$$

The tensor $h^{k\bar{l}}$ is positive definite and its inverse defines a Hermitian metric on M. The operator $\tilde{\triangle}$ is, up to a constant factor, the Laplacian associated to this Hermitian metric.

By the maximum principle for parabolic equations, (2.2) implies that

$$\inf_M(\Lambda_{\chi_0}\omega) \leq \Lambda_{\chi}\omega \leq \sup_M(\Lambda_{\chi_0}\omega), \tag{2.3}$$

which gives a lower bound for χ,

$$\chi \geq \frac{1}{\sup_M(\Lambda_{\chi_0}\omega)}\,\omega. \tag{2.4}$$

The J-functional [C1] is defined by

$$J_{\omega,\chi_0}(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t}\,\frac{\omega \wedge \chi_{\phi_t}^{n-1}}{(n-1)!}\,dt,$$

where $\{\phi_t\}$ is a path in $\mathcal{H}$ between 0 and ϕ. The functional is independent of the choice of path. We will need the following formula for the functional in the case $n = 2$. Taking the path $\phi_t = t\phi$, we see that

$$J_{\omega,\chi_0}(\phi) = \frac{1}{2}\int_M \phi\,\omega \wedge (\chi_0 + \chi). \tag{2.5}$$

Chen also makes use of the I-functional,

$$I_{\omega,\chi_0}(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \frac{\chi^n_{\phi_t}}{n!}\, dt.$$

This is a well-known functional in Kähler geometry (see [Ma]). Notice that $I(\phi) = 0$ along the flow. For $n = 2$, this functional is given by

$$I_{\omega,\chi_0}(\phi) = \frac{1}{6} \int_M \phi\, (\chi_0^2 + \chi \wedge \chi_0 + \chi^2). \tag{2.6}$$

In the course of the paper, $C_0, C_1, \ldots$ will denote constants depending only on the initial data ω and χ_0. Curvature expressions such as $R_{i\bar{j}k\bar{l}}$ will always refer to the metric $g_{i\bar{j}}$.

3. Second order estimate.

We use the maximum principle to obtain an estimate on the second derivative of ϕ in terms of ϕ. We choose to calculate the evolution of $(\log \Lambda_\omega \chi - A\phi)$ for some constant A (compare to [Y1], [Au] or [Si] for the analogous estimate for the well-known Monge-Ampère equation, and [Ca] for the Kähler-Ricci flow.)

Theorem 3.1. *Suppose that (M, ω) has dimension $n = 2$ and that*

$$\chi_0 - \omega > 0. \tag{3.1}$$

Let $\phi = \phi_t$ be a solution of the J-flow (2.1) on $[0, \infty)$. Then there exist constants $A > 0$ and $C > 0$ depending only on the initial data such that for any time $t \geq 0$, $\chi = \chi_{\phi_t}$ satisfies

$$\Lambda_\omega \chi \leq C e^{A(\phi - \inf_{M\times[0,t]} \phi)}. \tag{3.2}$$

Proof. We will calculate

$$(\tilde{\triangle} - \frac{\partial}{\partial t})(\log(\Lambda_\omega \chi) - A\phi).$$

Using normal coordinates for ω, first calculate

$$\begin{aligned} \tilde{\triangle}(\Lambda_\omega \chi) &= \frac{1}{n} h^{k\bar{l}} \partial_k \partial_{\bar{l}} (g^{i\bar{j}} \chi_{i\bar{j}}) \\ &= \frac{1}{n} h^{k\bar{l}} R_{k\bar{l}}{}^{i\bar{j}} \chi_{i\bar{j}} + \frac{1}{n} h^{k\bar{l}} g^{i\bar{j}} \partial_k \partial_{\bar{l}} \chi_{i\bar{j}}. \end{aligned}$$

And

$$\begin{aligned}
\frac{\partial}{\partial t}(\Lambda_\omega \chi) &= \frac{\partial}{\partial t}(g^{i\bar{j}} \partial_i \partial_{\bar{j}} \phi) \\
&= -\frac{1}{n} g^{i\bar{j}} \partial_i \partial_{\bar{j}} (\chi^{k\bar{l}} g_{k\bar{l}}) \\
&= \frac{1}{n}(g^{i\bar{j}} \partial_i (\chi^{p\bar{l}} \partial_{\bar{j}} \chi_{p\bar{q}} \chi^{k\bar{q}}) g_{k\bar{l}} + g^{i\bar{j}} \chi^{k\bar{l}} R_{i\bar{j}k\bar{l}}) \\
&= \frac{1}{n}(g^{i\bar{j}} h^{p\bar{q}} \partial_i \partial_{\bar{j}} \chi_{p\bar{q}} - g^{i\bar{j}} h^{r\bar{q}} \chi^{p\bar{s}} \partial_i \chi_{r\bar{s}} \partial_{\bar{j}} \chi_{p\bar{q}} \\
&\quad - g^{i\bar{j}} h^{p\bar{s}} \chi^{r\bar{q}} \partial_i \chi_{r\bar{s}} \partial_{\bar{j}} \chi_{p\bar{q}} + \chi^{k\bar{l}} R_{k\bar{l}}).
\end{aligned}$$

Now

$$\tilde{\triangle} \log(\Lambda_\omega \chi) = \frac{\tilde{\triangle}(\Lambda_\omega \chi)}{\Lambda_\omega \chi} - \frac{|\tilde{\nabla}(\Lambda_\omega \chi)|^2}{(\Lambda_\omega \chi)^2},$$

where

$$|\tilde{\nabla}(\Lambda_\omega \chi)|^2 = \frac{1}{n} h^{k\bar{l}} \partial_k (\Lambda_\omega \chi) \partial_{\bar{l}} (\Lambda_\omega \chi).$$

Note that by the Kähler property of χ, we have

$$\partial_i \partial_{\bar{j}} \chi_{k\bar{l}} = \partial_k \partial_{\bar{l}} \chi_{i\bar{j}}.$$

Then

$$\begin{aligned}
&(\tilde{\triangle} - \frac{\partial}{\partial t}) \log(\Lambda_\omega \chi) \\
&= \frac{1}{n\Lambda_\omega \chi}(h^{k\bar{l}} R_{k\bar{l}}{}^{i\bar{j}} \chi_{i\bar{j}} - n\frac{|\tilde{\nabla}(\Lambda_\omega \chi)|^2}{\Lambda_\omega \chi} + g^{i\bar{j}} h^{r\bar{q}} \chi^{p\bar{s}} \partial_i \chi_{r\bar{s}} \partial_{\bar{j}} \chi_{p\bar{q}} \\
&\quad + g^{i\bar{j}} h^{p\bar{s}} \chi^{r\bar{q}} \partial_i \chi_{r\bar{s}} \partial_{\bar{j}} \chi_{p\bar{q}} - \chi^{k\bar{l}} R_{k\bar{l}}).
\end{aligned}$$

We need the following lemma to deal with the second term on the right hand side.

Lemma 3.2.

$$n|\tilde{\nabla}(\Lambda_\omega \chi)|^2 \leq (\Lambda_\omega \chi) g^{i\bar{j}} h^{r\bar{q}} \chi^{p\bar{s}} \partial_i \chi_{r\bar{s}} \partial_{\bar{j}} \chi_{p\bar{q}}.$$

Proof. Using normal coordinates for ω in which χ is diagonal, and making use of the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
n|\tilde{\nabla}(\Lambda_\omega\chi)|^2 &= \sum_{i,j,k} \chi^{k\bar{k}}\chi^{k\bar{k}}\partial_k\chi_{i\bar{i}}\partial_{\bar{k}}\chi_{j\bar{j}} \\
&\leq \sum_{i,j}\left(\sum_k (\chi^{k\bar{k}})^2|\partial_k\chi_{i\bar{i}}|^2\right)^{1/2}\left(\sum_k (\chi^{k\bar{k}})^2|\partial_k\chi_{j\bar{j}}|^2\right)^{1/2} \\
&= \left(\sum_i \left(\sum_k (\chi^{k\bar{k}})^2|\partial_k\chi_{i\bar{i}}|^2\right)^{1/2}\right)^2 \\
&= \left(\sum_i \sqrt{\chi_{i\bar{i}}}\left(\sum_k (\chi^{k\bar{k}})^2\chi^{i\bar{i}}|\partial_k\chi_{i\bar{i}}|^2\right)^{1/2}\right)^2 \\
&\leq \sum_i \chi_{i\bar{i}} \sum_{i,k} (\chi^{k\bar{k}})^2\chi^{i\bar{i}}|\partial_k\chi_{i\bar{i}}|^2 \\
&= (\Lambda_\omega\chi)\sum_{i,k}(\chi^{k\bar{k}})^2\chi^{i\bar{i}}\partial_k\chi_{i\bar{i}}\partial_{\bar{k}}\chi_{i\bar{i}} \\
&= (\Lambda_\omega\chi)\sum_{i,k}(\chi^{k\bar{k}})^2\chi^{i\bar{i}}\partial_i\chi_{k\bar{i}}\partial_{\bar{i}}\chi_{i\bar{k}} \\
&\leq (\Lambda_\omega\chi)\sum_{i,j,k}(\chi^{k\bar{k}})^2\chi^{i\bar{i}}\partial_j\chi_{k\bar{i}}\partial_{\bar{j}}\chi_{i\bar{k}} \\
&= (\Lambda_\omega\chi)g^{i\bar{j}}h^{r\bar{q}}\chi^{p\bar{s}}\partial_i\chi_{r\bar{s}}\partial_{\bar{j}}\chi_{p\bar{q}}.
\end{aligned}$$

Let C_0 be a constant satisfying

$$R_{k\bar{l}}{}^{i\bar{j}} \geq -C_0 g_{k\bar{l}}g^{i\bar{j}}.$$

Then,

$$\begin{aligned}
(\tilde{\triangle} - \frac{\partial}{\partial t})\log(\Lambda_\omega\chi) &\geq \frac{1}{n\Lambda_\omega\chi}(-C_0h^{k\bar{l}}g_{k\bar{l}}g^{i\bar{j}}\chi_{i\bar{j}} - \chi^{k\bar{l}}R_{k\bar{l}}) \\
&= \frac{1}{n}(-C_0h^{k\bar{l}}g_{k\bar{l}} - \frac{1}{\Lambda_\omega\chi}\chi^{k\bar{l}}R_{k\bar{l}}).
\end{aligned}$$

Now calculate

$$\begin{aligned}(\tilde{\triangle} - \frac{\partial}{\partial t})\phi &= \frac{1}{n}(h^{k\bar{l}}\partial_k\partial_{\bar{l}}\phi + \chi^{i\bar{j}}g_{i\bar{j}} - 1)\\ &= \frac{1}{n}(\chi^{k\bar{j}}\chi^{i\bar{l}}g_{i\bar{j}}\chi_{k\bar{l}} - h^{k\bar{l}}\chi_{0\,k\bar{l}} + \chi^{i\bar{j}}g_{i\bar{j}} - 1)\\ &= \frac{1}{n}(2\chi^{i\bar{j}}g_{i\bar{j}} - h^{k\bar{l}}\chi_{0\,k\bar{l}} - 1).\end{aligned}$$

At this point we must choose our value of A. From our assumption (3.1), we can choose $0 < \epsilon < 1/3$ to be sufficiently small so that

$$\chi_0 \geq (1+3\epsilon)\omega. \tag{3.3}$$

Let A be given by

$$A = \frac{C_0}{\epsilon}.$$

Fix a time $t > 0$. There is a point (x_0, t_0) in $M \times [0,t]$ at which the maximum of $(\log(\Lambda_\omega\chi) - A\phi)$ is achieved. We may assume that $t_0 > 0$. At this point, we have

$$\begin{aligned}0 &\geq (\tilde{\triangle} - \frac{\partial}{\partial t})(\log(\Lambda_\omega\chi) - A\phi)\\ &\geq \frac{1}{n}(-C_0h^{k\bar{l}}g_{k\bar{l}} - \frac{1}{\Lambda_\omega\chi}\chi^{k\bar{l}}R_{k\bar{l}} - 2A\chi^{i\bar{j}}g_{i\bar{j}} + Ah^{k\bar{l}}\chi_{0\,k\bar{l}} + A)\\ &\geq \frac{1}{n}(-C_0h^{k\bar{l}}g_{k\bar{l}} - \frac{1}{\Lambda_\omega\chi}\chi^{k\bar{l}}R_{k\bar{l}} - 2A\chi^{i\bar{j}}g_{i\bar{j}} + (1-\epsilon)Ah^{k\bar{l}}\chi_{0\,k\bar{l}}\\ &\quad + \epsilon Ah^{k\bar{l}}g_{k\bar{l}} + A)\\ &= \frac{1}{n}(-\frac{1}{\Lambda_\omega\chi}\chi^{k\bar{l}}R_{k\bar{l}} - 2A\chi^{i\bar{j}}g_{i\bar{j}} + (1-\epsilon)Ah^{k\bar{l}}\chi_{0\,k\bar{l}} + A).\end{aligned}$$

From the lower bound (2.4) on $\chi_{k\bar{l}}$, the term $\chi^{k\bar{l}}R_{k\bar{l}}$ is bounded above and hence at (x_0, t_0), we have

$$1 + (1-\epsilon)h^{k\bar{l}}\chi_{0\,k\bar{l}} - 2\chi^{i\bar{j}}g_{i\bar{j}} \leq \frac{C_1}{(\Lambda_\omega\chi)}.$$

From (3.3), we get

$$1 + (1+\epsilon)h^{k\bar{l}}g_{k\bar{l}} - 2\chi^{i\bar{j}}g_{i\bar{j}} \leq \frac{C_1}{(\Lambda_\omega\chi)}. \tag{3.4}$$

We will compute in normal coordinates at x_0 for ω in which χ is diagonal and has eigenvalues λ_1, λ_2. From (2.4), λ_1 and λ_2 are bounded below by a positive constant. We want to show that they are also bounded above. First, observe that for $n = 2$,

$$\frac{1}{\Lambda_\chi \omega} = \frac{\det \chi}{(\det \omega)(\Lambda_\omega \chi)},$$

and by (2.3), this is bounded along the flow.

Multiplying (3.4) by $(\det \chi / \det \omega)$ gives,

$$\lambda_1 \lambda_2 + (1+\epsilon)(\frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2}) - 2(\lambda_1 + \lambda_2) \leq C_2.$$

From (2.3), we may suppose that one of the eigenvalues, say λ_2, is bounded from above. Rewrite the inequality as

$$\lambda_1(\lambda_2 + (1+\epsilon)\frac{1}{\lambda_2} - 2) + (1+\epsilon)\frac{\lambda_2}{\lambda_1} - 2\lambda_2 \leq C_2.$$

Then, since the function $f: (0, \infty) \to \mathbf{R}$ defined by

$$f(x) = x + (1+\epsilon)\frac{1}{x} - 2,$$

is bounded below by a small positive constant depending on ϵ, we see that λ_1 must also be bounded above. Hence at the point (x_0, t_0), there exists C depending only on the initial data such that

$$\Lambda_\omega \chi \leq C.$$

Then, on $M \times [0, t]$,

$$\log(\Lambda_\omega \chi) - A\phi \leq \log C - A \inf_{M \times [0,t]} \phi.$$

Exponentiating gives

$$\Lambda_\omega \chi \leq C e^{A(\phi - \inf_{M \times [0,t]} \phi)},$$

completing the proof of the theorem.

4. Zero order estimate.

We prove an estimate on the C^0 norm of ϕ using a Moser iteration method applied to the exponential of the solution rather than a power of the solution (compare to [Y1]) and the estimate of Theorem 3.1.

Theorem 4.1. *Suppose that* (M, ω) *has dimension* $n = 2$ *and that*

$$\chi_0 - \omega > 0.$$

Let ϕ_t *be a solution of the J-flow* (2.1) *on* $[0, \infty)$. *Then there exists a constant* $\tilde{C}$ *depending only on the initial data such that*

$$\|\phi_t\|_{C^0(M)} \leq \tilde{C}.$$

Proof. Suppose first that $\inf_M \phi_t$ is bounded from below uniformly in time. We will show that this implies the above estimate. Since the functional J_{ω,χ_0} decreases along the flow, there exists a constant C_0 such that

$$\int_M \phi_t \, \omega \wedge (\chi_0 + \chi_{\phi_t}) \leq C_0,$$

using (2.5). Let C_1 be a positive constant satisfying

$$\omega^2 \leq C_1 \omega \wedge \chi_0.$$

Then

$$\begin{aligned}
\int_M \phi_t \, \omega^2 &= \int_M (\phi_t - \inf_M \phi_t)\omega^2 + \int_M \inf_M \phi_t \, \omega^2 \\
&\leq C_1 \int_M (\phi_t - \inf_M \phi_t)\omega \wedge \chi_0 + \inf_M \phi_t \int_M \omega^2 \\
&\leq C_1 C_0 - C_1 \int_M \phi_t \, \omega \wedge \chi_{\phi_t} + \inf_M \phi_t \left(\int_M \omega^2 - C_1 \int_M \omega \wedge \chi_0 \right) \\
&= C_1 C_0 - C_1 \int_M (\phi_t - \inf_M \phi_t)\omega \wedge \chi_{\phi_t} \\
&\quad + \inf_M \phi_t \left(\int_M \omega^2 - 2C_1 \int_M \omega \wedge \chi_0 \right) \\
&\leq C_1 C_0 + \inf_M \phi_t \left(\int_M \omega^2 - 2C_1 \int_M \omega \wedge \chi_0 \right).
\end{aligned}$$

This gives an upper bound for $\int_M \phi_t \, \omega^2$ depending on the lower bound for $\inf_M \phi_t$. Since $\triangle_\omega \phi_t > -\Lambda_\omega \chi_0$ along the flow, it follows from the existence of a lower bound on the Green's function of ω that $\sup_M \phi_t$ is bounded from above, giving us the required estimate.

Now suppose that no such lower bound for $\inf_M \phi_t$ exists. Then we can assume that there is a sequence of times $t_i \to \infty$ such that

(i) $\inf_M \phi_{t_i} = \inf_{t \in [0,t_i]} \inf_M \phi_t$

(ii) $\inf_M \phi_{t_i} \to -\infty$.

We will seek a contradiction. For a fixed i, write

$$\psi_{t_i} = \phi_{t_i} - \sup_M \phi_{t_i}.$$

Notice that $\sup_M \phi_{t_i}$ is bounded from below by zero from (2.6) and the fact that $I(\phi_t) = 0$. Hence

$$\|\psi_{t_i}\|_{C^0} \to \infty.$$

The following proposition is the key result of this section.

Proposition 4.2. *Let M be a compact complex surface with two Kähler metrics χ_0 and ω. Suppose that $\psi \in C^\infty(M)$ satisfies the conditions*

$$\chi_\psi = \chi_0 + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\psi > 0, \qquad \sup_M \psi = 0,$$

and

$$\Lambda_\omega \chi_\psi \leq Ce^{A(\psi - \inf_M \psi)}.$$

Then there exists a constant C' depending only on M, ω, χ_0 and the constants A and C such that

$$\|\psi\|_{C^0} \leq C'.$$

We apply this proposition to $\psi = \psi_{t_i}$ and obtain a contradiction since

$$\begin{aligned} \Lambda_\omega \chi_{\psi_{t_i}} &= \Lambda_\omega \chi_{\phi_{t_i}} \\ &\leq Ce^{A(\phi_{t_i} - \inf_{t\in[0,t_i]} \inf_M \phi_t)} \\ &= Ce^{A(\psi_{t_i} - \inf_M \psi_{t_i})}, \end{aligned}$$

where we have used Theorem 3.1 and condition (i) above. It remains to prove the proposition.

Proof of Proposition 4.2 Let δ be a small positive constant, to be determined later. Set $B = A/(1-\delta)$ and let $u = e^{-B\psi}$.

Now, for $\beta = n/(n-1) = 2$, the Sobolev inequality for functions f on (M, ω) is

$$\|f\|^2_{2\beta} \leq C_2(\|\nabla f\|^2_2 + \|f\|^2_2),$$

for C_2 depending on ω. We will apply this to $u^{p/2}$ for $p \geq 1$. This gives

$$\left(\int_M e^{-Bp\beta\psi} \frac{\omega^2}{2} \right)^{1/\beta} \leq C_2 \left(\int_M |\nabla e^{-Bp\psi/2}|^2 \frac{\omega^2}{2} + \int_M e^{-Bp\psi} \frac{\omega^2}{2} \right). \quad (4.1)$$

Now calculate

$$\begin{aligned}
\int_M |\nabla e^{-Bp\psi/2}|^2 \frac{\omega^2}{2} &= \sqrt{-1}\int_M \partial e^{-Bp\psi/2} \wedge \overline{\partial} e^{-Bp\psi/2} \wedge \omega \\
&= \frac{B^2p^2}{4}\sqrt{-1}\int_M e^{-Bp\psi}\partial\psi \wedge \overline{\partial}\psi \wedge \omega \\
&= -\frac{Bp}{4}\sqrt{-1}\int_M \partial(e^{-Bp\psi}) \wedge \overline{\partial}\psi \wedge \omega \\
&= \frac{Bp}{2}\int_M e^{-Bp\psi}\frac{\sqrt{-1}}{2}\partial\overline{\partial}\psi \wedge \omega \\
&= \frac{Bp}{2}\int_M e^{-Bp\psi}(\chi_\psi - \chi_0) \wedge \omega \\
&= \frac{Bp}{2}\int_M e^{-Bp\psi}(\Lambda_\omega\chi_\psi - \Lambda_\omega\chi_0)\frac{\omega^2}{2} \\
&\leq \frac{CBp}{2}\int_M e^{-Bp\psi}e^{A(\psi - \inf_M \psi)}\frac{\omega^2}{2} \\
&= \frac{CBp}{2}e^{-A\inf_M \psi}\int_M e^{-(p-(1-\delta))B\psi}\frac{\omega^2}{2},
\end{aligned}$$

where we have used the estimate

$$\Lambda_\omega\chi_\psi \leq Ce^{A(\psi - \inf_M \psi)}.$$

Then in (4.1),

$$\left(\int_M u^{p\beta}\frac{\omega^2}{2}\right)^{1/\beta} \leq C_3 p e^{-A\inf_M \psi}\int_M u^{p-(1-\delta)}\frac{\omega^2}{2}.$$

Raising to the power $1/p$ and writing $\gamma = 1-\delta$ gives

$$\|u\|_{p\beta} \leq C_3^{1/p}p^{1/p}e^{-(A/p)\inf_M \psi}\|u\|_{p-\gamma}^{(p-\gamma)/p}.$$

Take the logarithm of both sides to get

$$\log\|u\|_{p\beta} \leq \frac{1}{p}\log C_3 + \frac{1}{p}\log p + \frac{1}{p}\sup_M(-A\psi) + \frac{(p-\gamma)}{p}\log\|u\|_{p-\gamma}.$$

We now apply the iteration. First, replace p with $p\beta+\gamma$ to get

$$\begin{aligned}
\log\|u\|_{p\beta^2+\gamma\beta} \leq\ & \frac{1+\beta}{p\beta+\gamma}\log C_3 + \frac{1}{p\beta+\gamma}(\beta\log p + \log(p\beta+\gamma)) \\
& + \frac{1+\beta}{p\beta+\gamma}\sup_M(-A\psi) + \frac{\beta(p-\gamma)}{p\beta+\gamma}\log\|u\|_{p-\gamma}.
\end{aligned}$$

Repeat this procedure, replacing p with $p\beta+\gamma$ to obtain for any positive integer k,

$$\begin{aligned}
&\log \|u\|_{p\beta^{k+1}+\gamma(\beta+\beta^2+\ldots+\beta^k)} \\
\leq\ & \frac{1+\beta+\beta^2+\ldots+\beta^k}{p\beta^k+\gamma(1+\beta+\beta^2+\ldots+\beta^{k-1})}\log C_3 \\
&+\frac{1}{p\beta^k+\gamma(1+\beta+\ldots+\beta^{k-1})}\left(\ \beta^k\log p+\beta^{k-1}\log(p\beta+\gamma)+\ldots\right. \\
&\qquad \left.\ldots+\log(p\beta^k+\gamma(1+\beta+\ldots+\beta^{k-1})\ \right) \\
&+\frac{1+\beta+\beta^2+\ldots+\beta^k}{p\beta^k+\gamma(1+\beta+\beta^2+\ldots+\beta^{k-1})}\sup_M(-A\psi) \\
&+\frac{\beta^k(p-\gamma)}{p\beta^k+\gamma(1+\beta+\beta^2+\ldots+\beta^{k-1})}\log\|u\|_{p-\gamma}.
\end{aligned} \tag{4.2}$$

Now set $p=1+\delta$. Then, since $\beta=2$ we have

$$p\beta^k+\gamma(1+\beta+\beta^2+\ldots+\beta^{k-1})=1+\beta+\beta^2+\ldots+\beta^k+\delta.$$

Notice that the second term on the right hand side of (4.2) is bounded by

$$\begin{aligned}
\log p+\frac{1}{\beta}\log\beta^2+\ldots+\frac{1}{\beta^k}\log(\beta^{k+1}) &\leq \log p+\log\beta(\sum_{i=1}^{k}\frac{i+1}{\beta^i}) \\
&\leq C_4.
\end{aligned}$$

Then

$$\begin{aligned}
&\log\|u\|_{p\beta^{k+1}+\gamma(\beta+\beta^2+\ldots+\beta^k)} \\
&\qquad\qquad \leq \log C_3+C_4+\sup_M(-A\psi)+2\delta\max(\log\|u\|_{2\delta},0).
\end{aligned}$$

Using the fact that $A=(1-\delta)B$ and $-B\psi=\log u$, and letting k tend to infinity,

$$\log\|u\|_{C_0}\leq C_5+2\max(\log\|u\|_{2\delta},0).$$

Hence we get the following inequality for ψ,

$$\|\psi\|_{C^0}\leq C_6+C_7\max\left(\log\left(\int_M e^{-2\delta B\psi}\frac{\omega^2}{2}\right)^{1/2\delta},0\right). \tag{4.3}$$

We can now finish the estimate. First, define

$$P(M,\chi_0)=\{\Phi\in C^2(M)\mid \chi_0+\frac{\sqrt{-1}}{2}\partial\overline{\partial}\Phi\geq 0,\ \sup_M\Phi=0\}.$$

Then Proposition 2.1 of [T1] (see section 4.4, [Ho]) states that there exist constants $\alpha > 0$ and C_8 depending only on (M, χ_0) such that

$$\int_M e^{-\alpha \Phi} \frac{\chi_0^n}{n!} \leq C_8 \qquad \text{for all } \Phi \in P(M, \chi_0).$$

Define δ to be

$$\delta = \min\{\frac{\alpha}{4A}, \frac{1}{2}\} > 0.$$

Then the required estimate follows from (4.3), since ψ belongs to $P(M, \chi_0)$.

5. Convergence of the flow.

In this section we complete the proof of the main theorem. We assume, using the result of [C2], that a solution $\phi = \phi_t$ for the J-flow exists for all time. From Theorem 3.1 and Theorem 4.1 we have uniform estimates on ϕ and the derivatives $\partial_i \partial_{\overline{j}} \phi$, using the fact that

$$\chi_{i\overline{j}} = \chi_{0\, i\overline{j}} + \partial_i \partial_{\overline{j}} \phi > 0.$$

Since the operator

$$\frac{1}{n}(1 - \Lambda_\chi \omega),$$

is concave in the $\chi_{i\overline{j}}$, it is well known that, by the work of Evans [E1, E2] and Krylov [Kr] (see also [Tr]), one can deduce a uniform Hölder estimate on the second derivatives $\partial_i \partial_{\overline{j}} \phi$. By differentiating the equation (2.1) and applying standard Schauder estimates for parabolic equations (see [LSU] for example), one can obtain uniform estimates on all of the derivatives of ϕ. It then follows that there is a sequence of times $t_j \to \infty$ such that ϕ_{t_j} converges in C^∞ to some smooth function ϕ_∞. In order to show that we have convergence without having to pass to a subsequence, we will use a modification of the argument in [Ca].

Notice that $\partial \phi / \partial t$ satisfies the heat equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) = \tilde{\triangle}\left(\frac{\partial \phi}{\partial t}\right).$$

Since we have uniform bounds for $\chi_{i\overline{j}}$ from above and away from zero, and bounds on $\frac{\partial}{\partial t}\chi_{i\overline{j}}$ and all the covariant derivatives of $\chi_{i\overline{j}}$ and $\frac{\partial}{\partial t}\chi_{i\overline{j}}$, it follows from the Harnack inequality of Li and Yau [LY] and the argument in [Ca] that there exist positive constants C_0 and η, which are independent of t, such that

$$\sup_M \left(\frac{\partial \phi}{\partial t}\right) - \inf_M \left(\frac{\partial \phi}{\partial t}\right) \leq C_0 e^{-\eta t}.$$

Since

$$\int_M \frac{\partial \phi}{\partial t}\chi^2 = 0,$$

$\partial\phi/\partial t$ must take on the value zero somewhere on M for each t, and so

$$\left|\frac{\partial \phi}{\partial t}\right| \leq C_0 e^{-\eta t}.$$

Hence for any $0 < s < s'$, and any $x \in M$,

$$\begin{aligned} |\phi(x,s') - \phi(x,s)| &= |\int_s^{s'} \frac{\partial \phi}{\partial t}(x,t)dt| \\ &\leq \int_s^{s'} |\frac{\partial \phi}{\partial t}(x,t)|dt \\ &\leq C_0 \int_s^{s'} e^{-\eta t} dt \\ &= C_0 \frac{1}{\eta}(e^{-\eta s} - e^{-\eta s'}), \end{aligned}$$

which tends to zero as s and s' tend to infinity. Hence ϕ_t converges in the C_0 norm to ϕ_∞. It must converge also in the C^∞ topology, since otherwise there would exist an integer N, an $\epsilon > 0$ and a sequence $t_j \to \infty$ with

$$\|\phi_{t_j} - \phi_\infty\|_{C^N} \geq \epsilon.$$

Since ϕ is bounded in all the C^k norms, one could pass to a subsequence of the ϕ_{t_j} which would converge to some $\phi'_\infty \neq \phi_\infty$, giving the contradiction. This completes the proof.

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