The Apollonian Inner Metric

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In this paper we derive an explicit formula for the inner metric
of the Apollonian metric and prove that for most domains there
exists a geodesic connecting two arbitrary points. We also give a
necessary and sufficient condition for the Apollonian inner metric
to be bilipschitz equivalent to the quasihyperbolic metric.

1. Introduction.

In this paper we consider the inner metric of the Apollonian metric from
[3]. Some basic features of this metric are that it is Möbius invariant, equals
the hyperbolic metric in balls and half-spaces and has a geodesic connecting
an arbitrary pair of points (in most domains). We start by defining the
Apollonian metric and its inner metric in order to state our main results.
The notation used conforms largely to that of [2] and [19], the reader can
consult Section 2.1 of this paper, if necessary.

We will be considering domains (open connected non-empty sets) G in
the Möbius space $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$. The Apollonian metric is defined for
$x, y \in G \subsetneq \mathbb{R}^n$ by

$$\alpha_G(x,y) := \sup_{a,b \in \partial G} \log \frac{|a-x|}{|a-y|} \frac{|b-y|}{|b-x|} \tag{1.1}$$

(with the understanding that $|\infty - x|/|\infty - y| = 1$). It is in fact a met-
ric if only if the complement of G is not contained in a hyperplane and a
pseudometric otherwise, as was noted in [3, Theorem 1.1]. This metric was
introduced in [3] and has also been considered in [5, 8–14, 16, 17]. It should
also be noted that the same metric has been considered from a different
perspective under the name of the Barbilian metric for instance in [4] and
[15].

Let $\gamma\colon [0,1] \to G \subset \mathbb{R}^n$ be a path, i.e. a continuous function. If d is a metric in G, then the d-length of γ is defined by

$$d(\gamma) := \sup \sum_{i=0}^{k-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over $k < \infty$ and all sequences $\{t_i\}$ satisfying $0 = t_0 < t_1 < \ldots < t_k = 1$. All the paths in this paper are assumed to be rectifiable, that is have finite Euclidean length. The inner metric of d is defined by the formula

$$\tilde{d}(x,y) := \inf_{\gamma} d(\gamma),$$

where the infimum is taken over all paths connecting x and y in G. We denote the inner metric of the Apollonian metric by $\tilde{\alpha}_G$ and call it the Apollonian inner metric.

Since the Apollonian metric is only a pseudometric in some domains it is not reasonable to expect the Apollonian inner metric to be a metric in every domain. The next theorem shows that it is a proper metric in a somewhat larger class of domains than is the Apollonian metric.

Theorem 1.2. *The Apollonian inner metric is a pseudometric in every domain $G \subsetneq \mathbb{R}^n$ and a metric if and only if the complement of G is not contained in an $(n-2)$-dimensional plane.*

In order to present an integral formula for the Apollonian inner metric we need the Apollonian spheres and the density of the Apollonian metric, both from [8].

Definition 1.3. Let $G \subsetneq \mathbb{R}^n$, $x \in G$ and $\theta \in \mathbb{R}^n \setminus \{0\}$.

1. Let $r := r(x,\theta) := \sup\{s\colon B^n(x + s\theta/|\theta|, s) \subset G\}$. We define the *Apollonian sphere through x in direction θ* to be $S^{n-1}(x + r\theta/|\theta|, r)$ if $r < \infty$ and the limiting half-space otherwise.

2. We denote $\overline{\alpha}_G(x;\theta) := 1/(2r(x,\theta)) + 1/(2r(x,-\theta))$ and call this function the *density* of the Apollonian metric.

The main result of the present paper is the following integral formula for the Apollonian inner metric. Piecewise continuously differentiable means continuously differentiable except in a finite number of points.

Theorem 1.4. *If $x, y \in G \subsetneq \mathbb{R}^n$, then*

$$\tilde{\alpha}_G(x, y) = \inf_\gamma \int \overline{\alpha}_G(\gamma(t); \gamma'(t))|\gamma'(t)|dt,$$

where the infimum is taken over all paths connecting x and y in G that are piecewise continuously differentiable (with the understanding that $\overline{\alpha}_G(z; 0)0 = 0$ for all $z \in G$, even when $\overline{\alpha}_G(z; 0)$ is not defined).

The next result shows that the infimum in the definition of the Apollonian inner metric is a minimum in almost all cases, namely, when α_G is a metric.

Theorem 1.5. *Let $x, y \in G \subsetneq \mathbb{R}^n$ and G be such that G^c is not contained in a hyperplane. Then there exists a path γ connecting x and y in G such that $\tilde{\alpha}_G(\gamma) = \tilde{\alpha}_G(x, y)$.*

In the last section of this paper we consider inequalities between the Apollonian inner metric, the j_G metric and the quasihyperbolic metric, k_G. Let $G \subsetneq \mathbb{R}^n$ be a domain and $x, y \in G$. The j_G metric, which is a modification from [18] of a metric from [6], is defined by

$$j_G(x, y) := \log\left\{ 1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}} \right\}.$$

The quasihyperbolic metric from [7] is defined by

$$k_G(x, y) := \inf_\gamma \int \frac{|dz|}{d(z, \partial G)},$$

where the infimum is taken over all paths γ joining x and y in G. The results of this section are summarized in the following theorem.

Theorem 1.6. *For $G \subsetneq \mathbb{R}^n$ the following conditions are equivalent:*

1. *There exists $C > 0$ such that $\overline{\alpha}_G(x; r)d(x, \partial G) \geq C$ for all $x \in G$ and $r \in S^{n-1}$;*

2. *There exists $C > 0$ such that $\frac{1}{C}k_G \leq \tilde{\alpha}_G \leq Ck_G$;*

3. *There exists $C > 0$ such that $j_G \leq C\tilde{\alpha}_G$.*

The structure of the rest of this paper is as follows. In the next section we review the notation and terminology used. In Section 3 we prove Theorem 1.4 and in Section 4 we use this result to prove Theorema 1.2 and 1.5. In Section 5 we relate the Apollonian metric to the quasihyperbolic metric by means of the concept of quasi-isotropy and prove Theorem 1.6.

2. Notation and terminology.

The notation used conforms largely to that in [2] and [19], as was mentioned
in the introduction, and is reviewed in Section 2.1. Sections 2.2 and 2.3, on
the other hand, contain material specific to the Apollonian metric.

2.1. Common notation.

We denote by $\{e_1, e_2, \ldots, e_n\}$ the standard basis of $\mathbb{R}^n$ and by n the dimen-
sion of the Euclidean space under consideration and assume that $n \geq 2$. For
$x \in \mathbb{R}^n$ we denote by x_i its ith coordinate. The following notation is used
for balls, spheres and the upper half-space ($x \in \mathbb{R}^n$ and $0 < r < \infty$):

$$B^n(x, r) := \{y \in \mathbb{R}^n \colon |x - y| < r\}, \ S^{n-1}(x, r) := \{y \in \mathbb{R}^n \colon |x - y| = r\},$$

$$B^n := B^n(0, 1), \ S^{n-1} := S^{n-1}(0, 1), \ H^n := \{y \in \mathbb{R}^n \colon y_n > 0\}.$$

We use the notation $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$ for the one point compactification
of $\mathbb{R}^n$. If $G \subset \overline{\mathbb{R}^n}$ we denote by ∂G, G^c and $\overline{G}$ its boundary, complement
and closure, respectively, all with respect to $\overline{\mathbb{R}^n}$. In contrast to topological
operations, we always consider metric operations with respect to the ordinary
Euclidean metric. For $x \in G \subsetneq \mathbb{R}^n$ we denote $\delta(x) := d(x, \partial G) := \min\{|x -
z| \colon z \in \partial G\}$.

2.2. The Apollonian balls approach.

In this subsection we present the Apollonian balls approach which has previ-
ously been used in [3, Section 2], [4] and [17, Theorem 4.1]. This presentation
is from Section 5.1 of [11].

For $x, y \in G \subsetneq \mathbb{R}^n$ we define

$$q_x := \sup_{a \in \partial G} \frac{|a - x|}{|a - y|}, \ q_y := \sup_{b \in \partial G} \frac{|b - y|}{|b - x|}.$$

The numbers q_x are called the *Apollonian parameters* of x and y (with respect
to G) and by definition $\alpha_G(x, y) = \log\{q_x q_y\}$. The balls or half-spaces

$$B_x := \{z \in \mathbb{R}^n \colon \tfrac{|z-x|}{|z-y|} < \tfrac{1}{q_x}\} \text{ and } B_y := \{z \in \mathbb{R}^n \colon \tfrac{|z-y|}{|z-x|} < \tfrac{1}{q_y}\},$$

are called the *Apollonian balls* about x and y (in G). We denote the centers
and radii of B_x and B_y by x_0, y_0, r_x and r_y. We collect some immediate
results regarding these balls, similar results obviously hold with x and y
interchanged.

1. $B_x \subset G$ and $\overline{B_x} \cap \partial G \neq \emptyset$.

2. If i_x and i_y denote the inversions in the spheres ∂B_x and ∂B_y, then $y = i_x(x) = i_y(x)$.

3. Since $\infty \notin G$ we have $q_x \geq 1$, if moreover $\infty \notin \overline{G}$, then $q_x > 1$.

4. If $q_x > 1$, x_0 denotes the center of B_x and r_x its radius, then

$$|x - x_0| = \frac{|x - y|}{q_x^2 - 1} = \frac{r_x}{q_x}.$$

2.3. The Apollonian spheres and quasi-isotropy.

Let us start by defining the concept of quasi-isotropy, which was introduced [8] and was studied in depth in [10].

Definition 2.1. We say that a metric space (G, d) with $G \subset \mathbb{R}^n$ is K–quasi-isotropic if

$$\limsup_{r \to 0} \frac{\sup\{d(x, z) \colon |x - z| = r\}}{\inf\{d(x, y) \colon |x - y| = r\}} \leq K$$

for every $x \in G$. A 1-quasi-isotropic metric space is called *isotropic*.

Zair Ibragimov calls a point $x \in G$ for which

$$\limsup_{r \to 0} \frac{\sup\{\alpha_G(x, z) \colon |x - z| = r\}}{\inf\{\alpha_G(x, y) \colon |x - y| = r\}} = 1$$

a point of conformality of the Apollonian metric. In [13, Theorem 2] he showed that for the Apollonian metric either none, one or all points are points of conformality. The middle case provides an interesting link to convex sets of constant width, see [13, Theorem 3].

We say that a domain $G \subsetneq \mathbb{R}^n$ is quasi-isotropic if (G, α_G) is, similarly for isotropic. We define the function qi on the set of proper subdomains of $\mathbb{R}^n$ so that $qi(G)$ is the infimum over constants for which G is quasi-isotropic or $qi(G) = \infty$ if G is not quasi-isotropic for any K.

Recall that the density of the Apollonian metric was defined in the introduction. With this concept we can give the following alternative characterization of quasi-isotropy.

Lemma 2.2 (Lemma 3.5, [10]). *For $G \subsetneq \mathbb{R}^n$ we have*

$$qi(G) = \sup_{x \in G} \frac{\sup_{r \in S^{n-1}} \overline{\alpha}_G(x; r)}{\inf_{r \in S^{n-1}} \overline{\alpha}_G(x; r)},$$

with the understanding that if $\overline{\alpha}_G(x;r) = 0$ for some $x \in G$ and $r \in S^{n-1}$ then $qi(G) = \infty$.

When we do not need the exact value of the quasi-isotropy constant the following lemma is often more convenient to use.

Lemma 2.3 (Lemma 4.2, [8]). *Let $G \subsetneq \mathbb{R}^n$ be L–quasi-isotropic. Then $\overline{\alpha}_G(x;r)\delta(x) \geq 1/L$ for every $x \in G$ and $r \in S^{n-1}$.*

If conversely $\overline{\alpha}_G(x;r)\delta(x) \geq 1/L$ for every $x \in G$ and $r \in S^{n-1}$, then G is $2L$–quasi-isotropic.

3. An integral formula for the Apollonian inner metric.

In this section we prove that we can evaluate the Apollonian inner metric by taking the integral over the density on piecewise continuously differentiable paths.

The idea with the integral formulae for inner metrics is that the density is the derivative of the metric considered as a function of one variable and the inner metric is essentially the Riemann integral sum over this derivative. In fact for continuously differentiable paths this is strictly speaking true both for the j_G and the Apollonian metric (the latter claim is proved in Lemma 3.2). Since the density of the j_G metric, $1/\delta(x)$, does not depend on the direction we can then use a density argument of continuously differentiable paths in the set of all paths to conclude that $\tilde{j}_G = k_G$.

For the Apollonian metric, whose density depends on the direction, we would have to approximate not only a path but also its derivative, however, the latter need not even be defined for an arbitrary path. In view of this difficulty, we will show explicitly that Apollonian length of a path is about the same as the Apollonian length of a piecewise linear approximation of the path. This will be done in two steps, first for quasi-isotropic domains in Lemma 3.5 and then for general domains in Corollary 3.11. With this sketch of the proof in mind we proceed with the details.

Unless specified to the contrary we assume throughout this paper that every path γ is defined on the set $[0,1]$ and we call a sequence $\{t_i\}_{i=0}^k$ with $0 = t_0 < t_1 < \ldots < t_k = 1$ a *length sequence*. Let us denote, for the time being,

$$\tilde{\alpha}'_G(\gamma) = \int_0^1 \overline{\alpha}_G(\gamma(t); \gamma'(t))|\gamma'(t)|dt, \tag{3.1}$$

where $\gamma \subset G$ is a piecewise differentiable (as before, we define $\overline{\alpha}_G(x;0)0 = 0$). We start by showing that the integral formula is valid for segments.

Many of the arguments that follow are based on the fact that "α_G is continuous" which is an immediate consequence of the definition of the metric.

Lemma 3.2. *For every segment $[x,y] \subset G$ we have $\alpha_G([x,y]) = \tilde{\alpha}'_G([x,y])$.*

Proof. We may obviously assume that $x \neq y$. Denote $r := y - x$ and choose $\gamma(t) := x + rt$, from which it follows that the kernel of the integral in (3.1) equals $\overline{\alpha}_G(x+rt; r)$. Since $\alpha_G(x, x+rt)/t$ is continuous and tends to $\overline{\alpha}_G(x; r)$ as $t \to 0$ (shown in Lemma 5.8, [8]) and since $[x,y]$ is compact we may, for $\epsilon > 0$, choose $\delta > 0$ such that

$$\alpha_G(z,w)/(1+\epsilon) \leq \overline{\alpha}_G(z;r)|z-w| \leq (1+\epsilon)\alpha_G(z,w) \qquad (3.3)$$

for $z, w \in [x,y]$ with $|z-w| < \delta$. Moreover, there exists a $\delta' > 0$ such that

$$\left| \int_0^1 \overline{\alpha}_G(x+rt; r)|r|dt - \sum_{i=0}^{k-1} \overline{\alpha}_G(x+rt_i; r)|r||t_{i+1} - t_i| \right| \leq \epsilon,$$

for every length sequence $\{t_i\}$ satisfying $t_{i+1} - t_i < \delta'$ for all $0 \leq i < k$, since $z \mapsto \overline{\alpha}_G(z; r)$ is continuous and hence integrable. Taking the supremum over all length sequences with

$$\max\{t_{i+1} - t_i\} \leq \min\{\delta', \delta/|r|\}$$

and using the first inequality of (3.3) we find that

$$\begin{aligned}
\alpha_G([x,y])/(1+\epsilon) &\leq \sup \sum_{i=0}^{k-1} \overline{\alpha}_G(x+rt_i; r)|r||t_{i+1} - t_i| \\
&\leq \int_0^1 \overline{\alpha}_G(x+rt; r)|r|dt + \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, the inequality $\alpha_G([x,y]) \leq \tilde{\alpha}'_G([x,y])$ is thus proved. The converse inequality follows similarly. $\square$

Remark 3.4. The previous proof also shows that the integral formula holds for paths that are continuously differentiable almost everywhere, in particular for piecewise linear paths.

We now proceed to the harder part of the proof – showing that we do not gain anything by allowing (say) widely oscillating nondifferentiable paths in the infimum. We first consider domains in which the Apollonian metric is quasi-isotropic. The intuitive reason for this restriction is that we can then estimate the Apollonian metric in a uniform manner.

Lemma 3.5. *Let G be L–quasi-isotropic and $x, y \in G$. Then*

$$\inf_{\gamma \in A} \alpha_G(\gamma) = \inf_{\gamma \in B} \alpha_G(\gamma),$$

where A is the set of all paths joining x and y in G and B is the set of all piecewise linear paths joining x and y in G.

Proof. Since $B \subset A$, it is clear that

$$\inf_{\gamma \in A} \alpha_G(\gamma) \leq \inf_{\gamma \in B} \alpha_G(\gamma),$$

so it remains only to prove the opposite inequality. Fix $\gamma \in A$. We will show that for every $\epsilon > 0$ there exists a length sequence such that we have

$$(1 + \epsilon)\alpha_G(\gamma(t_i), \gamma(t_{i+1})) \geq \alpha_G([\gamma(t_i), \gamma(t_{i+1})]) \tag{3.6}$$

for every $0 \leq i < k$. Adding these inequalities we see that

$$
\begin{aligned}
(1 + \epsilon)\alpha_G(\gamma) &\geq (1 + \epsilon) \sum_{i=0}^{k-1} \alpha_G(\gamma(t_i), \gamma(t_{i+1})) \\
&\geq \sum_{i=0}^{k-1} \alpha_G([\gamma(t_i), \gamma(t_{i+1})]) = \alpha_G(\gamma'),
\end{aligned}
$$

where γ' is the path formed by concatenating the segments $[\gamma(t_i), \gamma(t_{i+1})]$. Since ϵ was arbitrary, this shows that the opposite inequality holds too.

It remains to construct a length sequence such that (3.6) holds. Let us denote $d(\gamma, \partial G) =: c > 0$. We will show that there exists a constant $u \in (0, 0.15)$ depending on L and ϵ such that

$$(1 + \epsilon)\alpha_G(x, y) \geq \alpha_G([x, y]) \tag{3.7}$$

for every $x, y \in \gamma$ with $|x - y| \leq uc$. It follows from this that it suffices to choose $\{t_i\}_{i=0}^{k}$ to be some sequence with $|\gamma(t_i) - \gamma(t_{i+1})| \leq uc$ for all $0 \leq i < k$, which is possible (for some k), since γ is rectifiable.

Let us assume then that $x, y \in \gamma$ with $|x - y| \leq uc$. Let B_x and B_y be the Apollonian balls about x and y and let q_x and q_y be the Apollonian parameters. We note that

$$\max\{q_x, q_y\} \leq q_x q_y = \exp \alpha_G(x, y) \leq \exp\{2j_G(x, y)\} \leq (1 + u)^2 \leq 1.4,$$

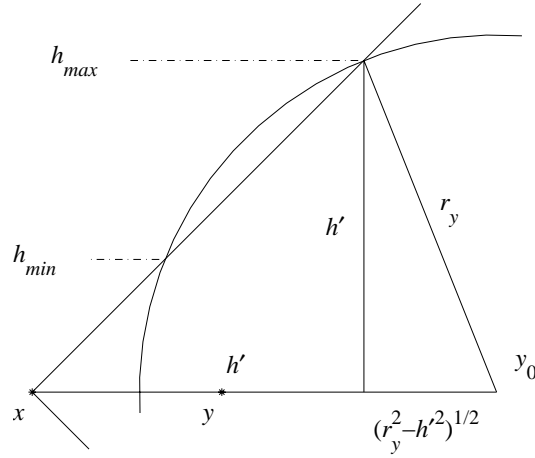where the second inequality is from [3, Theorem 3.2].

Figure 1: Calculating h when the mantle of the cylinder is the limiting factor.

It is clear that the balls B_x, B_y, $B^n(x,c)$ and $B^n(y,c)$ all lie in G. Define $h := \min\{r_y, \sqrt{c^2 - |x-y|^2/4}\}$ and let C be the cylinder around the segment $[x_0, y_0]$ (recall that x_0 is the center of B_x and y_0 of B_y) with radius h. We easily see that

$$G' := \big(B_x \cup B_y \cup C\big) \subset \big(B_x \cup B_y \cup B^n(x,c) \cup B^n(y,c)\big) \subset G$$

(the domain G' is indicated in Figure 2 by the heavy line). Since the Apollonian spheres about x and y are the same in G and G' we find that $\alpha_G(x,y) = \alpha_{G'}(x,y) = \log\{q_x q_y\}$. Since the Apollonian metric is monotone in the domain of definition (see [3, Introduction (3)]) we have $\alpha_G([x,y]) \le \alpha_{G'}([x,y])$ and thus it suffices to show that $(1+\epsilon)\alpha_{G'}(x,y) \ge \alpha_{G'}([x,y])$ in order to prove (3.7).

Let $z \in [x,y]$ and define $t := |y-z|$. The Apollonian sphere in G' through z in direction $y-x$ is limited by either the mantle of $C \setminus B_y$ or by $(\partial C) \cap (\partial B_y)$ when $z \notin B_y$ and by the sphere ∂B_y when $z \in B_y$. We will derive an estimate for the radius of the Apollonian sphere in each case.

Suppose first that the Apollonian sphere through z in direction $y-x$ is limited by the mantle of $C \setminus B_y$ for some z. It is clear that so is the Apollonian sphere through x in the same direction. This means that the cone with vertex x, direction $y-x$ and angle $\pi/4$ intersects the mantle of C. We calculate where this cone intersects ∂B_y. Let h' denote the distance of the points of intersection from the line xy. We have $h' + \sqrt{r_y^2 - h'^2} = |x - y_0| = q_y r_y$, see

Figure 1. This second degree equation in h' has roots (we used the formulae from observation 4 in Section 2.2)

$$h' = \frac{r_y}{2}\big(q_y \pm \sqrt{2 - q_y^2}\big).$$

Since this cone is supposed to intersect the mantel of C in B_y^c it is clear that either h (the radius of the cone) is smaller than the smaller root or larger than the larger one. Consider first the smaller of root. We find

$$h_{min} = \frac{|x-y|}{2}\frac{q_y}{q_y^2 - 1}\frac{q_y^2 - (2 - q_y^2)}{q_y + \sqrt{2 - q_y^2}} = \frac{q_y}{q_y + \sqrt{2 - q_y^2}}|x-y|$$

and we see that this root is smaller than $|x - y|$. On the other hand we have

$$h = \min\{r_y, \sqrt{c^2 - |x-y|^2/4}\} \geq \min\left\{\frac{|x-y|}{q_y - 1/q_y}, |x-y|\sqrt{u^{-2} - 1}\right\}$$
$$\geq \min\{1.45, 6.5\}|x-y|,$$

where we used $q_y \leq 1.4$ and $u \leq 0.15$. Hence we see that h is not smaller than the smaller root, and so we conclude that it is greater than the larger root, which we denote by h_{max}.

Suppose next that z is such that the Apollonian sphere through z in direction $y - x$ is limited by $(\partial C) \cap (\partial B_y)$. With the variables indicated in Figure 2 we find the radius of this sphere equals $w/(2\cos\theta) = w^2/(2s)$, where $s := t + r_y/q_y - \sqrt{r_y^2 - h^2}$, $w := \sqrt{s^2 + h^2}$ and $\cos\theta := s/w$. Using the expressions for s and w we find that

$$\frac{s}{w^2} = \frac{t + r_y/q_y - \sqrt{r_y^2 - h^2}}{\big(t + r_y/q_y - \sqrt{r_y^2 - h^2}\big)^2 + h^2}$$

$$\leq \frac{|x-y|}{\big(\frac{r_y}{q_y} - \sqrt{r_y^2 - h^2}\big)^2 + h^2} + \frac{r_y - \sqrt{r_y^2 - h^2}}{\big(r_y - \frac{r_y}{q_y}\big)^2 + \frac{2r_y}{q_y}\big(r_y - \sqrt{r_y^2 - h^2}\big)}$$

$$\leq \frac{|x-y|}{h^2} + \frac{q_y}{2r_y}.$$

Here we used $t \leq |x - y|$ and $r_y/q_y \leq r_y$ in the first inequality and threw away some positive terms from the denominators for the second inequality.

Finally, in the case when the Apollonian sphere through z in direction $y - x$ is limited by ∂B_y we find that its radius equals $(r_y + |y_0 - z|)/2 \geq (r_y + |y_0 - y|)/2 = r_y(1 + 1/q_y)/2$.
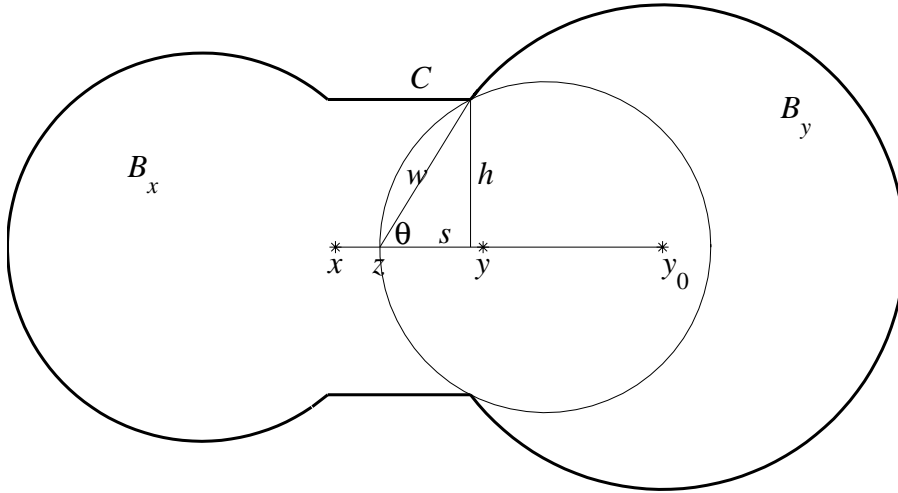
Figure 2: Calculating the radius of the Apollonian sphere when $(\partial C) \cap (\partial B_y)$ is the limiting factor.

Let r_+ denote the radius of the Apollonian sphere thorough z in direction $y - x$. Combining the estimates from the previous three paragraphs we see that

$$
\begin{aligned}
\frac{1}{2r_+} &\leq \max\left\{ \frac{1}{2h_{max}}, \frac{s}{w^2}, \frac{1}{r_y + |y_0 - z|} \right\} \\
&\leq \max\left\{ \frac{1}{\left(q_y + \sqrt{2 - q_y^2}\right)r_y}, \frac{q_y}{2r_y} + \frac{|x - y|}{h^2}, \frac{1}{(1 + 1/q_y)r_y} \right\} \\
&= \frac{q_y}{2r_y} + \frac{|x - y|}{h^2}.
\end{aligned}
$$

Similarly we derive an estimate for the radius r_- of the Apollonian sphere through z in direction $x - y$ and find that

$$
\frac{1}{2r_-} \leq \frac{q_x}{2r_x} + \frac{|x - y|}{h_x^2}
$$

where $h_x := \min\{r_x, \sqrt{c^2 - |x - y|^2/4}\}$.

It follows from these estimates and Lemma 3.2 that

$$\begin{aligned} \alpha_G([x,y]) &= \int_{[x,y]} \frac{1}{2r_+} + \frac{1}{2r_-} ds \\ &\leq \frac{q_x|x-y|}{2r_x} + \frac{q_y|x-y|}{2r_y} + 2\rho^2 \\ &= q_x^2/2 + q_y^2/2 - 1 + 2\rho^2, \end{aligned}$$

where we have denoted $\rho := |x-y|/m$ for $m := \min\{r_x, r_y, \sqrt{c^2 - |x-y|^2/4}\}$. We have now shown that

$$\frac{\alpha_G([x,y])}{\alpha_G(x,y)} \leq \frac{q_x^2/2 + q_y^2/2 - 1 + 2\rho^2}{\log\{q_x q_y\}}.$$

We next show that we can choose u so that the left hand side of the previous inequality is bounded from above by $1 + \epsilon$.

We start by deriving a bound for ρ. Since G is L–quasi-isotropic we have, by [8, Lemma 4.5],

$$\log\{1 + |x-y|/(cL')\} \leq j_G(x,y)/L' \leq \alpha_G(x,y) = \log\{q_x q_y\},$$

where L' depends on L and c. If $m = \sqrt{c^2 - |x-y|^2/4}$ this implies that

$$\rho = \frac{|x-y|}{\sqrt{c^2 - |x-y|^2/4}} \leq \frac{|x-y|/c}{\sqrt{1 - u^2/4}} \leq 1.1\frac{|x-y|}{c} \leq 1.1L'(q_x q_y - 1).$$

In case $m = r_x$, we have $\rho = q_x - 1/q_x \leq 2(q_x - 1)$ and for $m = r_y$ we have $\rho \leq 2(q_y - 1)$. Hence in every case $\rho \leq 2L'(q_x q_y - 1)$. We then find that

$$\frac{2\rho^2}{\log\{q_x q_y\}} \leq \frac{8L'^2(q_x q_y - 1)^2}{\log\{q_x q_y\}} \leq \frac{72L'^2 u^2}{\log\{1 + 3u\}},$$

where we used $q_x q_y \leq (1+u)^2 \leq 1 + 3u$ and the fact that $z \mapsto z^2/\log\{1+z\}$ is increasing in z for the second inequality. Since $72L'^2 u^2/\log\{1 + 3u\} \to 0$ as $u \to 0$, we can choose $u_0 > 1$ so that $72L'^2 u^2/\log\{1 + 3u\} \leq \epsilon/2$ for $u < u_0$.

We move on to the other terms. Since

$$\lim_{q_x \to 1+} \frac{q_x^2 - 1}{2\log q_x} = 1,$$

there exists a $q > 1$ such that $q_x^2 - 1 \leq 2(1 + \epsilon/2)\log q_x$ holds for $q_x < q$. Then for $u < \min\{\sqrt{q} - 1, u_0\}$ it follows that $\max\{q_x, q_y\} \leq (1+u)^2 < q$,

and so we have

$$
\frac{(q_x^2 - 1)/2 + (q_y^2 - 1)/2 + 2\rho^2}{\log\{q_x q_y\}} \quad \leq \quad \frac{(1 + \epsilon/2)(\log q_x + \log q_y)}{\log\{q_x q_y\}} + \frac{72 L'^2 u^2}{\log\{1 + 3u\}}
$$

$$
\leq \quad 1 + \epsilon,
$$

which completes the proof.   □

In the previous proof the claim that every length sequence with $|\gamma(t_i) - \gamma(t_{i+1})| < uc$ satisfies (3.7) is false if we drop the assumption that the domain is quasi-isotropic. On the other hand, (3.7) holds for "most" points even in this case, a fact that we will exploit. We develop a method for removing the assumption of quasi-isotropy, which is based on constructing a sequence of domains approaching a given domain with larger and larger quasi-isotropy constants.

Definition 3.8. For $G \subsetneq \mathbb{R}^n$ we define $G_r := \{x \in G \colon d(x, \partial G) > r\}$.

Lemma 3.9. *Let $G \subsetneq \mathbb{R}^n$ be a domain and $x \in G$. For every $r > 0$ and $R > 0$ the set $G_r \cap B^n(x, R)$ is quasi-isotropic.*

Proof. We have

$$
G_r = G \setminus \bigcup_{z \in \partial G} \overline{B^n}(z, r),
$$

hence for every boundary point w of G_r there exists a ball $B^n(z, r)$ which is not in G_r for which $w \in S^{n-1}(z, r)$. This is clearly also true for all boundary points of $G_r \cap B^n(x, R)$. Hence it follows from [11, Theorem 4.3] that this domain has the approximation property (the ball condition derived above is called the $(1, r)$–EB condition in that paper) and is therefore quasi-isotropic. □

Lemma 3.10. *Let $x, y \in G \subsetneq \mathbb{R}^n$ and γ_0 be an α_G-rectifiable path connecting x and y. Define $G' := G_r \cap B^n(x, R)$ where $c := \alpha_G(\gamma_0)$, $r := \delta(x)/(e^c + 2)$ and $R := (e^c + 1)\delta(x)$. Then*

$$
\inf_{\gamma \in A} \alpha_G(\gamma) = \inf_{\gamma \in B} \alpha_G(\gamma),
$$

where A is the set of all paths joining x and y in G and B is the set of all paths joining x and y in G'.

Proof. If γ is a path connecting x to a point $z \in G$ with $\delta(z) < \delta(x)/3$, then

$$\alpha_G(\gamma) \geq \alpha_G(x, z) \geq \log\{|x - z|/\delta(z) - 1\} \geq \log\{\delta(x)/\delta(z) - 2\},$$

as we see by choosing $a = \infty$ and b such that $|z - b| = \delta(z)$ in the supremum in the definition of the Apollonian metric for a lower bound. If $\delta(z) < \delta(x)/(e^c + 2)$ then this means that $\alpha_G(\gamma) > c$. Let next γ be a path such that $\gamma \cap S^{n-1}(x, R) \neq \emptyset$ for some $R > 3\delta(x)$ and let $z \in \gamma \cap S^{n-1}(x, R)$. Then we have

$$\alpha_G(\gamma) \geq \alpha_G(x, z) \geq \log\{|x - z|/\delta(x) - 1\}.$$

Now if $|x - z| = R > (e^c + 1)\delta(x)$, then again $\alpha_G(\gamma) > c$. We have thus seen that if $\gamma \in A \setminus B$, then $\alpha_G(\gamma) > c$ and so we see that such paths have no bearing on the infimum which is less than or equal to c, anyway. $\square$

Using these auxiliary lemmata we show that the assumption of quasi-isotropy was not essential in Lemma 3.5.

Corollary 3.11. *For every domain $G \subsetneq \mathbb{R}^n$ we have*

$$\inf_{\gamma \in A} \alpha_G(\gamma) = \inf_{\gamma \in C} \alpha_G(\gamma),$$

where A is the set of all paths joining x and y in G and C is the set of all piecewise linear paths joining x and y in G.

Proof. Since $C \subset A$ we have

$$\inf_{\gamma \in A} \alpha_G(\gamma) \leq \inf_{\gamma \in C} \alpha_G(\gamma)$$

and so it remains only to consider the opposite inequality.

Let $x \in G$ and define $G(c) := G_c \cap B^n(x, 1/c)$. Using Lemma 3.9, we see that $G(c)$ is quasi-isotropic for $c > 0$. Since $\tilde{\alpha}_G \leq 2k_G$ by [3, Theorem 3.2], we see that there exists a path connecting x and y in G which is $\tilde{\alpha}_G$-rectifiable. Let γ_0 be the one such path and let G' be as in Lemma 3.10 and define $K := \overline{G'}$. Since $\alpha_G(x, y)$ is continuous in the domain of definition G, it follows that for every $\epsilon > 0$ there exists a $c_0 > 0$ such that

$$\alpha_{G(c)}(z, w)/(1 + \epsilon) \leq \alpha_G(z, w) \tag{3.12}$$

holds for all $z, w \in K$ and every $c < c_0$. Let B be as in Lemma 3.10. We have

$$\inf_{\gamma \in A} \alpha_G(\gamma) = \inf_{\gamma \in B} \alpha_G(\gamma) \geq \tfrac{1}{1+\epsilon} \inf_{\gamma \in B} \alpha_{G(c)}(\gamma) = \tfrac{1}{1+\epsilon} \inf_{\gamma \in C} \alpha_{G(c)}(\gamma)$$
$$\geq \tfrac{1}{1+\epsilon} \inf_{\gamma \in C} \alpha_G(\gamma).$$

Here the first equality follows from Lemma 3.10, the first inequality from (3.12), the second equality from Lemma 3.5 and the last inequality since α_G is monotone in the domain of definition by [3, Introduction (3)]. Since ϵ can be arbitrarily small this implies that $\inf_{\gamma \in A} \alpha_G(\gamma) \geq \inf_{\gamma \in C} \alpha_G(\gamma)$, which was to be shown. $\qquad \square$

Proof of Theorem 1.4. Fix $x, y \in G$ and let A be the set of all paths joining x and y in G, B be the set of piecewise linear paths in A and C be the set of paths in A which are piecewise continuously differentiable. It is clear that $B \subset C \subset A$ and so

$$\inf_{\gamma \in A} \alpha_G(\gamma) \leq \inf_{\gamma \in C} \alpha_G(\gamma) \leq \inf_{\gamma \in B} \alpha_G(\gamma).$$

By Corollary 3.11 we have $\inf_{\gamma \in A} \alpha_G(\gamma) = \inf_{\gamma \in B} \alpha_G(\gamma)$, which combined with the previous inequality gives

$$\inf_{\gamma \in A} \alpha_G(\gamma) = \inf_{\gamma \in C} \alpha_G(\gamma) = \inf_{\gamma \in B} \alpha_G(\gamma).$$

Finally, it follows from Remark 3.4 that

$$\inf_{\gamma \in C} \tilde{\alpha}_G(\gamma) = \inf_{\gamma \in C} \tilde{\alpha}'_G(\gamma),$$

which concludes the proof. $\qquad \square$

4. Pseudometrics and geodesics.

In this section we consider two basic aspects of the Apollonian inner metric, namely, in which domains it is a metric and in which domains there exist geodesics connecting arbitrary points.

Proof of Theorem 1.2. Since α_G satisfies the triangle inequality, is nonnegative and symmetric it is clear that so does and is $\tilde{\alpha}_G$. It thus remains to

consider whether or not the Apollonian inner metric is positive definite for a given domain, i.e whether it is a pseudometric or a metric.

Suppose first that G^c is contained in an $(n-2)$-dimensional plane, and assume without loss of generality that this plane is $\{x \in \mathbb{R}^n : x_{n-1} = x_n = 0\}$. Define $\gamma(t) = \sin(\pi t)e_{n-1} + \cos(\pi t)e_n$ for $t \in [0,1]$. For every $x, y \in \gamma([0,1])$ we have $\alpha_G(x,y) = 0$ since every boundary point of G is at an equal distance from x and y. This means that $\alpha_G(\gamma) = 0$ and hence $\tilde{\alpha}_G(e_n, -e_n) = 0$ which means that $\tilde{\alpha}_G$ is a pseudometric.

It remains to check that for distinct $x, y \in G$ we have $\tilde{\alpha}_G(x,y) > 0$ if G^c is not contained in an $(n-2)$-dimensional plane. If G^c is not contained in a hyperplane, then α_G is a metric and it is clear that so is $\tilde{\alpha}_G$, since $\tilde{\alpha}_G(x,y) \geq \alpha_G(x,y)$ always holds. It remains to consider the case when G^c is contained in a hyperplane but not in an $(n-2)$-dimensional plane. For definiteness say $\partial G \subset \partial H^n$. It is clear that we need only consider the Apollonian inner metric for distinct $x, y \in G$ such that $\alpha_G(x,y) = 0$, which means that x and y are each others mirror images in the plane ∂H^n. Assume without loss of generality that $x_n > 0$. This means that $\overline{\alpha}_G(z;\theta) > 0$ for every $z \in \overline{B^n}(x, x_n/2) =: B$ and $\theta \in S^{n-1}$, since $\overline{\alpha}_G(z;\theta) = 0$ implies that ∂G is contained in the hyperplane P through z with θ as normal, which would imply that ∂G would be contained in the $(n-2)$-dimensional plane $P \cap \partial H^n$, contrary to assumption. Thus

$$\overline{\alpha}_G(z;\theta) \geq \min_{z \in B} \min_{\theta \in S^{n-1}} \overline{\alpha}_G(z;\theta) =: a > 0$$

and so $\tilde{\alpha}_G(x,y) \geq ax_n > 0$ since every path connecting x and y has Euclidean length at least $x_n/2$ in B and another $x_n/2$ in $\overline{B^n}(y, x_n/2)$.   □

We then move on to consider the geodesics.

Proof of Theorem 1.5.   Fix $x, y \in G$ and consider the set A of piecewise linear paths connecting x and y in G. Since $\tilde{\alpha}_G \leq 2k_G$ by [3, Theorem 3.2], we see that there exists a path in A which is $\tilde{\alpha}_G$-rectifiable. Let c_0 be the length of one such path.

Let us define $r := \delta(x)/(e^{c_0}+2)$, $R := (e^{c_0}+1)\delta(x)$, $G' := G_r \cap B^n(x,R)$ and $K := \overline{G'}$. It follows from Lemma 3.10 that we need only consider paths connecting x and y in K. Since $\overline{\alpha}_G(x;\theta) > 0$ (here we use that α_G is a metric) and continuous in $x \in G$ and $\theta \in S^{n-1}$, it follows that it has a minimum in the compact set $K \times S^{n-1}$, say c_1, and so we see that we need only consider paths of Euclidean length $\ell(\gamma) \leq c_0/c_1$, since for longer paths we have $\alpha_G(\gamma) > \ell(\gamma)c_1 > c_0$. Let B be the set of piecewise linear paths

connecting x and y in K of Euclidean length at most c_0/c_1.

We have shown that

$$\inf_{\gamma \in A} \alpha_G(\gamma) = \inf_{\gamma \in B} \alpha_G(\gamma)$$

and so we can choose a sequence (γ_i') of paths in B parameterized by arc-length such that $\lim_{i \to \infty} \alpha_G(\gamma_i') = \tilde{\alpha}_G(x, y)$. Define the path γ_i by $\gamma_i(t) := \gamma_i'(t\ell(\gamma_i'))$ for $t \in [0, 1]$ and every i. It is clear that $\lim_{i \to \infty} \alpha_G(\gamma_i) = \tilde{\alpha}_G(x, y)$, since the parameterization of the path does not affect its length.

Let next $f \colon \mathbb{N} \to [0, 1] \cap \mathbb{Q}$ be a surjection. Since $\gamma_i(f(0))$ is a sequence of points in the compact set K it follows that there exists a subsequence (γ_i^0) of (γ_i) such that $\gamma_i^0(f(0))$ converges to a point in K. Continuing like this we construct a subsequence (γ_i^k) of (γ_i^{k-1}) such that $\gamma_i^k(f(j))$ converges for $0 \le j \le k$. Let then $\gamma_\infty = \lim_{k \to \infty} \gamma_k^k$. For every i we have

$$|\gamma_i(t) - \gamma_i(s)| \le \ell(\gamma_i)|s - t| \le c_0|s - t|/c_1,$$

for $s, t \in [0, 1]$, where the first inequality follows since the mapping γ_i' is a contraction and the second one since we are considering paths of length less than or equal to c_0/c_1. Hence we see that the family $\{\gamma_i\}$ is equicontinuous which implies that γ_∞ is continuous and hence a path connecting x and y in K.

Suppose then that $\alpha_G(\gamma_\infty) - \inf_{\gamma \in A} \alpha_G(\gamma) =: \delta > 0$. Let $\{t_i\}$ be a sequence with $0 = t_0 < \ldots < t_l = 1$ and $t_i \in \mathbb{Q}$ such that

$$\sum_{i=0}^{l-1} \alpha_G(\gamma_\infty(t_{i+1}), \gamma_\infty(t_i)) > \alpha_G(\gamma_\infty) - \delta/2.$$

Let us denote $m := \inf_{\gamma \in A} \alpha_G(\gamma)$ and set $\epsilon := \delta/(4m + \delta)$. Let $r_i > 0$ be such that

$$\alpha_G(\gamma_\infty(t_{i+1}), \gamma_\infty(t_i)) < (1 + \epsilon)\alpha_G(z, w)$$

for $z \in B^n(\gamma_\infty(t_{i+1}), r_i)$ and $w \in B^n(\gamma_\infty(t_i), r_i)$ and define $r := \min r_i$. Since $\gamma_k^k(t_i) \to \gamma_\infty(t_i)$ we can find a k_0 such that $|\gamma_k^k(t_i) - \gamma_\infty(t_i)| < r$ for all $k > k_0$ and all i. It then follows that for $k > k_0$ we have

$$
\begin{aligned}
\alpha_G(\gamma_k^k) &\ge \sum_{i=0}^{l-1} \alpha_G(\gamma_k^k(t_{i+1}), \gamma_k^k(t_i)) \ge \frac{1}{1+\epsilon} \sum_{i=0}^{l-1} \alpha_G(\gamma_\infty(t_{i+1}), \gamma_\infty(t_i)) \\
&> \frac{\alpha_G(\gamma_\infty) - \delta/2}{1 + \epsilon} = \frac{m + \delta/2}{1 + \delta/(4m + \delta)} = m + \delta/4.
\end{aligned}
$$

This contradicts the fact that γ_k^k is a sequence with $\alpha_G(\gamma_k^k) \to m$ as $k \to \infty$, which means that the assumption $\delta > 0$ was false and so $\alpha_G(\gamma_\infty) = m$, which was to be shown. $\qquad\square$

5. Quasi-isotropy.

Recall that the concept of quasi-isotropy was defined in Section 2.3. In this section we study bilipschitz equivalences of the Apollonian inner metric, the quasihyperbolic metric and the j_G metric.

We denote $d \approx d'$ if there exists a constant $C > 0$ such that $\frac{1}{C}d(x,y) \leq d'(x,y) \leq Cd(x,y)$ for all x, y in the domain of definition of the metrics. We denote $d \lesssim d'$ if there exists a constant $C > 0$ such that $d(x,y) \leq Cd'(x,y)$ for all x, y in the domain of definition of the metrics.

Proposition 5.1. *For $G \subsetneq \mathbb{R}^n$ we have $\tilde{\alpha}_G \approx k_G$ if and only if G is quasi-isotropic.*

Proof. If G is quasi-isotropic, then $\tilde{\alpha}_G \approx k_G$ by [8, Corollary 5.4].

Suppose then that $\tilde{\alpha}_G \approx k_G$. This means that there exists a constant K such that $\tilde{\alpha}_G(x,y) \geq k_G(x,y)/K$ for every $x, y \in G$. Fix $x \in G$ and $e \in S^{n-1}$. We have $\tilde{\alpha}_G([x, x+re]) \geq k_G([x, x+re])/K$ for every $r < \delta(x)$. Since these two metrics are integrals over continuous kernels we see by considering $r \to 0$ that $\overline{\alpha}_G(x; e) \geq 1/(K\delta(x))$. This implies that $\overline{\alpha}_G(x; e)\delta(x) \geq 1/K$ for all $x \in G$, $e \in S^{n-1}$ and so it follows from Lemma 2.3 that G is quasi-isotropic. $\square$

Lemma 5.2. *Let $x \in G$ and $y \in B^n(x, \delta(x))$. Then $\log\{2\}|x - y|/\delta(x) \leq j_G(x,y)$.*

Proof. It suffices to show that

$$\log\{2\}|x - y|/\delta(x) \leq \log\{1 + |x - y|/\delta(x)\}$$

since the right hand side is less than or equal to $j_G(x,y)$. Since the function $z \mapsto \log\{1+z\} - z \log 2$ is increasing for $z \leq (\log 2)^{-1} - 1$ and then decreasing it suffices to check this inequality for the maximal $|x-y|/\delta(x)$, i.e 1, in which case the inequality reduces to $\log 2 \leq \log 2$. $\square$

Proposition 5.3. *If $G \subsetneq \mathbb{R}^n$ is not quasi-isotropic then $j_G \not\lesssim \tilde{\alpha}_G$.*

Proof. For every $K > 2$ there exists a point $x \in G$ and a direction $e \in S^{n-1}$ such that $\overline{\alpha}_G(x; e)\delta(x) < 1/K$, since otherwise G would be quasi-isotropic, by Lemma 2.3. By continuity there exists a $0 < r < \delta(x)$ such

that $\overline{\alpha}_G(y;e)\delta(x) < 2/K$ for all $y \in B^n(x,r)$. Let $z, w \in B^n(x,r)$ be such that $z - x = x - w = re/2$. Then

$$\tilde{\alpha}_G(z,w) \leq \alpha_G([z,w]) < \frac{2|z-w|}{K\delta(x)} \leq \frac{2}{K\log 2}\frac{\delta(z)}{\delta(x)}j_G(z,w) \leq \frac{3}{K\log 2}j_G(z,w).$$

Here the second to last inequality follows from Lemma 5.2. Since $K > 2$ can be arbitrarily large, we see that $j_G \lesssim \tilde{\alpha}_G$ does not hold, as claimed. $\square$

By Lemma 2.2, the following corollary is equivalent to Theorem 1.6.

Corollary 5.4. *For $G \subsetneq \mathbb{R}^n$ the following conditions are equivalent:*

1. *G is quasi-isotropic;*

2. *$k_G \approx \tilde{\alpha}_G$;*

3. *$j_G \lesssim \tilde{\alpha}_G$.*

Proof. That items 1 and 2 are equivalent was show in Lemma 5.1. In Lemma 5.3 we showed that 3 implies 1. It thus suffices to prove that 2 implies 3. But if $\frac{1}{C}k_G \leq \tilde{\alpha}_G$, then it is clear that $j_G \leq C\tilde{\alpha}_G$ since $j_G \leq 2k_G$ always holds. $\square$

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