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The Theorem of Busemann-Feller-Alexandrov in Carnot Groups

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1. Introduction.

A classical theorem states that a convex function in $\mathbb{R}^n$ admits second derivatives at *a.e.* point. This result was first proved by Busemann and Feller [BF] for functions in the plane, and subsequently generalized by A.D. Alexandrov [A] to arbitrary dimensions. The theorem of Busemann-Feller-Alexandrov plays a basic role in analysis and in pde's, especially in the theory of fully nonlinear equations. For instance, in the proof of uniqueness of viscosity solutions, see Theorems 5.1 and 5.3 in [CC], a quantitative version of such result (see Theorem 6.4.1 in [EG]) plays an essential role.

In this paper we prove a version of the Busemann-Feller-Alexandrov theorem for the class of weakly H-convex functions in Carnot groups introduced in [DGN]. Here is our main result [3] .

Theorem 1.1. *Let* $\mathbf{G}$ *be a Carnot group of step* $r = 2$*, with a system* $X_1, ..., X_m$ *of bracket generating left-invariant vector fields. If* $u \in C(\mathbf{G})$ *is a weakly* H*-convex function, then the horizontal second derivatives* X_iX_ju *exist at a.e. point in* $\mathbf{G}$*. More precisely, for* dg*-a.e. point* $g_o \in \mathbf{G}$ *there exists a polynomial of weighted degree* ≤ 2*,* $P_u(g; g_o)$*, such that*

$$\lim_{g \to g_o} \frac{u(g) - P_u(g; g_o)}{d(g, g_o)^2} \ = \ 0 \ .$$

For the relevant definitions we refer the reader to Section 2. Here, we recall that, given a Carnot group $\mathbf{G}$, a function $u : \mathbf{G} \to \mathfrak{R}$ is called weakly H-convex if for every $g \in \mathbf{G}$, and $0 \leq \lambda \leq 1$, the following inequality holds

$$u(g\delta_\lambda(g^{-1}g')) \ \leq \ (1-\lambda)u(g) \ + \ \lambda u(g') \ , \qquad \text{for every} \quad g' \in H_g \ , \quad (1.1)$$

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where H_g indicates the horizontal plane through $g \in \mathbf{G}$. In (1.1) we have denoted by $\delta_\lambda : \mathbf{G} \to \mathbf{G}$ the anisotropic dilations on $\mathbf{G}$. The point $g\delta_\lambda(g^{-1}g')$ denotes the twisted convex combination of g and g' based at g. The geometric notion of convexity (1.1) was introduced in [DGN]. In the same paper it was proved that $u \in \Gamma^2(\mathbf{G})$ is weakly H-convex if and only if the symmetrized horizontal Hessian $Hess_X(u) = [u_{,ij}]$, defined by

$$u_{,ij} \overset{def}{=} \frac{X_iX_ju + X_jX_iu}{2} , \qquad i,j = 1,...,m , \tag{1.2}$$

is semi-definite positive at every point, see Theorem 2.3. It was also shown in [DGN] that for every L^1_{loc} weakly H-convex function u, the distributional derivatives $u_{,ij}$ are signed Radon measures. This interesting property, however, says nothing concerning the unsymmetrized second derivatives X_iX_ju. Since

$$X_iX_ju \ = \ u_{,ij} \ + \ \frac{1}{2}[X_i, X_j]u \qquad \text{in} \quad \mathcal{D}'(\mathbf{G}) , \tag{1.3}$$

it is clear that the central open question here is whether the commutators $[X_i, X_j]u$ are Radon measures.

The main contribution of the present paper is proving that in a Carnot group $\mathbf{G}$ of step two if $u \in C(\mathbf{G})$ is a weakly H-convex function, then the commutators

$$[X_i, X_j]u \ \in \ L^2_{loc}(\mathbf{G}) , \qquad i,j = 1...,m , \tag{1.4}$$

In particular, they are Radon measures. Because of (1.3), this implies that X_iX_ju are Radon measures. This is equivalent to saying that u belongs to the Banach space $BV^2_{H,loc}(\mathbf{G})$ of functions with horizontal gradient Xu locally of bounded H-variation (we recall here that a classical result of Rešetnjak [R] shows that a convex function in $\mathbb{R}^n$ belongs to $BV^2_{loc}(\mathbb{R}^n)$). We can thus appeal to the following recent result of Ambrosio and Magnani which states: *if $u \in BV^2_{H,loc}(\mathbf{G})$, then for dg-a.e. $g_o \in \mathbf{G}$ there exists a polynomial $P(g_o;\cdot)$ of weighted degree ≤ 2, such that*

$$\lim_{r\to 0} \frac{1}{r^2} \frac{1}{|B(g_o,r)|} \int_{B(g_o,r)} |u(g) \ - \ P(g_o,g)| \, dg \ = \ 0 . \tag{1.5}$$

This tells us that, at least in the average, the second horizontal derivatives of u exist at dg-a.e. point. To bridge the gap from this information and the actual pointwise statement in Theorem 1.1, we use the following compactness estimate, which is Theorem 9.2 in [DGN]: *let u be a continuous weakly H-convex function in $\mathbf{G}$, then there exists $C(\mathbf{G}) > 0$ such that for every gauge* *ball $B(g,r)$ one has*

$$\sup_{B(g,r)} |u| \ \leq\ C(\mathbf{G})\, \frac{1}{|B(g,5r)|} \int_{B(g,5r)} |u|\ dg' \ , \tag{1.6}$$

and

$$ess \sup_{B(g,r)} |Xu| \ \leq\ \frac{C(\mathbf{G})}{r}\, \frac{1}{|B(g,15r)|} \int_{B(g,15r)} |u|\ dg' \ . \tag{1.7}$$

Combining (1.4) with (1.5), (1.6), (1.7), we can close the circle and establish Theorem 1.1. We mention that for the Heisenberg group $\mathbb{H}^n$, $n = 1, 2$, Theorem 1.1 was recently proved by two of us in [GT], and our present work is motivated by the approach there, and by the results in [DGN]. In [GT] the crucial property (1.4) was deduced from the following monotonicity result and from an adaption of an idea in [TW]: *Let u and v be (smooth) weakly H-convex functions in a domain $\Omega \subset \mathbb{H}^n$, $n = 1, 2$, $u \geq v$ in Ω, and $u = v$ on $\partial\Omega$, then*

$$\int_\Omega S_{ma}(u)\ dg \ \leq\ \int_\Omega S_{ma}(v)\ dg\ . \tag{1.8}$$

Here, we have denoted by $S_{ma}(u)$ the fully nonlinear operator acting on a function u on $\mathbb{H}^2$ as follows

$$\mathcal{S}_{ma}(u) = \det\ Hess_X(u) + \frac{3}{4}\Bigg\{ \det \begin{pmatrix} u_{,22} & u_{,24} \\ u_{,24} & u_{,44} \end{pmatrix} + 2\ \det \begin{pmatrix} u_{,12} & u_{,14} \\ u_{,23} & u_{,34} \end{pmatrix} + \det \begin{pmatrix} u_{,11} & u_{,13} \\ u_{,13} & u_{,33} \end{pmatrix} \Bigg\}(Tu)^2 + \frac{5}{16}(Tu)^4. \tag{1.9}$$

The expression within curly brackets represents a suitable combination of 2×2 minors of the 4×4 matrix $Hess_X(u)$. For functions on $\mathbb{H}^1$ the operator in (1.9) takes the simpler form

$$S_{ma}(u) \ =\ \det Hess_X(u)\ +\frac{3}{4}\ (Tu)^2\ ,$$

and in this setting (1.8) was first proved by Gutierrez and Montanari in [GM], with the different purpose of proving a maximum principle and generalizing some of the results in [TW]. Although in [GM] the authors did not explicitly make the connection with the Busemann-Feller-Alexandrov theorem, they did establish (1.4) for the first Heisenberg group $\mathbb{H}^1$.

Extending the monotonicity property (1.8) to arbitrary Carnot groups is a difficult task. On the other hand, an analysis of the proof of (1.4) reveals that having the horizontal Monge-Ampère operator $\det Hess_X(u)$ in (1.8) is not necessary, and that a simpler monotonicity result would suffice. Since in [TW] the authors extend Krylov's monotonicity property for the Monge-Ampère operator [K] to what they call k-Hessian measures, it is natural to ask whether, for any fixed $r = 1, ..., m$, the monotonicity property (1.8) continues to be valid for the fully nonlinear operators $\mathcal{F}_r[u]$ associated with the lower symmetric functions (2.8) of the eigenvalues of $Hess_X(u)$. As it is well known, such functions play an important role in geometry as they interpolate between the arithmetic mean and the determinant, see e.g. [Sp]. For the precise description of the operators $\mathcal{F}_r[u]$ we refer to equations (2.8), (2.9) in Section 2. Following the classification introduced for the classical case in [TW], a function $u \in \Gamma^2(\mathbf{G})$ will be called $(H)_r$-*convex* if $\mathcal{F}_k[u] \geq 0$ for every $k = 1, ..., r$. It is worth noting that, when the dimension of the bracket generating layer in the Lie algebra of $\mathbf{G}$ is $m = 2$, then the relevant operator $\mathcal{F}_2[u]$ coincides with $\det Hess_X(u)$. This is the case for instance in the first Heisenberg group $\mathbb{H}^1$, or in the 4-dimensional Engel group $\mathfrak{E}$ of step three, and in such cases one respectively recovers Theorem 3.1 in [GM] and Theorem 9.1 in [GT]. Of course, it is easier to work with the lower symmetric functions, rather than with the determinant of the horizontal Hessian, since the computations involving the commutators are more manageable.

In Section 3 we prove a version of (1.8) for arbitrary Carnot groups and for $(H)_2$-convex functions, i.e., when $\det Hess_X(u)$ is replaced by $\mathcal{F}_2[u]$, see Theorem 3.2. This type of result is reminiscent of the monotonicity theorems in [K], [TW], except that because of the non-trivial commutations the basic null-Lagrangian property of the symmetric forms fails (for such property see Proposition 2.1 in [Re]), and we had to find the appropriate substitute for it.

In Section 4 we prove Theorem 1.1. Here, we focus on groups of step two, and the reason for this is twofold. First, in these ambients all higher-order commutators in Theorem 3.2 vanish, and we obtain the notable Corollary 3.3. Secondly, in order to adapt an idea in [TW] we need to produce a smooth $(H)_2$-convex function to use in Corollary 3.3 whose level sets are compact and generate the topology of $\mathbf{G}$. For a group of step two it was proved in [DGN] that the function N^4, where N is the anisotropic gauge, is weakly H-convex. This is quite remarkable since the gauge itself is almost certainly not weakly H-convex (it is so in groups of Heisenberg type, but such groups enjoy some symmetry properties which are lacking in general Carnot groups of step two). Since N^4 is smooth, from its weak H-convexity we infer that it is, in particular $(H)_2$-convex, and clearly it possesses the two above mentioned additional properties. The existence of related functions in groups of steps ≥ 3 is a challenging question which we plan to address in a future study.

In closing, we mention that, similarly to its classical predecessor, Theorem 1.1 will play a key role in proving the uniqueness of viscosity solutions for fully nonlinear equations in Carnot groups. Such area is presently undergoing a rapid development, see [B], [DGN], [BC], [LMS], [Wa1], [Wa2], [GM], [GT], [BR], [Wa3], [M]. In particular, in the Heisenberg group a notion of convexity in the viscosity sense of [CIL] (called v-convexity) has been recently set forth in [LMS]. While it is easy to see that every weakly H-convex function is also v-convex, the more delicate reverse implication has been recently established in the papers [BR], [Wa3], [M]. As a consequence, one now knows that the geometric notion of weak H-convexity is in fact equivalent to that of v-convexity.

As a final comment, we note that thanks to the recent works [BR], [Wa3] and [M], the assumption $u \in C(\mathbf{G})$ in Theorem 1.1 can be somewhat relaxed. For instance, it suffices to assume that u is locally bounded from above, see Remark 4.7. However, in the Heisenberg group $\mathbb{H}^n$ even this hypothesis can be dispensed with altogether since it has been proved in [BR] that weakly H-convex functions are in L^∞_{loc}.

2. Preliminaries.

We begin by introducing the relevant geometric framework. A Carnot group of step r is a simply-connected Lie group whose Lie algebra is graded, i.e., $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = V_1 \oplus ... \oplus V_r$, with $[V_1, V_j] = V_{j+1}$, for $j = 1, ..., r-1$, and $\mathfrak{g}$ is r-nilpotent, i.e., $[V_1, V_r] = \{0\}$, see [FS], [S], [Be]. We assume that a scalar product $< \cdot, \cdot >$ is given on $\mathfrak{g}$ for which the $V_j's$ are mutually orthogonal. We let $m_j = dim\, V_j$, $j = 1, ..., r$, and denote by $N = m_1 + ... + m_r$ the topological dimension of $\mathbf{G}$. The notation $\{e_{j,1}, ..., e_{j,m_j}\}$, $j = 1, ..., r$, will indicate a fixed orthonormal basis of the j-th layer V_j. Elements of V_j are assigned the formal degree j. As a rule, we will use letters g, g', g_o for points in $\mathbf{G}$, whereas we will reserve the letters X, Y, Z, for elements of the Lie algebra $\mathfrak{g}$. We will denote by $L_{g_o}(g) = g_o g$ the left-translations on $\mathbf{G}$ by an element $g_o \in \mathbf{G}$. Recall that the exponential map $exp : \mathfrak{g} \to \mathbf{G}$ is a global analytic diffeomorphism [V]. It allows to define analytic maps $\xi_i : \mathbf{G} \to V_i$, $i = 1, ..., r$, by letting $g = \exp(\xi_1(g) + ... + \xi_r(g))$. The mapping $\xi : \mathbf{G} \to \mathfrak{g}$ defined by

$$\xi(g) \ = \ \xi_1(g) + ... + \xi_r(g) \ ,$$

is the inverse of the exponential mapping. For $g \in \mathbf{G}$, the projection of the exponential coordinates of g onto the layer V_j, $j = 1, ..., r$, are defined as follows

$$x_{j,s}(g) \ = \ < \xi_j(g), e_{j,s} >, \qquad s = 1, ..., m_j. \tag{2.1}$$

In the sequel it will be convenient to have a separate notation for the first two layers V_1 and V_2. For simplicity, we set $m = m_1$, $k = m_2$, and indicate

$$\{e_1, \ ... \ , \ e_m\} \ = \ \{e_{1,1}, \ ... \ , \ e_{1,m}\} \ , \qquad \{\epsilon_1, \ ... \ , \ \epsilon_k\} \ = \ \{e_{2,1}, \ ..., \ e_{2,k}\} \ . \tag{2.2}$$

We indicate with

$$x_i(g) \ = < \xi_1(g), e_i >, \quad i = 1, ..., m, \quad y_s(g) \ = < \xi_2(g), \epsilon_s >, \quad s = 1, ..., k. \tag{2.3}$$

the projections of the exponential coordinates of g onto V_1 and V_2. Letting $x(g) = (x_1(g), ..., x_m(g))$, $y(g) = (y_1(g), ..., y_k(g))$, we will routinely identify $g \in \mathbf{G}$ with its exponential coordinates

$$g \ = \ (x(g), y(g), ...), \tag{2.4}$$

where the dots indicate the $(N - (m + k))$-dimensional vector

$$(x_{3,1}(g), ..., x_{3,m_3}(g), ..., x_{r,1}(g), ..., x_{r,m_r}(g)).$$

When $\mathbf{G}$ is a group of step 2, then (2.4) simply becomes $g = (x(g), y(g))$. Such identification of $\mathbf{G}$ with its Lie algebra is justified by the Baker-Campbell-Hausdorff formula, see, e.g., [V]

$$\begin{aligned} exp\ Z \ \exp \ Z' = exp\,(Z + Z' + \frac{1}{2}[Z, Z'] \\ + \frac{1}{12}\big\{[Z, [Z, Z']] - [Z', [Z, Z']]\big\}...) \qquad Z, Z' \in \mathfrak{g} \ , \end{aligned} \tag{2.5}$$

where the dots indicate a finite linear combination of terms containing commutators of order three and higher.

We denote by X and Y the systems of left-invariant vector fields on $\mathbf{G}$ defined by

$$X_i(g) \ = \ (L_g)_*(e_i) \ , i = 1, ..., m, \qquad Y_s(g) \ = \ (L_g)_*(\epsilon_s) \ , \ s = 1, ..., k \ ,$$

where $(L_g)_*$ denotes the differential of L_g. The system X defines a basis for the so-called horizontal subbundle $H\mathbf{G}$ of the tangent bundle $T\mathbf{G}$. For a given function $f : \mathbf{G} \to \mathbb{R}$, the action of X_j on f is specified by the equation

$$X_i f(g) \ = \ \lim_{t \to 0} \frac{f(g\ exp\ (tX_i)) \ - \ f(g)}{t} \ = \ \frac{d}{dt}\ f(g\ exp\ (tX_i))\big|_{t=0}. \tag{2.6}$$

A Carnot group of step r is naturally equipped with a family of non-isotropic dilations defined by

$$\delta_\lambda(g) \;=\; exp \circ \Delta_\lambda \circ exp^{-1}(g), \quad g \in \mathbf{G}, \tag{2.7}$$

where $\Delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ is defined by $\Delta_\lambda(Z_1+...+Z_r) = \lambda Z_1+...+\lambda^r Z_r$. We denote by dg the push-forward of Lebesgue measure on $\mathfrak{g}$ via the exponential map. Such dg defines a bi-invariant Haar measure on $\mathbf{G}$. One has $d(g \circ \delta_\lambda) = \lambda^Q dg$, so that the number

$$Q \;=\; m_1 \;+\; 2\,m_2 \;+\; ... \;+\; r\,m_r$$

plays the role of a dimension with respect to the group dilations.

For this reason Q is called the homogeneous dimension of $\mathbf{G}$. Such number is larger than the topological dimension N of $\mathbf{G}$ defined above.

Henceforth, for a given open set $\Omega \subset \mathbf{G}$ we denote by $\Gamma^k(\Omega)$ the Folland-Stein class of functions having continuous derivatives up to order k with respect to the vector fields $X_1, ..., X_m$. The most basic partial differential operator in a Carnot group is the sub-Laplacian associated with X is the second-order partial differential operator on $\mathbf{G}$ given on a function $u \in \Gamma^2(\mathbf{G})$ by

$$\mathcal{L} \;=\; \sum_{i=1}^{m} X_i^2.$$

If $\lambda(Hess_X(u)) = (\lambda_1(Hess_X(u)), ..., \lambda_m(Hess_X(u)))$ denote the eigenvalues of the symmetrized horizontal Hessian of u, defined by (1.3), we clearly have $\mathcal{L}u = S_1(\lambda(Hess_X(u)))$, where for $r = 1, ..., m$, the r-th elementary symmetric function is defined by

$$S_r(x) \;=\; \sum_{i_1<...<i_r} x_{i_1} \,...\, x_{i_r}\ , \qquad\qquad 1 \le r \le m\ . \tag{2.8}$$

When $r > 1$ we can use such functions to form the fully nonlinear differential operators

$$\mathcal{F}_r[u] \;=\; S_r(\lambda_1(Hess_X(u)), ..., \lambda_m(Hess_X(u)))\ . \tag{2.9}$$

One easily recognizes that

$$\mathcal{F}_1[u] \;=\; S_1(\lambda) \;=\; \mathcal{L}u \;=\; \sum_{i=1}^{m} X_i X_i u\ , \qquad\qquad \text{(sub-Laplacian of } u) \tag{2.10}$$

$$\mathcal{F}_2[u] \;=\; S_2(\lambda) \;=\; \frac{1}{2}\left\{(\mathcal{L}u)^2 \;-\sum_{i,j=1}^{m} u_{,ij}^2\right\} \;=\; \sum_{i<j}\left(u_{,ii}\, u_{,jj} \;-\; u_{,ij}^2\right)\;, \tag{2.11}$$

$$\mathcal{F}_m[u] \;=\; S_m(\lambda) \;=\; \det\; Hess_X(u) \qquad \text{(horizontal Monge-Ampère)}\;. \tag{2.12}$$

Following [TW] we make the definition.

Definition 2.1. *For $r = 1, ..., m$, a function $u \in \Gamma^2(\mathbf{G})$ is called $(H)_r$-convex, if $\mathcal{F}_k(u) \geq 0$ for $k = 1, ..., r$.*

Remark 2.2. *It is important to observe that H_1-convex functions correspond to $\mathcal{L}$-subharmonic functions, whereas a function u is H_m-convex if and only if $Hess_X(u) \geq 0$. According to the following result, which is Theorem 5.12 in [DGN], this is equivalent to saying that u is weakly H-convex.*

Theorem 2.3. *In a Carnot group* $\mathbf{G}$ *a function $u \in \Gamma^2(\Omega)$ is weakly H-convex if and only if $Hess_X(u) \geq 0$.*

For later purposes we record here that $\mathcal{F}_2[\cdot]$ is (degenerate) elliptic on $(H)_2$-convex functions, which means that if z is $(H)_2$-convex, then

$$\sum_{i,j=1}^{m} \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}}\, \zeta_i\, \zeta_j \;\geq\; 0\;, \tag{2.13}$$

for every $\zeta \in \mathbb{R}^m$. The proof of this property is the same as for the classical case, for which we refer the reader to [Sp].

The Heisenberg group. An important model of Carnot group of step $r = 2$ is the Heisenberg group $\mathbb{H}^n$, see [S]. The underlying manifold of this Lie group is simply $\mathbb{R}^{2n+1}$, with the non-commutative group law

$$g\, g' \;=\; (x, y, t)\,(x', y', t') \;=\; (x+x', y+y', t+t' + \frac{1}{2}(< x', y > - < x, y' >))\;, \tag{2.14}$$

where we have let $x, x', y, y' \in \mathbb{R}^n$, $t, t' \in \mathbb{R}$. Let $(L_g)_*$ be the differential of the left-translation (2.14). A simple computation shows that

$$(L_g)_* \left(\frac{\partial}{\partial x_i}\right) \stackrel{def}{=} X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial t}, \qquad i = 1, ..., n, \tag{2.15}$$
$$(L_g)_* \left(\frac{\partial}{\partial y_i}\right) \stackrel{def}{=} X_{n+i} = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial t}, \qquad i = 1, ..., n,$$
$$(L_g)_* \left(\frac{\partial}{\partial t}\right) \stackrel{def}{=} T = \frac{\partial}{\partial t}$$

We note that the only non-trivial commutator is

$$[X_i, X_{n+j}] = T\, \delta_{ij}, \qquad i, j = 1, ..., n, \tag{2.16}$$

therefore the vector fields $\{X_1, ..., X_{2n}\}$ generate the Lie algebra $\mathfrak{h}_n = \mathbb{R}^{2n+1} = V_1 \oplus V_2$, where $V_1 = \mathbb{R}^{2n} \times \{0\}_t$, $V_2 = \{0\}_{(x,y)} \times \mathbb{R}$. The non-isotropic group dilations associated with this grading are

$$\delta_\lambda(g) = (\lambda x, \lambda y, \lambda^2 t), \tag{2.17}$$

with relative homogeneous dimension $Q = 2n + 2$.

Carnot groups of step two. For a Carnot group $\mathbf{G}$ of step $r = 2$ we denote by b^s_{ij} the group constants defined by the formula

$$[e_i, e_j] = \sum_{s=1}^{k} b^s_{ij}\, \epsilon_s,$$

see (2.2). The following useful lemma for the first and second derivatives along the vector fields X_j in exponential coordinates holds. For its proof see [DGN].

Lemma 2.4. *Let* $\mathbf{G}$ *be a Carnot group of step* 2*, then for every* $i, j = 1, ..., m$*, one has*

$$X_i = \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{s=1}^{k} < [\xi_1, e_i], \epsilon_s > \frac{\partial}{\partial y_s} \tag{2.18}$$
$$= \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{s=1}^{k}\sum_{j=1}^{m} b^s_{ji}\, x_j(g)\, \frac{\partial}{\partial y_s}.$$

$$\begin{aligned} X_iX_j \ = \ & \frac{\partial^2}{\partial x_i \partial x_j} \ + \ \frac{1}{2} \sum_{s=1}^{k} < [e_i, e_j], \epsilon_s > \frac{\partial}{\partial y_s} \\ & + \ \frac{1}{2} \sum_{s=1}^{k} < [\xi_1, e_j], \epsilon_s > \frac{\partial^2}{\partial x_i \partial y_s} \ + \ \frac{1}{2} \sum_{s=1}^{k} < [\xi_1, e_i], \epsilon_s > \frac{\partial^2}{\partial x_j \partial y_s} \\ & + \ \frac{1}{4} \sum_{s,s'=1}^{k} < [\xi_1, e_i], \epsilon_s >< [\xi_1, e_j], \epsilon_{s'} > \frac{\partial^2}{\partial y_s \partial y_{s'}} \ . \end{aligned} \tag{2.19}$$

From (2.19) *we obtain the commutator formula*

$$[X_i, X_j] \ u \ = \ \sum_{s=1}^{k} b_{ji}^s \ \frac{\partial}{\partial y_s} \ . \tag{2.20}$$

The Engel group of step $r = 3$. We next describe the four-dimensional cyclic or Engel group. This group is important in many respects since it represents the next level of difficulty with respect to the Heisenberg group and provides an ideal framework for testing whether results which are true in step 2 generalize to step 3 or higher. The reader unfamiliar with the cyclic group can consult [CGr], or also [Mon]. The Engel group $\mathfrak{E} = K_3$, see ex. 1.1.3 in [CGr], is the Lie group whose underlying manifold can be identified with $\mathbb{R}^4$, and whose Lie algebra is given by the grading,

$$\mathfrak{e} \ = \ V_1 \oplus V_2 \oplus V_3 \ ,$$

where $V_1 = span\{e_1, e_2\}$, $V_2 = span\{e_3\}$, and $V_3 = span\{e_4\}$, so that $m_1 = 2$ and $m_2 = m_3 = 1$. We will denote with (x, y), t and s respectively the variables in V_1, V_2 and V_3, so that any $Z \in \mathfrak{e}$ can be written as $Z = xe_1 + ye_2 + te_3 + se_4$. If $g = \exp(Z)$, we will identify $g = (x, y, t, s)$. For the corresponding left-invariant vector fields on $\mathfrak{E}$ given by $X_i(g) = (L_g)_*(e_i)$, $i = 1, ..., 4$, we assign the commutators

$$[X_1, X_2] \ = \ X_3 \qquad [X_1, X_3] \ = \ [X_1, [X_1, X_2]] \ = \ X_4 \ , \tag{2.21}$$

all other commutators being assumed trivial. We observe right-away that the homogeneous dimension of $\mathfrak{E}$ is

$$Q \ = \ m_1 \ + \ 2\, m_2 \ + \ 3\, m_3 \ = \ 7 \ .$$

The group law in $\mathfrak{E}$ is given by the Baker-Campbell-Hausdorff formula [V]. In exponential coordinates, if $g = \exp(Z)$, $g' = \exp(Z')$, we have

$$g \ \circ \ g' \ = \ Z \ + \ Z' \ + \frac{1}{2} \ [Z, Z'] \ + \ \frac{1}{12} \left\{ [Z, [Z, Z']] \ - \ [Z', [Z, Z']] \right\}.$$

A computation based on (2.21) gives (see also ex. 1.2.5 in [CGr])

$$g \circ g' = \Big(x + x', y + y', t + t' + P_3, s + s' + P_4\Big) ,$$

where

$$P_3 = \frac{1}{2}\,(xy' - yx') ,$$

$$P_4 = \frac{1}{2}(xt' - tx') + \frac{1}{12}\Big(x^2y' - xx'(y + y') + yx'^2\Big) .$$

Using the Baker-Campbell-Hausdorff formula we find the following expressions for the vector fields $X_1, ..., X_4$

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial t} - \left(\frac{t}{2} + \frac{xy}{12}\right)\frac{\partial}{\partial s} , \qquad (2.22)\\ X_2 &= \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial t} + \frac{x^2}{12}\frac{\partial}{\partial s} ,\\ X_3 &= \frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial s} ,\\ X_4 &= \frac{\partial}{\partial s} . \end{aligned}$$

3. Monotonicity.

To introduce the results in this section we continue to denote with $\mathfrak{E}$ the Engel group discussed above. We observe that since $\mathfrak{E}$ has step $r = 3$, if $u \in \Gamma^3(\overline{\Omega})$, then $u \in C^1(\overline{\Omega})$.

Theorem 3.1. *Let $\Omega \subset \mathfrak{E}$ be a C^1 bounded open set, and consider two weakly H-convex functions $u, v \in \Gamma^3(\overline{\Omega})$ such that $u \geq v$ in Ω and $u = v$ on $\partial\Omega$. For $0 \leq \tau \leq 1$ we set*

$$z = z(g, \tau) \overset{def}{=} (1 - \tau)u(g) + \tau v(g) , \qquad g \in \Omega .$$

We have

$$\frac{d}{d\tau}\int_\Omega \left\{\det Hess_X(z) + \frac{3}{4}\,(X_3z)^2 + \frac{1}{2}\,(X_2z)(X_4z)\right\} dg \geq 0 .$$

In particular,

$$\begin{aligned} &\int_\Omega \left\{\det Hess_X(u) + \frac{3}{4}\,(X_3u)^2 + \frac{1}{2}\,(X_2u)(X_4u)\right\} dg \\ &\leq \int_\Omega \left\{\det Hess_X(v) + \frac{3}{4}\,(X_3v)^2 + \frac{1}{2}\,(X_2v)(X_4v)\right\} dg . \end{aligned}$$

Theorem 3.1 was proved in [GT]. When the functions u and v do not depend on the variable in the last layer V_4, then $X_4 z = 0$, and the corresponding statement about weakly H-convex functions in the first Heisenberg group $\mathbb{H}^1$ was proved in [GM]. We note that in $\mathfrak{E}$ the notions of $(H)_2$- and weak H-convexity coincide.

The aim of this section is to obtain a version of Theorem 3.1 for arbitrary Carnot groups. Here is our main result. For a C^1 domain $\Omega \subset \mathbf{G}$, with Riemannian outer unit normal ν, we introduce the horizontal normal to $\partial\Omega$,

$$\nu_X \ = \ (\nu_{X,1}, ..., \nu_{X,m})^T \ , \tag{3.1}$$

whose components are defined by

$$\nu_{X,i} \ = < X_i, \nu > \ . \tag{3.2}$$

Theorem 3.2. *Let $\mathbf{G}$ be a Carnot group of arbitrary step, and $\Omega \subset \mathbf{G}$ be a C^1 bounded open set. Consider $(H)_2$-convex functions $u, v \in \Gamma^3(\overline{\Omega}) \cap C^1(\overline{\Omega})$ such that $u \geq v$ in Ω, and $u = v$ on $\partial\Omega$. For $0 \leq s \leq 1$, we set*

$$z \ = \ z(g, s) \ \overset{def}{=} \ (1 - s)u(g) + sv(g) \ , \qquad g \ \in \Omega \ , \tag{3.3}$$

and indicate with z_s the partial derivative

$$\frac{\partial z}{\partial s} \ = \ v - u \ .$$

With ν_X defined by (3.2), one has

$$\begin{aligned} &\frac{d}{ds} \int_\Omega \Big\{ \mathcal{F}_2[z] + \frac{3}{4} \sum_{i<j} ([X_i, X_j]z)^2 + \frac{1}{2} \sum_{j=1}^m \sum_{i \neq j} [[X_j, X_i], X_i]z \ X_j z \\ &+ \frac{1}{4} \sum_{j=1}^m \sum_{i \neq j} [X_j, [[X_j, X_i], X_i]](z^2) \Big\} dg \\ &= \int_{\partial\Omega} \sum_{i,j=1}^m \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} \ \nu_{X,i} \ \nu_{X,j} \ |\nabla z_s| \ d\sigma \ \geq \ 0 \ , \end{aligned} \tag{3.4}$$

where in the last inequality we have used (2.13). *In particular, one obtains* *from* (3.4)

$$\int_\Omega \Bigg\{ \mathcal{F}_2[u] + \frac{3}{4}\sum_{i<j}([X_i,X_j]u)^2 + \frac{1}{2}\sum_{j=1}^m\sum_{i\neq j}[[X_j,X_i],X_i]u\ X_ju \tag{3.5}$$

$$+\frac{1}{4}\sum_{j=1}^m\sum_{i\neq j}[X_j,[[X_j,X_i],X_i]](u^2)\Bigg\}\,dg$$

$$\leq \int_\Omega \Bigg\{ \mathcal{F}_2[v] + \frac{3}{4}\sum_{i<j}([X_i,X_j]v)^2 + \frac{1}{2}\sum_{j=1}^m\sum_{i\neq j}[[X_j,X_i],X_i]v\ X_jv$$

$$+\frac{1}{4}\sum_{j=1}^m\sum_{i\neq j}[X_j,[[X_j,X_i],X_i]](v^2)\Bigg\}\,dg\ .$$

Theorem 3.2 has the following important consequence.

Corollary 3.3. *Let* $\mathbf{G}$ *be a Carnot group of step* $r = 2$*, and* $\Omega \subset \mathbf{G}$ *be a* C^1 *bounded open set. Consider* $(H)_2$*-convex functions* $u, v \in \Gamma^3(\overline{\Omega})$ *such that* $u \geq v$ *in* Ω *and* $u = v$ *on* $\partial\Omega$*. For* $z = z(g,s)$ *as in* (3.3)*, one has*

$$\frac{d}{ds}\int_\Omega \left\{ \mathcal{F}_2[z] \ +\ \frac{3}{4}\ \sum_{i<j}([X_i,X_j]z)^2 \right\}\ dg$$

$$= \int_{\partial\Omega}\sum_{i,j=1}^m \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}}\ \nu_{X,i}\ \nu_{X,j}\ |\nabla z_s|\ d\sigma\ \geq\ 0\ ,$$

which gives in particular

$$\int_\Omega \left\{ \mathcal{F}_2[u] \ +\ \frac{3}{4}\ \sum_{i<j}([X_i,X_j]u)^2 \right\}\ dg \tag{3.6}$$

$$\leq\ \int_\Omega \left\{ \mathcal{F}_2[v] \ +\ \frac{3}{4}\ \sum_{i<j}([X_i,X_j]v)^2 \right\}\ dg\ .$$

Proof. It suffices to observe that if the step of $\mathbf{G}$ is $r = 2$, then we have trivially $[[X_j,X_i],X_i] = [X_j,[[X_j,X_i],X_i]] = 0$ for all $i,j = 1,...,m$, so that the conclusion follows immediately from Theorem 3.2. □

In particular, using (2.16) one obtains from Corollary 3.3.

Corollary 3.4. *Under the same assumptions of Corollary* 3.3*, when* $\mathbf{G} = \mathbb{H}^n$*, the Heisenberg group, one has*

$$\int_\Omega \left\{ \mathcal{F}_2[u] \ + \ \frac{3}{4} n \ (Tu)^2 \right\} \ dg \ \leq \ \int_\Omega \left\{ \mathcal{F}_2[v] \ + \ \frac{3}{4} n \ (Tu)^2 \right\} \ dg \ . \tag{3.7}$$

We also note that since in the Engel group $\mathfrak{E}$ a function is $(H)_2$-convex if and only if it is weakly H-convex, and since in $\mathfrak{E}$ one has $[X_j, [[X_j, X_i], X_i]] = 0$, Theorem 3.2 contains Theorem 3.1.

To establish Theorem 3.2 we begin with a simple calculus lemma.

Lemma 3.5. *Let* $\mathbf{G}$ *be a Carnot group and* $\Omega \subset \mathbf{G}$ *be a* C^1 *bounded open set. Consider two functions* $u, v \in \Gamma^3(\overline{\Omega}) \cap C^1(\overline{\Omega})$ *such that* $u \geq v$ *in* Ω *and* $u = v$ *on* $\partial\Omega$*. For* $0 \leq s \leq 1$*, let* $z = z(g, s)$ *be as in* (3.3)*, and define*

$$f(s) \ = \ \int_\Omega \mathcal{F}_2[z] \ dg \ . \tag{3.8}$$

One has

$$f'(s) \ = \ \int_{\partial\Omega} \sum_{i,j=1}^m \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} \ \nu_{X,i} \ \nu_{X,j} \ |\nabla z_s| \ d\sigma \ - \ \int_\Omega \sum_{i,j=1}^m X_i \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} \ X_j(z_s) \ dg \ , \tag{3.9}$$

where we have indicated with $d\sigma$ *the Riemannian volume measure on* $\partial\Omega$*.*

Proof. In the sequel we will tacitly use the summation convention over repeated indices. Also, we will indicate with z_s the partial derivative

$$\frac{\partial z}{\partial s} \ = \ v - u \ .$$

We note explicitly that $z_s \leq 0$ in Ω, and $z_s = 0$ on $\partial\Omega$. Thereby, the Riemannian unit normal ν to $\partial\Omega$ satisfies the relation

$$\nabla z_s \ = \ \nu \ |\nabla z_s| \ , \tag{3.10}$$

where ∇ indicates the Riemannian gradient in $\mathbf{G}$. A differentiation now gives

$$f'(s) \ = \ \int_\Omega \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} \ \frac{\partial z_{,ij}}{\partial s} \ dg \ = \ \int_\Omega \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} \ (z_s)_{,ij} \ dg \ .$$

Using the definition (1.2), and integrating by parts, we find

$$f'(s) = \frac{1}{2} \int_{\partial\Omega} \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} X_j(z_s) < X_i, \nu > \ d\sigma \ - \ \frac{1}{2} \int_{\Omega} X_i \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} \ X_j(z_s) \ dg \tag{3.11}$$

$$+ \ \frac{1}{2} \int_{\partial\Omega} \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} X_i(z_s) < X_j, \nu > \ d\sigma \ - \ \frac{1}{2} \int_{\Omega} X_j \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} \ X_i(z_s) \ dg \ .$$

Next, using (3.10) we see that

$$X_i(z_s) \ = < X_i, \nu > |\nabla z_s| \ = \ \nu_{X,i} \ |\nabla z_s| \ . \tag{3.12}$$

Substitution in (3.11) gives

$$f'(s) \ = \ \int_{\partial\Omega} \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} \ \nu_{X,i} \ \nu_{X,j} \ |\nabla z_s| \ d\sigma \ - \ \int_{\Omega} X_i \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} \ X_j(z_s) \ dg \ , \tag{3.13}$$

which is (3.9). □

We now turn to the

Proof of Theorem 3.2. We claim that the following formula holds

$$\int_{\Omega} X_i \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} \ X_j(z_s) \ dg \ = \ \frac{d}{ds} \left\{ \frac{3}{4} \int_{\Omega} \sum_{i<j} ([X_i, X_j]z)^2 \right. \tag{3.14}$$

$$\left. + \ \frac{1}{2} \sum_{j=1}^{m} \sum_{i \neq j} [[X_j, X_i], X_i]z \ X_j z \ + \ \frac{1}{4} \sum_{j=1}^{m} \sum_{i \neq j} [X_j, [[X_j, X_i], X_i]](z^2) \right\} .$$

Assuming (3.14) valid for a moment, then from it, and from (3.9) of Lemma 3.5, we would obtain

$$\frac{d}{ds} \int_{\Omega} \mathcal{F}_2[z] = \int_{\partial\Omega} \sum_{i,j=1}^{m} \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} \ \nu_{X,i} \ \nu_{X,j} |\nabla z_s| d\sigma - \frac{d}{ds} \left\{ \frac{3}{4} \int_{\Omega} \sum_{i<j} ([X_i, X_j]z)^2 \right.$$

$$\left. + \ \frac{1}{2} \sum_{j=1}^{m} \sum_{i \neq j} [[X_j, X_i], X_i]z \ X_j z \ + \ \frac{1}{4} \sum_{j=1}^{m} \sum_{i \neq j} [X_j, [[X_j, X_i], X_i]](z^2) \right\} ,$$

and this would complete the proof of the theorem. We are thus left with proving (3.14).

We write

$$X_i \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} X_j(z_s) \ = \ X_i \frac{\partial \mathcal{F}_2[z]}{\partial z_{,i1}} X_1(z_s) \ + \ ... \ + \ X_i \frac{\partial \mathcal{F}_2[z]}{\partial z_{,im}} X_m(z_s) \ . \tag{3.15}$$

In the sequel, we will need the following simple formulas

$$\frac{\partial \mathcal{F}_2[u]}{\partial u_{,ii}} \; = \; \sum_{j\neq i}^{m} u_{,jj} \; , \qquad \frac{\partial \mathcal{F}_2[u]}{\partial u_{,ij}} \; = \; - \, u_{,ij} \qquad \text{when} \quad i \neq j \; . \tag{3.16}$$

If we consider the first addend in the right-hand side of (3.15), using (3.16) we obtain

$$\begin{aligned} X_i \frac{\partial \mathcal{F}_2[z]}{\partial z_{,i1}} \; &= \; X_1 \frac{\partial \mathcal{F}_2[z]}{\partial z_{,11}} \; + \sum_{i=2,\ldots,m} X_i \frac{\partial \mathcal{F}_2[z]}{\partial z_{,i1}} \\ &= \; \sum_{i=2}^{m} X_1 z_{,ii} \; - \; \sum_{i=2}^{m} X_i z_{,i1} \; = \; \sum_{i=2}^{m} \{ X_1 z_{,ii} \; - \; X_i z_{,i1} \} \; . \end{aligned} \tag{3.17}$$

We now observe that

$$\begin{aligned} & X_1 z_{,ii} \; - \; X_i z_{,i1} \; = \; X_1 X_i X_i z \; - \; \frac{1}{2} X_i X_1 X_i z \; - \; \frac{1}{2} X_i X_i X_1 z \\ & = \; X_1 X_i X_i z \; - \; \frac{1}{2} X_i X_1 X_i z \; - \; \frac{1}{2} X_i X_1 X_i z \; + \; \frac{1}{2} X_i [X_1, X_i] z \\ & = \; [X_1, X_i] X_i z \; + \; \frac{1}{2} X_i [X_1, X_i] z \; = \; \frac{3}{2} X_i [X_1, X_i] z \; + \; [[X_1, X_i], X_i] z \; . \end{aligned} \tag{3.18}$$

Replacing (3.18) in (3.17), we conclude

$$X_i \frac{\partial \mathcal{F}_2[z]}{\partial z_{,i1}} \; = \; \frac{3}{2} \sum_{i=2}^{m} X_i [X_1, X_i] z \; + \; \sum_{i=2}^{m} [[X_1, X_i], X_i] z \; . \tag{3.19}$$

The same considerations allow to establish analogous formulas for the remaining addends in (3.15), obtaining

$$\begin{aligned} X_i \frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}} X_j(z_s) = \frac{3}{2} \; & \sum_{j=1}^{m} \left(\sum_{i \neq j} X_i [X_j, X_i] z \right) X_j(z_s) \\ & + \sum_{j=1}^{m} \left(\sum_{i \neq j} [[X_j, X_i], X_i] z \right) X_j(z_s) \; . \end{aligned} \tag{3.20}$$

We consider the first term in the right-hand side of (3.20). Integrating by parts, we find

$$
\begin{aligned}
&\frac{3}{2}\sum_{j=1}^{m}\sum_{i\neq j}\int_{\Omega} X_i[X_j,X_i]z\ X_j(z_s)\ dg \qquad (3.21)\\
&= \frac{3}{2}\sum_{j=1}^{m}\sum_{i\neq j}\int_{\partial\Omega}[X_j,X_i]z\ <X_i,\nu>X_j(z_s)\ d\sigma\\
&\quad - \frac{3}{2}\sum_{j=1}^{m}\sum_{i\neq j}\int_{\Omega}[X_j,X_i]z\ X_iX_j(z_s)\ dg\\
&= \frac{3}{2}\sum_{j=1}^{m}\sum_{i\neq j}\int_{\partial\Omega}[X_j,X_i]z\ <X_i,\nu><X_j,\nu>|\nabla z_s|\ d\sigma\\
&\quad - \frac{3}{2}\sum_{j=1}^{m}\sum_{i\neq j}\int_{\Omega}[X_j,X_i]z\ X_iX_j(z_s)\ dg\\
&= (I)\ +\ (II)\ ,
\end{aligned}
$$

where in the last equality we have used (3.12). We now claim that

$$
(I)\ =\ 0\ . \qquad (3.22)
$$

This easily follows from the identity $[X_i,X_j]=-[X_j,X_i]$, and from

$$
\begin{aligned}
(I) = \frac{3}{2}\int_{\partial\Omega}\Big\{&[X_1,X_2]z<X_1,\nu><X_2,\nu>+...\\
&+[X_1,X_{m-1}]z<X_1,\nu><X_{m-1},\nu>\\
&+[X_1,X_m]z<X_1,\nu><X_m,\nu>\\
&+[X_2,X_1]z<X_2,\nu><X_1,\nu>+...\\
&+[X_{m-1},X_1]z<X_{m-1},\nu><X_1,\nu>+...\\
&+[X_{m-1},X_m]z<X_{m-1},\nu><X_m,\nu>\\
&+[X_m,X_1]z<X_m,\nu><X_1,\nu>+...\\
&+[X_m,X_{m-1}]z<X_m,\nu><X_{m-1},\nu>\Big\}|\nabla z_s|d\sigma.
\end{aligned}
$$

Just notice that the terms in the above boundary integral cancel in pairs. Finally, we claim that

$$
(II)\ =\ \frac{d}{ds}\left\{\frac{3}{4}\sum_{i<j}\int_{\Omega}([X_i,X_j]z)^2\ dg\right\}\ . \qquad (3.23)
$$

To prove (3.23) we proceed as follows

$$
\begin{aligned}
(II) \;=\; & -\frac{3}{2} \sum_{j=1}^{m} \sum_{i\neq j} \int_\Omega [X_j, X_i]z \; X_i X_j(z_s)\; dg \\
=\; & -\frac{3}{2} \int_\Omega \Big\{ [X_1, X_2]z \; X_2 X_1(z_s) \;+\; ...+\; [X_1, X_m]z \; X_m X_1(z_s) \\
& +\; [X_2, X_1]z \; X_1 X_2(z_s) \;+\; ...+\; [X_m, X_{m-1}]z \; X_{m-1} X_m(z_s) \Big\}\; dg \\
=\; & -\frac{3}{2} \int_\Omega \Big\{ [X_1, X_2]z \; \big(X_2 X_1(z_s) - X_1 X_2(z_s)\big) + ... \\
& +\; [X_1, X_m]z \; \big(X_m X_1(z_s) - X_1 X_m(z_s)\big) \\
& +\; ... \;+\; [X_{m-1}, X_m]z \; \big(X_m X_{m-1}(z_s) - X_{m-1} X_m(z_s)\big) \Big\}\; dg \\
=\; & \frac{d}{ds} \left\{ \frac{3}{4} \sum_{i<j} \int_\Omega ([X_i, X_j]z)^2 \;\; dg \right\} .
\end{aligned}
$$

We next consider the second term in the right-hand side of (3.20). An integration by parts gives

$$
\begin{aligned}
& \sum_{j=1}^{m} \sum_{i\neq j} \int_\Omega [[X_j, X_i], X_i]z \; X_j(z_s)\; dg \qquad (3.24) \\
=& \sum_{j=1}^{m} \sum_{i\neq j} \int_{\partial\Omega} z < [[X_j, X_i], X_i], \nu > X_j(z_s) d\sigma - \sum_{j=1}^{m} \sum_{i\neq j} \int_\Omega z [[X_j, X_i], X_i] X_j(z_s) dg.
\end{aligned}
$$

We now consider $\sum_{j=1}^{m} \sum_{i\neq j} X_j z [[X_j, X_i], X_i](z_s)$, and integrate this function by parts, obtaining

$$
\begin{aligned}
& \sum_{j=1}^{m} \sum_{i\neq j} \int_\Omega X_j z \; [[X_j, X_i], X_i](z_s)\; dg \\
=& \sum_{j=1}^{m} \sum_{i\neq j} \int_{\partial\Omega} z < X_j, \nu > [[X_j, X_i], X_i](z_s) d\sigma - \sum_{j=1}^{m} \sum_{i\neq j} \int_\Omega z X_j [[X_j, X_i], X_i](z_s) dg \\
=& \; \sum_{j=1}^{m} \sum_{i\neq j} \int_{\partial\Omega} z \;\; < X_j, \nu >< [[X_j, X_i], X_i], \nu > |\nabla z_s| \; d\sigma \\
& - \sum_{j=1}^{m} \sum_{i\neq j} \int_\Omega z [[X_j, X_i], X_i] X_j(z_s) dg - \sum_{j=1}^{m} \sum_{i\neq j} \int_\Omega z [X_j, [[X_j, X_i], X_i]](z_s) dg.
\end{aligned}
$$

The latter equality gives

$$\begin{aligned}
&- \sum_{j=1}^{m}\sum_{i\neq j}\int_{\Omega} z\ [[X_j,X_i],X_i]X_j(z_s)\ dg \qquad (3.25)\\
&= \sum_{j=1}^{m}\sum_{i\neq j}\int_{\Omega} X_j z\ [[X_j,X_i],X_i](z_s)\ dg\\
&\quad - \sum_{j=1}^{m}\sum_{i\neq j}\int_{\partial\Omega} z\ < X_j,\nu >< [[X_j,X_i],X_i],\nu > |\nabla z_s|\ d\sigma\\
&\quad + \sum_{j=1}^{m}\sum_{i\neq j}\int_{\Omega} z\ [X_j,[[X_j,X_i],X_i]](z_s)\ dg\ .
\end{aligned}$$

We now substitute (3.25) into (3.24), obtaining

$$\begin{aligned}
&\sum_{j=1}^{m}\sum_{i\neq j}\int_{\Omega} [[X_j,X_i],X_i]z\ X_j(z_s)\ dg \qquad (3.26)\\
&= \sum_{j=1}^{m}\sum_{i\neq j}\int_{\partial\Omega} z\ < [[X_j,X_i],X_i],\nu > X_j(z_s)\ d\sigma\\
&\quad - \sum_{j=1}^{m}\sum_{i\neq j}\int_{\partial\Omega} z\ < X_j,\nu >< [[X_j,X_i],X_i],\nu > |\nabla z_s|\ d\sigma\\
&\quad + \sum_{j=1}^{m}\sum_{i\neq j}\int_{\Omega} X_j z\ [[X_j,X_i],X_i](z_s)\ dg\\
&\quad + \sum_{j=1}^{m}\sum_{i\neq j}\int_{\Omega} z\ [X_j,[[X_j,X_i],X_i]](z_s)\ dg\\
&= \sum_{j=1}^{m}\sum_{i\neq j}\int_{\Omega} X_j z\ [[X_j,X_i],X_i](z_s)\ dg\\
&\quad + \sum_{j=1}^{m}\sum_{i\neq j}\int_{\Omega} z\ [X_j,[[X_j,X_i],X_i]](z_s)\ dg\ ,
\end{aligned}$$

where we have been able to eliminate the boundary integrals because of (3.10). We now use (3.26) to find

$$\begin{aligned}
&\frac{d}{ds}\left(\sum_{j=1}^{m}\sum_{i\neq j}\int_\Omega [[X_j,X_i],X_i]z\ X_jz\ dg\right) \qquad (3.27)\\
&= \sum_{j=1}^{m}\sum_{i\neq j}\int_\Omega [[X_j,X_i],X_i](z_s)\ X_jz\ dg\\
&\quad + \sum_{j=1}^{m}\sum_{i\neq j}\int_\Omega [[X_j,X_i],X_i]z\ X_j(z_s)\ dg\\
&= 2\sum_{j=1}^{m}\sum_{i\neq j}\int_\Omega [[X_j,X_i],X_i]z\ X_j(z_s)\ dg\\
&\quad - \sum_{j=1}^{m}\sum_{i\neq j}\int_\Omega z\ [X_j,[[X_j,X_i],X_i]](z_s)\ dg\\
&= 2\sum_{j=1}^{m}\sum_{i\neq j}\int_\Omega [[X_j,X_i],X_i]z\ X_j(z_s)\ dg\\
&\quad - \frac{1}{2}\frac{d}{ds}\left(\sum_{j=1}^{m}\sum_{i\neq j}\int_\Omega [X_j,[[X_j,X_i],X_i]](z^2)\ dg\right).
\end{aligned}$$

From (3.27) we finally obtain

$$\begin{aligned}
\sum_{j=1}^{m}\sum_{i\neq j}\int_\Omega [[X_j,X_i],X_i]z\ X_j(z_s)dg &= \frac{d}{ds}\Bigg\{\frac{1}{2}\sum_{j=1}^{m}\sum_{i\neq j}\int_\Omega [[X_j,X_i],X_i]z\ X_jzdg \qquad (3.28)\\
&+ \frac{1}{4}\sum_{j=1}^{m}\sum_{i\neq j}\int_\Omega [X_j,[[X_j,X_i],X_i]](z^2)\ dg\Bigg\}.
\end{aligned}$$

We now integrate (3.20) over Ω, and use (3.21), (3.22), (3.23) and (3.28) to conclude

$$\begin{aligned}
\int_\Omega X_i\frac{\partial \mathcal{F}_2[z]}{\partial z_{,ij}}\ X_j(z_s)\ dg = \frac{d}{ds}\int_\Omega \Bigg\{&\frac{3}{4}\sum_{i<j}([X_i,X_j]z)^2\\
&+\frac{1}{2}\sum_{j=1}^{m}\sum_{i\neq j}[[X_j,X_i],X_i]z\ X_jz + \frac{1}{4}\sum_{j=1}^{m}\sum_{i\neq j}[X_j,[[X_j,X_i],X_i]](z^2)\Bigg\}dg.
\end{aligned}$$

This finally establishes (3.14), thus completing the proof of the theorem. □

4. The theorem of Busemann-Feller and Alexandrov.

Throughout this section $\mathbf{G}$ represents a Carnot group of step $r = 2$. Our primary objective in this section is to prove Theorem 1.1. Our first step will be to obtain a local control from above of the fully nonlinear operator appearing in Theorem 3.2 in terms of the oscillation of the function u. Here, we adapt an idea in the paper by Trudinger and Wang [TW] which has already been exploited for the generalized Monge-Ampère operator in the Heisenberg group in [GM] and [GT]. To implement this idea we need to provide a suitable smooth $(H)_2$-convex test function to insert in Theorem 3.2. It turns out that we can use for this purpose a suitable power of the gauge N. That this is possible at all is quite remarkable, since the weak H-convexity of the gauge itself in an arbitrary Carnot group of step 2 is very much in doubt. However, in [DGN] it was proved that the function $u = N^4$ is weakly H-convex in any such group. For completeness we record the short proof in the next lemma.

Lemma 4.1. *In a Carnot group of step 2 the function* $u(g) = N(g)^4 = |x(g)|^4 + 16|y(g)|^2$ *is weakly* H*-convex, hence in particular it is* $(H)_r$*-convex for* $r = 1, ..., m$ *(see Remark 2.2).*

Proof. Consider the two functions $\psi(g) = |x(g)|^4$, $\chi(g) = |y(g)|^2$. It suffices to show that ψ and χ are weakly H-convex in $\mathbf{G}$. Since ψ does not depend on the variables $(y_1(g), ..., y_k(g))$, we easily obtain from (2.19) in Lemma 2.4

$$\psi_{,ij} \;=\; 4\;|x(g)|^2\;\delta_{ij} \;+\; 8\;x_i(g)\;x_j(g)\;. \tag{4.1}$$

From this formula we easily infer for every $\zeta \in V_1$

$$< Hess_X(\psi)(g)\zeta, \zeta > = \;4\;|x(g)|^2\;|\zeta|^2 \;+\; 8\;< \xi_1(g), \zeta >^2 \geq\; 0\;,$$

which thanks to Theorem 2.3 guarantees the weak H-convexity of ψ. Next, we look at χ. Again, from (2.19) in Lemma 2.4 we have

$$\begin{aligned} X_iX_j\chi &= \sum_{s=1}^{k} < [e_i, e_j], \epsilon_s > y_s + \frac{1}{2}\;\sum_{s,s'=1}^{k} < [\xi_1, e_i], \epsilon_s >< [\xi_1, e_j], \epsilon_{s'} > \;\delta_{ss'} \\ &= \sum_{s=1}^{k} < [e_i, e_j], \epsilon_s > y_s + \frac{1}{2}\;\sum_{s=1}^{k} < [\xi_1, e_i], \epsilon_s >< [\xi_1, e_j], \epsilon_s > \;. \end{aligned}$$

Since $[e_i, e_j] = -[e_j, e_i]$, we obtain

$$\chi_{,ij} \;=\; \frac{1}{2}\big(X_i X_j \chi \;+\; X_j X_i \chi\big) \;=\; \frac{1}{2}\;\sum_{s=1}^{k} < [\xi_1, e_i], \epsilon_s >< [\xi_1, e_j], \epsilon_s > \;\; . \tag{4.2}$$

This gives for every $\zeta \in V_1$

$$\begin{aligned} < Hess_X(\chi)(g)\zeta, \zeta > &= \frac{1}{2}\sum_{s=1}^{k}\sum_{i,j=1}^{m} < J(\epsilon_s)\xi_1(g), e_i > \zeta_i < J(\epsilon_s)\xi_1(g), e_j > \zeta_j \\ &= \frac{1}{2}\sum_{s=1}^{k} < J(\epsilon_s)\xi_1(g), \zeta >^2 \;\geq\; 0 \;, \end{aligned} \tag{4.3}$$

where we have denoted by $J : V_2 \to End(V_1)$ the Kaplan mapping defined by the equation

$$< J(\eta)\xi, \xi' > \, = \, < [\xi, \xi'], \eta > \;\; .$$

From Theorem 2.3 we conclude that χ is weakly H-convex. $\square$

Lemma 4.2. *Let* $\mathbf{G}$ *and* u *be as in Lemma* 4.1, *then there exists a constant* $C(\mathbf{G}) > 0$ *such that*

$$\left| \mathcal{F}_2[u] \;+\; \frac{3}{4}\;\sum_{i<j}([X_i, X_j]u)^2 \right| \;\leq\; C(\mathbf{G})\; u \;.$$

Proof. From (4.1), (4.2) we obtain

$$\begin{aligned} u_{,ij}(g) = \;& 4\;|x(g)|^2\;\delta_{ij} \;+\; 8\;x_i(g)\;x_j(g) \\ & +\;\frac{1}{2}\;\sum_{s=1}^{k} < J(\epsilon_s)\xi_1(g), e_i >< J(\epsilon_s)\xi_1(g), e_j > \;\; . \end{aligned} \tag{4.4}$$

From (4.4) and (2.11) we easily infer

$$|\mathcal{F}_2[u]| \;\leq\; C(\mathbf{G})\;|x(g)|^4 \;. \tag{4.5}$$

Next, we use (2.20) to find

$$\sum_{i<j}([X_i, X_j]u|)^2 \;\leq\; C(\mathbf{G})\;|y|^2 \;. \tag{4.6}$$

Combining (4.5), (4.6) we reach the desired conclusion. □

The following is the second main result of this paper. We emphasize that it represents a kind of Caccioppoli inequality, but for the fully nonlinear operator appearing in Corollary 3.3.

Theorem 4.3. *Consider a bounded open set Ω in a group of step two $\mathbf{G}$. Let $u \in \Gamma^3(\Omega)$ be a $(H)_2$-convex function. For any $D \subset\subset \Omega$ we have for some constant $C > 0$ depending on $\mathbf{G}, \Omega$, and D,*

$$\int_D \left\{ \mathcal{F}_2[u] \;+\; \frac{3}{4} \sum_{i<j} ([X_i, X_j]u)^2 \right\} dg \;\leq\; C \; \left(\underset{\Omega}{osc}\; u \right)^2 .$$

Proof. We observe preliminarily that since $\mathbf{G}$ has step $r = 2$, then the assumption $u \in \Gamma^3(\Omega)$ implies that for any $\omega \subset\subset \Omega$ one has $u \in C^1(\overline{\omega})$. We now fix a gauge ball $B = B(g_o, R) \subset \Omega$, and without loss of generality we assume that $g_o = 0$, the group identity. By considering instead of u the function $v = u - \sup_B u - \epsilon$, we can assume that $u \leq -\epsilon$ in B, for some $\epsilon > 0$. If we set $m_o = \inf_B \; v < 0$, we next define the C^∞ function

$$\psi(g) \;=\; \frac{|m_o|}{\sigma} \; \left\{ \frac{N(g)^4}{R^4} \;-\; 1 \right\} \; ,$$

where $\sigma \in (0, 1)$ is fixed. From Lemma 4.1 we know that the function N^4 is weakly H-convex, hence, in particular, it is $(H)_2$-convex. We thus conclude that ψ is a smooth $(H)_2$-convex function. Furthermore, we have

$$\psi(e) \;=\; -\; \frac{|m_o|}{\sigma} \;<\; m_o \; .$$

We apply (3.6) in Corollary 3.3 to v and ψ on the open set $\tilde{B} = \{g \in \Omega \mid \psi(g) < v(g)\}$, obtaining

$$\begin{aligned} \int_{\tilde{B}} \left\{ \mathcal{F}_2[v] + \frac{3}{4} \sum_{i<j} ([X_i, X_j]v)^2 \right\} dg &\leq \int_{\tilde{B}} \left\{ \mathcal{F}_2[\psi] + \frac{3}{4} \sum_{i<j} ([X_i, X_j]\psi)^2 \right\} dg \qquad (4.7) \\ &\leq \int_{B} \left\{ \mathcal{F}_2[\psi] + \frac{3}{4} \sum_{i<j} ([X_i, X_j]\psi)^2 \right\} dg. \end{aligned}$$

We next observe that $\{g \in \Omega \mid \psi(g) < m_o\} \subset \tilde{B}$. This being said we now claim that there exists $\delta = \delta(\sigma) \in (0,1)$, independent of u, such that

$$B(0, \delta R) \ \subset \ \{g \in \Omega \mid \psi(g) < m_o\} \ .$$

The proof of this property easily follows from the definition of ψ, provided that we choose $\delta = (1-\sigma)^{1/4}$. From these considerations and from (4.7) we conclude

$$\int_{B(0,\delta R)} \left\{ \mathcal{F}_2[v] + \frac{3}{4} \sum_{i<j} ([X_i, X_j]v)^2 \right\} dg \leq \int_B \left\{ \mathcal{F}_2[\psi] + \frac{3}{4} \sum_{i<j} ([X_i, X_j]\psi)^2 \right\} dg. \tag{4.8}$$

At this point we appeal to Lemma 4.2 to conclude

$$\int_B \left\{ \mathcal{F}_2[\psi] + \frac{3}{4} \sum_{i<j} ([X_i, X_j]\psi)^2 \right\} dg \ \leq \ C(\mathbf{G}) \ \frac{m_o^2}{\sigma^2} \ R^{-8} \ \int_B N^4(g) \ dg \ . \tag{4.9}$$

Using Proposition 1.15 in [FS], or a rescaling, we find

$$\int_B N^4(g) \ dg \ = \ \alpha(\mathbf{G}) \ R^{Q+4} \ ,$$

where $\alpha(\mathbf{G}) > 0$, and $Q = m + 2k$ is the homogeneous dimension of $\mathbf{G}$. Substituting this information in (4.9), and then using such inequality in (4.8), and letting $\epsilon \to 0$, we finally obtain

$$\int_{B(0,\delta R)} \left\{ \mathcal{F}_2[u] + \frac{3}{4} \sum_{i<j} ([X_i, X_j]u)^2 \right\} dg \tag{4.10}$$
$$\leq \ C(\mathbf{G}) \ \frac{R^{Q-4}}{\sigma^2} \ (osc_B \ u)^2 \ \leq \ C'(\mathbf{G}) \ R^{Q-4} \ (osc_\Omega \ u)^2 \ .$$

To complete the proof, we simply cover $D \subset\subset \Omega$ with a finite number of balls $B(g_j, \sigma R)$, and apply (4.10) to each of these balls. ☐

We now present an important consequence of Theorem 4.3, namely that the commutators of a weakly H-convex function are locally in L^2_{loc}.

Corollary 4.4. *Let $u \in C(\Omega)$ be a weakly H-convex function in an open set $\Omega \subset \mathbf{G}$, where $\mathbf{G}$ is a Carnot group of step two, then for every $i, j = 1, ..., m$ one has*

$$[X_i, X_j]u \ \in \ L^2_{loc}(\Omega) \ .$$

Proof. Fix $D \subset\subset D' \subset\subset \Omega$. Let $K \in C_o^\infty(\mathbb{H}^n)$ be such that $K \geq 0$, $supp\ K \subseteq \overline{B}(0,1)$, $\int_{\mathbb{H}^n} K(g)dg = 1$, and let $K_\epsilon(g) = \epsilon^{-Q} K(\delta_{\epsilon^{-1}} g)$ be the approximation to the identity associated with K. By Remark 5.9 in [DGN], for sufficiently small ϵ, depending on $dist(D', \Omega)$, the function $u_\epsilon = K_\epsilon \star u$ is weakly H-convex in D' and C^∞. In particular, u_ϵ is $(H)_2$-convex in D'. Furthermore, since $u_\epsilon \to u$ uniformly on compact subsets of Ω, we clearly have

$$\underset{D'}{osc}\ u_\epsilon \ \leq\ C\ \underset{\Omega}{osc}\ u\ ,$$

for some constant $C > 0$ depending only on $dist(D', \Omega)$, but not on ϵ. From the latter inequality, and from Theorem 4.3, we find

$$\int_D \left\{ \mathcal{F}_2[u_\epsilon] + \frac{3}{4} \sum_{i<j} ([X_i, X_j]u_\epsilon)^2 \right\} dg \leq C \left(\underset{\Omega}{osc} u \right)^2 = C(\Omega, \Omega', n, u) < \infty\ . \tag{4.11}$$

By (4.11), and by the $(H)_2$-convexity of u_ϵ, we infer

$$\int_D ([X_i, X_j]u_\epsilon)^2\ dg\ \leq\ C(\Omega, \Omega', n, u)\ . \tag{4.12}$$

In particular, (4.12) says that $||[X_i, X_j]u_\epsilon||_{L^2(D)} \leq C(\Omega, \Omega', n, u)$, and therefore there exists $v \in L^2(D)$ such that $[X_i, X_j]u_\epsilon \rightharpoonup v$. Denoting by $[X_i, X_j]u$ the distributional derivative of u along the commutator $[X_i, X_j]$, one easily recognizes that $[X_i, X_j]u = v \in L^2(D)$. This proves the theorem. $\square$

We now recall a basic result, which is Theorem 8.1 in [DGN], see also Theorem 4.2 in [LMS] for a similar result in the special case of the Heisenberg group.

Theorem 4.5. *Let* $\mathbf{G}$ *be a Carnot group* $\mathbf{G}$ *and consider a weakly* H*-convex function* $u \in L^1_{loc}(\mathbf{G})$. *For* $i, j = 1, ..., m$, *there exist signed Radon measures* $\nu_H^{ij} = \nu_H^{ji}$ *such that for every* $\phi \in C_o^\infty(\mathbf{G})$ *one has*

$$\int_{\mathbf{G}} u(g)\ \phi_{,ij}(g)\ dg = \int_{\mathbf{G}} u(g)\ \frac{X_i X_j \phi(g) + X_j X_i \phi(g)}{2}\ dg = \int_{\mathbf{G}} \phi(g)\ d\ \nu_H^{ij}(g)\ .$$

In addition, the measures ν_H^{ii} *are nonnegative.*

With Theorem 4.5 we can establish the following important consequence of Corollary 4.4.

Theorem 4.6. *Let $u \in C(\mathbf{G})$ be weakly H-convex in a group $\mathbf{G}$ of step two, then the non-symmetrized distributional second derivatives X_iX_ju, $i, j = 1, ..., m$, are signed Radon measures.*

Proof. Clearly, $u \in L^1_{loc}(\Omega)$. We now observe that

$$X_iX_ju \ = \ u_{,ij} \ + \ \frac{1}{2}[X_i, X_j]u \qquad\qquad \text{in} \quad \mathcal{D}'(\Omega) \ .$$

From Corollary 4.4 we conclude that $[X_i, X_j]u \in L^2_{loc}(\Omega)$, hence in particular all first commutators are Radon measures. The conclusion thus follows from the above identity and from Theorem 4.5. □

Remark 4.7. *By the recent results in [BR], [Wa3] and [M], we know that if u is weakly H-convex and locally bounded from above, then in fact $u \in L^\infty_{loc}(\mathbf{G})$. Therefore, the conclusion of Theorem* 4.6 *continues to hold under the weaker assumption that u is locally bounded from above. In the special case of $\mathbb{H}^n$, even such weaker assumption is not needed, see [BR].*

In the sequel, we denote by $\Gamma^{0,1}_{loc}(\mathbf{G})$ the space of functions which are locally Lipschitz with respect to the Carnot-Carathéodory metric in $\mathbf{G}$. We will need the following result which is contained in Theorem 1.3 and Theorem 2.7 in [GN].

Theorem 4.8. *Let $\mathbf{G}$ be a Carnot group and $h \in \Gamma^{0,1}_{loc}(\mathbf{G})$. There exists $C = C(\mathbf{G}) > 0$ such that for every g', $g'' \in B(g, r)$ one has*

$$|h(g') - h(g'')| \ \leq \ C \ d(g', g'') \ \|Xh\|_{\mathcal{L}^\infty(B(g,3r))} \ .$$

We also need the next result, which is Theorem 9.1 in [DGN].

Theorem 4.9. *Let $\mathbf{G}$ be a Carnot group and $u \in L^\infty_{loc}(\mathbf{G})$ be a weakly H-convex function, then u can be modified on a set of measure zero so that for some constant $C = C(\mathbf{G}) > 0$ one has for every $g_o \in \mathbf{G}$ and every $R > 0$*

$$||Xu||_{L^\infty(B(g_o,R))} \ \leq \ \frac{C}{R} \ ||u||_{L^\infty(B(g_o,3R))} \ ,$$

$$|u(g) - u(g')| \ \leq \ \frac{C}{R} \ ||u||_{L^\infty(B(g_o,3R))} \ d(g, g') \ , \qquad\qquad g, g' \in B(g_o, R) \ .$$

To state our next result we recall the notion of horizontal bounded variation introduced in [CDG]. Let $\Omega \subset \mathbf{G}$ be an open set in a Carnot group $\mathbf{G}$, and $u \in L^1_{loc}(\Omega)$. Denote by $\zeta = \sum_{i=1}^m \zeta_i X_i$ an element of $C^1_o(\Omega; H\mathbf{G})$. Let

$$\mathcal{F}_H(\Omega) \ = \ \left\{ \zeta \in C^1_o(\Omega; H\mathbf{G}) \ \mid \ ||\zeta||_\infty \leq 1 \right\}.$$

The $H-$variation of u in Ω is defined as follows

$$Var_H(u;\Omega) \ = \ \sup_{\zeta \in \mathcal{F}_H(\Omega)} \int_\Omega \ u \ \sum_{i=1}^m X_i \zeta_i \ dg \ .$$

A function $u \in L^1(\Omega)$ is called of bounded $H-$variation if $Var_H(u;\Omega) < \infty$. In such case, we write $u \in BV_H(\Omega)$, and the collection of all such functions becomes a Banach space when endowed with the norm

$$||u||_{BV_H(\Omega)} \ = \ ||u||_{L^1(\Omega)} \ + \ Var_H(u;\Omega) \ .$$

The notation $BV_{H,loc}(\Omega)$ indicates the collection of functions $u \in L^1_{loc}(\Omega)$, such that $u \in BV_H(\omega)$, for every $\omega \subset\subset \Omega$. We denote with $BV^2_{H,loc}(\Omega)$ the Banach space of functions $u \in \mathcal{L}^{1,1}_{loc}(\Omega)$ such that $X_i u \in BV_{H,loc}(\Omega)$, $i = 1, ..., m$.

Theorem 4.10. *Let $u \in C(\mathbf{G})$ be weakly H-convex in a Carnot group of step two, $\mathbf{G}$, then $u \in BV^2_{H,loc}(\mathbf{G})$.*

Proof. By Theorem 4.9 we know that $u \in \Gamma^{0,1}_{loc}(\mathbf{G})$. By Theorem 4.8 we infer that $X_j u \in L^\infty_{loc}(\Omega)$, hence in particular, $X_j u \in L^1_{loc}(\Omega)$, $j = 1, ..., m$. Let $\omega \subset\subset \Omega$, and consider $\zeta \in \mathcal{F}_H(\omega)$. For any $i = 1, ..., m$ we have

$$\int_\omega X_i u \ \sum_{j=1}^m X_j \zeta_j \ dg \ = \ - \sum_{j=1}^m \int_\omega u \ X_i X_j \zeta_j \ dg \ . \tag{4.13}$$

$$= \ - \ 2 \ \sum_{j=1}^m \int_\omega u \ \frac{X_i X_j \zeta_j + X_j X_i \zeta_j}{2} \ dg \ + \ \sum_{j=1}^m \int_\omega u \ X_j X_i \zeta_j \ dg$$

Using Theorem 4.5 we obtain from (4.13)

$$\int_\omega X_i u \ \sum_{j=1}^m X_j \zeta_j \ dg \ = \ - \ 2 \ \sum_{j=1}^m \int_{\mathbf{G}} \zeta_j(g) \ d\nu_H^{ij}(g) \ + \ \sum_{j=1}^m (X_i X_j u, \zeta_j) \ ,$$

where we have denoted by $(\cdot,\cdot)$ the duality between $\mathcal{D}'(\mathbf{G})$ and $\mathcal{D}(\mathbf{G})$. By Theorem 4.6 we know that also X_iX_ju are Radon measures, therefore we conclude

$$\int_\omega X_iu \sum_{j=1}^m X_j\zeta_j \, dg \ \le\ 2\sum_{j=1}^m \nu_H^{ij}(\omega) \ +\ \sum_{j=1}^m X_iX_ju(\omega) \ <\ \infty\ .$$

Taking the supremum on all $\zeta \in \mathcal{F}_H(\omega)$ we reach the conclusion that for every $i = 1, ..., m$, $X_iu \in B_H(\omega)$, hence $u \in BV_H^2(\omega)$. This completes the proof. □

Theorem 4.10 now allows to close the gap between the integral version of the Busemann-Feller-Alexandrov theorem in (1.5), and the estimates (1.6), (1.7) from [DGN]. We proceed to proving Theorem 1.1 following the approach in [EG]. In connection with this part of the paper, we mention that, after this work was completed, we have received the preprint [M] from Magnani in which the author, assuming Theorem 4.10 as valid, has also derived the following arguments.

We need the following proposition.

Proposition 4.1. *In a Carnot group* $\mathbf{G}$ *let* $h \in \Gamma^{0,1}_{loc}(\mathbf{G})$ *be such that*

$$\lim_{r\to 0^+} \frac{1}{r^2|B(g,r)|}\int_{B(g,r)} |h(g')|\, dg' \ =\ 0\ . \tag{4.14}$$

For every $\eta, \epsilon > 0$ *there exists* $r_o = r_o(g,\eta,\epsilon) > 0$ *such that for* $0 < r < r_o$ *one has*

$$\sup_{B(g,r)} |h| \ \le \epsilon\, r^2 \ +\ 4\eta^{\frac{1}{Q}}\, \|Xh\|_{L^\infty(B(g,3r))}\, r\ . \tag{4.15}$$

Proof. Fix $g \in \mathbf{G}$. Given $\eta > 0$, $\epsilon > 0$ we use assumption (4.14) and Chebyshev's inequality, to obtain $r_o = r_o(g,\eta,\epsilon) > 0$ such that for $0 < r < r_o$ we have

$$\frac{|\{g' \in B(g,r)\,|\,|h(g')| > \epsilon\, r^2\}|}{|B(g,r)|} \ \le\ \frac{1}{\epsilon\, r^2|B(g,r)|}\int_{B(g,r)} |h(g')|\, dg' \ \le\ \eta\ . \tag{4.16}$$

Let $\sigma = 4\eta^{\frac{1}{Q}}$. Then for every $g' \in B(g,r/2)$, there exists $g'' = g''(g',r) \in B(g,r)$ such that

$$d(g',g'') \ \le\ \sigma\, r \qquad \text{and} \qquad |h(g'')| \ \le\ \epsilon\, r^2\ . \tag{4.17}$$

For if not, then there is a $g_o' \in B(g, r/2)$ such that for every $g'' \in B(g,r)$ either $|h(g'')| > \epsilon\, r^2$, or $d(g', g'') > \sigma\, r$. This implies $B(g_o', \sigma\, r) \subset \{g' \in B(g,r) \,|\, |h(g')| > \epsilon\, r^2\}$ and hence using (4.16) we infer

$$|B(g_o', \sigma\, r)| \;\le\; \eta |B(g,r)| \,.$$

On the other hand, by choice of σ we have

$$\eta |B(g,r)| \;\le\; \eta\, |B(g_o', 2r)| \;=\; \eta 2^Q \sigma^{-Q} |B(g_o', \sigma\, r)| < |B(g_o', \sigma\, r)| \,.$$

This contradiction proves (4.17). Hence, for every $g_o' \in B(g, r/2)$ we have

$$\begin{aligned} |h(g')| \;&\le\; |h(g'')| \;+\; |h(g') \;-\; h(g'')| \\ \text{(by Theorem 4.8)} \;&\le\; \epsilon\, r^2 \;+\; d(g', g'') \|Xh\|_{L^\infty(B(g,3r))} \\ &\le\; \epsilon\, r^2 \;+\; \sigma \,\|Xh\|_{L^\infty(B(g,3r))}\, r \,. \end{aligned}$$

□

We are finally ready to complete the

Proof of Theorem 1.1. Let u be an upper semicontinuos weakly H-convex function in a Carnot group group $\mathbf{G}$ of step two. Appealing to Theorem 4.10, we know that $u \in BV^2_{H,loc}(\mathbf{G})$. We fix $g_o \in \mathbf{G}$ such that (1.5) holds for a polynomial of weighted degree 2, $P_u(g; g_o)$. By Theorem 4.9 $u \in \Gamma^{0,1}_{loc}(\mathbf{G})$, hence also the function $h(g) \;= u(g) \;-\; P_u(g_o, g)$ belongs to $\Gamma^{0,1}_{loc}(\mathbf{G})$. Let $\eta > 0$ be given, $\epsilon > 0$ will be chosen later. Applying Proposition 4.1 to the function h we obtain $r_o > 0$ such that for every $r < \frac{1}{45} min(r_o, dist(g, \partial\Omega))$

$$\sup_{B(g_o, r/2)} |h(g)| \;\le\; \epsilon\, r^2 \;+\; 4\eta^{\frac{1}{Q}} \|Xh\|_{L^\infty(B(g,3r))}\, r \,. \tag{4.18}$$

Our next task is to estimate $\|Xh\|_{L^\infty(B(g,3r))}$. To this end, we make the observation that (1.5) implies $P_u(g_o; g_o) = u(g_o)$. We write $P_u(g; g_o) = P_1 + P_2$ where

$$P_1(x,y) \;=\; u(g_o) \;+\; \sum_{i=1}^{m} a_i (x_i(g) - x_i(g_o))$$

and

$$P_2(x,y) \;=\; \sum_{i,j=1}^{m} a_{i,j} (x_i(g) - x_i(g_o))(x_j(g) - x_j(g_o)) \;+\; \sum_{s=1}^{k} c_s (y_s(g) - y_s(g_o)) \,.$$

Lemma 2.4 then gives

$$\begin{aligned} X_{i_o} P_2(x,y) &= \sum_{j=1}^{m} (a_{i_o,j} + a_{j,i_o})(x_j(g) - x_j(g_o)) \\ &\quad + \frac{1}{2} \sum_{s=1}^{k} \sum_{j=1}^{m} c_s b^s_{j,i_o} (x_j(g) - x_j(g_o)) \\ X_{j_o} X_{i_o} P_2(x,y) &= a_{i_o,j_o} + a_{j_o,i_o} + \frac{1}{2} \sum_{s=1}^{k} c_s b^s_{j_o,i_o} \ . \end{aligned}$$

Let

$$M = 2\, m \max_{i_o,j_o=1\ldots m} \left| a_{i_o,j_o} + a_{j_o,i_o} + \frac{1}{2} \sum_{s=1}^{k} c_s b^s_{j_o,i_o} \right|$$

and define

$$Q(x,y) = M\left((x_1(g) - x_1(g_o))^2 + \cdots + (x_m(g) - x_m(g_o))^2\right) \ .$$

Since

$$X_{j_o} X_{i_o} (P_2(x,y) + Q(x,y)) = a_{i_o,j} + a_{j,i_o} + \frac{1}{2} \sum_{s=1}^{k} c_s b^s_{j_o,i_o} + 2\delta_{i_o,j_o} M \ ,$$

the entries D_{ij} of the symmetric matrix $Hess_X(P_2 + Q)$, satisfy

$$D_{ii} > \sum_{l=1}^{m} \frac{D_{il} D_{li}}{D_{ll}}$$

and hence this matrix is positive definite on $\mathbf{G}$. It is obvious that $u - P_1$ is weakly H-convex. Theorem 2.3 allows us to conclude that $u - P_u - Q$ is weakly H-convex. Therefore, if $r < \frac{1}{45} min(r_o, dist(g, \partial\Omega))$, then

$$\begin{aligned} \sup_{B(g_o,3r)} |Xh| &\le \sup_{B(g_o,3r)} |X(u - P_u - Q)| + \sup_{B(g_o,3r)} |XQ| \qquad (4.19) \\ \text{(by (1.7))} &\le \frac{\tilde{C}}{r} \frac{1}{|B(g_o,45r)|} \int_{B(g_o,45r)} |u - P_u - Q|\, dg' + C_1 r \\ &\le \frac{\tilde{C}}{r} \frac{1}{|B(g_o,45r)|} \int_{B(g_o,45r)} |u - P_u|\, dg' + C_2 r + C_1 r \\ \text{(by (1.5))} &\le \tilde{C} \eta r + C_1 r + C_2 r \end{aligned}$$

In the above chain of inequalities, we have used also the fact that the polynomial Q is homogeneous of weighted degree 2 with respect to the point g_o. Using (4.19) in (4.18) we obtain

$$\sup_{B(g_o,r/2)} |h(g)| \le \epsilon r^2 + 4\eta^{\frac{1}{Q}} r(\tilde{C}\eta r + C_1 r + C_2 r)$$

$$= (\epsilon + 4\eta^{\frac{1}{Q}}(\tilde{C}\eta + C_1 + C_2)) r^2 = 2\epsilon r^2 ,$$

provided we choose $\epsilon = 4\eta^{\frac{1}{Q}}(\tilde{C}\eta + C_1 + C_2)$. The last estimate establishes the theorem. □

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