

Decorated Teichmüller Theory of Bordered Surfaces

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This paper extends the decorated Teichmüller theory developed before for punctured surfaces to the setting of “bordered” surfaces, i.e., surfaces with boundary, and there is non-trivial new structure discovered. Beyond this, the main new result of this paper identifies an open dense subspace of the arc complex of a bordered surface up to proper homotopy equivalence with a certain quotient of the moduli space, namely, the quotient by the natural action of the positive reals by homothety on the hyperbolic lengths of geodesic boundary components. One tool in the proof is a homeomorphism between two versions of a “decorated” moduli space for bordered surfaces. The explicit homeomorphism relies upon points equidistant to suitable triples of horocycles.

Introduction.

Complexes of arc families in surfaces arise in several related contexts in mathematics. Poincaré dual to a suitable arc family in a punctured surface is a graph embedded in the surface, so complexes of arc families are also manifest as suitable spaces of graphs. Such arc or graph complexes arise in many related mathematical contexts in the works of Culler-Vogtmann, Harer, Kontsevich, the author, Strebel, and others (and an overview of this combinatorial structure, from both the conformal and hyperbolic points of view, is given at the end of this Introduction). Up to this point, these graphical techniques for Riemann surfaces have been utilized in the setting of punctured surfaces without boundary. One principal purpose of this paper is to present the analogous theory for surfaces with boundary, or so-called “bordered” surfaces, and in effect, we present the relative version of the established theory.

In the punctured case, the quotient of an open dense subspace of the arc complex by the mapping class group is homeomorphic to moduli space. Our main results in this paper analogously extend to the bordered case and give a proper homotopy equivalence between an open dense subspace of the arc complex of a bordered surface and an \mathbf{R}_+ -quotient of the moduli space of

the bordered surface. An essential difference between the punctured and bordered cases of arc complexes is that in the latter case the arcs in an arc family come in a natural linear ordering; in effect, this kills all finite isotropy in the action of the mapping class group.

The arc complex itself thus forms a combinatorial compactification of the moduli space in the punctured case and of its quotient by \mathbf{R}_+ in the bordered case. These compactifications were studied in [8], where it was conjectured that this compactification is an orbifold in the punctured case and a sphere in the bordered case. (In the established theory for punctured surfaces, there are other known sphericity results, but these are on the level of families of arcs in a fixed surface, not its quotient by the mapping class group as in the current case of bordered surfaces.) This conjecture was proved for the case of multiply punctured spheres in [8], as follows from the positive resolution of the sphericity conjecture for the relatively simple case of multiply punctured disks, but the conjectures have recently been disproven in general.

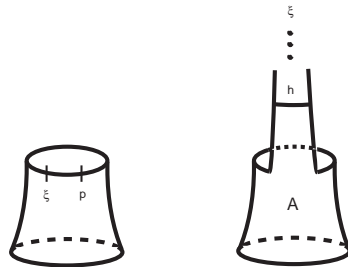
The proper homotopy equivalence mentioned before is produced from an identification of two versions of a “decorated” moduli space; one version is based upon surfaces with horocycles about cusps in analogy to the punctured case, and the other is based upon pairs of distinct labeled points in the geodesic boundary of a hyperbolic surface. The identification of the two versions depends upon a construction based upon equidistant points to triples of horocycles, which is the heart of this paper.

Let $F = F_{g,r}^s$ denote a smooth surface of genus $g \geq 0$ with $r \geq 0$ labeled boundary components and $s \geq 0$ labeled punctures, where $6g - 7 + 4r + 2s \geq 0$ (so we exclude only the surfaces $F_{1,0}^0$ and $F_{0,1}^1$). The *pure mapping class group* $PMC = PMC(F)$ of F is the group of all isotopy classes of orientation-preserving homeomorphisms of F pointwise fixing each boundary component and each puncture, where the isotopy is likewise required to pointwise fix these sets.

To begin with the case $r = 0$, let us define the classical *Teichmüller space* T_g^s of F to be the space of all complete finite-area metrics of constant Gauss curvature -1 (so-called “hyperbolic metrics”) on F modulo push-forward by diffeomorphisms fixing each puncture which are isotopic to the identity relative to the punctures. As is well known, T_g^s is homeomorphic to an open ball of real dimension $6g - 6 + 2s$, PMC acts on T_g^s by push-forward of metric under a representative diffeomorphism, and the quotient $M_g^s = T_g^s / PMC$ is *Riemann’s moduli space* of F . There is a trivial \mathbf{R}_+^s -bundle $\tilde{T}_g^s \rightarrow T_g^s$,

where the fiber over a point is the collection of all s -tuples of horocycles in F , one horocycle about each puncture, and this leads to the corresponding trivial bundle $\widetilde{T}_g^s/PMC = \widetilde{M}_g^s \rightarrow M_g^s$; the total spaces of these bundles are respectively called the *decorated Teichmüller* and *decorated moduli spaces*, which are studied in [6-10].

Turning to the case $r \neq 0$ of bordered surfaces, there are two essentially different geometric treatments of a distinguished point ξ in a boundary component C of a surface, which lead to two different versions for the corresponding (decorated) moduli space for bordered surfaces. In the first treatment, we take a hyperbolic metric so that $C \ni \xi$ is a geodesic as illustrated in Figure 1a, and this leads to one version of the moduli space of a bordered surface as we shall see (in §1). In the second treatment, we remove the distinguished point ξ from C and take a hyperbolic metric so that $C - \{\xi\}$ is totally geodesic as illustrated in Figure 1b (with a more hyperbolically realistic depiction given in Figure 2), and this leads to alternate versions of moduli space as we shall also see (in §1). We enrich the structure in the first treatment by “decorating” with a second point $p \in C$, where $p \neq \xi$, and in the second treatment by specifying a horocyclic segment h centered at ξ , as are also illustrated in Figure 1, and in this way we define two versions for the decorated moduli space of a bordered surface. After a small retraction, we prove that these two versions of decorated moduli space are homeomorphic.



1a Geodesic boundary. **1b** Totally geodesic boundary.

Figure 1 Different treatments for points in the boundary of a surface.

The proof that the two versions are homeomorphic depends upon new applications (in §5) of the constructions and calculations in [7], where we studied points in hyperbolic space which are equidistant to tuples of horocycles. The

proof further involves the extension of the decorated Teichmüller theory for punctured unbordered surfaces, which was developed in [6], to the setting of bordered surfaces; we shall find that many of the arguments in [6] extend painlessly to the current case. There is no doubt that the serious reader must consult the paper [6] (and perhaps [7] as well) for complete details of some of the arguments given here, but we have endeavored to keep this note logically self-contained and complete by recalling here enough of the relevant material.

One purpose of this paper is to present the decorated Teichmüller theory for bordered surfaces (in §3): We shall give both global “lambda length” coordinates on the decorated Teichmüller space as well as a *PMC*-invariant cell decomposition of it based on a “convex hull construction”. In the literature for the case of bordered surfaces, Kojima has previously described a related convex hull construction in [3], and Chekhov and Fock [1] have previously given analogous global coordinates on Teichmüller space. We shall also in an extended side-remark comment on further extensions for bordered surfaces of the decorated Teichmüller theory which are interesting but are not needed here.

Another purpose of this paper is to understand the arc complex of a bordered surface (defined in §6). There is a distinguished subspace of the arc complex which corresponds to arc families which “fill” F in a precise sense. We shall prove (in §6) that this subspace is proper homotopy equivalent to the quotient of the moduli space of the bordered surface (in the version with geodesic boundary) by the natural \mathbf{R}_+ -action by homothety on the hyperbolic lengths of the boundary geodesics.

This paper is organized as follows. §1 defines two versions for decorated moduli space, and §2 contains relevant definitions regarding arcs in bordered surfaces. The extension of the decorated Teichmüller theory to bordered surfaces is described in §3, where we both recall certain arguments from [6] for completeness and sketch extensions of other arguments from [6] with technical details. §4 is dedicated to the study of points equidistant to triples of horocycles, with elementary calculations from [7] simply recalled and not re-proved here, as well as a deformation retraction of one version of decorated moduli space which is required in the sequel. The previous material is applied in §5 to give the real-analytic homeomorphism between the two versions of decorated moduli space. Circle actions are studied in §6, and the arc complex is defined; the proper homotopy equivalence between the “filling” subspace of the arc complex and the quotient of moduli space by the homothetic

\mathbf{R}_+ -action is also presented in §6.

To close this Introduction before turning to bordered surfaces in the sequel, we shall contrast some of the main constructions and results in [6-10] (“in the hyperbolic setting”) for unbordered punctured surfaces with results and constructions (“in the conformal setting”) using quadratic differentials. In the conformal setting, Riemann’s moduli space of F is regarded as the space of all equivalence classes of conformal structures on F .

We begin in the conformal setting and describe the $PMC(F)$ -invariant cell decomposition of \tilde{T}_g^s , which is due to Harer-Mumford-Thurston [2], and relies on the Jenkins-Strebel theory [12]. The formulation of the combinatorics given here relies on graphs with extra structure as in [10], [5].

A “fatgraph” or “ribbon graph” G is a graph with vertices at least tri-valent together with a cyclic ordering on the half-edges about each vertex. G may be “fattened” to a bordered surface in the following way: begin with disjoint planar neighborhoods of the vertices of G , where the cyclic ordering agrees with that induced by the orientation of the plane; glue orientation-preserving bands, one band for each edge of G , in the natural way to these neighborhoods. This produces a topological surface $F_G \supseteq G$, where G is a spine of F_G . The complement $F_G - G$ is a collection of topological annuli $A_i \subseteq F_G$, $i = 1, \dots, s \geq 1$; each annulus A_i has one boundary component ∂'_i lying in G and the other ∂_i in ∂F_G .

Let $E = E(G)$ denote the set of edges of G . A “metric” on a fatgraph is a function $w \in (\mathbf{R}_+ \cup \{0\})^E$. The metric w is “positive” if $w \in \mathbf{R}_+^E$, and w is “non-degenerate” provided the subgraph $G^w = \cup\{e \in E : w(e) = 0\}$ of G is a forest, i.e., each component of G^w is a tree. Let $\sigma(G)$ denote the set of non-degenerate metrics on G . In particular, a positive metric is automatically non-degenerate, and furthermore, given a non-degenerate metric $w \in \sigma(G)$, we may Whitehead collapse each component of G^w to a vertex in the natural way to produce another fatgraph G_w ; the non-degenerate metric w on G induces a positive metric, still denoted w , on G_w . A closed edge path on G has a “length” defined by $\sum w(e)$, where the sum is over the edges e counted with multiplicity, and in particular for each boundary component ∂_i of F_G , there is a corresponding closed edge-path on G whose length is denoted ℓ_i , for $i = 1, \dots, r$.

Given a positive metric $w \in \sigma(G)$ with corresponding lengths $\ell_i > 0$, for $i = 1, \dots, s$, we may construct a metric surface homeomorphic to F_g^s as follows, where $2-2g-s$ is the Euler characteristic of G . Give each $A_i = A_i(w)$

the structure of a flat cylinder with circumference ℓ_i and height unity, so A_i has modulus ℓ_i ; isometrically identify these cylinders in the natural way along common edges in $\cup\{\partial'_i\}_1^s$ as dictated by the metric and fatgraph. This produces a metric structure on F_G , where the boundary component ∂_i of F_G is a standard circle of circumference ℓ_i ; glue to each ∂_i a standard flat disk of circumference ℓ_i with puncture $*_i$, where the boundary of the disk is concentric with $*_i$, for each $i = 1, \dots, s$. This produces a conformal structure on a surface which we may identify with F_g^s .

An analytic fact is:

Theorem A [12;§23.5] *Given a positive metric w on a fatgraph G with lengths $\ell_i = \ell_i(w)$, for $i = 1, \dots, s$, there is a unique meromorphic quadratic differential q on F_g^s so that for each $i = 1, \dots, s$:*

the non-critical horizontal trajectories of q in F foliate $A_i(w) \subseteq F_g^s$ by curves homotopic to the cores;

*the residue of \sqrt{q} at $*_i$ is ℓ_i .*

Let μ_q denote the conformal structure on F_g^s determined by q . We may think of Theorem A abstractly as a mapping

$$(G, w) \mapsto \mu_q \times (\ell_i)_1^s \in T_g^s \times \mathbf{R}_+^s,$$

where the effective construction of $\mu_q \times (\ell_i)_1^s$ was described before.

Now, fix a surface $F = F_g^s$ and consider the collection C_g^s of all homotopy classes of inclusions $G \subseteq F$, where G is a strong deformation retraction of F . If $G \subseteq F$ and $w \in \sigma(G)$ is non-degenerate, then we may produce another $G_w \subseteq F$ by contracting each edge $e \in E(G)$ with $w(e) = 0$ to produce a positive metric w on G_w .

Define

$$U_g^s = \left[\coprod_{(G \subseteq F) \in C_g^s} \sigma(G) \right] / \sim,$$

where \coprod denotes disjoint union, and $(G^1, w^1) \sim (G^2, w^2)$ if and only if $G_{w^1}^1 \subseteq F$ agrees with $G_{w^2}^2 \subseteq F$ as members of C_g^s and $w^1 \in \sigma(G^1)$ agrees with $w^2 \in \sigma(G^2)$. $PMC(F)$ acts on U_g^s in the natural way induced by $(\phi : G \rightarrow F) \mapsto (f \circ \phi : G \rightarrow F)$ if $f : F \rightarrow F$ is a homeomorphism. Define the “fatgraph complex” $G_g^s = U_g^s / PMC$ and let $[G, w] \in G_g^s$ denote the class of $(G, w) \in U_g^s$. Thus, the fatgraph complex is roughly the space

of equivalence classes of non-degenerate (or equivalently positive) metric fatgraphs in F_g^s up to homotopy.

Theorem B [12;§25.6] [11] *The mapping $(G, w) \mapsto \mu_q \times (\ell_i)_1^s$ induces real-analytic homeomorphisms $U_g^s \rightarrow T_g^s \times \mathbf{R}_+^s$ and $G_g^s \rightarrow M_g^s \times \mathbf{R}_+^s$.*

Theorem A says that the mapping $G_g^s \rightarrow M_g^s \times \mathbf{R}_+^s$ is well-defined and one-to-one, while the further analytic content of Theorem B is that this mapping is moreover onto. The inverse map $T_g^s \times \mathbf{R}_+^s \rightarrow U_g^s$ is transcendental and highly non-computable. Theorem B gives a *PMC*-invariant cell decomposition of $T_g^s \times \mathbf{R}_+^s$ induced by the cell structure of G_g^s .

In the hyperbolic setting, we begin with a (conjugacy class of) Fuchsian group Γ uniformizing a point of T_g^s . Specifying also a collection of horocycles, one horocycle about each puncture of $F = F_g^s$ (called a “decoration”) furthermore uniquely determines a point $\tilde{\Gamma} \in \tilde{T}_g^s$ by definition. We shall next describe the “convex hull construction”, which assigns to $\tilde{\Gamma} \in \tilde{T}_g^s$ a corresponding point $(G_{\tilde{\Gamma}}, w_{\tilde{\Gamma}}) \in U_g^s$; this assignment is effectively computable as we shall see.

Here is a sketch of the convex hull construction from [6]. One may identify the open positive light-cone L^+ in Minkowski three-space with the space of all horocycles in the hyperbolic plane. Via this identification, we find a Γ -invariant set B of points in L^+ corresponding to the decoration, where we regard Γ as acting via Minkowski isometries. One can show that B is discrete in L^+ and consider the convex hull H of B in the vector space structure underlying Minkowski space. The extreme edges of the resulting Γ -invariant convex body H project to a collection of disjointly embedded arcs $\alpha_{\tilde{\Gamma}}$ connecting punctures; furthermore, each component of $F - \cup \alpha_{\tilde{\Gamma}}$ is simply connected, and we say that $\alpha_{\tilde{\Gamma}}$ “fills” F . Given any arc family α filling F , we may define a subset

$$C(\alpha) = \{ \tilde{\Gamma} \in \tilde{T}_g^s : \alpha_{\tilde{\Gamma}} \text{ is homotopic to } \alpha \}.$$

The Poincaré dual of the cell decomposition $F - \cup \alpha_{\tilde{\Gamma}}$ of F is a fatgraph G embedded as a spine of F . An explicit formula in terms of Minkowski geometry (which will be given in §3) for the “simplicial coordinates”, gives a positive metric $w_{\tilde{\Gamma}}$ on G .

In the conformal setting, the effective construction maps $G_g^s \rightarrow (M_g^s \times \mathbf{R}_+^s)$; in contrast in the hyperbolic setting, the effective construction (namely, the

convex hull construction) maps $\tilde{M}_g^s \rightarrow G_g^s$ in the opposite direction! Just as the conformal setting has a non-computable inverse $(M_g^s \times \mathbf{R}_+^s) \rightarrow G_g^s$, there is a non-computable (or at least, very difficult to compute) inverse $G_g^s \rightarrow \tilde{M}_g^s$ in the hyperbolic setting. (In fact, the paper [7] is dedicated to the study of exactly these “arithmetic problems”, which may be thought of as the computation of the hyperbolic geometric data from the combinatorial data.)

The conformal and hyperbolic treatments of the cell decomposition of decorated Teichmüller space are thus “inverses” in this sense, and each setting has its difficult theorem: surjectivity of the effective construction. There is no known way to use one such difficult theorem to prove the other. (One can, however, show that there is bounded distortion from identifying simplicial coordinates with the fatgraph metric, but we shall take this up elsewhere.)

Thus, the difficult theorem in decorated Teichmüller theory is that the putative cells $C(\alpha) \subseteq \tilde{\mathcal{T}}_g^s$ are in fact cells. This putative cellularity is proven in [6] (independent of the Jenkins-Strebel theory) by introducing an “energy functional” on a Euclidean space containing $\tilde{\mathcal{T}}_g^s$ and analyzing its gradient flow in order to apply the Poincaré-Hopf Theorem. Technical details of the extension of this theorem to the bordered case are discussed in §3.

There is further structure in the hyperbolic setting (for instance, global coordinates on $\tilde{\mathcal{T}}_g^s$ coming from Minkowski lengths, which has no analogue in the conformal setting), as is discussed in [6] and extended to the bordered case in §3.

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1. Two versions of decorated moduli space.

Having defined in the Introduction the Teichmüller and moduli spaces for $r = 0$, both decorated and classical, this section is dedicated solely to the definitions of our two versions of decorated moduli spaces in the bordered case.

Let us henceforth assume that $r \neq 0$, enumerate the (smooth) boundary components of F as ∂_i , where $i = 1, \dots, r$, and set $\partial = \cup\{\partial_i\}_1^r$.

Let $Hyp(F)$ be the space of all hyperbolic metrics on F with geodesic bound-

ary and define the first version of *moduli space* to be

$$M = M(F) = [Hyp(F) \times (\prod_1^r \partial_i)] / PF,$$

where PF denotes the equivalence relation of push-forward by orientation-preserving diffeomorphism

$$f_*(\Gamma, (\xi_i)_1^r) = (f_*(\Gamma), (f(\xi_i)_1^r)),$$

where $\Gamma \mapsto f_*(\Gamma)$ is the usual push-forward of metric on $Hyp(F)$.

Remark We shall not require in this paper a version of the corresponding Teichmüller space, but comment on it here for completeness. In certain mathematical circles, the “standard” definition of the Teichmüller space $T = T(F)$ of the bordered surface is as follows: Choose a point $\xi_i \in \partial_i$, for each $i = 1, \dots, r$ and let $T(F)$ be the quotient $Hyp(F)$ by push-forward by diffeomorphisms fixing each ξ_i , where the diffeomorphisms are isotopic to the identity with the isotopy likewise required to fix each ξ_i . T can be shown to be homeomorphic to an open ball of real dimension $6g - 6 + 4r + 2s$. PMC acts on T by push-forward with a quotient T/PMC which is non-naturally homeomorphic to the moduli space M just defined. To define a homeomorphism $T/PMC \rightarrow M$ requires using the geometry of Γ to concoct well-defined basepoints in the universal cover of each ∂_i , and it is delicate.

Letting $\ell_i(\Gamma)$ denote the hyperbolic length of ∂_i for $\Gamma \in Hyp(F)$, define the first version of *decorated moduli space* to be

$$\begin{aligned} \tilde{M} &= \tilde{M}(F) \\ &= \{(\Gamma, (\xi_i)_1^r, (t_i)_1^r) : \Gamma \in Hyp(F), \xi_i \in \partial_i, 0 < t_i < \ell_i(\Gamma), i = 1, \dots, r\} / PF, \end{aligned}$$

where PF denotes push-forward by diffeomorphisms on $(\Gamma, (\xi_i)_1^r)$ as before, extended by the trivial action on $(t_i)_1^r$. Thus, a point of \tilde{M} is represented by $\Gamma \in Hyp(F)$ together with a pair of points $\xi_i \neq p_i$ in each ∂_i , where p_i is the point at hyperbolic distance t_i along ∂_i from ξ_i in the orientation on ∂_i^* as a boundary component of F^* . There is one special case, namely $g = 0 = s = r - 2$, so F is an annulus; in this case, we define $\tilde{M}(F)$ to be the collection of all configurations of two distinct labeled points in a circle of some radius.

This completes the definition of the first version, and we turn now to the second version. Begin with a smooth surface F with smooth boundary,

choose one distinguished point $d_i \in \partial_i$ in each boundary component, and set $D = \{d_i\}_1^r$. (We could take $d_i = \xi_i$, for instance, but it would be confusing notation in the sequel.) Define a *quasi hyperbolic metric* on F to be a hyperbolic metric on $F^\times = F - D$ so that each $\partial_i^\times = \partial_i - \{d_i\}$ is totally geodesic, for each $i = 1, \dots, r$, and set $\partial^\times = \cup\{\partial_i^\times\}_1^r$. To explain this, consider a hyperbolic metric on a once-punctured annulus A and the simple geodesic arc a in it asymptotic in both directions to the puncture which separates the two boundary components; the induced metric on a component of $A - a$ gives a model for the structure near a component of ∂^\times ; see Figure 2.

The *decorated Teichmüller space* is the space $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}(F)$ of all quasi hyperbolic metrics on $F - D$, where we furthermore specify for each d_i a segment of a horocycle centered at d_i , modulo push-forward by diffeomorphisms of $F - D$ which are isotopic to the identity; diffeomorphisms of F act trivially on hyperbolic lengths of horocyclic segments by definition. We shall see in Theorem 1 that $\tilde{\mathcal{T}}$ is homeomorphic to an open ball of dimension $6g - 6 + 5r + 2s$. The second version of *decorated moduli space* is $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}(F) = \tilde{\mathcal{T}}(F)/PMC(F)$.

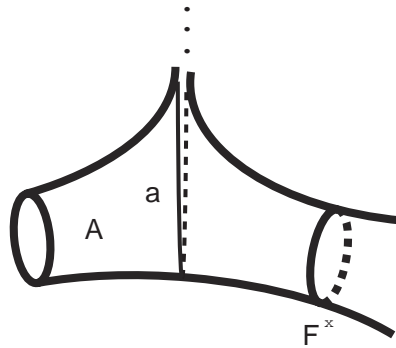


Figure 2 The model for F^\times .

Fix some quasi hyperbolic metric on F . In the homotopy class of ∂_i is a unique separating geodesic $\partial_i^* \subseteq F$. We may excise from $F - \cup\{\partial_i^*\}_1^r$ the components containing points of ∂^\times to produce a surface F^* , which inherits a hyperbolic metric. In the special case of an annulus, the surface F^* collapses to a circle. As a point of notation, $\tilde{\Gamma} \in \tilde{\mathcal{T}}$ has its underlying hyperbolic metric given by a conjugacy class of Fuchsian group Γ for F^* .

2. Arc families.

Define an (*essential*) *arc* in F to be a smooth path a embedded in F whose endpoints lie in D and which meets ∂F transversely, where we demand that a is not isotopic rel endpoints to a path lying in $\partial F - D$. Two arcs are said to be *parallel* if there is an isotopy between them which fixes D pointwise. An *arc family* in F is the isotopy class of a collection of disjointly embedded essential arcs in F , no two of which are parallel.

If α is a collection of arcs representing an arc family in F , we shall say that an embedded arc or curve C in F *meets α efficiently* if there are no bigons in F complementary to $\alpha \cup C$.

Suppose that α is an arc family in F so that each component of $F - \cup\alpha$ is either a polygon or a once-punctured polygon; in this case, we shall say that α *quasi fills* the surface F . In the extreme case that each component is a triangle or a once-punctured monogon, then α is called a *quasi triangulation*.

3. Lambda lengths and simplicial coordinates.

We begin with a global coordinatization of $\tilde{\mathcal{T}}$ and recall that if h_0 and h_1 are two horocycles in the hyperbolic plane, then their *lambda length* is $\sqrt{2 \exp \delta}$, where δ is the signed hyperbolic distance between h_0 and h_1 (and the sign is positive if and only if h_0 and h_1 are disjoint). Via the canonical identification of the open positive light-cone in Minkowski space with the space of all horocycles in the hyperbolic plane (cf. [6;§1]), the square of the lambda length is simply the negative of the Minkowski inner product (cf. [6; Lemma 2.1]).

Theorem 1 *Fix any quasi triangulation τ of F . Then the assignment of lambda lengths defines a real-analytic homeomorphism*

$$\tilde{\mathcal{T}}_{g,r}^s \rightarrow \mathbf{R}_+^{\tau \cup \partial^\times}.$$

Proof As in the proof of Theorem 3.1 of [6], we may choose a triangle complementary to τ in F and a triple of rays in the light-cone in Minkowski space and uniquely realize a triple of putative lambda lengths on a chosen triangle of $F - \tau$ with a triple of points in these rays. One may then uniquely and inductively construct lifts of adjacent triangles to the light-cone realizing the putative lambda lengths in order to produce a tessellation. Finally

(and following Poincaré), one explicitly constructs the underlying Fuchsian group as the group of hyperbolic symmetries of this tessellation, which leaves invariant the corresponding set of horocycles by construction. *q.e.d.*

Side-Remark In fact, one can give a representation of PMC as a group of rational functions acting on lambda lengths as follows. Fix a quasi triangulation τ , and adopt lambda lengths coordinates for $\tilde{\mathcal{T}}$ with respect to τ . If $f \in PMC$, then there is a sequence of “elementary transformations” (i.e., replacing one diagonal of a quadrilateral with the other) which carries $f(\tau)$ to τ . In order to describe the action of f , we must calculate the length of the other diagonal from the one. If e is an arc in a decorated quasi triangulation τ which separates two triangles with respective sides a, b, e and c, d, e and f is the diagonal other than e of the quadrilateral with sides a, b, c, d , then it can be shown [6; Proposition 2.6] that $f = e^{-1}(ac + bd)$, where we have identified an arc with its lambda length for convenience. Such a transformation of lambda lengths is called a “Ptolemy transformation” owing to its kinship with the classical theorem of Ptolemy on Euclidean quadrilaterals which inscribe in a circle. Thus, after a permutation induced by f , the action of f on lambda lengths with respect to τ is given by a composition of Ptolemy transformations. For instance, we calculate the representation of the braid groups in the addendum to [6]. There is also a simple expression for the Weil-Petersson Kähler two-form in lambda lengths [9; Theorem 3.3.6] in the case of surfaces without boundary; the invariance of this expression under Ptolemy transformations, which devolves to a simple calculation, shows that this same expression provides a PMC -invariant two-form on $\tilde{\mathcal{T}}$ and hence a two-form on $\tilde{\mathcal{M}}$ itself. Finally, [6; §6] shows that “centers” of cells, corresponding to setting the lambda lengths identically equal to unity, are uniformized by Fuchsian groups $\Gamma < PSL_2(\mathbf{R})$ that are arithmetic in the sense that there is a representative of the conjugacy class with $\Gamma < PSL_2(\mathbf{Z})$. An analogous statement holds in the bordered case as well, where there is a representative $\Gamma \in PSL_2(\mathbf{R})$ so that each $\gamma \in \Gamma$ lies in $PSL_2(\mathbf{Z})$ *except* those hyperbolics corresponding to boundary geodesics; there is further interesting arithmetic structure associated with these exceptional covering transformations which deserves further study.

For the second coordinatization, recall that if e is an arc in a decorated quasi triangulation τ which separates two triangles with respective sides a, b, e and

c, d, e , then the *simplicial coordinate* of e is

$$E = \frac{a^2 + b^2 - e^2}{abe} + \frac{c^2 + d^2 - e^2}{cde},$$

where we have identified an arc with its lambda length for convenience. In the special case that e bounds a once-punctured monogon, define its simplicial coordinate to vanish; in the special case that $e \in \partial^\times$, it bounds a triangle on only one side, say with edges a, b, e , and we define its simplicial coordinate to be $E = 2 \frac{a^2+b^2-e^2}{abe}$ (and are thus taking the usual simplicial coordinate in the double of F).

Fix a quasi triangulation τ of F , and define the subspace

$$\begin{aligned} \tilde{C}(\tau) &= \{(\vec{y}, \vec{x}) \in \mathbf{R}^{\partial^\times} \times (\mathbf{R}_+ \cup \{0\})^\tau : \text{there are no vanishing cycles or arcs}\} \\ &\subseteq \mathbf{R}^{\tau \cup \partial^\times}, \end{aligned}$$

where the coordinate functions are taken to be the simplicial coordinates (rather than lambda lengths as in Theorem 1). By “no vanishing cycles”, we mean there is no essential simple closed curve $C \subseteq F$ meeting a representative τ efficiently so that

$$0 = \sum_{p \in C \cap \tau} x_p,$$

where $p \in C \cap a$, for $a \in \tau$, contributes to the sum the coordinate x_p of a . By “no vanishing arcs”, we mean there is no essential simple arc $A \subseteq F$ meeting τ efficiently and properly embedded in F with its endpoints disjoint from D so that

$$0 = \sum_{p \in A \cap \partial^\times} y_p + \sum_{p \in A \cap \tau} x_p,$$

where again the x_p and y_p denote the coordinate on a at an intersection point $p = A \cap a$ or $p = C \cap a$ for $a \in \tau$. (There are always two terms in the former sum.) We may think of this as a convex constraint on \vec{y} given \vec{x} .

Fix a quasi triangulation τ of F , and let σ_i denote the triangle in F complementary to τ which contains ∂_i^\times , for $i = 1, \dots, r$. We shall require the following analogue of Lemma 5.2 of [6].

Lemma 2 *Suppose that $(\vec{y}, \vec{x}) \in \tilde{C}(\tau)$. If the strict triangle inequality on the lambda length of ∂_i^\times in σ_i holds, for each $i = 1, \dots, r$, then all three strict triangle inequalities hold on the lambda lengths of any triangle complementary to τ .*

Proof Adopt the notation in the definition of simplicial coordinates for the lambda lengths near an edge e . If $c + d \leq e$, then $c^2 + d^2 - e^2 \leq -2cd$, so the non-negativity of the simplicial coordinate E gives $0 \leq cd[(a - b)^2 - e^2]$, and we find a second edge-triangle pair so that the triangle inequality fails. Define an *arc of triangles* $(t_j)_1^n$ to be a collection of triangles complementary to τ so that $t_j \cap t_{j+1} = e_j$, for each $j = 1, \dots, n-1$, and likewise define a *cycle of triangles* when $t_j \cap t_{j+1} = e_j$, for each $j = 1, \dots, n$, taking the index j to be cyclic (so that $t_{n+1} = t_1$). In either case, if the edges of t_j are $\{e_j, e_{j-1}, b_j\}$, for $j = 1, \dots, n$, then the collection $\{b_j\}_1^n$ is called the *boundary* of the cycle, and the edges $\{e_j\}$ are called the *consecutive edges* of the cycle. It follows that if there is any such triangle t so that the triangle inequalities do fail for t , then there must be a cycle of triangles of such failures or an arc of triangles of such failures whose boundary begins and ends with elements of ∂^\times . The former possibility is untenable since if $e_{j+1} \geq b_j + e_j$, for $j = 1, \dots, n$, where we again identify an arc with its lambda length, then upon summing and canceling like terms, we find $0 \geq \sum_{j=1}^n b_j$, which is absurd since lambda lengths are positive. *q.e.d.*

As discussed in the Introduction, Theorem 5.4 of [6] is our version of the reverse Jenkins-Strebel Theorem in decorated Teichmüller theory (and is proved independently of the usual Jenkins-Strebel Theorem), and it gives a *PMC*-invariant cell decomposition of $\tilde{\mathcal{T}}$. In effect, $\tilde{\Gamma}$ gives rise to a quasi filling arc family $\alpha_{\tilde{\Gamma}}$ via the convex hull construction; fixing the topological type of $\alpha_{\tilde{\Gamma}}$ and varying $\tilde{\Gamma}$ gives a cell in the decomposition of $\tilde{\mathcal{T}}$. The extension of the convex hull construction and its associated cell decomposition to bordered surfaces is given by

Theorem 3 *There is a real-analytic homeomorphism of the decorated Teichmüller space $\tilde{\mathcal{T}}$ of F with $[\bigcup_\tau \tilde{C}(\tau)] / \sim$, where $(\tau_1, \vec{y}_1, \vec{x}_1) \sim (\tau_2, \vec{y}_2, \vec{x}_2)$ if $\vec{y}_1 = \vec{y}_2$ and \vec{x}_1 agrees with \vec{x}_2 on $\tau_1 - \{a \in \tau_1 : x^1 = 0\} = \tau_2 - \{a \in \tau_2 : x^2 = 0\}$, where x^j denotes the \vec{x} coordinate on a , for $j = 1, 2$. Indeed, a point $\tilde{\Gamma} \in \tilde{\mathcal{T}}$ gives rise to the quasi filling arc family $\alpha_{\tilde{\Gamma}}$ via the convex hull construction as well as a tuple of simplicial coordinates $(\vec{y}, \vec{x}) \in \tilde{C}(\tau)$ for any quasi triangulation $\tau \supseteq \alpha_{\tilde{\Gamma}}$, where \vec{x} vanishes on $\tau - \alpha_{\tilde{\Gamma}}$.*

Proof The proof closely follows that of Theorem 5.4 of [6] in the double \mathcal{F} of F , where one takes the convex hull in Minkowski space of the set of all horocycles in F to produce an invariant convex body.

A more technical discussion of the extension to our present situation is as

follows. As in Theorem 5.4, the argument involves an “energy functional”, which is defined exactly as in the proof of Theorem 5.4 (p. 322) but on the double \mathcal{F} . From the very definition, notice that there is no new contribution to the functional for the edges lying in ∂^\times since the coupling equations automatically hold for these edges in the double; there is therefore no need for further computational elaboration beyond the cases considered in [6; Cases 1-8, pp. 324-327]. Furthermore, the cycle of triangles argument in Claim 1 (p. 322) again extends to a cycle or arc of triangles argument (just as in Lemma 2). The remaining proof of Theorem 5.4 now holds verbatim. Consider a face of the convex body corresponding to a triangle complementary to a quasi triangulation; if this triangle contains points of ∂^\times , then it is not necessarily the case that the support plane of this face is elliptic, but any other support plane is either elliptic as in [6; p. 320] or parabolic as in [6; p. 336]. *q.e.d.*

In particular, for a “generic” point $\tilde{\Gamma} \in \tilde{\mathcal{T}}$, the arc family $\alpha_{\tilde{\Gamma}}$ arising from the convex hull construction is a quasi triangulation.

To close this section, we recall an elementary fact which is useful in §6, where we study certain asymptotic problems, and to formulate this fact, we recall the notion of “h-length”. Consider a decorated ideal triangle with lambda lengths of consecutive edges given by a, b, e (where we use these same symbols also for the edges themselves). The geodesics a, b cut off a finite horocyclic segment from the horocycle opposite e , and a calculation [6; Proposition 2.8] shows that half the horocyclic length, or *h-length* of this segment is given by e/ab . Notice that the simplicial coordinate of an edge is by definition a six-term linear combination of the h-lengths near the edge.

Given a cycle of triangles in an ideal triangulation of F , each arc in its boundary is opposite a well defined horocyclic segment, called an “included” horocyclic segment.

Lemma 4 *Given any cycle of triangles, the sum of the simplicial coordinates of the consecutive edges is twice the sum of the included h-lengths.*

Proof Adding the six-term linear relations for consecutive edges in a cycle of triangles, the formula follows from elementary cancellation. *q.e.d.*

4. Equidistant points to horocycles.

In [7], we studied equidistant points to horocycles in the hyperbolic plane and next recall the results of attendant elementary and explicit calculation.

Lemma 5 *Given three horocycles h_0, h_1, h_2 with distinct centers, let λ_j denote the lambda length of h_k and h_ℓ , where $\{j, k, \ell\} = \{0, 1, 2\}$.*

a) [7; Proposition 2.3] *There is a point ζ in the hyperbolic plane which is equidistant from h_0, h_1 , and h_2 if and only if $\lambda_0, \lambda_1, \lambda_2$ satisfy all three possible strict triangle inequalities; in this case, ζ is unique, and fixing the centers and varying only the decorations, all points of the hyperbolic plane arise. Finally, the exponential ρ of the common hyperbolic distance from ζ to h_0, h_1 , or h_2 is given by*

$$\rho^2 = \frac{2\lambda_0^2\lambda_1^2\lambda_2^2}{(\lambda_0 + \lambda_1 + \lambda_2)(\lambda_0 + \lambda_1 - \lambda_2)(\lambda_0 + \lambda_2 - \lambda_1)(\lambda_1 + \lambda_2 - \lambda_0)}.$$

b) [7; Proposition 2.5] *If σ is the geodesic connecting the centers of h_k and h_ℓ , then the signed hyperbolic length of the horocyclic segment between $\sigma \cap h_k$ and the central projection of ζ to h_k is given by*

$$\frac{\lambda_k^2 + \lambda_\ell^2 - \lambda_j^2}{4\lambda_j\lambda_k\lambda_\ell},$$

where the sign is positive if and only if σ does not separate ζ from the center of h_j . In particular, if σ does separate ζ from the center of h_j , then $\lambda_j^2 > \lambda_k^2 + \lambda_\ell^2$.

c) *Suppose that e is a diagonal of a decorated quadrilateral where the lambda lengths satisfy all three strict triangle inequalities on each triangle complementary to e , and let ζ, ζ' denote the corresponding equidistant points from part a). Choose an endpoint of e and centrally project ζ, ζ' to the horocycle centered at this endpoint. The simplicial coordinate of e vanishes if and only if these projections coincide.*

Proof The reader is referred to [7] for the computational proofs of parts a) and b). Part c) follows directly from part b) and the definition of simplicial coordinates as in [7]. *q. e. d.*

In order to guarantee the existence of equidistant points and apply Lemma 5, we must pass to a strong deformation retract $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}(F) \subseteq \widetilde{\mathcal{M}}$ of $\widetilde{\mathcal{M}}$. The

subspace $\widehat{\mathcal{M}}$ is most easily defined as the *PMC*-quotient of another space

$$\widehat{\mathcal{T}} = \widehat{\mathcal{T}}(F) = [\bigcup_{\tau} \widehat{C}(\tau)] / \sim,$$

where \sim is as in Theorem 3, and for membership in $\widehat{C}(\tau)$ we demand not only that there are no vanishing cycles or arcs, but we also require that for any triangle $t \subseteq F$ complementary to τ , the lambda lengths on the edges of t satisfy all three possible strict triangle inequalities. This defines the subspace $\widehat{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}$.

Lemma 6 $\widehat{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}$ is a strong deformation retraction.

Proof Again consider the triangle σ_i containing ∂_i^\times which is complementary to some quasi triangulation τ . If the triangle inequality on lambda lengths fails for ∂_i^\times in σ_i , then we may simply decrease the lambda length of ∂_i^\times in order to ensure that the strict triangle inequality holds on the resulting lambda length of ∂_i^\times in σ_i , for each $i = 1, \dots, r$. According to Lemma 2, there can then be no failure of strict triangle inequality on the lambda lengths for any triangle complementary to τ . This homotopy of $\widetilde{\mathcal{T}}$ to $\widehat{\mathcal{T}}$ descends to give the asserted strong deformation retraction of $\widetilde{\mathcal{M}}$ to $\widehat{\mathcal{M}}$. *q.e.d.*

We close this section with a lemma which will be useful in §6.

Lemma 7 Suppose α is an arc family arising from the convex hull construction for some point of $\widehat{\mathcal{M}}$ and an arc in α has corresponding lambda length e and simplicial coordinate E . Then $eE \leq 4$.

Proof Observe that in the notation of the definition of simplicial coordinate, we have

$$eE = (a^2 + b^2 - e^2)/ab + (c^2 + d^2 - e^2)/cd.$$

Since the underlying decorated hyperbolic structure lies in $\widehat{\mathcal{M}}$, each triple a, b, e and c, d, e satisfy all three possible triangle inequalities. By the Euclidean law of cosines, the righthand side of the previous equation can be interpreted as twice the sum of two cosines, and is therefore at most four. *q.e.d.*

5. Identification of the two versions.

Given a generic point $\tilde{\Gamma} \in \hat{\mathcal{T}}$, let $\tau = \alpha_{\tilde{\Gamma}}$ denote the quasi triangulation arising from the convex hull construction. We may take τ to consist entirely of Γ -geodesics.

Insofar as τ is a quasi triangulation, ∂_i^\times lies in the frontier of an ideal triangle (which was called σ_i before) of $F - \tau$, and we may choose a lift t_0 of this ideal triangle to the universal cover U of F^\times . U is a proper subset of the hyperbolic plane which is bounded by lifts of the various ∂_i^\times . Consider the orbit of t_0 under the primitive hyperbolic covering transformation γ corresponding to ∂_i^* whose axis G meets t_0 , and adopt the notation illustrated in Figure 3 for the edges and vertices in $\{\gamma^j(t_0)\}_{-\infty}^\infty$: the vertices of t_0 are u_0, u_1, v_0 , the edge c_0 of t_0 covers ∂_i^\times with endpoints u_0, u_1 , the remaining edges of t_0 are a_0 which has endpoints u_0, v_0 and b_0 which has endpoints u_1, v_0 , and $z_j = \gamma^j(z_0)$, for $z = u, v, a, b, c$ and $j \in \mathbf{Z}$.

Notice that at each such vertex there is a well-defined horocycle derived from the decoration. Since ∂_i^* inherits an orientation from that of F , $\gamma \in \{\gamma^{\pm 1}\}$ can be well-defined, and we suppose that a_1 separates t_0 from the attracting fixed point at infinity of γ .

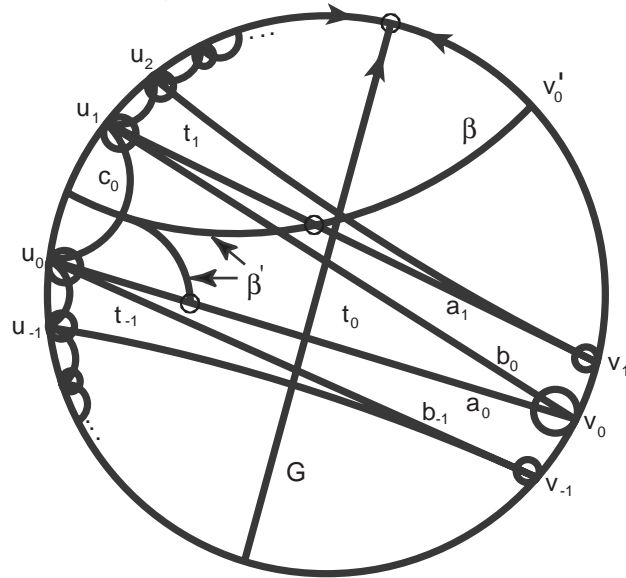


Figure 3 The orbit of t_0 .

The horocycles h_0, h_1, k_0 centered respectively at u_0, u_1, v_0 admit a unique

equidistant point ζ according to part a) of Lemma 4 since $\tilde{\Gamma} \in \widehat{\mathcal{T}}$. Since the horocycles h_0 and h_1 both cover the same horocyclic arc, there is a curve β in the hyperbolic plane of possible points equidistant to them, and β is simply the perpendicular bisector of c_0 ; that is, $\beta \cap c_0$ is equidistant to the horocycles h_0 and h_1 , and β is perpendicular to c_0 ; β is furthermore asymptotic to $v'_0 = \delta v_0$, for some $\delta \in \Gamma$. (The arc β' will be explained later.)

We may define a projection

$$\pi : U \rightarrow \partial_i^*$$

as follows: In each t_j , π is induced by central projection from v_j , and on the component of $U - \cup\{t_j\}_{-\infty}^{\infty}$ lying between t_{k-1} and t_k , π is induced by central projection from u_k ; these combine to give a continuous surjection $\pi = \pi_\tau$ which will be useful later.

For any geodesic a in the hyperbolic plane which is disjoint from the interior of t_0 , let $H(a)$ denote the half-plane of a containing t_0 .

Lemma 8 *In the notation above, $\zeta \in H(a_1) \cap H(b_{-1})$.*

Proof Suppose for instance that $\zeta \notin H(a_1)$. By part b) of Lemma 5, the triangle inequality on squares of lambda lengths fails on the edges of t_0 , and by part c) of Lemma 5, each of the equidistant points of the triangles of $U - \tau$ lying in $H(a_1)$ which contain u_1 as vertex also must lie in the complement of $H(a_1)$. There is thus a cycle of triangles of such failures, and the argument of Lemma 2 again applies to derive a contradiction. (Thus, whereas we apply the logic of Lemma 5.2 in [6] to the lambda lengths themselves in Lemma 2, here we apply part of this logic to the squares of the lambda lengths.) The argument for $H(b_{-1})$ is analogous. *q.e.d.*

Let f_i denote the Dehn twist along ∂_i^* with induced action \tilde{f}_i on $\widehat{\mathcal{T}}$, for $i = 1, 2, \dots, r$. Pushing forward by f_i has the effect of moving v'_0 to $\delta v_{\pm 1}$. Thus, in the \mathbf{Z} -orbit $\tilde{f}_i^j(\tilde{\Gamma})$ generated by this Dehn twist, there is some least j , call it J , so that v'_0 lies in the complement of $H(a_1)$.

Passing now to the *PMC*-orbit of $\tilde{\Gamma}$, we may replace $\tilde{\Gamma}$ by $\tilde{f}_J(\tilde{\Gamma})$ as these represent the same point of $\widehat{\mathcal{M}}$.

The universal cover $\tilde{\mathcal{F}}$ of the double \mathcal{F} of F is obtained by gluing together copies of U along the lifts of the various ∂_i^\times and may be identified with the hyperbolic plane. Let ω denote the Möbius transformation which interchanges u_0 and u_1 and maps $\tilde{\iota}(v_0)$ to v_0 , where $\tilde{\iota}$ denotes reflection in c_0 ;

that is, $\tilde{\iota}$ is the lift of the canonical involution $\iota : \mathcal{F} \rightarrow \mathcal{F}$ which setwise fixes c_0 .

Notice that if $\zeta \notin H(c_0)$, then $\omega(\zeta) \in H(a_0) \cap H(b_0)$ by part b) of Lemma 5 since the simultaneous failure of two triangle inequalities (one weak and one strict inequality on the squares of lambda lengths) among positive numbers is absurd. Thus, $\zeta \notin H(c_0)$ implies that $\omega(\zeta) \in t_0$.

In any case, $\zeta' \in U \cap \{\zeta, \omega(\zeta)\}$ is a well-defined point in U which lies on the piecewise-smooth arc

$$\beta' = [\beta \cap H(a_1) \cap H(c_0)] \cup [\omega(\beta \cap \tilde{\iota}(t_0))]$$

illustrated in Figure 3. In fact, ζ' lies in the interior of β' by Lemma 8 and the previous paragraph.

Define

$$\begin{aligned} A_i &= A_i(\tilde{\Gamma}) = \pi_\tau(a_1 \cap \beta') = \pi_\tau(a_0 \cap \beta'), \\ B_i &= B_i(\tilde{\Gamma}) = \pi_\tau(\zeta'), \\ C_i &= C_i(\Gamma) = \pi_\tau(\beta \cap G), \end{aligned}$$

where G is the axis of γ , i.e., G is the lift of ∂_i^* depicted in Figure 3.

Lemma 9 *B_i is a well-defined point in $\partial_i^* - \{A_i\}$, for $i = 1, \dots, r$, and $C_i \in \partial_i^*$ is well-defined independent of decoration, where τ is any completion to an ideal triangulation of the arc family $\alpha_{\tilde{\Gamma}}$ associated to $\tilde{\Gamma}$ via the convex hull construction.*

Proof Suppose that the convex hull construction assigns an arc family $\alpha = \alpha_{\tilde{\Gamma}}$ to $\tilde{\Gamma}$ which is not a quasi triangulation. Complete α in any manner to a quasi triangulation τ , and extend simplicial coordinates by setting them to zero on the arcs of $\tau - \alpha$. According to part c) of Lemma 5, the vanishing of simplicial coordinates precisely guarantees the independence of B_i from the choice of τ . C_i is well-defined independent of this choice by construction and of the decoration since the horocycles h_0, h_1 cover a common horocycle in the surface. *q.e.d.*

Notice that if all three horocycles centered at the vertices of t_0 cover the same horocyclic arc in F , for instance when $r = 1$, then as we vary the length of this horocyclic arc, the equidistant point ζ does not change. On the other hand, when we vary the length of the horocyclic arc corresponding to the common horocycles at u_0 and u_1 , the distance to the equidistant point varies continuously in any case.

Define δ_i to be the signed distance from ζ' to h_0 , where the sign is positive if and only if h_0 and h_1 are disjoint; the exponential e^{δ_i} of this distance δ_i is expressed in lambda lengths in part a) of Lemma 5. Furthermore, δ_i gives an abstract coordinate on β' in any case for $\delta_i^- < \delta_i < \delta_i^+$, where δ_i^+ corresponds to $\zeta^+ = \beta' \cap a_1$ and δ_i^- corresponds to $\zeta^- = \beta' \cap a_0$. (One can in fact compute δ_i^\pm in terms of lambda lengths and the entries of γ , but the formula is complicated and unnecessary.)

Define the “oriented distance” $d_o(X, Y)$ between two distinct points $X, Y \in \partial_i^*$ to be the distance along ∂_i^* from X to Y in the orientation as a boundary component of F^* . Notice that $d_o(X, Y) + d_o(Y, X) = \ell_i$ for any distinct $X, Y \in \partial_i^*$. We shall also say that the point $X \in \partial_i^*$ is at “signed oriented distance” d from $Y \in \partial_i^*$, if $d_o(Y, X) = d$ or equivalently $d_o(X, Y) = \ell_i - d$.

Define $\xi_i = C_i \in \partial_i^*$; we shall define p_i by specifying its signed oriented distance from ξ_i . Define $\ell_i^+ = d_o(A, B)$ and $\ell_i^- = -d_o(B, A)$, so $\ell_i = \ell_i^+ - \ell_i^-$, and let $f_i : [\delta_i^-, \delta_i^+] \rightarrow [\ell_i^-, \ell_i^+]$ be the orientation-preserving affine homeomorphism. Define $p_i \in \partial_i^*$ to be the point in ∂_i^* at signed oriented distance from $\xi_i = C_i$ given by

$$g_i(\delta_i) = (\delta_i^+ - \delta_i^-)f_i(\delta_i).$$

Theorem 10 *The mapping $F \mapsto F^*$ together with the assignment of points $p_i, \xi_i \in \partial_i^*$, for $i = 1, 2, \dots, r$, gives a well-defined real-analytic homeomorphism*

$$\Psi : \widehat{\mathcal{M}} \rightarrow \widetilde{M}$$

provided F is not the annulus $F_{0,2}^0$.

Proof We have already proven that the mapping $\Psi : \widehat{\mathcal{M}} \rightarrow \widetilde{M}$ is well-defined, and it is obviously real-analytic. We must still show that Ψ is a bijection.

To this end, first notice that by construction, a change in only the hyperbolic length of the horocyclic arc at d_i leaves ξ_i invariant, moves p_i monotonically relative to ξ_i , and leaves invariant each p_j, ξ_j , for $i \neq j = 1, \dots, r$. Furthermore, the distance of p_i from any fixed point of ∂_i^* varies at a uniform rate $\ell_i = \ell_i^+ - \ell_i^-$ as a function of δ_i .

In contrast, the Fenchel-Nielsen deformation [13] along ∂_i^* moves the endpoint of β in $H(c_0)$ monotonically along the circle at infinity by definition,

so $\xi_i = C_i$ moves monotonically along ∂_i^* in the deformation parameter by construction. Thus, the Fenchel-Nielsen deformation monotonically varies ξ_i along ∂_i^* while fixing each p_j, ξ_j for $j \neq i$; p_i also moves under this Fenchel-Nielsen deformation, and there is a compensatory deformation of horocyclic segment at d_i so that p_i and ξ_i both move keeping $d_o(\xi_i, p_i)$ invariant.

As to surjectivity, first observe that the composition $\widehat{\mathcal{M}} \xrightarrow{\Psi} \tilde{M} \rightarrow M$ is surjective for $F \neq F_{0,2}^0$ since given a hyperbolic metric, we may always adjoin a unique hyperbolic surface of type $F_{0,1}^{0,1}$ to get a quasi hyperbolic metric on F lying in $\widehat{\mathcal{M}}$. Since the compensated Fenchel-Nielsen deformations along ∂_i^* attain all possible $\xi_i \in \partial_i^*$, it remains only to alter the horocyclic segment at d_i to vary p_i .

To see that Ψ is injective, again use that there is a unique hyperbolic surface of type $F_{0,1}^{0,1}$ to conclude that if $\Psi(\tilde{\Gamma}_1) = \Psi(\tilde{\Gamma}_2)$, then the quasi hyperbolic metrics underlying $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ must agree up to a Fenchel-Nielsen deformation on ∂_i^* . Again, ξ_i is monotone in the deformation parameter, and $d_o(\xi_i, p_i)$ is monotone in the size of the horocycle by construction, completing the proof of injectivity. *q.e.d.*

Notice that for each $i = 1, \dots, r$, scaling the lambda length of each edge $e \in \tau \cup \partial_i^*$ by a factor $t \in \mathbf{R}$ raised to the power of the number of ends of e asymptotic to d_i fixes ξ_i , moves p_i monotonically around ∂_i^* , and leaves invariant the underlying hyperbolic metric as well as leaving invariant each p_j, ξ_j , for $i \neq j = 1, \dots, r$. In particular, overall scaling of all the lambda lengths fixes the hyperbolic metric and each ξ_i and moves each p_i at uniform speed, i.e., $\frac{d}{d \ell n t} g_i(\delta_i + \ell n t) = \ell_i$.

6. Circle actions and the arc complex.

We first discuss circle actions on and quotients of \tilde{M} and M which will be required in the sequel. There is a natural \mathbf{R}_+^r -action on \tilde{M} , where $(t_i)_1^r \in \mathbf{R}_+^r$ replaces the vector $(\ell_i)_1^r$ of hyperbolic lengths of boundary components by $(t_i \ell_i)_1^r$. There is a corresponding diagonal \mathbf{R}_+ -action on \tilde{M} , and we shall let \tilde{M}/\mathbf{R}_+ denote the quotient; this action descends to M , and we shall also let M/\mathbf{R}_+ denote the quotient.

There is also a natural $(S^1)^r$ -action on \tilde{M} , where the i^{th} factor S^1 moves only p_i and ξ_i uniformly along ∂_i^* at a speed given by the hyperbolic length ℓ_i of ∂_i^* . This action is *not* fixed-point free, and the quotient $\tilde{M}/(S^1)^r$ is homotopy

equivalent to the usual moduli space M_g^{r+s} of the unbordered surface $F_{g,0}^{r+s}$. On the other hand, there is a subgroup $(S^1)^{r-1}$ which preserves the relative positions of pairs of ξ_i which evidently does act without fixed-points. This action descends to a well-defined action of $(S^1)^r$ on M itself, where the i^{th} factor uniformly moves only ξ_i .

Now, let us inductively build a simplicial complex $Arc'(F)$, where there is one p -simplex $\sigma(\alpha)$ for each arc family α in F of cardinality $p+1$. The simplicial structure of $\sigma(\alpha)$ is the natural one, where faces of $\sigma(\alpha)$ correspond to sub arc families of α . We begin with a vertex in $Arc'(F)$ for each isotopy class of essential arc in F to define the 0-skeleton. Having thus inductively constructed the $(p-1)$ -skeleton of $Arc'(F)$, for $p \geq 1$, let us adjoin a p -simplex for each arc family α consisting of $(p+1)$ essential arcs, where we identify the proper faces of $\sigma(\alpha)$ with simplices in the $(p-1)$ -skeleton in the natural way. Identifying the open standard p -simplex with the collection of all real-projective $(p+1)$ -tuples of positive reals assigned to the vertices, $Arc'(F)$ is identified with the collection of all arc families in F together with a real-projective weighting of non-negative real numbers, one such number assigned to each component of α .

$PMC(F)$ acts on $Arc'(F)$ in the natural way, and we define the *arc complex* of F to be the quotient

$$Arc(F) = Arc'(F)/PMC(F).$$

If α is an arc family in F with corresponding simplex $\sigma(\alpha)$ in $Arc'(F)$, then we shall let $[\alpha]$ denote the $PMC(F)$ -orbit of α and $\sigma[\alpha]$ denote the quotient of $\sigma(\alpha)$ in $Arc(F)$.

(The sphericity conjecture from [8] that $Arc(F)$ is piecewise-linearly homeomorphic to a sphere has recently been disproven. Furthermore, [3] introduces and studies a new topological operad whose underlying spaces are homeomorphic to open subspaces of $Arc(F)$.)

For each $i = 1, \dots, r$, there is a natural S^1 -action on $Arc(F)$ itself as follows. Suppose that the projective class of a positive weight w on α represents a point of $Arc(F)$ for some quasi-filling arc family α . Imagine fattening each component arc α to a band whose width is given by the weight of w assigned to the component; thus, a positively weighted arc family is regarded as a collection of bands of positive widths running between the boundary components, and conversely. Certain of the arcs in α meet some ∂_i at the

point d_i (or rather, are asymptotic to d_i in the surface F^\times with totally geodesic boundary); let a_1, \dots, a_k denote this set of arcs meeting d_i with corresponding weights w_1, \dots, w_k . Now, given $t \in S^1 = I/(0 \sim 1)$, alter the family of bands corresponding to $\{a_i\}_1^k$ in a neighborhood of ∂_i^\times as follows: twist a total width $t \sum_1^k w_i$ of the bands to the right around the boundary as illustrated in Figure 4. The resulting bands then determine a weighted arc family, and this gives $Arc(F)$ the structure of an $(S^1)^r$ -space.

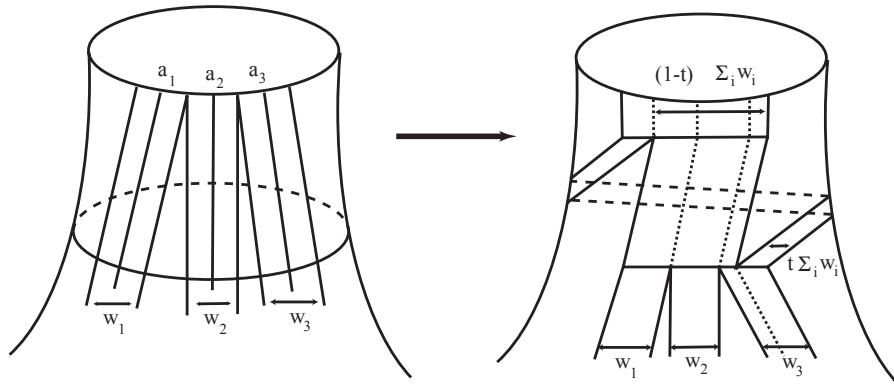


Figure 4 Circle action on Arc .

A subspace of $Arc(F)$ of special interest to us here and elsewhere in the general theory is

$$Arc_{\#}(F) = \{\sigma[\alpha] : \alpha \text{ quasi - fills } F\}.$$

The $(S^1)^r$ -action on $Arc(F)$ preserves $Arc_{\#}(F)$. It is useful to have a “deprojectivized” $Arc_{\#}(F)$, and we define $dArc_{\#}(F) = Arc_{\#}(F) \times \mathbf{R}_+$ which we identify with all positive-real weightings on the components of an arc family which quasi fills F .

For each quasi triangulation τ , define

$$\tilde{D}(\tau) = \{\vec{x} : \text{there are no vanishing cycles for } \vec{x}\},$$

and notice that by definition

$$dArc_{\#}(F) = \left(\left[\bigcup_{\tau} \tilde{D}(\tau) \right] / \sim \right) / PMC(F),$$

where the equivalence relation is as in Theorem 3.

Lemma 13 *The natural mapping $q : \widehat{\mathcal{M}} \rightarrow dArc_{\#}(F)$ induced by projection $(\vec{y}, \vec{x}) \mapsto \vec{x}$ is a homotopy equivalence. The fibers of q are homeomorphic to the interior of a cone over a finite-sided convex polyhedron.*

Proof Let $\vec{0}$ denote the r -dimensional vector consisting entirely of entries zero, so if $\vec{x} \in \tilde{D}(\tau)$, then $(\vec{0}, \vec{x}) \in \tilde{C}(\tau)$. As noted before Lemma 2, $q^{-1}(\vec{x})$ is convex and hence strong deformation retracts to $(\vec{0}, \vec{x})$. *q.e.d.*

Remark Forgetting the simplicial coordinates on ∂^{\times} is a violent operation: Fixing the simplicial coordinates \vec{x} on each arc in some quasi triangulation τ and altering only the simplicial coordinates \vec{y} of arcs in ∂^{\times} changes the underlying hyperbolic metric in an extremely complicated and non-computable way (cf. [7]). One characteristic shared by $[\vec{y}_1, \vec{x}], [\vec{y}_2, \vec{x}] \in \tilde{C}(\tau)$ is as follows: For any essential simple closed curve C in F , we may assume that C meets τ efficiently and consider the sum over $z \in C \cap \tau$ (with multiplicity and without sign) of the simplicial coordinate of \vec{x} on the arcs meeting C at z , which is a kind of length of C (as seen from the horocycles) as in [7; Lemma 1.2]. For each curve C , these lengths coincide for $[\vec{y}_1, \vec{x}]$ and $[\vec{y}_2, \vec{x}]$. A similar combinatorics in [1] captures the hyperbolic lengths of geodesics.

Theorem 14 *For any bordered surface $F \neq F_{0,(2)}^0$, $Arc_{\#}(F)$ is proper homotopy equivalent to $M(F)/\mathbf{R}_+$ as $(S^1)^r$ -spaces.*

Proof According to Lemma 13, the fiber of $q : \widehat{\mathcal{M}} \rightarrow dArc_{\#}$ is the interior of a cone over a finite-sided convex polyhedron, and the fiber of the induced map $\widehat{\mathcal{M}}/\mathbf{R}_+ \rightarrow Arc_{\#}$ is an open finite-sided convex polyhedron. In particular, $Arc_{\#}, dArc_{\#}, \widehat{\mathcal{M}}, \widehat{\mathcal{M}}/\mathbf{R}_+$ all have the same homotopy type. Likewise, $M, M/\mathbf{R}_+, \tilde{M}, \tilde{M}/\mathbf{R}_+$ all have the same homotopy type. Using the homeomorphism $\Psi : \widehat{\mathcal{M}} \rightarrow \tilde{M}$ of Theorem 10, it follows that $Arc_{\#}(F)$ is indeed homotopy equivalent to $M(F)/\mathbf{R}_+$. Notice that these two spaces furthermore have the same dimension.

To explicitly describe the map $Arc_{\#}(F) \rightarrow M/\mathbf{R}_+$, given a projective weight w on an arc family α representing a point of $\sigma[\alpha]$, regard w as projective simplicial coordinates on $\alpha \cup \partial^{\times}$, where the simplicial coordinate on each component of ∂^{\times} is taken to be zero. This point of $\widehat{\mathcal{M}}$ gives rise via the construction in the proof of Theorem 8 to a point of \tilde{M} ; we finally forget the decoration and projectivize the hyperbolic lengths of the boundary geodesics

to describe a point of M/\mathbf{R}_+ . The standard Fenchel-Nielsen deformations about the boundary curves act on each of the spaces $\widehat{\mathcal{M}}, \widetilde{M}, M/\mathbf{R}_+$; taking the point ξ_i as the initial point of an arc family meeting ∂_i^* for each i identifies the corresponding $(S^1)^r$ -action on $Arc_{\#}$ with the action described before on $Arc(F)$. Thus, the homotopy equivalence between $Arc_{\#}$ and M/\mathbf{R}_+ is indeed a map of $(S^1)^r$ -spaces.

It remains only to prove properness. To this end, since we mod out by the homothetic \mathbf{R}_+ -action on $Arc_{\#} = Arc_{\#}(F)$, we may and shall assume that all lambda lengths are bounded below, or in other words by homogeneity, that all simplicial coordinates are bounded above. In the same manner, since we mod out by the \mathbf{R}_+ -action on $M = M(F)$, we may and shall assume that all hyperbolic lengths of geodesic boundaries are likewise bounded above.

To prove properness, first imagine a path tending to infinity in M/\mathbf{R}_+ . As is well-known, there must either be an essential and non-boundary parallel curve g in F whose hyperbolic length is tending to zero, or the hyperbolic length of some geodesic boundary component is tending to zero. In the former case, the lambda lengths of all arcs meeting g must tend to infinity. It follows from Lemma 7, that the simplicial coordinates must tend to zero; that is, there is a vanishing cycle, so the corresponding path also tends to infinity in $Arc_{\#}$.

In the latter case, re-consider Figure 3. Not drawn in the figure are the ideal polygons complementary to $\cup\{t_i\}_{-\infty}^{+\infty}$. There is a unique such polygon with vertex u_i , for each i , and these polygons are generically triangulated in the special manner where each edge has u_i among its two endpoints. To fix a particular lift, consider the triangulated polygon P with vertex u_1 . The hyperbolic transformation γ discussed before with axis G can be described as follows: it is the composition $\gamma = \sigma \circ \tau$ of two parabolic transformations, where σ is the parabolic fixing v_0 mapping a_0 to b_0 , and τ is the parabolic with fixed point u_1 which maps b_0 to a_1 . Indeed, $\sigma \circ \tau$ maps $u_{-1} \mapsto u_1$ and $v_0 \mapsto v_1$ by definition. Since the lambda lengths on the edges of t_0 agree with those of t_1 (and lambda lengths are Möbius-invariant), it follows that also $u_1 \mapsto u_2$; this uniquely determines $\sigma \circ \tau$, which is the unique Möbius transformation γ preserving the triangulation and correctly mapping t_0 to t_1 .

The number of sides of P is uniformly bounded in terms of only the topology of F ; we shall refer to this fact as “bounded combinatorics”. Thus, the trace of γ is bounded away from ± 2 provided also the h -lengths are bounded away

from zero. In other words, if the hyperbolic length of a geodesic boundary component tends to zero, then a cycle of h-lengths must tend to zero, i.e., the corresponding path again tends to infinity in $Arc_{\#}$.

In the other direction, consider a path tending to infinity in $Arc_{\#}$, so there is some cycle C of triangles with vanishing simplicial coordinates. Suppose first that the cycle c in F dual to C is boundary parallel; if c is parallel to ∂_i , all the simplicial coordinates along C vanish, so by Lemma 4, all of the included h-lengths likewise vanish. Again by bounded combinatorics, the trace of γ tends to ± 2 , so the corresponding path also tends to infinity in M/\mathbf{R}_+ . When c is not boundary parallel, the trace of a matrix representing c likewise tends to ± 2 , so in any case, the corresponding path also tends to infinity in M/\mathbf{R}_+ . *q.e.d.*

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