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K Energy and K Stability on Hypersurfaces

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1. Introduction.

Suppose that M is a compact Fano manifold. That is, M is a compact Kähler manifold with positive first Chern class. One of the most important problems in Kähler geometry is the existence of Kähler metrics of constant scalar curvature. It is believed that the problem is related to certain notion of stability in the sense of Geometric Invariant Theory.

In Tian [17] and Donaldson [4], the notion of K stability was introduced. In the first three sections of this paper, we use the notations in [17] to derive our theorems. In the last section, we discuss the definition of [4] and some observations motivated by that paper.

Let M be a Fano manifold that is embedded in CP^n by the k-th power of the anticanonical line bundle, where k is a positive integer. Let $\sigma(t)$ be a one parameter family of automorphisms of CP^n. We write

$$\sigma(t)[Z_0, \cdots, Z_n] = [t^{\lambda_0} Z_0, \cdots, t^{\lambda_n} Z_n]$$

for integers $\lambda_0, \cdots, \lambda_n$ with $\sum \lambda_i = 0$. Then we can define a family of metrics $\omega_t = \sigma(t)^* \omega_{FS}$ on M such that $\alpha\omega_t \in c_1(M)$, where α is a rational number. Let $\mathcal{M}(\omega, \omega_t)$ be the K energy with respect to the metric $\alpha\omega$ and $\alpha\omega_t$ (for the definition of the K energy, see next section). It is known that

$$\lim_{t\to 0} t \frac{d}{dt} \mathcal{M}(\omega, \omega_t) = A \tag{1.1}$$

exists [17]. If $\mathcal{M}(\omega, \omega_t)$ has a lower bound, then $A \leq 0$. Since the one parameter family of automorphisms $\sigma(t)$ is generated by the holomorphic vector field $X = \sum \lambda_i Z_i \frac{\partial}{\partial Z_i}$, we come up with the following definition [17]:

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Definition 1.1. *We say that M is K stable if for any holomorphic vector field X on CP^n with $\lambda_0, \cdots, \lambda_n$ integers and $\lambda_0^2 + \cdots + \lambda_n^2 \neq 0$,*

$$\lim_{t\to 0} t \frac{d}{dt} \mathcal{M}(\omega, \omega_t) < 0.$$

If the above quantity is nonpositive for all vectors X on CP^n, we say M is K semistable.

The general setting which relates the K energy and the Futaki invariant (in the case of hypersurface) is as follows: Let M be a hypersurface of CP^n. Let X be a vector field of CP^n. Suppose M is defined by a polynomial $F = 0$ and let $F_t = \sigma(-t)^* F$. The degeneration of M by X is defined as the hypersurface in $\mathbb{C} \times CP^n$ by $G(t, Z) = F_t(Z) = 0$. The central fiber of the degeneration is defined as the intersection of the degeneration with the set $\{0\} \times CP^n$, excluding the factor $t = 0$.

In [2] or [17], the quantity A in (1.1) is represented as the (real part) of the (generalized) Futaki invariant of the central fiber if the central fiber is a normal variety. It is not hard to see that the exact same proof can go through if we assume that the central fiber does not have multiplicity greater than 1. In particular, we can define the Futaki invariant on algebraic cycles with multiplicity 1 pretty much the same way as in the smooth case.

Remark 1.1. For the sake of simplicity, we don't distinguish the notations of K stability and K semi-stability in this paper. K stable in this paper means either K stable or K semi-stable. On the other side, for the applications in Geometric Invariant Theory, we just need to assume that t is a real number and $\lambda_0, \cdots, \lambda_n$ are rational numbers, although the main idea of this paper extends to the case where $t, \lambda_0, \cdots, \lambda_n$ are complex numbers.

The motivation of our work is to find an effective way to verify the K stability for hypersurfaces. In general, this is a harder problem than the problem of finding an effective way to compute the Futaki invariant, because the K energy is the nonlinear version of the "Futaki" invariant(see [10]). By the work of [2] or [17], if the central fiber is normal, the quantity A is the real part of the corresponding Futaki invariant. However, the technical difficulty in the proof is that the degeneration of a hypersurface under a one parameter subgroup is "generically" an algebraic cycle of multiplication greater than 1. If that is the case, we would not be able to generalize the argument in [2] directly. In fact, our result shows that the limit may *not* depend on the central fiber alone. To see this, we consider the "generic" case where the central fiber is represented by the algebraic cycle $Z_0^{i_0} \cdots Z_n^{i_n} = 0$. All the information of the central fiber is contained in the vector $(i_0, \cdots, i_n)$. However, (1.6) shows that we have the other term

$$\sum_{i=0}^{n} \int_0^\infty \varphi_i'(x)(\varphi_i'(x) - 1)dx$$

that can't be recovered from the central fiber. Thus the left hand side of (1.6) depends not only on the central fiber, but also on the whole degeneration F_t.

In this paper, we overcome the above difficulty in the case that the central fiber is of multiplicity greater than one. We first represent the K energy into an explicitly formula(Theorem 2.5). Then we carefully analyze the integrand in the formula by using some analytic techniques and a recent result of Phong and Sturm [14] to get the conclusion.

This paper can be viewed as a nonlinearized version of the paper [5] of the author, where the Futaki invariant of a hypersurface of CP^n was computed.

Phong and Sturm studied K stability for arbitrary smooth manifolds in CP^N [13]. They also studied the linearized version, i.e., the computation of the Futaki invariant, in [15]. In order to establish the result in the general case, they make use of the Chow point and Deligne pair. The Chow point, which is a hypersurface of some Grassmannian, contains all the information of the original manifold.

The Chow stability (which is closely related to K stability) was studied by Paul [11] and more recently by Paul and Tian [12], where they have formulated the stability in terms of double Chow points.

Before stating the main result, we setup notations: let M be defined by the zeros of the polynomial

$$F = \sum_{i=0}^{p} a_i Z_0^{\alpha_0^i} \cdots Z_n^{\alpha_n^i} \tag{1.2}$$

of degree d. Let $(\lambda_0, \cdots, \lambda_n)$ be rational numbers satisfying $\sum \lambda_i = 0$. Let

$$\lambda = \operatorname*{Max}_{0 \le i \le p} (\sum_{k=0}^{n} \lambda_k \alpha_k^i). \tag{1.3}$$

Let

$$\varphi(x_0, \cdots, x_n) = \operatorname*{Min}_{0 \le i \le p} (-\sum_{k=0}^{n} \lambda_k \alpha_k^i + \sum_{k=0}^{n} \alpha_k^i x_k), \tag{1.4}$$

and let

$$\varphi_i(x) = \varphi(0, \cdots, \underset{i}{x}, \cdots, 0). \tag{1.5}$$

Then we have the following

Theorem 1.1. *For "generic" (See section 3 for details)* $(\lambda_0, \cdots, \lambda_n)$*, we have*

$$
\begin{aligned}
&\lim_{r\to 0} t\frac{d}{dt}\mathcal{M}(t)\\
&= \frac{2}{d}\left(-\frac{\lambda(d-1)(n+1)}{n} + \sum_{i=0}^{n}\int_0^\infty \varphi_i'(x)(\varphi_i'(x)-1)dx\right).
\end{aligned}
\tag{1.6}
$$

Since for a Kähler-Einstein manifold, the K energy has a lower bound, we have the following:

Theorem 1.2. *If* M *is a Kähler-Einstein hypersurface with positive first Chern class, then we have*

$$
-\frac{\lambda(d-1)(n+1)}{n} + \sum_{i=0}^{n}\int_0^\infty \varphi_i'(x)(\varphi_i'(x)-1)dx \le 0
$$

for any $\lambda_0 \cdots, \lambda_n \in \mathbb{R}$ *with* $\sum \lambda_i = 0$.

Proof of Theorem 1.2**.** The expression in the theorem is continuous and homogeneous with respect to $\lambda_0, \cdots, \lambda_n$. So by taking the limit, we proved that the inequality is valid for any choice of $\lambda_0, \cdots, \lambda_n \in \mathbb{R}$. □

This paper is a refinement of the paper [7]. We rewrite the introduction of this paper in order to cite the important papers of Donaldson [4], Phong-Sturm [13, 15]. We also give some new observations in the last section motivated by the work [4].

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2. An explicit formula for the K energy.

In this section, we give an explicit formula for the K energy of smooth hypersurfaces of CP^n.

First, let's recall the definition of the K energy [10]. Let M be a compact Kähler manifold with positive first Chern class $c_1(M)$. Let $\omega_0, \omega_1 \in c_1(M)$ and let $\omega_1 = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\xi$ for a smooth function ξ. We put $\omega_s = \omega_0 + s\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\xi$ and define

$$\mathcal{M}(\omega_0, \omega_1) = -\frac{1}{V}\int_0^1 \left(\int_X \xi(R(\omega_s) - m)\omega_s^m\right) ds, \tag{2.1}$$

where $R(\omega_s)$ is the scalar curvature of the metric, m is the complex dimension of M, and V is the volume of X with respect to ω_0. The functional $\mathcal{M}$, which is called the K energy by Mabuchi, has the properties:

Proposition 2.1. *Using the notations as above, we have*

1. $\mathcal{M}(\omega_0, \omega_1) = -\mathcal{M}(\omega_1, \omega_0)$,
2. $\mathcal{M}(\omega_0, \omega_1) + \mathcal{M}(\omega_1, \omega_2) = \mathcal{M}(\omega_0, \omega_2)$,

where $\omega_0, \omega_1, \omega_2 \in c_1(X)$, and are Kähler metrics.

From now on, let's assume that ω is the Kähler form of the Fubini-Study metric of CP^n. Let M be a hypersurface in CP^n defined by the polynomial $F = 0$ of degree d. Of course, we need $d \leq n$ to insure that M is Fano. Let $\lambda_0, \cdots, \lambda_n$ be integers such that $\sum_{i=0}^n \lambda_i = 0$. Let F_t be the polynomial defined by

$$F_t(Z_0, \cdots, Z_n) = F(t^{-\lambda_0}Z_0, \cdots, t^{-\lambda_n}Z_n),$$

and let M_t be the hypersurface defined by the zero set of F_t. Geometrically, M_t is the image of M under the automorphism $\sigma(t)$ generated by the holomorphic vector field $X = \sum_{i=0}^n \lambda_i Z_i \frac{\partial}{\partial Z_i}$. The automorphisms $\sigma(t)$ can be written as $\sigma(t)([Z_0, \cdots, Z_n]) = [t^{\lambda_0}Z_0, \cdots, t^{\lambda_n}Z_n]$. Using these automorphisms, one can define a family of Kähler forms $\omega_t = \sigma(t)^*\omega$ on M. It is easy to see that both $(n-d+1)\omega$ and $(n-d+1)\omega_t$ are Kähler forms of M in the cohomological class $c_1(M)$. Define $\mathcal{M}(t) = \mathcal{M}((n-d+1)\omega, (n-d+1)\omega_t)$. It is a well known result [10] that if M admits a Kähler-Einstein metric, then $\mathcal{M}(t)$ has a lower bound.

Proposition 2.2. *Using the notations as above, we have*

$$t\frac{d}{dt}\mathcal{M}(t) = \frac{2(n-1)}{d}\int_{M_t}(\mathrm{Ric}(\omega|_{M_t}) - (n-d+1)\omega|_{M_t})\theta\omega^{n-2},$$

where θ is defined as

$$\theta = -\frac{\sum_{i=0}^n \lambda_i |Z_i|^2}{\sum_{i=0}^n |Z_i|^2}, \tag{2.2}$$

and $\mathrm{Ric}(\omega|_{M_t})$ *is the Ricci form of* $\omega|_{M_t}$.

Proof. It basically follows from Proposition 2.1. Also see [2, Lemma 2.1] for details. □

The following lemma can be found in [16], we include the proof here for the sake of completeness.

Lemma 2.1. *Let M be the smooth hypersurface defined as the zero of $\{F = 0\}$. We use ω to denote the Fubini-Study metric on CP^n as well as the Kähler form on M, which is the restriction of ω on M. Let*

$$\xi = \log \frac{|\nabla F|^2}{(\sum_{i=0}^n |Z_i|^2)^{(d-1)}}, \tag{2.3}$$

where $[Z_0, \cdots, Z_n]$ is the homogeneous coordinate in CP^n. Then we have

$$Ric(\omega) - (n-d+1)\omega = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\xi. \tag{2.4}$$

Proof. Without losing generality, we prove the above lemma on the open set

$$U_0 = \{[Z_0, \cdots, Z_n] | |Z_0| > \frac{1}{2}|Z_j|, j = 1, \cdots, n\}$$

in CP^n. The local coordinate system on U_0 is $z = (z_1, \cdots, z_n)$ where $z_i = Z_i/Z_0$ for $i = 1, \cdots n$. Under this coordinate system, the Fubini-Study metric can be written as

$$\omega = \frac{\sqrt{-1}}{2\pi} g_{i\overline{j}} dz_i \wedge d\overline{z}_j = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^n \left(\frac{\delta_{ij}}{1+|z|^2} - \frac{z_j \overline{z}_i}{(1+|z|^2)^2}\right) dz_i \wedge d\overline{z}_j, \tag{2.5}$$

where $|z|^2 = \sum |z_i|^2$. Let's further assume that in a small open set V of U_0, from the equation $F = 0$, we can solve z_1. Namely,

$$z_1 = z_1(z_2, \cdots, z_n) \tag{2.6}$$

for a holomorphic function z_1. Let the Kähler form ω on V, under the local coordinate system $(z_2, \cdots, z_n)$, be written as

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=2}^n \tilde{g}_{i\overline{j}} dz_i \wedge d\overline{z}_j,$$

and let $a_i = \frac{\partial z_1}{\partial z_i}, i = 2, \cdots, n$. Then by (2.5) and (2.6), we have

$$\begin{aligned}\tilde{g}_{i\bar{j}} = & \frac{\delta_{ij}}{1+|z|^2} - \frac{z_j \overline{z}_i}{(1+|z|^2)^2} - \frac{z_j \overline{z_1} a_i}{(1+|z|^2)^2} - \frac{z_1 \overline{z}_i \overline{a}_j}{(1+|z|^2)^2} \\ & + \frac{a_i \overline{a}_j}{1+|z|^2} - \frac{|z_1|^2 a_i \overline{a}_j}{(1+|z|^2)^2},\end{aligned}$$

for $i, j = 2, \cdots, n$. We want to compute the determinant $\det(\tilde{g}_{i\bar{j}})$. In order to do this, we let

$$K_{i\bar{j}} = \delta_{ij} + a_i \overline{a}_j - \frac{1}{1+|z|^2}(\overline{z}_i + \overline{z}_1 a_i)\overline{(\overline{z}_j + \overline{z}_1 a_j)}.$$

Then

$$\tilde{g}_{i\bar{j}} = \frac{1}{1+|z|^2} K_{i\bar{j}}, \quad i, j = 2, \cdots, n. \tag{2.7}$$

Let

$$\begin{aligned}A &= (a_2, \cdots, a_n); \\ B &= (\overline{z}_2 + \overline{z}_1 a_2, \cdots, \overline{z}_n + \overline{z}_1 a_n).\end{aligned}$$

Then the matrix $K = (K_{i\bar{j}})$ can be represented by

$$K = I + A^T \overline{A} - \frac{1}{1+|z|^2} B^T \overline{B}.$$

A straightforward computation gives

$$\begin{aligned}KA^T &= (1+|a|^2)A^T - \frac{1}{1+|z|^2}(\overline{B}A^T)B^T; \\ KB^T &= (\overline{A}B^T)A^T + (1 - \frac{|B|^2}{1+|z|^2})B^T.\end{aligned}$$

Thus the vector space spanned by the vectors A, B is K-invariant. Furthermore, on the complement of the vector space, K is the identity. So we have

$$\begin{aligned}\det K &= (1+|a|^2)(1 - \frac{|B|^2}{1+|z|^2}) + \frac{1}{1+|z|^2}|\overline{B}A^T|^2 \\ &= \frac{1}{1+|z|^2}(1+|a|^2 + |\sum_{i=2}^n a_i z_i - z_1|^2).\end{aligned} \tag{2.8}$$

Let f be the defining function of M on U_0, i.e.

$$f = F(1, \frac{Z_1}{Z_0}, \cdots, \frac{Z_n}{Z_0}) = \frac{F}{Z_0^d}.$$

Then

$$\frac{\partial z_1}{\partial z_k} = -\frac{\frac{\partial f}{\partial z_k}}{\frac{\partial f}{\partial z_1}} = -\frac{F_k}{F_1}, \quad (k = 2, \cdots, n) \tag{2.9}$$

where we define $F_k = \frac{\partial F}{\partial Z_k}$ for $k = 0, \cdots, n$. Thus by the homogeneity of F, we have

$$\begin{aligned}(\sum_{i=2}^n a_i z_i) - z_1 &= -\left(\sum_{i=2}^n \frac{Z_i}{Z_0}\frac{F_i}{F_1}\right) - \frac{Z_1}{Z_0} \\ &= -\frac{1}{Z_0 F_1}(\sum_{i=1}^n Z_i F_i) = \frac{F_0}{F_1}\end{aligned} \tag{2.10}$$

on M. Using (2.7), (2.8), and (2.10), we have

$$\det \tilde{g}_{i\bar{j}} = \frac{1}{(1+|z|^2)^n}\frac{1}{|F_1|^2}(\sum_{k=0}^n |F_k|^2). \tag{2.11}$$

Then by (2.3)

$$\det \tilde{g}_{i\bar{j}} = \frac{1}{(1+|z|^2)^{n-d+1}} \cdot \frac{1}{|\frac{\partial f}{\partial z_1}|^2} \cdot e^{\xi}.$$

(2.4) follows from the formula of the Ricci curvature and the above equation. □

In order to represent the K energy in terms of the polynomial F, we need the following purely algebraic lemma:

Lemma 2.2. *With the same notations as above, let η be a $(1,1)$ form on CP^n. Let $\pi : C^{n+1} \to CP^n$ be the projection. Let*

$$\pi^*\eta = \frac{\sqrt{-1}}{2\pi}\sum_{i,j=0}^n \tilde{a}_{i\bar{j}} dZ_i \wedge d\overline{Z}_j. \tag{2.12}$$

Then on M,

$$\eta \wedge \omega^{n-2} = \frac{|Z|^2}{n-1}\left(\sum_{i=0}^n \tilde{a}_{i\bar{i}} - \frac{\sum_{i,j=0}^n \tilde{a}_{i\bar{j}} F_j \overline{F}_i}{|\nabla F|^2}\right)\omega^{n-1} \tag{2.13}$$

for $|Z|^2 = \sum_{i=0}^n |Z_i|^2$.

Remark 2.1. The righthand side of (2.13) is well defined because $\tilde{a}_{i\bar{j}}$ for $i, j = 0, \cdots, n$ are homogeneous functions of order (-2).

Proof. As in the proof of the previous lemma, we can consider the problem only on $U_0 \cap \{\frac{\partial F}{\partial Z_1} \neq 0\}$, without losing generality. Define $A_{i\bar{j}}$ on CP^n as follows:

$$\eta \wedge \omega^{n-2} = (\frac{\sqrt{-1}}{2\pi})^{n-1}(-1)^{\frac{1}{2}(n-1)(n-2)} \cdot \sum_{i,j=1}^{n}(-1)^{i+j}A_{i\bar{j}}dz_1 \wedge \cdots \wedge \hat{dz_i} \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge \hat{d\overline{z}_j} \cdots \wedge d\overline{z}_n, \tag{2.14}$$

where the 'hat' symbol " ˆ " over dz_j and $d\overline{z}_j$ means these two differential forms are deleted from the expression. Define $b = (b_1, \cdots, b_n)$ by

$$b = (1, -a_2, \cdots, -a_n) = (1, -\frac{\partial z_1}{\partial z_2}, \cdots, -\frac{\partial z_1}{\partial z_n}) = (1, \frac{F_2}{F_1}, \cdots, \frac{F_n}{F_1}).$$

Then by (2.14), we have

$$\eta \wedge \omega^{n-2} = (\frac{\sqrt{-1}}{2\pi})^{n-1}(-1)^{\frac{1}{2}(n-1)(n-2)} \cdot \sum_{i,j=1}^{n} A_{i\bar{j}}b_i\overline{b}_j dz_2 \wedge \cdots \wedge dz^n \wedge d\overline{z}^2 \wedge \cdots \wedge d\overline{z}^n \tag{2.15}$$

on M. Thus in order to prove (2.13), we just need to compute $\sum A_{i\bar{j}}b_i\overline{b}_j$. To this end, let

$$\eta = \frac{\sqrt{-1}}{2\pi}\sum_{k,l=1}^{n} a_{k\bar{l}}dz_k \wedge d\overline{z}_l, \tag{2.16}$$

and fix r, s. By (2.14), we have

$$\frac{\sqrt{-1}}{2\pi}dz_r \wedge d\overline{z}_s \wedge \frac{\sqrt{-1}}{2\pi}\sum_{k,l=1}^{n} a_{k\bar{l}}dz_k \wedge d\overline{z}_l \wedge \omega^{n-2} = (\frac{\sqrt{-1}}{2\pi})^{n}(-1)^{\frac{1}{2}(n-1)(n-2)}(-1)^{n-1}A_{r\bar{s}}dz_1 \wedge \cdots \wedge d\overline{z}_n. \tag{2.17}$$

We also have the following algebraic fact:

$$\frac{\sqrt{-1}}{2\pi}dz_r \wedge d\overline{z}_s \wedge \frac{\sqrt{-1}}{2\pi}\sum_{k,l=1}^{n} a_{k\bar{l}}dz_k \wedge d\overline{z}_l \wedge \omega^{n-2} = \frac{1}{n(n-1)}\left(\sum_{\alpha,\beta=1}^{n}(g^{\alpha\overline{\beta}}a_{\alpha\overline{\beta}})g^{r\overline{s}} - \sum_{\alpha,\beta=1}^{n} g^{\alpha\overline{s}}g^{r\overline{\beta}}a_{\alpha\overline{\beta}}\right)\omega^n. \tag{2.18}$$

By (2.5), we have

$$\omega^n = \left(\frac{\sqrt{-1}}{2\pi}\right)^n n!(-1)^{\frac{1}{2}n(n-1)} \frac{1}{(1+|z|^2)^{n+1}} dz_1 \wedge \cdots \wedge d\overline{z}_n. \tag{2.19}$$

Comparing (2.17), (2.18) and (2.19), we have

$$A_{r\overline{s}} = \frac{(n-2)!}{(1+|z|^2)^{n+1}} (\sum_{\alpha,\beta=1}^n (g^{\alpha\overline{\beta}} a_{\alpha\overline{\beta}}) g^{r\overline{s}} - \sum_{\alpha,\beta=1}^n g^{\alpha\overline{s}} g^{r\overline{\beta}} a_{\alpha\overline{\beta}}), \tag{2.20}$$

for $r, s = 1, \cdots, n$. By (2.20), we have

$$\begin{aligned} \sum_{i,j=1}^n A_{i\overline{j}} b_i \overline{b}_j &= \frac{(n-2)!}{(1+|z|^2)^{n+1}} \\ &\cdot (\sum_{\alpha,\beta=1}^n g^{\alpha\overline{\beta}} a_{\alpha\overline{\beta}} \sum_{i,j=1}^n g^{i\overline{j}} b_i \overline{b}_j - \sum_{i,j,\alpha,\beta=1}^n g^{\alpha\overline{j}} g^{i\overline{\beta}} a_{\alpha\overline{\beta}} b_i \overline{b}_j). \end{aligned} \tag{2.21}$$

We need the following

Lemma 2.3. *Using the same notations as above, we have*

$$\sum_{\alpha,\beta=1}^n g^{\alpha\overline{\beta}} a_{\alpha\overline{\beta}} = |Z_0|^2 (1+|z|^2) \sum_{i=0}^n \tilde{a}_{i\overline{i}}, \tag{2.22}$$

$$\sum_{i,j=1}^n g^{i\overline{j}} b_i \overline{b}_j = (1+|z|^2) \frac{|\nabla F|^2}{|F_1|^2}, \tag{2.23}$$

$$\sum_{i,j,\alpha,\beta=1}^n g^{\alpha\overline{j}} g^{i\overline{\beta}} a_{\alpha\overline{\beta}} b_i \overline{b}_j = |Z_0|^2 (1+|z|^2)^2 \frac{\sum_{\alpha,\beta=0}^n \tilde{a}_{\alpha\overline{\beta}} \overline{F}_\alpha F_\beta}{|F_1|^2}, \tag{2.24}$$

where $\tilde{a}_{i\overline{j}}$ *is defined in* (2.12).

Proof. Comparing (2.12) and (2.16), we have

$$\begin{cases} a_{k\overline{l}} = \tilde{a}_{k\overline{l}} \cdot |Z_0|^2, & k, l \neq 0; \\ \sum_{i=1}^n z_i a_{i\overline{l}} = -\tilde{a}_{0\overline{l}} \cdot |Z_0|^2, & l \neq 0; \\ \sum_{j=1}^n \overline{z}_j a_{k\overline{j}} = -\tilde{a}_{k\overline{0}} \cdot |Z_0|^2, & k \neq 0; \\ \sum_{i,j=1}^n z_i \overline{z}_j a_{i\overline{j}} = \tilde{a}_{0\overline{0}} \cdot |Z_0|^2. \end{cases} \tag{2.25}$$

Since $g^{\alpha\overline{\beta}} = (1+|z|^2)(\delta_{\alpha\beta} + z_\alpha \overline{z}_\beta)$, by (2.25) , we have

$$\sum_{\alpha,\beta=1}^n g^{\alpha\overline{\beta}} a_{\alpha\overline{\beta}} = (1+|z|^2) \sum_{\alpha,\beta=1}^n (\delta_{\alpha\beta} + z_\alpha \overline{z}_\beta) a_{\alpha\overline{\beta}} = |Z|^2 \sum_{\alpha=0}^n \tilde{a}_{\alpha\overline{\alpha}}.$$

This proves (2.22). By (2.10), we have

$$\sum_{i=1}^{n} z_i b_i = -\frac{F_0}{F_1}$$

on M. Thus (2.23) and (2.24) follow from a straightforward computation using the above equation. □

Continuation of the Proof of Lemma 2.2**.** By Lemma 2.3, we have

$$\begin{aligned}&\sum_{\alpha,\beta=1}^{n} g^{\alpha\overline{\beta}} a_{\alpha\overline{\beta}} \sum_{i,j=1}^{n} g^{i\overline{j}} b_i \overline{b}_j - \sum_{i,j,\alpha,\beta=1}^{n} g^{\alpha\overline{j}} g^{i\overline{\beta}} a_{\alpha\overline{\beta}} b_i \overline{b}_j \\ &= |Z_0|^2 (1+|z|^2)^2 \frac{|\nabla F|^2}{|F_1|^2} \left(\sum_{i=0}^{n} \tilde{a}_{i\overline{i}} - \sum_{i,j=0}^{n} \frac{\tilde{a}_{i\overline{j}} F_j \overline{F}_i}{|\nabla F|^2} \right).\end{aligned} \tag{2.26}$$

By (2.11),

$$\begin{aligned}\omega^{n-1} = &\left(\frac{\sqrt{-1}}{2\pi}\right)^{n-1} (-1)^{\frac{1}{2}(n-1)(n-2)} \frac{(n-1)!}{(1+|z|^2)^n} \frac{|\nabla F|^2}{|F_1|^2} \\ &\cdot dz_2 \wedge \cdots \wedge dz_n \wedge d\overline{z}_2 \wedge \cdots d\overline{z}_n.\end{aligned} \tag{2.27}$$

(2.13) follows from (2.15), (2.21), (2.26) and (2.27). □

Lemma 2.4. *Let ξ be the function defined in (2.3) and let θ be defined in (2.2). Then we have*

$$\begin{aligned}&\frac{\sqrt{-1}}{2\pi} \partial\xi \wedge \overline{\partial}\theta \wedge \omega^{n-2} \\ &= \frac{1}{n-1} \left(-\sum_{k=0}^{n} \left(\frac{XF}{|\nabla F|^2}\right)_k \overline{F_k} + \frac{\sum_{k=0}^{n} \lambda_k |F_k|^2}{|\nabla F|^2} - (d-1)\theta \right) \omega^{n-1}.\end{aligned} \tag{2.28}$$

Furthermore, we have

$$\begin{aligned}&\frac{\sqrt{-1}}{2\pi} \int_M \partial\xi \wedge \overline{\partial}\theta \wedge \omega^{n-2} \\ &= -\frac{1}{n-1} \int_M \sum_{k=0}^{n} \left(\frac{XF}{|\nabla F|^2}\right)_k \overline{F_k} \omega^{n-1} + \frac{n-d+1}{n-1} \int_M \theta \omega^{n-1}.\end{aligned} \tag{2.29}$$

Proof. Let $\eta = \frac{\sqrt{-1}}{2\pi}\partial\xi\wedge\overline{\partial}\theta$ and let

$$\pi^*\eta = \frac{\sqrt{-1}}{2\pi}\sum_{i,j=0}^{n}\tilde{a}_{i\bar{j}}dZ_i\wedge d\overline{Z}_j.$$

Then we have

$$\tilde{a}_{i\bar{j}} = \frac{\partial\xi}{\partial Z_i}\cdot\frac{\partial\theta}{\partial\overline{Z}_j}.$$

A straightforward computation gives

$$\sum_{i=0}^{n}\tilde{a}_{i\bar{i}} = \frac{-\sum_{k=0}^{n}(XF)_k\overline{F}_k + \sum_{k=0}^{n}\lambda_k|F_k|^2}{|Z|^2|\nabla F|^2} - (d-1)\frac{\theta}{|Z|^2},$$

and

$$\frac{\sum_{i,j=0}^{n}\tilde{a}_{i\bar{j}}F_j\overline{F}_i}{|\nabla F|^2} = -\frac{XF\cdot\sum_{i,k=0}^{n}F_{ik}\overline{F}_i\overline{F}_k}{|Z|^2|\nabla F|^4}$$

on M. Thus by Lemma 2.2, we got (2.28). Using Lemma 2.2 again for $\eta = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\theta$, we have

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\theta\wedge\omega^{n-2} = \frac{1}{n-1}\left(-n\theta + \frac{\sum_{k=0}^{n}\lambda_k|F_k|^2}{|\nabla F|^2}\right)\omega^{n-1}. \tag{2.30}$$

(2.29) follows from (2.28), (2.30) and the Stokes Theorem. □

Although not needed in this paper, we give a simple proof of the following formula for the Futaki invariant in [5] as an application of Lemma 2.1, Lemma 2.2 and Lemma 2.4.

Corollary 2.1. *Let M be a smooth hypersurface in CP^n defined by the homogeneous polynomial $F = 0$ of degree d. Let X be a vector in CP^n satisfying*

$$XF = \kappa F. \tag{2.31}$$

The Futaki invariant is defined as

$$\mathcal{F}(X) = -\int_M X(\xi)\omega^{n-1}.$$

Then

$$\mathcal{F}(X) = -\frac{(n+1)(d-1)}{n}\kappa. \tag{2.32}$$

Proof. We have

$$i(X)\omega = -\frac{\sqrt{-1}}{2\pi}\overline{\partial}\theta. \tag{2.33}$$

Since X leave M invariant, we have

$$0 = \int_M i(X)(\partial\xi \wedge \omega^{n-1}) = \int_M X\xi\omega^{n-1} + (n-1)\int_M \frac{\sqrt{-1}}{2\pi}\partial\xi \wedge \overline{\partial}\theta \wedge \omega^{n-2}.$$

By the above equation and (2.31), from Lemma 2.4, we have

$$\mathcal{F}(X) = -\kappa \int_M \omega^{n-1} + (n-d+1)\int_M \theta\omega^{n-1}.$$

By [5, Theorem 5.1], we have

$$\int_M \theta\omega^{n-1} = \frac{\kappa}{n}.$$

(2.32) follows from the above two equations. □

Finally, we have the following

Theorem 2.5. *The K energy $\mathcal{M}(t)$ can be represented as*

$$\begin{aligned}\mathcal{M}(t) = \frac{2}{d}\int_1^t \Bigg(\int_{M_\tau} \frac{1}{\tau}\Bigg(-\sum_{k=0}^n \left(\frac{XF_\tau}{|\nabla F_\tau|^2}\right)_k \overline{(F_\tau)_k}\omega^{n-1} \\ +(n-d+1)\int_{M_\tau} \theta\omega^{n-1}\Bigg)\Bigg) d\tau,\end{aligned} \tag{2.34}$$

where

$$F_\tau(Z_0, \cdots, Z_n) = F(\tau^{-\lambda_0}Z_0, \cdots, \tau^{-\lambda_n}Z_n),$$

and M_τ is the zero set of $F_\tau = 0$. In particular, we have

$$\begin{aligned}t\frac{d}{dt}\mathcal{M}(t) = \frac{2}{d}\Bigg(-\int_{M_t}\sum_{k=0}^n \left(\frac{XF_t}{|\nabla F_t|^2}\right)_k \overline{(F_t)_k}\omega^{n-1} \\ +(n-d+1)\int_{M_t}\theta\omega^{n-1}\Bigg).\end{aligned} \tag{2.35}$$

Proof. The theorem follows from Proposition 2.2, Lemma 2.1 and Lemma 2.4. □

3. The limit of the derivative of the K energy.

In this section, we compute the limit $\lim_{t\to 0} t\mathcal{M}'(t)$ using Theorem 2.5. First, we need some combinatoric preparations.

Let $(\delta_i, \sigma_i), i = 0, \cdots, p$ be a sequence of pair of nonnegative rational numbers. $\delta_0 = 0$. We assume that the sequence is "generic" in the sense that

1. All $\delta_i, (i = 0, \cdots, p)$ are distinct numbers(that implies $\delta_i > 0, i = 1, \cdots, p$);
2. None of the three lines defined by $\psi_i(x) = \delta_i + \sigma_i x, (i = 0, \cdots, p)$ intersect at the same point.

Define $(i_k, r_k), (k = 0, \cdots, m)$ inductively as follows: let $i_0 = 0, r_0 = 0$. If (i_k, r_k) has been defined, then

1. If for any $r > r_k$
$$\delta_{i_k} + \sigma_{i_k} r < \delta_i + \sigma_i r \qquad (i \neq i_k),$$
then let $m = k$ and stop;
2. If not, then define i_{k+1} and $r_{k+1} > r_k$ such that
$$\delta_{i_k} + \sigma_{i_k} r_{k+1} = \delta_{i_{k+1}} + \sigma_{i_{k+1}} r_{k+1} \leq \delta_i + \sigma_i r_{k+1}, \tag{3.1}$$
where $i = 1, \cdots, p$. Since $(\delta_i, \sigma_i), i = 0, \cdots, p$ are "generic", the choice of (i_k, r_k) is unique for $(k = 0, \cdots, m)$.

We have the following obvious

Lemma 3.1. $(i_k, r_k), (k = 0, 1, \cdots)$ *is a finite sequence. In particular, the sequence stops at* (i_m, r_m).

Proof. By the construction of i_k's, we have
$$\sigma_{i_0} > \sigma_{i_1} > \cdots > \sigma_{i_k} > \cdots .$$
Thus all i_k's must be distinct. But $0 \leq i_k \leq p$. So the length of the sequence is at most $p + 1$. □

Let
$$\psi(x) = \operatorname*{Min}_{i \geq 0} (\delta_i + \sigma_i x). \tag{3.2}$$
The function $\psi(x)$ is a piecewise linear function, whose derivative exists almost everywhere. $r_k, (k = 1, \cdots, m)$ are the non-smooth points of $\psi(x)$.

Lemma 3.2. *Assuming that* $\sigma_{i_m} = 0$*, we have*

$$\sum_{k=0}^{m-1}(-\delta_{i_k} + \delta_{i_{k+1}})(\sigma_{i_k} + \sigma_{i_{k+1}} - 1) = \int_0^\infty \psi'(x)(\psi'(x) - 1)dx. \tag{3.3}$$

Proof. First, let's remark that for x large enough, $\psi \equiv \delta_{i_m}$ is a constant. Thus the integral in the lemma is convergent.

By definition of $r_k (k = 0, \cdots, m)$ in (3.1), we have

$$-\delta_{i_k} + \delta_{i_{k+1}} = (\sigma_{i_k} - \sigma_{i_{k+1}})r_{k+1}$$

for $k = 0, \cdots, m-1$. Thus we have

$$\sum_{k=0}^{m-1}(-\delta_{i_k} + \delta_{i_{k+1}})(\sigma_{i_k} + \sigma_{i_{k+1}} - 1) = \sum_{k=0}^{m-1} r_{k+1}(\sigma_{i_k}^2 - \sigma_{i_{k+1}}^2) + (\delta_{i_0} - \delta_{i_m}).$$

The second term of the above equation is equal to

$$-\int_0^\infty \psi'(x)dx.$$

For the first term, using the summation by parts, we have

$$\sum_{k=0}^{m-1} r_{k+1}(\sigma_{i_k}^2 - \sigma_{i_{k+1}}^2) = r_1(\sigma_{i_0})^2 + \sum_{k=1}^{m-1}\sigma_{i_k}^2(r_{k+1} - r_k) = \int_0^\infty \psi'(x)^2 dx.$$

Combining the above two equations, we get (3.3). □

Consider the smooth hypersurface $M \subset CP^n$ defined by the polynomial $F = 0$ of degree d. Let $X = \sum_{i=0}^n \lambda_i Z_i \frac{\partial}{\partial Z_i}$ be the vector field for integers $(\lambda_0, \cdots, \lambda_n)$ such that $\sum \lambda_i = 0$. Let M_t be defined by the equation

$$F_t(Z_0, \cdots, Z_n) = F(t^{-\lambda_0} Z_0, \cdots, t^{-\lambda_n} Z_n). \tag{3.4}$$

We write F_t as

$$F_t = t^\delta \sum_{i=0}^p a_i t^{\delta_i} Z_0^{\alpha_0^i} \cdots Z_n^{\alpha_n^i}, \tag{3.5}$$

where $\delta_0 = 0$, and $\delta_i \geq 0, i = 1, \cdots, p$. By (3.4), we have

$$X(Z_0^{\alpha_0^i} \cdots Z_n^{\alpha_n^i}) = -(\delta_i + \delta) Z_0^{\alpha_0^i} \cdots Z_n^{\alpha_n^i} \tag{3.6}$$

for $i = 0, \cdots, p$.

In what follows we assume that the choice of $(\lambda_0, \cdots, \lambda_n)$ is "generic" in the following sense:

1. All δ_i's are distinct;

2. None of the three lines defined by $\delta_i+\alpha_k^i x$ for $i=0,\cdots,p$ intersect at the same points, where $k=0,\cdots,n$.

Without losing generality, we may assume that $a_0=1$, and $0=\delta_0<\delta_1<\delta_2<\cdots<\delta_p$. We also assume that $a_0,\cdots,a_p$ are all non-zero. Furthermore, since M is smooth, we see that for each $0\leq k\leq n$, there is an $0\leq i\leq p$ such that $\alpha_k^i=0$.

Let $U_i=\{[Z_0,\cdots,Z_n]\in CP^n||Z_i|>\frac{1}{2}|Z_j|, j=0,\cdots,n\}$. Then $\cup U_i=CP^n$. Let $P_i=\{Z_i=0\}$ and $P_{ij}=P_i\cap P_j$ for $i\neq j$ and $i,j=0,\cdots,n$. Let $\sigma>0$ be chosen so that $\sigma<\frac{1}{d}\underset{i\geq 1}{\mathrm{Min}}(\delta_i)$ (Note that $\underset{i\geq 1}{\mathrm{Min}}(\delta_i)>0$) and define

$$V_{ij}^t=\{z\in CP^n|d(z,P_{ij})<|t|^\sigma\},\ i\neq j, i,j=0,\cdots,n,$$

where $d(\cdot,\cdot)$ is the distance induced by the Fubini-Study metric on CP^n.

By (3.5), we see that $t^{-\delta}F_t\rightarrow Z_0^{\alpha_0^0}\cdots Z_n^{\alpha_n^0}$ as $t\rightarrow 0$. Intuitively, M_t goes to the hyperplanes defined by $Z_0^{\alpha_0^0}\cdots Z_n^{\alpha_n^0}=0$. This turns out to be essentially true by the following Lemma:

Lemma 3.3. *There is a $\sigma_1>\sigma$ such that for any $0\leq k\leq n$ and*

$$[Z_0,\cdots,Z_n]\in(M_t-\cup_{i,j=0}^n V_{ij}^t)\cap U_k,$$

one can find a unique $l\neq k$ such that

$$\left|\frac{Z_l}{Z_k}\right|<|t|^{\sigma_1}$$

for t small enough, where $[Z_0,\cdots,Z_n]\in M_t$.

Proof. By (3.5) we have

$$|Z_0^{\alpha_0^0}\cdots Z_n^{\alpha_n^0}|\leq 2^d\sum_{i=1}^p|a_i||t|^{\underset{i\geq 1}{\mathrm{Min}}(\delta_i)}|Z_k|^d. \tag{3.7}$$

Thus if for any $l\neq k$,

$$\left|\frac{Z_l}{Z_k}\right|\geq|t|^{\sigma_1},$$

we could have

$$|Z_0^{\alpha_0^0}\cdots Z_n^{\alpha_n^0}|\geq|t|^{\sigma_1 d}|Z_k|^d.$$

This is a contradiction since we choose σ_1 such that

$$\sigma < \sigma_1 < \frac{1}{d} \operatorname*{Min}_{i \geq 1}(\delta_i).$$

□

We are now going to prove that for t small enough, the connected components of $M_t \backslash \cup V_{ij}^t$ are graphs of functions over $\tilde{P}_i$, where

$$\tilde{P}_i = P_i - \cup_{j \neq i} V_{ij}^t.$$

To see this, we let

$$Q_i = \{[Z_0, \cdots, Z_i, \cdots, Z_n] | [Z_0, \cdots, Z_{i-1}, \underset{i}{0}, Z_{i+1}, \cdots, Z_n] \in \tilde{P}_i\},$$

for $i = 0, \cdots, n$. By (1.4) and (1.5), we have

$$\varphi(x_0, \cdots, x_n) = \operatorname*{Min}_{0 \leq i \leq p} (\delta + \delta_i + \alpha_0^i x_0 + \cdots + \alpha_n^i x_n), \tag{3.8}$$

and

$$\varphi_k(x) = \operatorname*{Min}_{0 \leq i \leq p} (\delta + \delta_i + \alpha_k^i x), \tag{3.9}$$

for $k = 0, \cdots, n$.

Remark 3.1. φ and $\varphi_k\,(k = 0, \cdots, n)$ are defined even if $\lambda_0, \cdots, \lambda_n$ are not choosing "generically". In the special case when

$$XF = \kappa F,$$

we have

$$\varphi_i(x) = -\kappa + (\operatorname*{Min}_{0 \leq j \leq p} \alpha_i^j) x$$

for $0 \leq i \leq n$ and $x \geq 0$. If M is a normal variety, we have

$$\operatorname*{Min}_{0 \leq j \leq p} \alpha_i^j = 0 \text{ or } 1.$$

In particular, in this case

$$\varphi_i'(x)(\varphi_i'(x) - 1) = 0$$

for $0 \leq i \leq n$. Using this and Theorem 1.1, we recover the main result in [2] for hypersurfaces.

Proposition 3.1. *Using the notations as above, we have*

$$\begin{aligned}&\int_{M_t\cap Q_i}\sum_{A=0}^{n}\left(\frac{XF_t}{|\nabla F_t|^2}\right)_A\overline{(F_t)_A}\omega^{n-1}\\&\to -\delta\alpha_i^0-\int_0^\infty \varphi_i'(x)(\varphi_i'(x)-1)dx,\end{aligned}\tag{3.10}$$

for $i=0,\cdots,n$ *as* $t\to 0$.

Proof. For the sake of simplicity, we omit unimportant constants in an inequality. Thus in the proof of this proposition, $A\leq B$ means there is a constant C independent of t such that $A\leq CB$.

We just need to prove the theorem for the case $i=1$. If $\alpha_1^0=0$, then the proposition is automatically true since $\varphi_1'\equiv 0$. Thus we assume that $\alpha_1^0\geq 1$. We work on $M_t\cap Q_1\cap U_0$, without losing generality.

We assume that $(z_1,\cdots,z_n)=(\frac{Z_1}{Z_0},\cdots,\frac{Z_n}{Z_0})$ on U_0. Then $F_t=0$ can be written as

$$f=\sum_{i=0}^{p}a_it^{\delta_i}z_1^{\alpha_1^i}\cdots z_n^{\alpha_n^i}=0\tag{3.11}$$

with $a_0=1$ and $\delta_0=0$(see (3.5)). The sequence $(\delta_i,\alpha_1^i),(i=0,\cdots,p)$ is assumed to be a "generic" sequence mentioned at the beginning of this section.

For $(z_1,\cdots,z_n)\in\tilde{P}_1\cap U_0$, we have

$$|z_i|\geq|t|^\sigma,$$

for $i=2,\cdots,n$. Let $\xi_i^k(i=1,\cdots,\alpha_1^{i_k}-\alpha_1^{i_{k+1}},k=0,\cdots,m)$ be the $(\alpha_1^{i_k}-\alpha_1^{i_{k+1}})$-th roots of

$$-\frac{a_{i_{k+1}}}{a_{i_k}}t^{\delta_{i_{k+1}}-\delta_{i_k}}z_2^{\alpha_2^{i_{k+1}}-\alpha_2^{i_k}}\cdots z_n^{\alpha_n^{i_{k+1}}-\alpha_n^{i_k}}.$$

In the following lemma, we give the solutions of $z_1=z_1(z_2,\cdots,z_n)$ of the equation $f=0$. Of course, they are multiple solutions.

Lemma 3.4. *For* $\sigma>0$ *small enough, there is a constant* $\varepsilon_0>0$ *such that the solutions of* z_1 *of* $f=0$ *satisfies*

$$|z_1-\xi_i^k|\leq|\xi_i^k|\cdot|t|^{\varepsilon_0}$$

for $(i=1,\cdots,\alpha_1^{i_k}-\alpha_1^{i_{k+1}},k=0,\cdots,m-1)$. *Furthermore, the balls* $B_i^k=\{z\in\mathbb{C}||z-\xi_i^k|\leq|\xi_i^k||t|^{\varepsilon_0}\}$ *for* $(i=1,\cdots,\alpha_1^{i_k}-\alpha_1^{i_{k+1}},k=0,\cdots,m-1)$ *do not intersect each other.*

Proof. In the proof, the scripts i, k are always running in $(i = 1, \cdots, \alpha_1^{i_k} - \alpha_1^{i_{k+1}}, k = 0, \cdots, m-1)$, unless otherwise stated. We choose $\varepsilon_1 > 0$ such that

$$\varepsilon_1 < \operatorname*{Min}_{0 \le k \le m} \operatorname*{Min}_{i \ne i_k, i_{k+1}} (\delta + \delta_i + \alpha_1^i r_{k+1} - \varphi_1(r_{k+1})).$$

Define f_k and g_k as follows

$$f_k = a_{i_k} t^{\delta_{i_k}} z_1^{\alpha_1^{i_k}} \cdots z_n^{\alpha_n^{i_k}} + a_{i_{k+1}} t^{\delta_{i_{k+1}}} z_1^{\alpha_1^{i_{k+1}}} \cdots z_n^{\alpha_n^{i_{k+1}}},$$

and

$$g_k = f - f_k.$$

By the definition of ξ_i^k, we have

$$|t|^{r_{k+1}+C\sigma} \le |\xi_i^k| \le |t|^{r_{k+1}-C\sigma}$$

for some constant C independent of t. We also have

$$|t|^\delta |g_k| \le |t|^{\varphi_1(r_{k+1})+\varepsilon_1 - d\sigma}$$

on B_i^k and

$$|t|^\delta |f_k| \ge |t|^{\varphi_1(r_{k+1})+\varepsilon_0 + d\sigma}$$

on ∂B_i^k. We choose σ small enough such that $\varepsilon_1 - d\sigma > \frac{3}{4}\varepsilon_1$ and ε_0 small enough such that $\varepsilon_0 \le \frac{1}{4}\varepsilon_1$ Thus we have

$$|f_k| > |g_k|$$

on ∂B_i^k. By the Rouché Theorem, f_k and $f = f_k + g_k$ have the same number of solutions in B_i^k. Since f_k has only one solution in B_i^k, we prove the first claim of the lemma. Next, if t is small enough, we have a constant C such that

$$|\xi_i^k - \xi_{i_1}^{k_1}| \ge C\text{Max}(|\xi_i^k|, |\xi_{i_1}^{k_1}|).$$

Thus if t is small enough, B_i^k's do not intersect each other. □

Continuation of the proof of Proposition 3.1**.** For simplicity, let $F = F_t$. For fixed i, k, attaching the B_i^k in the above lemma for each $p \in \tilde{P}_1 \cap U_0$, we get a bundle $\tilde{B}_i^k$. On each bundle $\tilde{B}_i^k$, since $|z_i| > |t|^\sigma$, we have

$$\sum_{A=0}^{n}\left(\frac{XF}{|\nabla F|^2}\right)_A(\overline{F_A}) = \frac{(XF)_1}{F_1} - \frac{(XF)F_{11}}{F_1^2} + o(1)$$
$$= \frac{-(\delta+\delta_{i_k})\alpha_1^{i_k} + (\delta+\delta_{i_{k+1}})\alpha_1^{i_{k+1}}}{\alpha_1^{i_k} - \alpha_1^{i_{k+1}}}$$
$$- \frac{(-\delta_{i_k}+\delta_{i_{k+1}})(\alpha_1^{i_k}(\alpha_1^{i_k}-1) - \alpha_1^{i_{k+1}}(\alpha_1^{i_{k+1}}-1))}{(\alpha_1^{i_k} - \alpha_1^{i_{k+1}})^2} + o(1)$$
$$= -\delta + \frac{-\delta_{i_k}\alpha_1^{i_k} + \delta_{i_{k+1}}\alpha_1^{i_{k+1}} + (\delta_{i_k} - \delta_{i_{k+1}})(\alpha_1^{i_k} + \alpha_1^{i_{k+1}} - 1)}{\alpha_1^{i_k} - \alpha_1^{i_{k+1}}} + o(1) \tag{3.12}$$

as $t \to 0$ for $k = 0, \cdots, m-1$, where $o(1)$ is a quantity that goes to zero as $t \to 0$. By the same argument, the above equation is also true for $p \in \tilde{P}_1 \cap U_l$ for $l \neq 0$. Thus the equation is true for $p \in \tilde{P}_1$. On the other hand, by (3.11), we have

$$\det \pi = o(1) \tag{3.13}$$

as $t \to 0$, where $\pi : Q_1 \to \tilde{P}_1$ is the projection. This is because $\frac{\partial z_1}{\partial z_k} = -F_k/F_1 \to 0$ for $x \in \tilde{P}_1$. Thus by (3.12) and (3.13), we have

$$\int_{M_t\cap Q_1}\sum_{A=0}^{n}\left(\frac{XF_t}{|\nabla F_t|^2}\right)_A\overline{(F_t)_A}\omega^{n-1}$$
$$= (-\delta\alpha_1^0 + \sum_{k=0}^{m-1}(\delta_{i_k} - \delta_{i_{k+1}})(\alpha_1^{i_k} + \alpha_1^{i_{k+1}} - 1))\mathrm{vol}(CP^{n-1}) + o(1)$$

as $t \to 0$. The proposition follows from Lemma 3.2 and the fact $\mathrm{vol}(CP^{n-1}) = 1$. □

Lemma 3.5. *Let p be a fixed point in M_t and let $d(x,p)$ be the distance from $x \in CP^n$ to p defined by the Fubini-Study metric. Let $B_p(\varepsilon) = \{x \in CP^n | d(x,p) < \varepsilon\}$. Then there are constants C, σ independent of p and t such that*

$$\int_{M_t\cap B_p(\varepsilon)} \omega^{n-1} \leq C\varepsilon^{2n-2}\log\varepsilon^{-1} \tag{3.14}$$

for t small enough, where $\varepsilon = |t|^\sigma$.

Proof. Consider the cut-off function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho \geq 0$ is smooth, $\rho \equiv 1$ on $[0, 1]$ and $\rho \equiv 0$ on $(-\infty, -1] \cup [2, +\infty)$. Then we have

$$\int_{M_t \cap B_p(\varepsilon)} \omega^{n-1} \leq \int_{M_t} \rho(\frac{d(x,p)}{\varepsilon})\omega^{n-1}.$$

Let F_t be the defining function of M_t. Then in the sense of distribution, we have

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{|F_t|^2}{(\sum_{i=0}^n |Z_i|^2)^d} = [M_t] - d\omega.$$

Thus we have

$$\begin{aligned}\int_{M_t} \rho(\frac{d(x,p)}{\varepsilon})\omega^{n-1} = d\int_{CP^n} \rho(\frac{d(x,p)}{\varepsilon})\omega^n \\ + \int_{CP^n} \rho\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{|F_t|^2}{(\sum_{i=0}^n |Z_i|^2)^d}\omega^{n-1}.\end{aligned} \tag{3.15}$$

We have an easy estimate for the first term of the right hand side of (3.15):

$$\int_{CP^n} \rho(\frac{d(x,p)}{\varepsilon})\omega^n \leq C\varepsilon^{2n}. \tag{3.16}$$

For the second term, assume that $p \in U_0 = \{[Z_0, \cdots, Z_n] | |Z_0| > \frac{1}{2}|Z_j|, j = 1, \cdots, n\}$. Then by (3.5)

$$F_t = t^\delta Z_0^d f_t,$$

where $f_t \to f_0 = z_1^{\alpha_1^0} \cdots z_n^{\alpha_n^0} \not\equiv 0$. Thus using integration by parts, we have

$$\begin{aligned}&\int_{CP^n} \rho(\frac{d(x,p)}{\varepsilon})\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{|F_t|^2}{(\sum_{i=0}^n |Z_i|^2)^d}\omega^{n-1} \\ &\leq C\varepsilon^{2n} + \frac{C}{\varepsilon^2}\left|\int_{|z|\leq 2\varepsilon} \log|f_t| dV_0\right|,\end{aligned} \tag{3.17}$$

where $dV_0 = (\frac{\sqrt{-1}}{2\pi})^n dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$ is the Euclidean measure and $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. Rescaling the second term of the above integral, we have

$$\frac{C}{\varepsilon^2}\left|\int_{|z|\leq 2\varepsilon} \log|f_t| dV_0\right| = C\varepsilon^{2n-2}\log\varepsilon^{-1} + C\varepsilon^{2n-2}\left|\int_{|z|\leq 2} \log|\tilde{f}_t| dV_0\right|, \tag{3.18}$$

where $\tilde{f}_t(z_1, \cdots, z_n) = f_t(\varepsilon z_1, \cdots, \varepsilon z_n)/\varepsilon^{\alpha_1^0+\cdots+\alpha_n^0}$. By (3.5), if σ is small enough, we have $\tilde{f}_t \to z_1^{\alpha_1^0} \cdots z_n^{\alpha_n^0}$. By a theorem of Phong and Sturm [14], we have

$$\int_{|z|\leq 2} \log |f_t|^{-1} dV_0 \leq C \tag{3.19}$$

for t small enough. (3.14) follows from (3.15), (3.16), (3.17), (3.18) and (3.19). □

Lemma 3.6. *There exists a constant $C > 0$ such that for t small*

$$\sum_{i\neq j} \int_{V_{ij}^t \cap M_t} \omega^{n-1} \leq C|t|^{2\sigma} \log |t|^{-1}.$$

Proof. Let $\varepsilon = |t|^\sigma$. Fixing i, j, there is a constant C_0 independent of ε such that one can find points $p_1, \cdots, p_m \in P_{ij}$ for $m = [\frac{C_0}{\varepsilon^{2n-4}}]$, satisfying

$$\bigcup_{k=1}^{m} B_{p_k}(\varepsilon) \supset P_{ij}.$$

Thus by the above lemma, we have

$$\int_{V_{ij}^t \cap M_t} \omega^{n-1} \leq \sum_{k=1}^{m} \int_{M_t \cap B_{p_k}(|t|^\sigma + \varepsilon)} \omega^{n-1}.$$

By Lemma 3.5, we have

$$\int_{V_{ij}^t \cap M_t} \omega^{n-1} \leq \frac{C}{\varepsilon^{2n-4}} \varepsilon^{2n-2} \log \varepsilon^{-1} = C\varepsilon^2 \log \varepsilon^{-1}.$$

The lemma follows since $\varepsilon = |t|^\sigma$. □

Lemma 3.7. *There exists a constant C independent of t such that for any measurable subset E of M_t*

$$\left| \int_E \partial\xi \wedge \overline{\partial}\theta \wedge \omega^{n-2} \right| \leq C\sqrt{\log |t|^{-1}} \cdot \sqrt{\mathrm{meas}(E)},$$

where the functions ξ and θ are defined in (2.3) *and* (2.2)*, respectively.*

Proof. Since M_t is a submanifold, the Ricci curvature has an upper bound. Thus from (2.3), we have a constant C such that

$$-\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial}\xi \leq C\omega. \tag{3.20}$$

On the other hand, since $[t^{\lambda_0}Z_0,\cdots,t^{\lambda_n}Z_n]\in M_t$ iff $[Z_0,\cdots,Z_n]\in M$, we have

$$|\nabla F_t|^2(t^{\lambda_0}Z_0,\cdots,t^{\lambda_n}Z_n)=\sum_{l=0}^n |t|^{-2\lambda_l}|F_l|^2(Z_0,\cdots,Z_n).$$

Since M is smooth, we have

$$-C\log|t|^{-1}\le|\xi|\le C\log|t|^{-1}$$

for some constant C. Using integration by parts, from (3.20), and the above estimate, we have

$$\int_{M_t}|\nabla\xi|^2\omega^{n-1}\le C\int_{M_t}(|\xi|+\log|t|^{-1})\omega^{n-1}\le C\log|t|^{-1}.$$

If E is a measurable subset of M_t, then we have

$$\left|\int_E\partial\xi\wedge\overline{\partial}\theta\wedge\omega^{n-2}\right|\le\int_E|\partial\xi|\le C\log|t|^{-1}\sqrt{\text{meas}(E)}$$

by the Cauchy inequality. □

Proof of Theorem 1.1**.** By Proposition 3.1, we have

$$\begin{aligned}&\sum_{i=0}^n\int_{M_t\cap\cup Q_i}\sum_{A=0}^n\left(\frac{XF_t}{|\nabla F_t|^2}\right)_A(\overline{F_t)_A}\omega^{n-1}\\&=-\delta d-\sum_{i=0}^n\int_0^\infty\varphi_i'(x)(\varphi_i'(x)-1)dx+o(1)\end{aligned}\tag{3.21}$$

as $t\to 0$. We are going to prove that

$$\int_{M_t\setminus\bigcup_{i=0}^n Q_i}\sum_{A=0}^n\left(\frac{XF_t}{|\nabla F_t|^2}\right)_A(\overline{F_t)_A}\omega^{n-1}=o(1)\tag{3.22}$$

as $t\to 0$. In order to see this, let's recall that we have

$$\begin{aligned}\int_{M_t\setminus\bigcup_{i=0}^n Q_i}\frac{\sqrt{-1}}{2\pi}\partial\xi\wedge\overline{\partial}\theta\wedge\omega^{n-2}=-\frac{1}{n-1}\int_{M_t\setminus\bigcup_{i=0}^n Q_i}\Bigg(&\sum_{A=0}^n\left(\frac{XF_t}{|\nabla F_t|^2}\right)_A(\overline{F_t)_A}\\&-\frac{\sum_{i=0}^n\lambda_i|(F_t)_i|^2}{|\nabla F_t|^2}+(d-1)\theta\Bigg)\omega^{n-1}\end{aligned}$$

by Lemma 2.4. Since θ and the function $\frac{\sum_{i=0}^n \lambda_i |(F_t)_i|^2}{|\nabla(F_t)|^2}$ are bounded, we have

$$\int_{M_t \backslash \overset{n}{\underset{i=0}{\cup}} Q_i} \left| \sum_{A=0}^n \left(\frac{XF_t}{|\nabla F_t|^2} \right)_A \overline{(F_t)_A} \right| \omega^{n-1} \leq \int_{M_t \backslash \overset{n}{\underset{i=0}{\cup}} Q_i} (|\partial \xi| + 1) \omega^{n-1},$$

by (2.28). By Lemma 3.7 the righthanded side of the above equation is less than or equal to

$$C\sqrt{\log |t|^{-1}} \sqrt{\text{meas}(M_t \backslash \overset{n}{\underset{i=0}{\cup}} Q_i)} + \text{meas}(M_t \backslash \overset{n}{\underset{i=0}{\cup}} Q_i).$$

If we can prove that there is a constant C such that

$$M_t \backslash \overset{n}{\underset{i=0}{\cup}} Q_i \subset \underset{i \neq j}{\cup} V_{ij}^{Ct}. \tag{3.23}$$

Then (3.22) will follow from Lemma 3.6. To see (3.23), let's consider a point $p \in M_t \backslash \overset{n}{\underset{i=0}{\cup}} Q_i$. Without losing generality, we assume that $p \in U_0$. By (3.7), we can find a $k \neq 0$ such that

$$|Z_k| \leq |t|^\sigma |Z_0|$$

for t small enough. By definition, $p \notin Q_k$. Thus there is a $j \neq 0, k$ such that

$$|Z_j| \leq |t|^\sigma |Z_0|$$

Thus $p \in V_{jk}^{Ct}$ for some constant C. (3.23) is proved.

Combining (3.21) and (3.22), we have

$$\int_{M_t} \sum_{A=0}^n \left(\frac{XF_t}{|\nabla F_t|^2} \right)_A (\overline{F}_t)_A \omega^{n-1} = -\delta d - \sum_{i=0}^n \int_0^\infty \varphi_i'(x)(\varphi_i'(x) - 1) dx + o(1)$$

as $t \to 0$. Finally, since θ is a bounded function

$$\int_{M_t} \theta \omega^{n-1} = \int_{M_0} \theta \omega^{n-1} + o(1)$$

as $t \to 0$, where M_0 is defined as the zero set of $Z_0^{\alpha_0^0} \cdots Z_n^{\alpha_n^0} = 0$ counting the multiplicity. In [5, Theorem 5.1], it is proved that

$$\int_{M_0} \theta \omega^{n-1} = -\frac{\delta}{n}.$$

By (2.35), we have

$$t\mathcal{M}'(t) = \frac{2}{d} \left(\frac{\delta(n+1)(d-1)}{n} + \sum_{i=0}^n \int_0^\infty \varphi_i'(x)(\varphi_i'(x) - 1) dx \right) + o(1)$$

as $t \to 0$ and Theorem 1.1 is proved. $\square$

4. Further Discussions.

In this section, we study the relations between the K-stability and the Chow-Mumford stability for algebraic varieties. First, we have the following

Definition 4.1. *Using our notations in the first section, then for a hypersurface M (not necessarily smooth) of CP^n defined by a polynomial F in* (1.2)*, it is Chow-Mumford stable if $\lambda > 0$ for any $(\lambda_1, \cdots, \lambda_n)$ with $\sum \lambda_i = 0$, where λ is defined in* (1.3)*. It is Chow-Mumford unstable, if $\lambda < 0$ for any $(\lambda_0, \cdots, \lambda_n)$ with $\sum \lambda_i = 0$.*

Using Theorem 1.1, we have the following

Proposition 4.1. *If M is K-stable, then M is not Chow-Mumford unstable.*

Proof. Let $(\lambda_0, \cdots, \lambda_n)$ be an arbitrary $(n+1)$ rational numbers with $\sum \lambda_i = 0$. Then we can perturb $(\lambda_0, \cdots, \lambda_n)$ so that it is "generic". By Theorem 1.1 we have

$$-\frac{\lambda(d-1)(n+1)}{n} + \sum_{i=0}^{n} \int_0^\infty \varphi_i'(x)(\varphi_i'(x) - 1)dx \leq 0.$$

Since the second term above is nonnegative, we have $\lambda \geq 0$. This is a contradiction. □

A well-known fact about the smooth hypersurface is that it is always Chow-Mumford stable. So the above proposition gives no new information about the stability. However, it would be interesting to generalize the notion of K stability into singular varieties.

In [16], Tian defined the generalized K energy on normal varieties. In the case of hypersurfaces, we can define the K energy for algebraic cycles of multiplicity one. The following lemma is interesting:

Lemma 4.1. *Let M be a divisor of CP^n defined by a homogeneous polynomial F of multiplicity one*[1]*. Then $\mathcal{M}(t)$ in* (2.34) *is well-defined and we call $\mathcal{M}(t)$ the generalized K energy on M.*

Remark 4.1. Clearly a normal hypersurface is defined by an irreducible polynomial. So the above result is a generalization of the result in [16] in the case of hypersurfaces. But the proof below is essentially the same as in that paper.

[1] A polynomial is of multiplicity one, if $\{F = 0\} \cap \{\nabla F = 0\}$ is of codimension at least 2.

By Lemma 2.4, Lemma 4.1 follows from the following

Lemma 4.2. *Let ξ be defined by Lemma* 2.1*. Then we have*

$$\int_M |\partial\xi|\omega^{n-1} < +\infty.$$

Proof. Let M_i be an irreducible component of M and let $\pi_i : \tilde{M}_i \to M_i$ be a smooth resolution of M_i. Then $\pi_i^*(e^\xi)$ is an analytic function on $\tilde{M}_i$. Thus we have

$$\int_{\tilde{M}_i} |\pi_i^*(\xi)|^2 \pi_i^*(\omega^{n-1}) < +\infty. \tag{4.1}$$

We also have

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\,\xi \geq -C\omega$$

for some constant C on M_i. Let $\sup\xi = C_1$. Then we have

$$\int_{\tilde{M}_i} \frac{1}{C_1 - \xi + 1}(\sqrt{-1}\partial\overline{\partial}\xi + C\omega) \wedge \omega^{n-2} \geq 0.$$

Consequently we have

$$\int_M \frac{1}{(C_1 - \xi + 1)^2} |\partial\xi|^2 \omega^{n-1} < +\infty. \tag{4.2}$$

Since M has only finitely many components. Using the Cauchy inequality, we have

$$(\int_M |\partial\xi|\omega^{n-1})^2 \leq \int_M \frac{1}{(C_1 - \xi + 1)^2}|\partial\xi|^2\omega^{n-1} \int_M (C_1 - \xi + 1)^2 \omega^{n-1}.$$

The lemma thus follows from (4.1) and (4.2). □

By the above lemma, we can define K stability for singular varieties. It is interesting to compare K stability with Chow-Mumford stability for these singular varieties. A more interesting and nonlinear problem would be that whether the K energy is proper and whether the Moser-Trudinger inequality is true for Fano hypersurfaces. The author strongly believe that they are true, regardless of the existence of Kähler-Einstein metrics.

We end up this paper by giving some observations related to the recent work of Donaldson [4]. We first setup the notations.

Let (M, L) be a polarised Kähler manifold. That is, L is an ample line bundle over the compact complex manifold M. We have the following setting of the metrics: let h be a Hermitian metric on the line bundle L with positive curvature $\mathrm{Ric}\,(h)$. We use $\omega_h = \mathrm{Ric}\,(h)$ to be the Kähler metric of the manifold X. The pair of metrics (h, ω_h) induces an L^2 metric on the complex vector space $H^0(M, L^m)$, where m is a large integer. Let $S_1, \cdots, S_d$ be an orthonormal basis of $H^0(M, L^m)$ under the L^2 metric. The quantity

$$\eta_m = \sum_{i=1}^{d} ||S_i||^2$$

plays an important role in complex geometry. In particular, integrating the quantity η_m, we get the Riemann-Roch Theorem:

$$\dim H^0(M, L^m) = \int_M \eta_m \omega^n = \int_M \mathrm{Td}(R) e^{m\omega},$$

where R is the curvature tensor of the metric ω, and $\mathrm{Td}(R)$ is the Todd polynomial of R.

By the result of Catlin [1] and Zelditch [18] (independently), there is an asymptotic expansion

$$\sum_{i=1}^{d} ||S_i||^2 \sim m^n (a_0 + \frac{a_1}{m} + \cdots + \frac{a_k}{m^k} + \cdots) \tag{4.3}$$

in the sense that

$$\left|\left| \sum_{i=1}^{d} ||S_i||^2 - m^n (a_0 + \frac{a_1}{m} + \cdots + \frac{a_k}{m^k}) \right|\right|_{C^l} \leq \frac{C(k, l, X)}{m^{k+1}},$$

where the constant $C(k, l, X)$ depends on k, l and the manifold M. If for some m, we can make the quantity η_m a constant, then the manifold M is Hilbert-Mumford stable [9, 19]. This result was used by Donaldson [3] in his work to prove the stability of manifolds with constant scalar curvature.

In [6], the author proved that

$$\begin{cases} a_0 = 1 \\ a_1 = \frac{1}{2}\rho \\ a_2 = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|Ric|^2 + 3\rho^2) \end{cases},$$

where R, Ric, ρ represent the curvature, the Ricci curvature and the scalar curvature of the Kähler metric ω_h. The author also proved in the same paper that a_k $(k \in Z)$ is a universal polynomial of the curvature and its derivatives.

Because of the above results, we can view (4.3) as a kind of *local* Riemann-Roch Theorem. It would be very interesting to find the general formula of a_k.

The first section of the paper [4] hints that there might be relations between the coefficients $\{a_k\}$ and the equivariant cohomology. The following observation of the author supports such a speculation.

We assume that M is embedded to CP^N and L is the restriction of the hyperplane bundle to M. Let X be a holomorphic vector field on CP^N that is tangent to X. Let θ be the Hamiltonian function of X. That is,

$$\iota(X)\omega = \frac{\sqrt{-1}}{2\pi}\bar\partial\theta,$$

where ω is the Kähler metric of M. Let $(z_1, \cdots, z_n)$ be a local holomorphic coordinates of M. We define

$$\nabla X = X^k_l \frac{\partial}{\partial z_k} \otimes dz_l,$$

where

$$X^k_l = \frac{\partial X^k}{\partial z_l} + \Gamma^k_{lm} X^m.$$

The following identity is straightforward

$$\iota(X)R = \frac{\sqrt{-1}}{2\pi}\bar\partial\nabla X,$$

where $R = R^i_{jk\bar l}$ is the curvature tensor. Then we have the following identities:

$$\begin{cases} \int_M a_0\theta dV_M = \frac{1}{(n+1)!}\int_M Td_0(R+\nabla X)(\omega+\theta)^{n+1} \\ \int_M a_1\theta dV_M = -\frac{1}{(n)!}\int_M Td_1(R+\nabla X)(\omega+\theta)^{n} \\ \int_M a_2\theta dV_M = -\frac{1}{(n-1)!}\int_M Td_2(R+\nabla X)(\omega+\theta)^{n-1} \\ \qquad\qquad\qquad +\frac{1}{n!}\int_M Td_1(R+\nabla X)\Delta\theta\omega^n \end{cases}, \tag{4.4}$$

where Td_k is the k-th homogeneous polynomial of the Todd invariant function. These identities can be verified directly. The third identity is complicated so that it should not be a coincident.

By the above observation, we have the following

Proposition 4.2. *Using the notations as above, then* $\int_M a_k\theta dV_M$ *for* $k = 0, 1, 2$ *are independent of the choice of the Kähler metric in the fixed polarization* $c_1(L)$*. They are generalized Futaki invariants.*

Proof. We have $(\overline{\partial} - \iota(X))(\omega + \theta) = 0$ and $(\overline{\partial} - \iota(X))(R + \nabla X) = 0$. Thus the integrand $a_k\theta$ is closed under the operator $\overline{\partial} - \iota(X)$. The proposition follows from this fact. □

There are several directions realted to this observation. First, it would be interesting to see that the expansion (4.3) is related to the (generalized) Futaki invariants; second it may be possible to find the general formula of $\{a_k\}$ in terms of equivariant cohomology; more importantly, in [8], we pointed out that $a_k = f$ is an elliptic equation, where f is a smooth function. In particular, $a_1 = const$ is the equation for finding Kähler metrics of constant scalar curvature. Thus prescribing a_k for $k > 1$ is interesting. We will study these questions in our subsequent papers.

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