On A-twisted Moduli Stack for Curves from Witten's Gauged Linear Sigma Models

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Witten's gauged linear sigma model [Wi1] is one of the universal frameworks or structures that lie behind stringy dualities. Its A-twisted moduli space at genus 0 case has been used in the Mirror Principle [L-L-Y] that relates Gromov-Witten invariants and mirror symmetry computations. In this paper the A-twisted moduli stack for higher genus curves is defined and systematically studied. It is proved that such a moduli stack is an Artin stack. For genus 0, it has the A-twisted moduli space of [M-P] as the coarse moduli space. The detailed proof of the regularity of the collapsing morphism by Jun Li in [L-L-Y: I and II] can be viewed as a natural morphism from the moduli stack of genus 0 stable maps to the A-twisted moduli stack at genus 0. Due to the technical demand of stacks to physicists and the conceptual demand of supersymmetry to mathematicians, a brief introduction of each topic that is most relevant to the main contents of this paper is given in the beginning and the appendix respectively. Themes for further study are listed in the end.

Introduction and outline.

Introduction.

Witten's gauged linear sigma model (GLSM) [Wi1] is one of the universal frameworks or structures that lie behind stringy dualities. There are many geometric data that are encoded in a GLSM, in particular a toric variety X. From a gauged linear sigma model, one can obtain two different field theories that have the same local but different global field-theoretic contents as the original theory. These descendant theories are called the A-twist and the B-twist of the original theory. The moduli space of vacua of the A-twisted theory is given by solutions to a system of vortex-type equations. Geometrically each solution corresponds to a system of line bundles-with-a-section on a curve with these sections satisfying some nondegeneracy or

nonvanishing conditions. (The background from field and string theory are summarized in Appendix for mathematicians.)

In general a system of line bundles-with-a-section has nontrivial automorphisms. Thus the correct language to describe and study the related moduli problem in the algebro-geometric setting is stack. Since the stack $\mathcal{AM}_g(X)$ in this moduli problem on one hand is related to curves and on the other hand arises from the A-twisted theory of a gauged linear sigma model, we will call it the *A-twisted moduli stack for curves* (from Witten's GLSM).

To really do geometry on a stack, usually one requires it be an algebraic (Artin or Deligne-Mumford) stack. For such stacks, by passing to a covering system of atlases, many (down-to-earth) concepts in algebraic geometry for schemes, notably cycles, intersection of cycles, coherent sheaves, and derived categories, can also be defined and studied. (A literature guide on stacks is given in Sec. 1 for physicists.) In Sec. 2, we spell out the definition of $\mathcal{AM}_g(X)$ and prove that it is indeed an Artin stack. Hence $\mathcal{AM}_g(X)$ is an object that one may hope to do geometries related to curves.

From the other end, recall the moduli stack $\overline{\mathcal{M}}_{g,n}(X)$ of stable maps studied in Behrend [Be1, Be2], Behrend-Manin [B-M], Fulton-Pandharipande [F-P], Li-Tian [L-T1, L-T2] ... and many others since Kontsevich that are related to Gromov-Witten invariants for algebraic varieties.

It is Witten's insight [Wi1] and Morrison-Plesser's later further push [M-P] with some foundation laid down by Cox [Cox2] that the two moduli stacks $\mathcal{AM}_g(X)$ and $\overline{\mathcal{M}}_{g,0}(X)$ should be closely related. In particular $\mathcal{AM}_g(X)$ could be as useful in the computation of Gromov-Witten invariants as $\overline{\mathcal{M}}_{g,0}(X)$ itself.

At the moment the detail of relations between these two stacks has been carried out for the genus 0 case. Indeed $\mathcal{AM}_0(X)$, or more precisely its coarse moduli space, has been used in the Mirror Principle [L-L-Y: I and II] that relates Gromov-Witten invariants and mirror symmetry computations. The two moduli stacks $\mathcal{AM}_0(X)$ and $\overline{\mathcal{M}}_{0,0}(X)$ are related by the natural morphisms

$$
\begin{array}{ccc}
& \coprod_d M_d(X) & \\
\swarrow & & \searrow \\
\overline{\mathcal{M}}_{0,0}(X) & & \mathcal{AM}_0(X)
\end{array} \quad ,
$$

where $\coprod_d M_d(X)$ is the moduli stack of genus 0 stable maps into $\mathbb{P}^1 \times X$ with the degree on the $\mathbb{P}^1$-component being equal to 1, $\coprod_d M_d(X) \to \overline{\mathcal{M}}_{0,0}(X)$ is the contracting morphism, and $\coprod_d M_d(X) \to \mathcal{AM}_0(X)$ is the collapsing morphism, whose regularity was proved by Jun Li. This is explained more

carefully in Sec. 3 and Sec. 4.

These notes lay down some foundations for several themes to be reported in the future.

Outline.

1. A brief tour and literature-guide on stacks.
2. From Cox functor to Witten's A-twisted moduli stack.
    2.1 The A-twisted moduli stack $\mathcal{AM}_g(X)$.
    2.2 $\mathcal{AM}_g(X)$ is an Artin stack.
3. The $g = 0$ case.
4. The collapsing morphism.

Appendix. Witten's gauged linear sigma models for mathematicians.

1. A brief tour and literature-guide on stacks.

Basic definitions on stacks needed for the discussions are collected in this section for introduction of notations and the convenience of readers/string-theorists.

• **Grothendieck topology and site.** Let (Sch/S_0) be the category of schemes over a base scheme S_0 ([Ha1]). See [G-M: Sec. II.4], [Kr], and [L-MB: Chapter 9] for the definition - and [Mu1] for why they are needed - of the following:

    ○ *Topology* = covering system.

        – *étale* topology.

        – *fppf* topology; fppf = faithfully flat (=flat+surjective), + locally of finite presentation.

        – fpqc topology; fpqc = faithfully flat + quasi-compact

    ○ *Site* on (Sch/S_0) = usual (Sch/S_0) + covering systems.

Notation. Let $f : U' \to U$ be a covering of U in the site (Sch/S_0). We shall adopt the following notations for the projection maps from fibered products:

$$U''' := U' \times_U U' \times_U U' \xrightarrow{\pi_{12},\, \pi_{13},\, \pi_{23}} U'' := U' \times_U U' \xrightarrow{p_1,\, p_2} U'.$$

Such compact notations are particularly useful for diagram-chasings.

• **Grothendieck's theory of descent.** Given a covering morphism $f : U' \to U$ in the site (Sch/S_0), *Grothendieck's theory of descent* studies (1) *when and how a geometric object* (e.g. a coherent sheaf) *on U' can be descended to a geometric object of the same kind on U* and (2) *when and how a morphism between descendable geometric objects on U' descends to a morphism between the descent geometric objects on U*. This is a big generalization of the *local-to-global constructions* in geometry. See [Kr : Lecture 4 and Lecture 5] for an introduction and references.

• **Stacks.** While *varieties* contain only closed points (= the usual geometric points when the ground field k is $\mathbb{C}$), *schemes* (e.g. [E-H] and [Mu4]) contain also nonclosed points to make doing geometry more natural. *Stacks* go one step further to contain "points" with nontrivial automorphisms. A "space" with such a feature is needed to parameterize geometric objects that can have nontrivial automorphisms, e.g. curves and coherent sheaves. Assuming the background on algebraic geometry in [Ha1], then [Mu1], [D-M], [Gó], and [Ed] (in suggested reading order) together give a concrete and solid introduction of algebraic stacks and their natural appearance in moduli problems in algebraic geometry; [L-MB] gives the final up-to-date polishment. See also [Art1], [Art2], [Be2], [Bry], [Gil], [H-M], [Mu3], and [Vi] for more details. Recall that a *groupoid* is a category in which all the morphisms are isomorphisms.

Definition 1.1 [(pre-)stack]. (Cf. [D-M], [Gó], and [Kr].) A *stack* $\mathcal{F}$ *over* (Sch/S_0) is a *category fibered in groupoids* $p_{\mathcal{F}} : \mathcal{F} \to (Sch/S_0)$ such that the assignment of the fiber $\mathcal{F}(U) := p_{\mathcal{F}}^{-1}(U)$ to each $U \in (Sch/S_0)$ is a sheaf of groupoids. I.e. it is an assignment of groupoids

$$U \in (Sch/S_0) \; \longrightarrow \; \mathcal{F}(U)$$

that satisfies the following sheaf axioms : Let $f : U' \to U$ be a covering of U in the site (Sch/S_0).

(1) (*Gluing of morphisms*) Let $\mathcal{E}_1$, $\mathcal{E}_2$ be objects in $\mathcal{F}(U)$ and $\varphi' : f^*\mathcal{E}_1 \to f^*\mathcal{E}_2$ be a morphism in $\mathcal{F}(U')$ such that there exists an isomorphism $\tau : p_1^* f^*\mathcal{E}_1 \to p_2^* f^*\mathcal{E}_2$ in $\mathcal{F}(U'')$. Then there exists a morphism $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ in $\mathcal{F}(U)$ such that $f^*\varphi = \varphi'$.

(2) (*Monopresheaf*) Let $\mathcal{E}_1$, $\mathcal{E}_2$ be objects in $\mathcal{F}(U)$ and $\varphi, \psi : \mathcal{E}_1 \to \mathcal{E}_2$ be morphisms in $\mathcal{F}(U)$ such that $f^*\varphi = f^*\psi$. Then $\varphi = \psi$.

(3) (*Gluing of objects*)   Let $\mathcal{E}'$ be an object in $\mathcal{F}(U')$ and $\tau : p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ be an isomorphism in $\mathcal{F}(U'')$ such that $\pi_{23}^*\tau \circ \pi_{12}^*\tau = \pi_{13}^*\tau$ in $\mathcal{F}(U''')$. Then there exists an object $\mathcal{E}$ in $\mathcal{F}(U)$ and an isomorphism $\sigma : f^*\mathcal{E} \to \mathcal{E}'$ in $\mathcal{F}(U')$ such that $p_2^*\sigma = \tau \circ p_1^*\sigma$.

If $\mathcal{F}$ satisfies only (1) and (2), then it is called a *prestack* over (Sch/S_0). In this case, *sheafification* (official term : *stackification*) of $\mathcal{F}$ gives a stack canonically associated to a prestack (cf. [Ha1 : II. Proposition-Definition 1.2], [Kr : Lecture 7], and [L-MB : Lemma 3.2].)

Remark 1.1.1 [*category fibered in groupoids*]. ([D-M : Sec. 4] and [Kr : Lecture 6].) Given a category $\mathcal{F}$ over (Sch/S_0), $p_\mathcal{F} : \mathcal{F} \to (Sch/S_0)$, it is *fibered in groupoids* over (Sch/S_0) if it satisfies the following morphism-lifting properties :

(1) For any $\varphi : U \to V$ in (Sch/S_0) and $y \in \mathcal{F}(V)$ there is a map $f : x \to y$ in $\mathcal{F}$ with $p_\mathcal{F}(f) = \varphi$.

(2) Given a diagram $\begin{array}{c} x \\ \searrow{}^{f} \\ \quad z \\ \nearrow_{g} \\ y \end{array}$ in $\mathcal{F}$ with its image $\begin{array}{c} U \\ \searrow{}^{\varphi} \\ \quad W \\ \nearrow_{\psi} \\ V \end{array}$ in (Sch/S_0).

Then for all $\chi : U \to V$ such that $\varphi = \psi \circ \chi$, there is a unique $h : x \to y$ such that $f = g \circ h$ and $p_\mathcal{F}(h) = \chi$.

Such lifting properties in many moduli problems, including the one studied in the notes, follow automatically by base-change or fibered products. Hence we will omit mentioning them.

Remark 1.1.2 [*algebraic S_0-spaces*].   For technical reasons in algebro-geometric study of moduli problems, it is natural to introduce the notion of *algebraic S_0-spaces* and use them, instead of schemes, to define atlas and representability of morphisms between stacks, cf. [L-MB : chapters 1 and 10]. Such spaces may be thought of as a collection of étale (instead of Zariski) local charts for a would-be (generally non-existing) scheme. To keep things down to earth, we do not adopt this convention in the notes.

● **Morphisms between stacks.** Let $p_\mathcal{F} : \mathcal{F} \to (Sch/S_0)$ and $p_\mathcal{G} : \mathcal{G} \to (Sch/S_0)$ be stacks over (Sch/S_0). A morphism from $\mathcal{F}$ to $\mathcal{G}$ is a functor $F : \mathcal{F} \to \mathcal{G}$ between the two categories such that $p_\mathcal{G} \circ F = p_\mathcal{F}$. Explicitly for moduli stacks, this means that F sends a flat family of one class of geometric objects to a flat family of another class of geometric objects in a way that

commutes with base change. F is *representable* if for all $X \in (Sch/S_0)$ and morphism $x : X \to \mathcal{G}$, the fibered product (cf. next item) $\mathcal{F} \times_{F,\mathcal{G},x} X$ is also in (Sch/S_0). Properties of schemes (e.g. *proper*, *separated*, *smooth*, etc.) that are stable under base change and of a local nature on the target can be defined for representable morphisms of stacks via fibered products with schemes: [D-M: Sec. 4], [Gó: Sec. 2], and [L-MB: Definitions (3.9) and (3.10.1)].

• **Fibered product.** [Be2: Lecture 1, Groupoids], [Gó: Sec. 2.2], and [L-MB: Sec. (2.2.2)]. Given two morphisms $F : \mathcal{X} \to \mathcal{Z}$ and $G : \mathcal{Y} \to \mathcal{Z}$ of stacks over (Sch/S_0), their *fibered product* $\mathcal{X} \times_{F,\mathcal{Z},G} \mathcal{Y}$ (or denoted $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ when F and G are clear from the text) is defined to be the stack over (Sch/S_0) with

$Objects$:   *Triples* (X, Y, α), where $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, and $\alpha : F(X) \to G(Y)$ is an isomorphism in $\mathcal{Z}$.

$Morphisms$:   A morphism from (X_1, Y_1, α_1) to (X_2, Y_2, α_2) is a *pair* $(\varphi_{\mathcal{X}}, \varphi_{\mathcal{Y}})$ of morphisms $\varphi_{\mathcal{X}} : X_1 \to X_2$, $\varphi_{\mathcal{Y}} : Y_1 \to Y_2$ over the same morphism $f : U \to V$ of schemes in (Sch/S_0) such that the following diagram commutes

$$
\begin{array}{ccc}
F(X_1) & \stackrel{F(\varphi_{\mathcal{X}})}{\longrightarrow} & F(X_2) \\
\alpha_1 \downarrow & & \downarrow \alpha_2 \\
G(Y_1) & \stackrel{G(\varphi_{\mathcal{Y}})}{\longrightarrow} & G(Y_2)
\end{array} \quad .
$$

• *Isom* and **Isom**. (Cf. [D-M: Definition (I.10)], [Gro: Sec. 4], and [Mu1: Sec. 3].) Given a pair of families of geometric objects, e.g. stable curves, $\pi_i : X_i \to S_i$, $i = 1, 2$, then each induces a family of geometric object, still denoted by π_i, over $S_1 \times_{S_0} S_2$ via pullback. Let $(Sets)$ be the category of sets. Then $\underline{Isom}(\pi_1, \pi_2)$ is the functor

$$
\begin{array}{rl}
\underline{Isom}(\pi_1, \pi_2) \ : \ (Sch/S_0) \to & (Sets) \\
S \quad \mapsto & \{(\alpha, \beta) \,|\, \alpha \in Hom(S, S_1 \times_{S_0} S_2) , \ \beta : \alpha^*\pi_1 \simeq \alpha^*\pi_2\} .
\end{array}
$$

In case $\underline{Isom}(\pi_1, \pi_2)$ is a representable functor, the scheme that represents $\underline{Isom}(\pi_1, \pi_2)$ will be denoted by **Isom**(π_1, π_2). Representability of *Isom* in many moduli problems boils down to the representability of the Hilbert functor or the Quot functor ([Gro]). When the moduli problem is described by a stack $\mathcal{X}$ over (Sch/S_0), then π_i correspond to morphisms $F_i : S \to \mathcal{X}$

and $\underline{Isom}(\pi_1, \pi_2) = S_1 \times_{F_1, \mathcal{X}, F_2} S_2$. Similarly for $\underline{Hom}$ and **Hom** that replace isomorphisms in the definition of $\underline{Isom}$ and **Isom** by morphisms.

Definition 1.2 [Artin stack]. ([Gó: Definition 2.22].) An *Artin stack* $\mathcal{F}$ over (Sch/S_0) is stack $\mathcal{F}$ over (Sch/S_0) that satisfies additional conditions:

(1) The diagonal morphism $\Delta_\mathcal{F} \to \mathcal{F} \times_{(Sch/S_0)} \mathcal{F}$ is representable, quasi-compact, and separated.

(2) There exists a scheme U - called an *atlas* - and a smooth and surjective morphism $u : U \to \mathcal{F}$.

See [L-MB: Definition (5.2)] for the definition of the *set of points* $|\mathcal{F}|$ of a stack $\mathcal{F}$.

Definiton 1.3 [coarse moduli space]. (Cf. [Gó: Definition 2.6] and [Vi: (2.1) Definition].) A *coarse moduli space for a stack* $\mathcal{F}$ is a scheme Z together with a morphism $\phi : \mathcal{F} \to Z$ such that

(i) if Z' is another scheme that admits a morphism $\phi' : \mathcal{F} \to Z'$ then there is a unique morphism of schemes $\eta : Z \to Z'$ with $\phi' = \eta \circ \phi$, (i.e. Z *corepresents* $\mathcal{F}$).

(ii) for any algebraically closed field k, the induced map on k-points $|\phi| : |\mathcal{F}|(k) \to Z(k)$ is bijective.

(Thus, when exists, Z is unique up to a canonical isomorphism.)

• **Quotient stack.** [Be2: Lecture 1, Example 18.3 and Example 20.4], [Gó: Example 2.14], and [L-MB: Sec. (2.4.2)]. For $S_0 = Spec\,k$, where k is a ground field, let G be an algebraic group over k. The *quotient stack* of a G-action on a k-scheme X is denoted by $[X/G]$. It is the stackification of the prestack $pre[X/G]$ over (Sch/k). An object of the groupoid $pre[X/G](U)$, $U \in (Sch/k)$, is a diagram

$$
\begin{array}{ccc}
P & \xrightarrow{f} & X \\
\downarrow & & \\
U & &
\end{array} \quad ,
$$

where P is a principal G-bundle over U and f is a G-equivariant k-morphism.

2. From Cox functor to Witten's A-twisted moduli stack.

Notations and terminologies of toric geometry used here follow mainly [Fu], see also [Oda].

2.1. The A-twisted moduli stack $\mathcal{AM}_g(X)$.

Let $N \simeq \mathbb{Z}^n$ be a lattice, M be its dual lattice, Δ be a fan in $N_\mathbb{R}$, $\Delta(1)$ be the 1-dimensional cones of Δ, and n_ρ be the generator of $\rho \cap N$ for $\rho \in \Delta(1)$. Let X be the smooth toric variety associated to Δ and Y be a scheme over S. Recall the following definition from [Cox2]:

Definition 2.1.1 [Δ-collection]. A Δ-*collection* $(L_\rho, u_\rho, c_m)_{\rho,m}$ on Y/S consists of line bundles L_ρ on Y flat over S, sections $u_\rho \in H^0(Y, L_\rho)$ indexed by $\rho \in \Delta(1)$, and a collection of isomorphisms $c_m : \otimes_\rho L_\rho^{\otimes \langle m, n_\rho \rangle} \simeq \mathcal{O}_Y$ indexed by $m \in M$ such that

  (i) *Compatibility*: $c_m \otimes c_{m'} = c_{m+m'}$ for all $m, m' \in M$.

  (ii) *Nondegeneracy*: The map

$$\sum_{\sigma \in \Delta_{max}} \otimes_{\rho \not\subset \sigma} u_\rho^* : \oplus_{\sigma \in \Delta_{max}} \otimes_{\rho \not\subset \sigma} L_\rho^{-1} \to \mathcal{O}_Y$$

is surjective, where $u_\rho^* : L_\rho^{-1} \to \mathcal{O}_Y$ is the dual morphism of $u_\rho : \mathcal{O}_Y \to L_\rho$.

An *isomorphism* $(L_\rho, u_\rho, c_m)_{\rho,m} \xrightarrow{\sim} (L_\rho', u_\rho', c_m')_{\rho,m}$ consists of isomorphisms $\gamma_\rho : L_\rho \xrightarrow{\sim} L_\rho'$ which carry u_ρ to u_ρ' and c_m to c_m'.

Explanation/Fact 2.1.2 [Cox]. For the application in this article, we will consider only the case when the set $\{n_\rho\}_\rho$ spans $N_\mathbb{R}$. In this case $X = (\mathbb{C}^{\Delta(1)} - V(I))/G$, where $\mathbb{C}^{\Delta(1)} = Spec\,\mathbb{C}[x_\rho : \rho \in \Delta(1)]$, I is the ideal generated by $\prod_{\rho \not\subset \sigma} x_\rho$, $\sigma \in \Delta_{max}$, and $G = Hom_\mathbb{Z}(\text{Pic}(X), \mathbb{C}^\times)$ acts on $\mathbb{C}^{\Delta((1)}$ via the exact sequence

$$1 \longrightarrow G \longrightarrow Hom_\mathbb{Z}(\mathbb{Z}^{\Delta(1)}, \mathbb{C}^\times) \longrightarrow T_N \longrightarrow 1\,.$$

The following statements either are explicitly in or follow immediately from [Cox2].

  (1) *The isomorphisms c_m.*

    (1.1) L_ρ are unrelated abstract line bundles on Y/S. To relate the collection of sections u_ρ to a map from Y to $\mathbb{C}^{\Delta(1)}$, some data is needed that enables one to compare sections in different L_ρ - more precisely, the induced sections from u_ρ on isomorphic tensor

products of L_ρ -. The data c_m gives exactly this information up to the G-action. Condition (i) (Compatibility) is the cocycle conditions that make sure this comparison of sections on different L_ρ is consistent among themselves.

(1.2) Given $\{L_\rho\}_\rho$ and two choices $\{c_m\}_m$ and $\{c'_m\}_m$, there exist automorphisms γ_ρ on L_ρ that carry $\{c_m\}_m$ to $\{c'_m\}_m$. Thus, up to isomorphisms, there is exactly one way to compare the line bundles L_ρ.

Reason. (Cf. [Cox2: Theorem 1.1, proof].) A pair of collections of isomorphisms $(\{c_m\}_m, \{c'_m\}_m)$ determines a morphism $\alpha : M \to H^0(Y, \mathcal{O}_Y^*)$. From the long exact sequence

$$0 \to Hom(\text{Pic } X,\ H^0(Y, \mathcal{O}_Y^*)) \to Hom(\mathbb{Z}^{\Delta(1)},\ H^0(Y, \mathcal{O}_Y^*)) \to Hom(M,\ H^0(Y, \mathcal{O}_Y^*))$$
$$\to Ext^1(\text{Pic } X,\ H^0(Y, \mathcal{O}_Y^*))\ (=0) \to \cdots$$

induced from the short exact sequence

$$0 \to M \to \mathbb{Z}^{\Delta(1)} \to \text{Pic } X \to 0\,,$$

one concludes that α can be lifted to a morphism $\widetilde{\alpha} : \mathbb{Z}^{\Delta(1)} \to H^0(Y, \mathcal{O}_Y^*)$. The morphism $\widetilde{\alpha}$ defines then a collection of automorphisms γ_ρ on L_ρ, $\rho \in \Delta(1)$, that carry $\{c_m\}_m$ to $\{c'_m\}_m$.

$\square$

(2) *The nondegeneracy condition.* Recall (e.g. [Ha1: Appendix A.3]) that, given a section $u_\rho : \mathcal{O}_Y \to L_\rho$, the zero-sheme of u_ρ is defined by the ideal sheaf $u_\rho^*(L_\rho^{-1})$. Thus, Condition (ii) (Nondegeneracy) of a Δ-collection means exactly that, for any open immersion $U \hookrightarrow Y$ such that $\mathcal{L}_\rho|_U$ is trivialized by an isomorphism $\mathcal{L}_\rho|_U \simeq \mathcal{O}_U$, the image of the induced map $U \to \mathbb{C}^{\Delta(1)}$ lies completely in $\mathbb{C}^{\Delta(1)} - V(I)$.

Explanation/Fact 2.1.2: Item (2), [Wi1] and [M-P] together lead to the following definitions.

Definition 2.1.3 [weak Δ-collection]. (1) A weak Δ-collection on Y/S is a set of data $(L_\rho, u_\rho, c_m)_{\rho,m}$ as in Definition 2.1.1 with Condition (ii) (Nondegeneracy) replaced by

(ii') *Nonvanishing*:   The map

$$\sum_{\sigma \in \Delta_{max}} \otimes_{\rho \not\subset \sigma} u_\rho^* \; : \; \oplus_{\sigma \in \Delta_{max}} \otimes_{\rho \not\subset \sigma} L_\rho^{-1} \; \to \; \mathcal{O}_Y$$

is not a zero-morphism when restricted to each ir-
reducible component of fibers Y_s over $s \in S$.

Isomorphisms of such data are defined the same as in Definition 2.1.1.

(2) Let (L_ρ, u_ρ, c_m) (resp. (L'_ρ, u'_ρ, c'_m)) be a weak Δ-collection on Y/S (resp. Y'/S'). Then a *morphism* from $(Y/S, (L_\rho, u_\rho, c_m))$ to $(Y'/S', (L'_\rho, u'_\rho, c'_m))$ is a pair (f, γ), where $f : Y \to Y'$ fits into a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\;f\;} & Y' \\ \downarrow & & \downarrow \\ S & \xrightarrow{\;\overline{f}\;} & S' \end{array}$$

and $\gamma : (L_\rho, u_\rho, c_m) \xrightarrow{\sim} f^*(L'_\rho, u'_\rho, c'_m)$ on Y/S.

Definition 2.1.4 [quasistable curves over S]. (Cf. [Ca].) A prestable (i.e. reduced connected nodal) curve is called *quasistable* if it is semistable and all its destabilizing chains have length 1. A *quasistable curve over S* is a flat family $\pi : \mathcal{C} \to S$ of quasistable curves over S. Define $\mathcal{QM}_g$ to be the category fibered in groupoids of quasistable curves over (Sch/S_0).

Definition/Lemma 2.1.5 [$\mathcal{AM}_g(X)$ stack]. *Let (Sch/S_0) be equipped with the fpqc or the fppf topology. Define $\mathcal{AM}_g(X)$ to be the category over (Sch/S_0) whose fiber over $U \in (Sch/S_0)$ is given by the groupoid*

$$\mathcal{AM}_g(X)(U) \; = \; \{ \text{weak } \Delta\text{-collections on quasistable curves } \mathcal{C} \text{ over } U \} \,.$$

Then $\mathcal{AM}_g(X)$ is a stack. We shall call it the A-twisted moduli stack asso-ciated to X for genus g curves.

Remark 2.1.6. Compared with [Wi1 : Sec. 3.4] and [M-P : Sec. 3.7] summa-rized in Appendix, $\mathcal{AM}_g(X)$ is related to the moduli space of the A-twisted gauged linear model in the higher genus case.

Definition/Lemma 2.1.5 follows from the proof of the effectiveness of a descent datum in the case of quasi-coherent sheaves on schemes in (Sch/S_0) and will be proven after the necessary Fact 2.1.7, Fact 2.1.8, and Corollary 2.1.9.

Recall from [Kr] the proof of the following fact: (See also [SGA1].)

Fact 2.1.7 [descent of quasi-coherent sheaves]. *Let $f : U' \to U$ be a fpqc or fppf morphism in (Sch/S_0). Recall the projection maps from Sec. 1*

$$U''' := U' \times_U U' \times_U U' \xrightarrow{\ \pi_{12},\ \pi_{13},\ \pi_{23}\ } U'' := U' \times_U U' \xrightarrow{\ p_1,\ p_2\ } U'\,.$$

(a) *Descent of quasi-coherent sheaves. Let $\mathcal{E}'$ be a quasi-coherent $\mathcal{O}_{U'}$-module and $\tau : p_1^* \mathcal{E}' \to p_2^* \mathcal{E}'$ be an isomorphism that satisfies $\pi_{23}^* \tau \circ \pi_{12}^* \tau = \pi_{13}^* \tau$. Then there exists a quasi-coherent $\mathcal{O}_U$-module $\mathcal{E}$ on U together with an isomorphism $\sigma : f^* \mathcal{E} \to \mathcal{E}'$ such that $p_2^* \sigma = \tau \circ p_1^* \sigma$. The sheaf $\mathcal{E}$ is unique up to a canonical isomorphism.*

(b) *Descent of morphisms. Let $(\mathcal{E}', \tau)$ and $(\mathcal{F}', \upsilon)$ be descent data and $(\mathcal{E}, \sigma)$ and $(\mathcal{F}, \rho)$ be their respective descent as in Item (a). Let $h' : \mathcal{E}' \to \mathcal{F}'$ be a morphism that satisfies $p_2^* h' \circ \tau = \upsilon \circ p_1^* h'$. Then there exists a unique morphism $h : \mathcal{E} \to \mathcal{F}$ such that $\rho \circ f^* h = h' \circ \sigma$.*

Sketch of proof. Consider the following stricter version of Statement (b):

(b^*) *Let $\mathcal{E}$ and $\mathcal{F}$ be quasi-coherent sheaves on U and $h' : f^* \mathcal{E} \to f^* \mathcal{F}$ be a morphism of $\mathcal{O}_{U'}$-modules such that $p_1^* h' = p_2^* h'$. Then there exists a unique morphism $h : \mathcal{E} \to \mathcal{F}$ such that $f^* h = h'$.*

Via diagram chasings, Statement (b) follows from Statement (b^*) and the existence part of Statement (a). The proof now consists of three steps.

(1) *Case: $f = $ faithfully flat morphism between affine schemes.*

(1.a) *Descent of quasi-coherent sheaves.*

Define $\overline{f} := f \circ p_1 \ (= f \circ p_2)$ and let $p_1^\sharp, p_2^\sharp : \mathcal{O}_{U'} \to \mathcal{O}_{U''}$ be the defining ring morphism of structure sheaves associated to p_1, p_2 respectively. Then their difference $p_1^\sharp - p_2^\sharp$ defines an $\mathcal{O}_U$-module morphism $q : f_* \mathcal{O}_{U'} \to \overline{f}_* \mathcal{O}_{U''}$. These morphisms induce natural morphisms $\widehat{p_1} : \mathcal{E}' \to p_1^* \mathcal{E}'$ and $\widehat{p_2} : \mathcal{E}' \to p_2^* \mathcal{E}'$ by pulling back global sections.

Let $\Theta := \tau \circ \widehat{p_1} - \widehat{p_2} : \mathcal{E}' \to p_2^* \mathcal{E}'$. Then the descent $\mathcal{E}$ of the descent datum $(\mathcal{E}', \tau)$ is given by $\mathcal{E} = f_* \, Ker\, \Theta$. By definition $\mathcal{E}$ fits into the following exact sequence of $\mathcal{O}_U$-modules

$$0 \longrightarrow \mathcal{E} \xrightarrow{\ \iota\ } f_* \mathcal{E}' \xrightarrow{\ \Theta\ } \overline{f}_* \, (p_2^* \mathcal{E}')\,.$$

τ determines an isomorphism $k : f^* \overline{f}_* p_2^* \mathcal{E}' \to p_{2*} \overline{f}^* f_* \mathcal{E}'$ and an automorphism h on $f^* f_* \mathcal{E}'$. These fit into a commutative square of $\mathcal{O}_{U'}$-modules

$$
\begin{array}{ccccccc}
0 & \longrightarrow & f^* \mathcal{E} & \xrightarrow{f^* \iota} & f^* f_* \mathcal{E}' & \xrightarrow{f^* \Theta} & f^* \overline{f}_* p_2^* \mathcal{E}' \\
& & & & \downarrow h & & \downarrow k \\
0 & \longrightarrow & \mathcal{E}' & \xrightarrow{\widetilde{f}} & f^* f_* \mathcal{E}' & \xrightarrow{\widetilde{q}} & p_{2*} \overline{f}^* f_* \mathcal{E}'
\end{array} ,
$$

where $\widetilde{f}$ and $\widetilde{q}$ are natural morphisms induced by f and q respectively and both horizontal complexes are exact. This determines an isomorphism $\sigma : f^* \mathcal{E} \to \mathcal{E}'$ that satisfies the conditions in Statement (a). It has the property that if s' is a global section in $Ker\,\Theta$ and s is its corresponding global section in $\mathcal{E}$, then $\sigma(f^* s) = s'$. Uniqueness of $\mathcal{E}$ up to a canonical isomorphism follows from Item (1.b*) below.

$(1.b^*)$ *Descent of morphisms.*

The following natural sequence of $\mathcal{O}_U$-modules induced by f and q is exact

$$
0 \longrightarrow \mathcal{F} \xrightarrow{\widehat{f}} f_* f^* \mathcal{F} \xrightarrow{\widehat{q}} \overline{f}_* \overline{f}^* \mathcal{F} .
$$

Hence for any global section s in $\mathcal{E}$, let $f^*(s)$ be the corresponding global section in $f^* \mathcal{E}$, then $h'(f^*(s)) = f^*(t)$ for a unique global section in $\mathcal{F}$.

One can now define the descent morphism $h : \mathcal{E} \to \mathcal{F}$ by setting $h(s) = t$. By definition $f^* h = h'$. Since f^* is an exact and faithful functor, such h is unique.

(2) *Statements for reductions.*

Let $R \xrightarrow{f_1} S \xrightarrow{f_2} T$ be a chain of morphisms of schemes in (Sch/S_0). By chasing the following two diagrams:

$$
\begin{array}{ccccc}
R \times_S R \times_S R & \longrightarrow & R \times_T R \times_T R & \longrightarrow & S \times_T S \times_T S \\
\Downarrow\Downarrow\Downarrow & & \Downarrow\Downarrow\Downarrow & & \Downarrow\Downarrow\Downarrow \\
R \times_S R & \longrightarrow & R \times_T R & \longrightarrow & S \times_T S \\
& \searrow & \Downarrow & & \Downarrow \\
& & R & \xrightarrow{f_1} & S \qquad \xrightarrow{f_2} T
\end{array}
$$

and

$$
\begin{array}{ccc}
(R \times_S R) \times_T (R \times_S R) & \simeq & (R \times_T R) \times_{S \times_T S} (R \times_T R) \\
\Downarrow & & \Downarrow \\
R \times_S R & \longrightarrow & R \times_T R ,
\end{array}
$$

where all the morphisms are natural projection maps from fibered products, one concludes the following statements for reduction:

Reduction (2.1). *Suppose that* (b^*) *holds for* f_1 *and that for any quasi-coherent sheaves* $\mathcal{A}$, $\mathcal{B}$ *on* $S \times_T S$, *the map*

$$\eta^* : Hom_{S \times_T S}(\mathcal{A}, \mathcal{B}) \to Hom_{R \times_T R}(\eta^*\mathcal{A}, \eta^*\mathcal{B})$$

is injective, then (b^*) *holds for* f_2 *if and only if* (b^*) *holds for* $f_2 \circ f_1$.

Reduction (2.2). *Suppose that* (b^*) *holds for* f_1 *as well as any pull-back of* f_1. *Suppose also that* (a) *and* (b) *hold for* f_1, *then* (a) *and* (b) *hold for* f_2 *if and only if* (a) *and* (b) *hold for* $f_2 \circ f_1$.

(3) *General case by reductions.*

For a general fpqc or fppf morphism $f : U' \to U$ between S_0-schemes, let $\{V_i\}_{i \in I}$ be a Zariski affine cover of U and for each $i \in I$ let $\{V'_{i,j}\}_{j \in J_i}$ be a Zariski affine cover of $f^{-1}(V_i)$. If f is fpqc, then J_i can be assumed to be a finite set. Then for each i the map $f_i : \coprod_j V'_{i,j} \to V_i$ is a faithfully flat affine morphism and hence (a) and (b) hold for f_i. Applying Reduction (2.1) and Reduction (2.2) to chains of morphisms

$$\coprod_{i,j} V'_{i,j} \xrightarrow{\amalg_i f_i} \coprod_i V_i \longrightarrow U \quad \text{and} \quad \coprod_{i,j} V'_{i,j} \longrightarrow U' \xrightarrow{f} U,$$

one concludes that (a) and (b) hold for f. If f is fppf, then $V_{i,j} := f(V'_{i,j})$ are open in U. For each i, $\{V_{i,j}\}_j$ is a Zariski open cover of V_i and each morphism $V'_{i,j} \to V_{i,j}$ is fpqc. Applying Reduction (2.1) and Reduction to (2.2) to chains of morphisms

$$\coprod_j V'_{i,j} \longrightarrow \coprod_j V_{i,j} \longrightarrow V_i, \qquad \coprod_{i,j} V'_{i,j} \longrightarrow \coprod_i V_i \longrightarrow U,$$

and

$$\coprod_{i,j} V'_{i,j} \longrightarrow U' \xrightarrow{f} U,$$

one concludes that (a) and (b) hold for f. This concludes the sketch. $\square$

Recall that $\mathcal{QM}_g$ is the category fibered in groupoids of quasistable curves over (Sch/S_0).

Fact 2.1.8 [$\mathcal{QM}_g$ Artin]. ([Be1].) $\mathcal{QM}_g$ *is an Artin stack for all* $g \geq 0$.

Explanation. This follows from [Be1 : Preliminaries on prestable curves] since $\mathcal{QM}_g$ is an open substack of the Artin stack of prestable curves. □

Corollary 2.1.9 [line bundle with a section]. *The category fibered in groupoids of line bundles with a section on quasistable curves over S_0-schemes is a stack.*

Proof. Since $\mathcal{QM}_g$ is an Artin stack, a descent datum for quasistable curves descends effectively. If $\underline{f} : S_1 \to S_2$ is a fppf morphism and $\mathcal{C}_2$ is a quasistable curve over S_2, then $\mathcal{C}_1 := \underline{f}^*\mathcal{C}_2 = S_1 \times_{S_2} \mathcal{C}_2$ is a quasistable curve over S_1 and the morphism $f : \mathcal{C}_1 \to \mathcal{C}_2$ from fibered product is also fppf. We shall apply Fact 2.1.7 and its proof to $f : \mathcal{C}_1 \to \mathcal{C}_2$.

Given a descent data $(\mathcal{E}', \tau)$ on $\mathcal{C}_1$ with $\mathcal{E}'$ invertible, let $\mathcal{E}$ be its descent on $\mathcal{C}_2$. Since $f^*\mathcal{E} \simeq \mathcal{E}'$ and f is fppf, $\mathcal{E}$ must be invertible as well.

$$
\begin{array}{ccc}
\mathcal{C}_1 & \xrightarrow{f} & \mathcal{C}_2 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
S_1 & \xrightarrow{\underline{f}} & S_2
\end{array}
$$

Thus Fact 2.1.7 remains true if quasi-coherent sheaves are replaced by invertible sheaves (cf. [Mu1 : Theorem 90 (Hilbert-Grothendieck)]).

In the construction of the descent of quasi-coherent sheaves and their morphisms for the affine case in the proof of Fact 2.1.7, one observes that if a global section s' is added to a descent datum $(\mathcal{E}', \tau)$ that satisfies the gluing condition given by τ, then it must lies in $Ker\,\Theta$ and hence descends to a global section s in $\mathcal{E}$. Furthermore, the statements in Item $(1.b^*)$ in that proof imply that such s is unique. I.e. the descent remains effective with a section added to the data. Consequently, Statement (a) and Statement (b) in Fact 2.1.7 hold for descent data of invertible sheaves with a section when f is a faithfully flat morphism between affine schemes. Now observe that in the remaining part of the proof of Fact 2.1.7, precisely two things are used repeatedly :

(i) *going-down* : effective descent by a morphism, for which Statement
                    (a) and Statement (b) are known to hold,

(ii) *going-up* :   pulling back a descent datum.

Whenever a going-down is employed, the existence and uniqueness of descent global section are known to hold by earlier reductions from the affine case while a going-up takes any part of descent data to the unique corresponding

part of descent data automatically. Thus the whole proof of Fact 2.1.7 goes through without change. This proves the lemma.   □

Proof of Definition/Lemma 2.1.5. Follow the notations from previous discussions. Since the nonvanishing condition of weak Δ-collections is an open condition, we only need to show that the data without this condition give a stack.

(a) *Descent of weak Δ-collections.* For each $\rho \in \Delta(1)$ the existence and uniqueness of descent of the descent data $(L'_\rho, u'_\rho ; \tau_\rho)$ on $\mathcal{C}_1$ to $(L_\rho, u_\rho ; \sigma_\rho)$ on $\mathcal{C}_2$ follow from Corollary 2.1.9. Similarly for morphisms between two such descent data. Each $(\otimes_\rho L'_\rho{}^{\otimes \langle m,n_\rho \rangle}, \otimes_\rho \tau_\rho{}^{\otimes \langle m,n_\rho \rangle})$, as well as $(\mathcal{O}_{\mathcal{C}_1}, Id)$, is a descent datum of line bundles on $\mathcal{C}_1$. Their descent on $\mathcal{C}_2$ are given by $(\otimes_\rho L_\rho{}^{\otimes \langle m,n_\rho \rangle}, \otimes_\rho \sigma_\rho{}^{\otimes \langle m,n_\rho \rangle})$ and $(\mathcal{O}_{\mathcal{C}_2}, Id)$ respectively. Thus $c'_m : \otimes_\rho L'_\rho{}^{\otimes \langle m,n_\rho \rangle} \to \mathcal{O}_{\mathcal{C}_1}$, as an isomorphism between two descent data of line bundles, descends to a unique isomorphism $c_m : \otimes_\rho L_\rho{}^{\otimes \langle m,n_\rho \rangle} \to \mathcal{O}_{\mathcal{C}_2}$. This shows that a descent datum of weak Δ-collection on quasistable curves descends effectively.

(b) *Descent of morphisms.* Since a descent datum of isomorphisms from a weak Δ-collection to another is really that for line bundles, it descends uniquely.

This concludes the proof.   □

2.2. $\mathcal{AM}_g(X)$ is an Artin stack.

Proposition 2.2.1 [$\mathcal{AM}_g(X)$ Artin].   *The A-twisted moduli stack $\mathcal{AM}_g(X)$ for genus g quasistable curves is an Artin stack.*

Proof. We check the properties that the diagonal morphism needs to satisfy and construct an atlas for $\mathcal{AM}_g(X)$ via a relative construction.

(a) *Representability, quasi-compactness, and separatedness of the diagonal morphism.*

These properties are reflected in the corresponding properties of the *Isom*-functor (cf. [Gó: Sec. 2.2]), which we will now check.

(a.1) *Representability of $\underline{Isom}_U(\mathcal{F}_1, \mathcal{F}_2)$.* Let $U \in (Sch/S_0)$ and $\mathcal{F}_1$, $\mathcal{F}_2 \in \mathcal{AM}_g(X)(U)$ with the underlying quasistable curves over U denoted by $\pi_1 : \mathcal{C}_1 \to U$, $\pi_2 : \mathcal{C}_2 \to U$ respectively. Let $\underline{Isom}_U(\mathcal{F}_1, \mathcal{F}_2)$ be the functor on (Sch/U) associating to each U-scheme $f : U' \to U$ the set of

U'-isomorphisms from $f^*\mathcal{F}_1$ to $f^*\mathcal{F}_2$. By passing through a standard limit, one may assume that the base schemes U and U' are Noetherian and of finite type over S_0 (cf. [L-MB: Théorème (4.6.2.1), proof]). There is a natural morphism of functors $\underline{Isom}_U(\mathcal{F}_1, \mathcal{F}_2) \to \underline{Isom}_U(\mathcal{C}_1, \mathcal{C}_2)$, whose fiber over an isomorphism $\varphi : f^*\mathcal{C}_1 \to f^*\mathcal{C}_2$ over $f : U' \to U$ is the set of isomorphisms from $f^*\mathcal{F}_1$ to $\varphi^*f^*\mathcal{F}_2$ on $f^*\mathcal{C}_1$. From [Gro] and Fact 2.1.8 the functor $\underline{Isom}_U(\mathcal{C}_1, \mathcal{C}_2)$ is represented by a scheme $\mathbf{Isom}_U(\mathcal{C}_1, \mathcal{C}_2)$ quasi-compact and separated over U. Let $h_0 : \mathbf{Isom}_U(\mathcal{C}_1, \mathcal{C}_2) \to U$ be the natural morphism, then there is a canonical isomorphism $\Phi : h_0^*\mathcal{C}_1 \xrightarrow{\sim} h_0^*\mathcal{C}_2$. Denote $h_0^*\mathcal{C}_1$ over $Y_0 := \mathbf{Isom}_U(\mathcal{C}_1, \mathcal{C}_2)$ by $\widetilde{\mathcal{C}}_1$. Consider the functor $\underline{Isom}_{\widetilde{\mathcal{C}}_1/Y_0}(h_0^*\mathcal{F}_1, \Phi^*h_0^*\mathcal{F}_2)$. From [Gro: Sec. 4] (also [L-MB: Théorème (4.6.2.1), proof]), if one just focuses on the part of collections of line bundles $\{L_\rho\}_\rho$ in weak Δ-collections, then the $\underline{Isom}$-functor is represented by an open subscheme Y_2 of a scheme Y_1 that is affine and of finite type over Y_0. The additional data: sections u_ρ and trivialization isomorphisms c_m of $\otimes_\rho L_\rho^{\otimes\langle m, n_\rho\rangle}$, specify a locally closed subscheme Y_3 of Y_2. Thus $\underline{Isom}_{\widetilde{\mathcal{C}}_1/Y_0}(h_0^*\mathcal{F}_1, \Phi^*h_0^*\mathcal{F}_2)$ is represented by Y_3 over Y_0. In summary,

$$Y_3 \xrightarrow{h_3} Y_2 \xrightarrow{h_2} Y_1 \xrightarrow{h_1} Y_0 = \mathbf{Isom}_U(\mathcal{C}_1, \mathcal{C}_2) \xrightarrow{h_0} U \,.$$

$$\begin{array}{ccccc} \text{locally} & \text{open} & \text{affine,} & \text{quasi-} & \\ \text{closed} & \text{immersion} & \text{of} & \text{compact,} & \text{Noetherian} \\ \text{immersion} & & \text{finite} & \text{separated} & \\ & & \text{type} & & \end{array}$$

Let $p : Y_3 \to Y_0$ be the composition $h_1 \circ h_2 \circ h_3$, then there is a canonical isomorphism $\Psi : p^*\mathcal{F}_1 \to p^*\Phi^*\mathcal{F}_2$ over $p^*h_0^*\mathcal{C}_1/Y_0$.

Claim. *The functor $\underline{Isom}_U(\mathcal{F}_1, \mathcal{F}_2)$ is represented by Y_3.*

Proof. One checks the functorial properties for a scheme that represents an $\underline{Isom}$-functor. Let $g : U' \to U$ be a U-scheme and $\widetilde{\gamma} : g^*\mathcal{F}_1 \to g^*\mathcal{F}_2$ be an isomorphism. Let $\gamma : g^*\mathcal{C}_1 \to g^*\mathcal{C}_2$ be the underlying isomorphism of quasistable curves over U'. Then one may rewrite $\widetilde{\gamma}$ as an isomorphism $\widetilde{\gamma} : g^*\mathcal{F}_1 \to \gamma^*g^*\mathcal{F}_2$ on $g^*\mathcal{C}_1$. Since Y_0 represents the functor $\underline{Isom}_U(\mathcal{C}_1, \mathcal{C}_2)$, there is a unique U-morphism $h : U' \to Y_0$ such that $(\gamma : g^*\mathcal{C}_1 \to g^*\mathcal{C}_2)$ is the pullback of the canonical isomorphism $(\Phi : h_0^*\mathcal{C}_1 \to h_0^*\mathcal{C}_2)$ over Y_0. Since $h_0 \circ h = g$, the data $(\widetilde{\gamma} : g^*\mathcal{F}_1 \to \gamma^*g^*\mathcal{F}_2)$ is the same as $\widetilde{\gamma} : h^*h_0^*\mathcal{F}_1 \to h^*\Phi^*h_0^*\mathcal{F}_2$. Since Y_3 is the scheme that represents the functor $\underline{Isom}_{\widetilde{\mathcal{C}}_1/Y_0}(h_0^*\mathcal{F}_1, \Phi^*h_0^*\mathcal{F}_2)$, there is a unique Y_0-morphism $\widetilde{h} : U' \to Y_3$ (i.e. a lifting of h) such that $\widetilde{\gamma}$ is the pullback of Ψ via $\widetilde{h}$. This shows that $\underline{Isom}_U(\mathcal{F}_1, \mathcal{F}_2) = Hom(\,\cdot\,, Y_3)$ and hence Y_3 represents $\underline{Isom}_U(\mathcal{F}_1, \mathcal{F}_2)$. $\qquad\square$

(*a*.2) *Separatedness.* Recall the schemes Y_i and the morphisms h_i between them from the discussion in Part (*a*.1). From Hartshorne [Ha]: (*i*) affine morphisms (cf. h_1) are separated [Ha: II. Exercise 5.17(b)], (*ii*) open and closed - and hence locally closed - immersions (cf. h_2, h_3) are separated [Ha: II. Corollary 4.6(a)], and (*iii*) composition of separated morphisms is separated, it follows that $Y_3 \to U$ is separated since h_0 is separated.

(*a*.3) *Quasi-compactness.* Recall again from Hartshorne [Ha: II. Exercise 3.3(a)] that a morphism of schemes is of finite type if and only if it is locally of finite type and quasi-compact. Since all the morphisms h_i are of finite type, so is their composition $Y_3 \to U$, which then must be quasi-compact.

These together justify that the diagonal morphism is representable, quasi-compact, and separated.

(*b*) *Construction of an atlas.*

We follow a relative construction, which is completed in three steps.

(*b*.1) *Atlas U_0 for $\mathcal{QM}_g$.* It follows from the discussion in [Be1] that an atlas for $\mathcal{QM}_g$ can be chosen as follows. Observe that for C a quasistable curve of genus $g \geq 0$, the number of unstable components of C is bounded strictly by 1, for $g = 0$, 1, and $3g - 3$, for $g \geq 2$, and the number of marked points on C needed to stabilize all these components is bounded by $n_0 = 3$ for $g = 0$, 1 for $g = 1$, and $3g - 3$ for $g \geq 2$. Let $\overline{\mathcal{M}}^q_{g,n_0}$ be the open substack of the Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n_0}$ that consists of stable curves of genus g with n_0 marked points such that when these marked points are forgotten, the underlying curves are quasistable. Then $\overline{\mathcal{M}}^q_{g,n_0}$ is a Deligne-Mumford stack and the morphism $F : \overline{\mathcal{M}}^q_{g,n_0} \to \mathcal{QM}_g$ induced by forgetting the marked points is representable, smooth, and surjective. (Indeed, if $V \in (Sch/S_0)$ and $(\pi : \mathcal{C} \to V) \in \mathcal{QM}_g(V)$, then the fibered-product morphism $\overline{\mathcal{M}}^q_{g,n_0} \times_{F,\mathcal{QM}_g,\pi} V \to V$ is the restriction of the morphism of S_0-schemes

$$(\underbrace{\mathcal{C} \times_V \cdots \times_V \mathcal{C}}_{n_0\text{-many times}})^{(0)} \longrightarrow V$$

to an open subscheme of $(\mathcal{C} \times_V \cdots \times_V \mathcal{C})^{(0)}$, where $(\mathcal{C} \times_V \cdots \times_V \mathcal{C})^{(0)}$ is the fibered product $\mathcal{C} \times_V \cdots \times_V \mathcal{C}$ with all the diagonals of the fibered product $\times_V$ and the locus of nodes of fibers of $\mathcal{C} \to V$ removed.) This re-justifies that $\mathcal{QM}_g$ is an Artin stack. Recall the Hilbert scheme construction in [F-P: Sec. 2] (with $\mathbb{P}^r$ therein set to $\mathbb{P}^0 = \{pt\}$) that realizes $\overline{\mathcal{M}}_{g,n_0}$ as a quotient stack of a quasi-projective variety U acted on by an algebraic group.

(Caution that a "quasistable curve" in [F-P] is a "prestable curve" in [Be1] and the current article.) There is an open subset U_0 in U whose geometric points corresponds to quasistable curves. This U_0 is then an atlas for $\mathcal{QM}_g$. Since it comes from a Hilbert scheme construction, the associated flat family of quasistable curves $\pi : \mathcal{C}_0 \to U_0$ is projective. We shall fix a relative very ample line bundle on $\mathcal{C}_0/U_0$.

(b.2) *Atlas U_1 for $\mathcal{WD}^X_{\mathcal{C}_0/U_0}$.* Consider now the stack $\mathcal{WD}^X_{\mathcal{C}_0/U_0}$ over (Sch/S_0) of weak Δ-collections on $\mathcal{C}_0/U_0$. Define an atlas V for a stack $\mathcal{S}$ in the same way as that for an Artin stack, namely a morphism $V \to \mathcal{S}$ that is representable, smooth, and surjective. Then an atlas for $\mathcal{WD}^X_{\mathcal{C}_0/U_0}$ can be constructed by a sequence of relative constructions given in the following steps.

   (b.2.1)   The Quot-scheme construction for an atlas V_1 for the Artin stack $Bun_1(\mathcal{C}_0/U_0)$ of line bundles on $\mathcal{C}_0$ flat over U_0 (cf. [Gómez: Sec. 2.3, Example 2.24] and [L-MB: Example (4.6.2)]). V_1 can be decomposed into a disjoint union of components labelled by the degree of the line bundles on fiber of $\mathcal{C}_0 \to U_0$ since the degree determines the Hilbert polynomial of line bundles on curves of fixed genus. By construction there are a natural quasi-projective morphism $V_1 \to U_0$ and a tauto-logical line bundle $\widetilde{\mathcal{L}}$ on $V_1 \times_{U_0} \mathcal{C}_0$ over V_1.

   (b.2.2)   An atlas V_2 for the stack $Bun_{1,s}(\mathcal{C}_0/U_0)$ of line bundles with a section on $\mathcal{C}_0/U_0$ is given by the scheme

$$V_2 \; := \; \mathbf{Hom}_{(V_1 \times_{U_0} \mathcal{C}_0)/V_1}(\mathcal{O}_{V_1 \times_{U_0} \mathcal{C}_0}, \, \widetilde{\mathcal{L}}) \, .$$

It is affine and surjective over V_1, ([Gro: Sec. 4] and [L-MB: Sec. (4.6.2)]). The fibered product

$$V_3 \; := \; \underbrace{V_2 \times_{U_0} \cdots \times_{U_0} V_2}_{|\Delta(1)|\text{-many times}}$$

gives an atlas for the stack of $|\Delta(1)|$-tuple of line bundles with a section on the same quasistable curve.

   (b.2.3)   The tensor product conditions $\otimes_\rho L_\rho^{\otimes\langle m, n_\rho\rangle} \simeq \mathcal{O}_C$ in a weak Δ-collection on a quasistable curve C are locally closed conditions. To-gether they determine a locally closed subscheme V_4 in V_3. Fix a basis for the M-lattice, then over $V_4 \times_{U_0} \mathcal{C}_0$ there is a *rank M*-tuple of line bundles $(\widetilde{\mathcal{L}}_m)_m$ defined by $(\otimes_\rho L_\rho^{\otimes\langle m, n_\rho\rangle})_m$, where m runs over

the fixed basis. To add in the data of the choices of trivialization $c_m : \otimes_\rho L_\rho^{\otimes\langle m, n_\rho\rangle} \simeq \mathcal{O}_C$, one takes the scheme

$$V_5 := \oplus_{m\in\text{basis}} \mathbf{Isom}_{(V_4\times_{U_0}\mathcal{C}_0)/V_4}(\widetilde{\mathcal{L}}_m, \mathcal{O}_{V_4\times_{U_0}\mathcal{C}_0}),$$

which is an affine bundle over V_4 with fiber the abelian group $\prod^{rank\, M} Spec\, k[t, t^{-1}]$.

(b.2.4) Finally, let U_1 be the open subscheme in V_5 that corresponds to the nonvanishing condition of a weak Δ-collection. Note that we start with V_1 that is smooth and surjective over the stack $Bun_1(\mathcal{C}_0/U_0)$. As we start to enlarge $Bun_1(\mathcal{C}_0/U_0)$ to tuples of line bundles, choice of sections, and so on or doing restriction by imposing open, closed or locally closed conditions, these extra data or restrictions do not have nontrivial automorphisms. Thus they do not influence the representability and the smoothness of the morphism of resulting V_i to the related stack in the discussion. Also, by construction they are surjective. Thus U_1 is an atlas for the stack $\mathcal{WD}_{\mathcal{C}_0/U_0}^X$. By construction there is a natural morphism $U_1 \to U_0$.

(b.3) U_1 as an atlas for $\mathcal{AM}_g(X)$. By construction there is a relative weak Δ-collection $\mathcal{F}_1$ on the quasistable curve $\mathcal{C}_1/U_1$. This gives a morphism $f_1 : U_1 \to \mathcal{AM}_g(X)$. Let $\mathcal{F}$ be a relative weak Δ-collection on a quasistable curve $\mathcal{C}_W$ over $W \in (Sch/S_0)$. This specifies a morphism $f_W : W \to \mathcal{AM}_g(X)$. By the functorial properties of $\mathbf{Isom}$ and the morphism $U_1 \to U_0$, one has the following natural morphisms

$$\mathbf{Isom}(\mathcal{C}_1/U_1, \mathcal{C}_W/W) = U_1 \times_{U_0} \mathbf{Isom}(\mathcal{C}_0/U_0, \mathcal{C}_W/W) \xrightarrow{\pi} \mathbf{Isom}(\mathcal{C}_0/U_0, \mathcal{C}_W/W)$$

and

$$\begin{array}{ccc} \pi_1^*\mathcal{C}_1 \simeq \pi^*\pi_{10}^*\mathcal{C}_0 \xrightarrow[]{\alpha := \pi^*(\alpha_0)}_{\sim} \pi_2^*\mathcal{C}_W \simeq \pi^*\pi_{20}^*\mathcal{C}_W & \qquad & \pi_{10}^*\mathcal{C}_0 \xrightarrow[]{\alpha_0}_{\sim} \pi_{20}^*\mathcal{C}_W \\ \diagdown \quad \diagup & & \diagdown \quad \diagup \\ \mathbf{Isom}(\mathcal{C}_1/U_1, \mathcal{C}_W/W) & \xrightarrow{\pi} & \mathbf{Isom}(\mathcal{C}_0/U_0, \mathcal{C}_W/W). \\ \pi_1 \diagup \quad \diagdown \pi_2 & & \pi_{10} \diagup \quad \diagdown \pi_{20} \\ U_1 \qquad W & & U_0 \qquad W \end{array}$$

It follows that

$$
\begin{aligned}
U_1 \times_{f_1,\, \mathcal{AM}_g(X),\, f_W} W \; &= \; \mathbf{Isom}\,(\mathcal{F}_1, \mathcal{F}_{\mathcal{C}_W/W}) \\
&= \; \mathbf{Isom}_{\,\pi_1^* \mathcal{C}_1/\mathbf{Isom}(\mathcal{C}_1/U_1,\, \mathcal{C}_W/W)}\,(\pi_1^* \mathcal{F}_1,\, \alpha^* \pi_2^* \,\mathcal{F}_W) \\
&= \; \mathbf{Isom}\,\Big((\mathcal{F}_1)_{\mathcal{C}_1/U_1/U_0}\,,\; (\alpha_0^* \pi_{20}^* \mathcal{F}_W)_{(\pi_{10}^* \mathcal{C}_0)/U_0} \Big) \\
&= \; U_1 \times_{\mathcal{WD}_{\mathcal{C}_0/U_0}^X} \mathbf{Isom}\,(\mathcal{C}_0/U_0,\, \mathcal{C}_W/W) \\[4pt]
&\overset{\text{smooth and surjective}}{\longrightarrow} \;\; \mathbf{Isom}\,(\mathcal{C}_0/U_0,\, \mathcal{C}_W/W) \; = \; U_0 \times_{\mathcal{QM}_g} W \\[4pt]
&\overset{\text{smooth and surjective}}{\longrightarrow} \;\; W \,.
\end{aligned}
$$

This shows that U_1 is an atlas for $\mathcal{AM}_g(X)$ and we conclude the proof. $\quad\square$

Remark 2.2.2. The above type of relative construction can be found also in the study of relative GIT construction of universal moduli spaces, e.g., [Hu] and [Pa].

3. The $g = 0$ case.

Since $\mathbb{CP}^1$ is rigid, the problem may be treated as in the study of bundles on a fixed variety. $\mathcal{AM}_0(X)$ is then the stackification of the prestack $pre\mathcal{AM}_0(X)$, whose fiber $pre\mathcal{AM}_0(X)$ over $S \in (Sch/S_0)$ is the groupoid

$$
pre\mathcal{AM}_0(X)(S) \; = \; \{ \text{ weak } \Delta\text{-collections on } S \times \mathbb{CP}^1 \text{ over } S \,\}\,.
$$

On the other hand one has the construction of Morrison-Plesser [M-P : Sec. 3.7], as is used in [L-L-Y : II, Sec. 2.4, Example 4]. In this section we shall discuss how Morrison-Plesser's construction is related to Cox's work and the stack $\mathcal{AM}_0(X)$ adapted from Sec. 2. We shall assume that $X = X_\Delta$ is convex throughout this section. In particular, this implies that every entry d_ρ of a multi-degree $d = (d_\rho)_\rho$ in the discussions are all nonnegative integers.

The small universal weak Δ-collection à la Morrison-Plesser.

Fix a presentation: $\mathbb{P}^1 = \mathrm{Proj}\,\mathbb{C}[z_0, z_1]$, $\mathbb{C}[z_0, z_1] = \oplus_{l \geq 0} M_l$, $\mathcal{O}_{\mathbb{P}^1}(l) = \mathbb{C}[z_0, z_1](l)^\sim$ and $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})(l) = M_l$ for $l \geq 0$, cf. [Ha1]. Then the graded $\mathbb{C}$-algebra structure $M_{l_1} \cdot M_{l_2} \to M_{l_1+l_2}$ induces a set of canonical isomorphisms of sheaves $\mathcal{O}_{\mathbb{P}^1}(l_1) \otimes \mathcal{O}_{\mathbb{P}^1}(l_2) \to \mathcal{O}_{\mathbb{P}^1}(l_1 + l_2)$. Since multiplication among M_l's is associative with respect to these isomorphisms, one has also canonical isomorphisms $\mathcal{O}(l_1) \otimes \cdots \otimes \mathcal{O}(l_s) \to \mathcal{O}(l_1 + \cdots + l_s)$. This

implies that, if one lets $\Delta(1) = \{\, \rho_{1i} \,\}_i \coprod \{\, \rho_{2j} \,\}_j$ such that $\sum_i m_{1i} d_{\rho_{1i}} = \sum_j m_{2j} d_{\rho_{2j}}$ for some m_{1i}, $m_{2j} \geq 0$, then there is a canonical isomorphism $\bigotimes_i \mathcal{O}_{\mathbb{P}^1}(d_{\rho_{1i}})^{\otimes m_{1i}} \simeq \bigotimes_j \mathcal{O}_{\mathbb{P}^1}(d_{\rho_{2j}})^{\otimes m_{2j}}$. (Cf. [Cox1 : Proposition 1.1].)

Definition 3.1 [set of canonical isomorphisms]. We shall call the above set of isomorphisms the set of *canonical isomorphisms* among tensor products of $\mathcal{O}_{\mathbb{P}^1}(l)$'s *with respect to the fixed presentation.*

Let $d = (d_\rho)_{\rho \in \Delta(1)}$, with d_ρ nonnegative integers, be a multi-degree and define

$$Y_d \ := \ \oplus_\rho H^0(\mathbb{P}^1, \mathcal{O}(d_\rho)) \quad \text{with the above fixed presentation}.$$

Recall the abelian group G and the quotient $X = (\mathbb{C}^{\Delta(1)} - V(I))/G$ from Explanation/Fact 2.1.2. Then each element of Y_d corresponds to a morphism $\mathbb{P}^1 \to \mathbb{C}^{\Delta(1)}$ up to a $\mathbb{C}^\times$-action on $\mathbb{C}^{\Delta(1)}$ by $t \cdot (x_\rho)_\rho = (t^{d_\rho} x_\rho)_\rho$. Define F_d to be the subvariety of Y_d that consists of elements whose corresponding map $\mathbb{P}^1 \to \mathbb{C}^{\Delta(1)}$ has image contained in $V(I)$. Since $V(I)$ is a union of coordinate subspaces in $\mathbb{C}^{\Delta(1)}$ and hence invariant under the above $\mathbb{C}^\times$-action, F_d is well-defined. The G-action on $\mathbb{C}^{\Delta(1)}$ induces a G-action on Y_d that leaves F_d invariant. Thus, one can define the quotient

$$W_d \ = \ \mathcal{M}_d := (Y_d - F_d)/G \,.$$

Cf. Appendix, [M-P : Sec. 3.7]; also [L-L-Y : II, Sec. 2.4, Example 4].

Let $\mathcal{F} := (L_\rho, u_\rho, c_m)_{\rho, m}$ be a weak Δ-collection on $\mathbb{P}^1$ of multi-degree d. Then $(c_m)_m$ determines isomorphisms $L_\rho \simeq \mathcal{O}_{\mathbb{P}^1}(d_\rho)$, compatible with the set of canonical isomorphisms, up to an ambiguity parameterized by G. Thus $\mathcal{F}$ corresponds to a G-orbit $O_\mathcal{F}$ in Y_d. The nonvanishing condition for $\mathcal{F}$ is that $\sum_{\sigma \in \Delta_{max}} \otimes_{\rho \not\subset \sigma} u_\rho^* \ : \ \oplus_{\sigma \in \Delta_{max}} \otimes_{\rho \not\subset \sigma} L_\rho^{-1} \ \to \ \mathcal{O}_{\mathbb{P}^1}$ is not a zero-morphism. Since $V(I)$ is defined by the ideal $I = (\prod_{\rho \not\subset \sigma} x_\rho \,|\, \sigma \in \Delta_{max})$ and the divisor on $\mathbb{C}^{\Delta(1)}$ defined by x_ρ corresponds to the subscheme on $\mathbb{P}^1$ defined by u_ρ, the nonvanishing condition means precisely that $O_\mathcal{F} \subset Y_d - F_d$.

Regard the sections of $\mathcal{O}_{\mathbb{P}^1}(d_\rho)$ as subschemes of the total space $Spec\, Sym^\bullet(\mathcal{O}_{\mathbb{P}^1}(d_\rho)^\vee)$ of $\mathcal{O}_{\mathbb{P}^1}(d_\rho)$. Then as in the case of Hilbert schemes one obtains the universal tuple of sections $(\widetilde{u}_\rho)_\rho$ of the line bundles $(\widetilde{\mathcal{O}}(d_\rho))_\rho$ over $(Y_d - F_d) \times \mathbb{P}^1$ from the pullback of the projection map $(Y_d - F_d) \times \mathbb{P}^1 \to \mathbb{P}^1$. The set of canonical isomorphisms in Definition 3.1 gives a canonical set of isomorphisms $(\widetilde{c}_m)_m$. Since $Y_d - F_d$ corresponds to tuples $(u_\rho)_\rho$ of sections that satisfy the nonvanishing condition, $(\widetilde{\mathcal{O}}(d_\rho), \widetilde{u}_\rho, \widetilde{c}_m)_{\rho, m}$ is a weak Δ-collection on $(Y_d - F_d) \times \mathbb{P}^1$ over $Y_d - F_d$.

Definition 3.2 [small universal weak Δ-collection].

$$\widetilde{\mathcal{F}}_d \; := \; (\, \widetilde{\mathcal{O}}(d_\rho), \widetilde{u}_\rho, \widetilde{c}_m \,)_{\rho,m}$$

is called the *small universal weak Δ-collection* on $(Y_d - F_d) \times \mathbb{P}^1$ over $Y_d - F_d$.

Remark 3.3 [*universal property*]. Since the line bundles $\widetilde{\mathcal{O}}(d_\rho)$ in $\widetilde{\mathcal{F}}_d$ are fixed and $\widetilde{c}_m$ is determined once a presentation of $\mathbb{P}^1$ is chosen, $\widetilde{\mathcal{F}}_d$ indeed comes from a restriction of the universal subscheme over a Hilbert scheme. It thus inherits a similar universal property as Hilbert schemes.

The $(\mathbb{C}^\times)^{|\Delta(1)|}$-action on $Y_d - F_d$ lifts to a $(\mathbb{C}^\times)^{|\Delta(1)|}$-action on $(Y_d - F_d) \times \mathbb{P}^1$ by acting on $\mathbb{P}^1$ by the identity. The latter then lifts to an action on each $\widetilde{\mathcal{O}}(d_\rho)$ that leaves $\widetilde{u}$ invariant. This is the unique lift that has this property. Since G is a subgroup of $(\mathbb{C}^\times)^{|\Delta(1)|}$, G lifts to a unique action on $(\widetilde{\mathcal{O}}(d_\rho), \widetilde{u}_\rho)_\rho$ as well. By the very definition of G, this G-action commutes with $(\widetilde{c}_m)_m$.

Definition 3.4 [canonical G-action]. The above G-action on $(\widetilde{\mathcal{O}}(d_\rho), \widetilde{u}_\rho, \widetilde{c}_m)_{\rho,m}$ is called the *canonical lifting* of the G-action on $Y_d - F_d$.

The big universal weak Δ-collection à la Cox.

We follow the notations in the previous theme. Fix a basis of the M-lattice. Define

$$\Xi_d \; = \; \oplus_m Isom_{\mathcal{O}_{\mathbb{P}^1}} (\, \otimes_\rho \mathcal{O}_{\mathbb{P}^1}(d_\rho)^{\otimes \langle m, n_\rho \rangle} \,,\, \mathcal{O}_{\mathbb{P}^1}) \quad \text{and} \quad \widehat{Y}_d \; := \; Y_d \oplus \Xi_d \,,$$

where m runs over the fixed basis of M. Let $\kappa : \widehat{Y}_d \to \Xi_d$ be the natural projection. Then, similar to the discussion in the previous theme, one has the $(\mathbb{C}^\times)^{\Delta(1)} := Hom_{\mathbb{Z}}(\mathbb{Z}^{\Delta(1)}, \mathbb{C}^\times)$-action on $\widehat{Y}_d$ induced from that on $\mathbb{C}^{\Delta(1)}$. Recall T_N from Definition/Fact 2.1.2, then T_N acts on Ξ_d freely and transitively and in such a way that κ is $((\mathbb{C}^\times)^{\Delta(1)}, T_N)$-equivariant and that the G-subaction on $\widehat{Y}_d$ leaves each preimage of κ invariant. Since the nonvanishing condition in a weak Δ-collection has nothing to do with the isomorphism data $(c_m)_m$, it specifies the open $(\mathbb{C}^\times)^{\Delta(1)}$-invariant subvariety $\widehat{Y}_d - \widehat{F}_d$, where $\widehat{F}_d := F_d \times \Xi_d$.

Following the same construction as in the previous theme, one has a big universal weak Δ-collection

$$\widetilde{\mathcal{F}}_d^{\,\mathrm{big}} \; := \; \left(\widetilde{\mathcal{O}}(d_\rho)^{\mathrm{big}} \,,\, \widetilde{u}_\rho^{\,\mathrm{big}} \,,\, \widetilde{c}_m^{\,\mathrm{big}} \right)_{\rho,m}$$

on $((\widehat{Y}_d - \widehat{F}_d) \times \mathbb{P}^1)/(\widehat{Y}_d - \widehat{F}_d)$ and the canonical lifting of the $(\mathbb{C}^\times)^{\Delta(1)}$-action on the total space of line bundles $(\widetilde{\mathcal{O}}(d_\rho)^{\mathrm{big}})_\rho$.

Remark 3.5 [Morrison-Plesser v.s. Cox]. From these very explicit constructions, one observes that a fixed presentation as in the previous theme selects a distinguished point $(c_m^{\mathrm{can}})_m$ in Ξ and $Y_d - F_d = \kappa^{-1}((c_m^{\mathrm{can}})_m)$. The small universal weak Δ-collection $\widetilde{\mathcal{F}}_d$ on $((Y_d - F_d) \times \mathbb{P}^1)/(Y_d - F_d)$ is the restriction of the big universal Δ-collection $\widetilde{\mathcal{F}}_d^{\mathrm{big}}$ on $((\widehat{Y}_d - \widehat{F}_d) \times \mathbb{P}^1)/(\widehat{Y}_d - \widehat{F}_d)$ to $((Y_d - F_d) \times \mathbb{P}^1)/(Y_d - F_d)$. The **Isom** construction in Sec. 2.2, adjusted for the fixed $\mathbb{P}^1$, gives $\widehat{Y}_d - \widehat{F}_d$.

Relation with $\mathcal{AM}_0(X)$.

The two quotient stacks $[(\widehat{Y}_d - \widehat{F}_d)/(\mathbb{C}^\times)^{\Delta(1)}]$ and $[(Y_d - F_d)/G]$ are isomorphic since the T_N-action on Ξ_d is transitive and free. The following lemma relates this quotient stack with $\mathcal{AM}_0(X)$.

Lemma 3.6 [$\mathcal{AM}_0(X)$]. *The Artin stack $\mathcal{AM}_0(X)$ is the quotient stack $\coprod_d [(Y_d - F_d)/G]$, for which $\coprod_d (Y_d - F_d)$ is an atlas and $\coprod_d W_d$ is the coarse moduli space. In particular, $\mathcal{AM}_0(X)$ is a smooth Artin stack.*

We check this at the prestack level. The statement then follows upon stackification.

Proof. The proof is divided in two parts.

(a) *$\mathcal{AM}_0(X)$ as a quotient stack.* Let $pre\mathcal{AM}_0(X) = \coprod_d pre\mathcal{AM}_0(X)_d$, where d runs over all the admissible multi-degrees. We shall construct morphisms of prestacks

$$J_d^{(1)} : pre\mathcal{AM}_0(X)_d \longrightarrow pre[(Y_d - F_d)/G]$$

and

$$J_d^{(2)} : pre[(Y_d - F_d)/G] \longrightarrow pre\mathcal{AM}_0(X)_d$$

so that $J_d^{(2)} \circ J_d^{(1)}$ and $J_d^{(1)} \circ J_d^{(2)}$ induce auto-equivalences of related fiber groupoids. (In other words, $J_d^{(1)}$ is an isomorphism of prestacks with inverse given by $J_d^{(2)}$.)

(a.1) *Construction of $J_d^{(1)}$.* Recall first the following standard construction that will be used repeatedly in the discussion:

Let $L^\times$ be the principal $\mathbb{C}^\times$-bundle on S from deleting the zero-section of a line bundle L on S. The projection map $L^\times \to S$ pulls back L to a line bundle $\widetilde{L}$ on the total space $Tot(L^\times)$ of $L^\times$. The natural inclusion map $L^0 \hookrightarrow L$ gives rise to a nowhere-zero global section in $\widetilde{L}$ over $Tot(L^\times)$ and hence a canonical trivialization of $\widetilde{L}$.

Given a weak Δ-collection $\mathcal{F} = (L_\rho, u_\rho, c_m)_{\rho,m}$ on $(S \times \mathbb{P}^1)/S$, each L_ρ defines a principal $\mathbb{C}^\times$-bundle $L_\rho^\times$ over $S \times \mathbb{P}^1$ by deleting the zero-section of L_ρ. The isomorphism of line bundles $c_m : \otimes_\rho L_\rho^{\otimes\langle m, n_\rho\rangle} \simeq \mathcal{O}_{S \times \mathbb{P}^1}$ induces an isomorphism of principal $\mathbb{C}^\times$-bundles over $S \times \mathbb{P}^1$ by the restriction $\otimes_\rho (L_\rho^\times)^{\otimes\langle m, n_\rho\rangle} \to \mathcal{O}_{S \times \mathbb{P}^1}^\times = (\mathbb{G}_m)_{S \times \mathbb{P}^1}$ of c_m. This then induces a morphism, still denoted by c_m, from the composition

$$c_m : \oplus_\rho L_\rho^\times \longrightarrow \otimes_\rho (L_\rho^\times)^{\otimes\langle m, n_\rho\rangle} \longrightarrow \mathcal{O}_{S \times \mathbb{P}^1}^\times .$$

This gives rise to a principal G-bundle on $S \times \mathbb{P}^1$ defined by the kernel (i.e. the preimage of the section $(1, \ldots, 1)$) of the morphism $(c_m)_m$ over $S \times \mathbb{P}^1$:

$$Ker \left((c_m)_m : \oplus_\rho L_\rho^\times \to (\mathcal{O}_{S \times \mathbb{P}^1}^\times)^{\oplus n} = (T_N)_{S \times \mathbb{P}^1} \right) ,$$

where m runs over elements in a fixed basis of M and n is the rank of M. A principal G-bundle over S, $p : P_S^G \to S$, is obtained by restricting the above principal G-bundle over $S \times \mathbb{P}^1$ to a horizontal slice, e,g. $S \times \{0\}$. (Note that any two such restrictions are isomorphic. The inverse of any such restriction of L_ρ gives the line bundles on S needed to twist L_ρ so that the result is a pullback line bundle from that on $\mathbb{P}^1$.) Consider the pullback weak Δ-collection $p^*\mathcal{F}$ on $(P_S^G \times \mathbb{P}^1)/P_S^G$. The identity morphism of line bundles $(L_\rho|_{S \times \{0\}})_\rho \to (L_\rho|_{S \times \{0\}})_\rho$ specifies a canonical trivialization $p^*(L_\rho|_{S \times \{0\}}) \simeq \mathcal{O}_{P_S^G \times \{0\}}$ on the horizontal slice $P_S^G \times \{0\}$ of $P_S^G \times \mathbb{P}^1$ over P_S^G. This implies that $p^*(L_\rho)_\rho$ is isomorphic to the pullback of $(\mathcal{O}_{\mathbb{P}^1}(d_\rho))_\rho$ by the projection map $P_S^G \times \mathbb{P}^1 \to \mathbb{P}^1$.

Furthermore, a generalization of the standard constructions recalled at the beginning to the tuple of line bundles $(L_\rho|_{S \times \{0\}})_\rho$, its associated principal $(\mathbb{C}^\times)^n$-bundles and its sub G-bundle, one deduces that the above trivialization over the slice $P_S^G \times \{0\}$ fixes an isomorphism $(p^*L_\rho)_\rho \simeq (\mathcal{O}_{\mathbb{P}^1}(d_\rho))_\rho$.

Pulling back now the sections u_ρ of L_ρ, one thus obtains a P_S^G-family $p^*(L_\rho, u_\rho)_\rho$ of line bundles on $\mathbb{P}^1$ with a section. It follows from the universal property of $Y_d - F_d$ inherited from that of Hilbert schemes that there exists a unique morphism $\zeta_\mathcal{F} : P_S^G \to (Y_d - F_d)$ with $p^*(L_\rho, u_\rho)_\rho = \zeta_\mathcal{F}^* \widetilde{\mathcal{F}}_d$. By construction $\zeta_\mathcal{F}$ is G-equivariant and $p^*(c_m)_m = \zeta_\mathcal{F}^* \widetilde{c}_m$. The correspondence

$\mathcal{F} \to \zeta_{\mathcal{F}}$ gives a morphism of prestacks

$$J_d^{(1)} : pre\mathcal{AM}_0(X)_d \to pre[(Y_d - F_d)/G] \,.$$

(a.2) *Construction of $J_d^{(2)}$.* To construct $J_d^{(2)} : pre[(Y_d - F_d)/G] \to pre\mathcal{AM}_0(X)$, observe that if P_S^G is a principal G-bundle on S with a G-equivariant morphism $\zeta : P_S^G \to (Y_d - F_d)$, (i.e. an object in the groupoid $pre[(Y_d - F_d)/G](S)$) then $\zeta^*\widetilde{\mathcal{F}}_d$ is a weak Δ-collection on $(P_S^G \times \mathbb{P}^1)/P_S^G$. Since there is a canonical G-action on $\widetilde{\mathcal{F}}_d$, the G-action on P_S^G also lifts canonically to $\zeta^*\widetilde{\mathcal{F}}_d$. The quotient by this action gives then a weak Δ-collection $\mathcal{F}$ on $(S \times \mathbb{P}^1)/S$. i.e. an object in $pre\mathcal{AM}_0(X)_d(S)$. This gives a morphism $J_d^{(2)} : pre[(Y_d - F_d)/G] \to pre\mathcal{AM}_0(X)_d$.

(a.3) *Isomorphisms of stacks.* It remains to show that $J_d^{(1)}$ (or $J_d^{(2)}$) is an isomorphism of stacks. This means that $J^{(2)} \circ J^{(1)}$ sends a weak Δ-collection on $(S \times \mathbb{P}^1)/S$ to an isomorphic weak Δ-collection on $(S \times \mathbb{P}^1)/S$, which follows from the very explicit construction of $J_d^{(1)}$ and $J_d^{(2)}$. Similarly for $J_d^{(1)} \circ J_d^{(2)}$.

(b) W_d *as the coarse moduli space.*

(b.1) *Construction of a morphism $\mathcal{AM}_0(X)_d \to W_d$.* A morphism

$$pre\mathcal{AM}_0(X)_d \longrightarrow W_d$$

is already given/hidden in [L-L-Y : II. Sec. 2.5, Lemma 2.7, proof] as follows. Let $\mathcal{O}(d_\rho)$ be the pullback of $\mathcal{O}_{\mathbb{P}^1}(d_\rho)$ to $S \times \mathbb{P}^1$ via the projection map. Given a weak Δ-collection $\mathcal{F} = (L_\rho, u_\rho, c_m)_{\rho,m}$ on $S \times \mathbb{P}^1$ over S of multi-degree d, since L_ρ is isomorphic to $\mathcal{O}(d_\rho)$ along each $\mathbb{P}^1$-fiber of the projection morphism $S \times \mathbb{P}^1 \to S$, L_ρ and $\mathcal{O}(d_\rho)$ differ by a twist from a line bundle on S. Let $\mathcal{U} := \{U_\alpha\}_{\alpha \in A}$ be an open cover of S such that all these twisting line bundles are trivial when restricted to any U_α. For each $U_\alpha \in \mathcal{U}$, the data $(L_\rho, u_\rho)_\rho$ determines non-uniquely a $(u'_\rho)_\rho \in \oplus_\rho H^0(S \times \mathbb{P}^1, \mathcal{O}(d_\rho))$ over $U_\alpha \times \mathbb{P}^1$ by looking at the zero-divisor/locus of u_ρ on $S \times \mathbb{P}^1$. The ambiguities of $(u'_\rho)_\rho$ come from the identification of $\mathcal{L}_\rho$ and $\mathcal{O}(d_\rho)$ when restricted to $U_\alpha \times \mathbb{P}^1$ and are parameterized by $(t_\rho)_\rho \in (\mathbb{C}^\times)^{|\Delta(1)|}$ that satisfy $\prod_\rho t_\rho^{\langle m, n_\rho \rangle} = 1$ for all $m \in M$. Thus, for each U_α, $(u'_\rho)_\rho$, though nonunique, determines a unique U_α-family of G-orbits on $\oplus_\rho H^0(\mathbb{P}^1, \mathcal{O}(d_\rho))$. The nonvanishing condition on $(u'_\rho)_\rho$ inherits from that of $\mathcal{F}$. By repeating the same argument for $U_\alpha \cap U_\beta$,

U_α, $U_\beta \in \mathcal{U}$, and consider the open immersions $U_\alpha \hookleftarrow U_\alpha \cap U_\beta \hookrightarrow U_\beta$, one concludes that these U_α-families of G-orbits coincide on intersections $U_\alpha \cap U_\beta$. Thus one obtains a well-defined morphism $S \to W_d$. Another construction can be obtained from the discussion of Part (a) as follows. $\mathcal{F}$ determined a principal G-bundle P_S^G on S with a unique G-equivariant morphism $P_S^G \to Y_d - F_d$. Taking quotient by G on both sides, one then obtains a morphism $S \to W_d$ determined by $\mathcal{F}$. Either way, one obtains a morphism

$$pre\,\phi \; : \; pre\mathcal{AM}_0(X)_d \; \longrightarrow \; W_d$$

and hence a morphism

$$\phi \; : \; \mathcal{AM}_0(X)_d \; \longrightarrow \; W_d \,.$$

(b.2) *The coarse moduli space conditions.* From the definition of W_d and points of a stack, $|\phi|(k) : |\mathcal{AM}_0(X)_d|(k) \to W_d(k)$ is bijective for any algebraically closed field k.

To see that W_d corepresents $\mathcal{AM}_0(X)_d$, first observe that there is a distinguished weak Δ-collection on $(W_d \times \mathbb{P}^1)/W_d$ constructed as follows. Consider the diagonal action of G on $(Y_d - F_d) \times (Y_d - F_d)$. The diagonal $\Delta_{(Y_d-F_d)}$ of $(Y_d - F_d) \times (Y_d - F_d)$ is invariant under this G-action. The quotient gives a fibration of $((Y_d - F_d) \times (Y_d - F_d))/G \to W_d$ with generic fiber $Y_d - F_d$. The diagonal $\Delta_{(Y_d-F_d)}$ descends to a section of this fibration, which corresponds to a weak Δ-collection $\widetilde{\mathcal{L}}$ on $W_d \times \mathbb{P}^1$. By construction $\phi(\widetilde{\mathcal{L}}) = Id_{W_d}$.

Now suppose that W_d' is another scheme with a morphism

$$\phi' : \mathcal{AM}_0(X)_d \; \longrightarrow \; W_d'\,.$$

Then a morphism of schemes $\eta : W_d \to W_d'$ can be defined via a morphism, also denoted by η, of their respective functor of points

$$\eta \; : \; Hom\,(-, W_d) \; \longrightarrow \; Hom\,(-, W_d')$$

defined by the composition

$$(f : S \to W_d) \; \longmapsto \; f^*\widetilde{\mathcal{L}} \in \mathcal{AM}_0(X)_d(S) \; \longmapsto \; (\phi'(f^*\widetilde{\mathcal{L}}) : S \to W_d')\,.$$

Given $\mathcal{L} \in \mathcal{AM}_0(X)_d(S)$, let $f = \phi(\mathcal{L}) \in Hom\,(S, W_d)$ and $f' = \phi'(\mathcal{L}) \in Hom\,(S, W_d')$. Observe that $\mathcal{L}$ and $f^*\widetilde{\mathcal{L}}$ are fiberwise isomorphic weak Δ-collections on $(S \times \mathbb{P}^1)/S$. Moreover, since $\mathcal{L}$ and $f^*\widetilde{\mathcal{L}}$ as bundles on $S \times \mathbb{P}^1$

differ only by a twist from line bundles on S, there is an étale chart $\pi : \widetilde{S} \to S$ of S such that $\mathcal{L}$ and $f^*\widetilde{\mathcal{L}}$ are isomorphic when lifted to the induced étale chart $\widetilde{S} \times \mathbb{P}^1$ of $S \times \mathbb{P}^1$. Since ϕ' induces $|\phi'|$ that sends points of $|\mathcal{AM}_0(X)_d|$ to closed points of W'_d that corresponds to $Spec\, k \to W'_d$, where k is an algebraically closed field, the two morphisms $\eta(f), \phi'(\mathcal{L}) : S \to W'_d$ coincides on the geometric points of S and their lifting to $\widetilde{S} \to W'_d$ are identical as scheme morphisms. Consequently, $\eta(\phi(\mathcal{L})) = \eta(f) = \phi'(\mathcal{L})$ as scheme morphisms. Since this applies for all S and $\mathcal{L}$, $\eta \circ \phi = \phi'$ and, hence, W_d corepresents $\mathcal{AM}_0(X)_d$.

(b.1) and (b.2) together show that $\coprod_d W_d$ is the coarse moduli space for $\mathcal{AM}_0(X)$ and we conclude the proof. $\qquad\square$

4. The collapsing morphism.

In this section, we re-run the proof of Jun Li of Lemma 2.7 in Mirror Principle II, with the A-twisted moduli stack $\mathcal{AM}_0(X)$ of Sec. 3 soldered into the discussion. All the schemes in the discussion are over $\mathbb{C}$.

Background.

Fact 4.1 [rank 1 sheaf]. ([Ha2]; also [Fr], [Huy-L], and [Od-S].) *Any rank 1 torsion-free coherent sheaf on a locally factorial scheme Y must be of the form $\mathcal{I}_Z \otimes \mathcal{L}$, where $\mathcal{I}_Z$ is the ideal sheaf of a subscheme Z of codimension ≥ 2 in Y and $\mathcal{L}$ is a line bundle on Y. Such a decomposition is unique up to isomorphisms of $\mathcal{O}_Y$-modules.*

Fact 4.2 [Hartogs extension theorem]. ([Ii].) *Let Y be a Noetherian normal scheme and Z be a closed subset of codimension ≥ 2 in Y. Then $H^0(Y - Z, \mathcal{O}_Y) = H^0(Y, \mathcal{O}_Y)$.*

Corollary 4.3 [determinant]. *Let $\mathcal{L}$ be a rank 1 coherent sheaf on a locally factorial scheme Y. Then there exists a canonical morphism $\mathcal{L} \to det\,\mathcal{L}$ of $\mathcal{O}_Y$-modules.*

Proof. Recall the definition and the relations of det and Div in [K-M]. From the exact sequence $0 \to Tor\,\mathcal{L} \to \mathcal{L} \to \mathcal{L}/Tor\,\mathcal{L} \to 0$, one has $det\,\mathcal{L} = det\,(Tor\,\mathcal{L}) \otimes det\,(\mathcal{L}/Tor\,\mathcal{L})$. Since $\mathcal{L}/Tor\,\mathcal{L}$ is torsion-free, $\mathcal{L}/Tor\,\mathcal{L} = \mathcal{I}_{\widehat{Z}} \otimes \widehat{\mathcal{L}}$ canonically, where $\mathcal{I}_Z$ is the ideal sheaf of the subscheme of codimension ≥ 2 in Y (from the flattening stratification, cf. [Mu3], of $\mathcal{L}/Tor\,\mathcal{L}$) on which the

fiber dimensions of $\mathcal{L}/Tor\mathcal{L}$ jump up and $\widehat{L}$ is a line bundle on Y. These give rise to a sequence of canonical morphisms/identifications of $\mathcal{O}_Y$-modules:

$$\begin{aligned}
\mathcal{L} &\rightarrow \mathcal{L}/Tor\mathcal{L} = \mathcal{I}_Z \otimes \widehat{\mathcal{L}} \rightarrow \widehat{\mathcal{L}} = det\left(\mathcal{I}_Z \otimes \widehat{\mathcal{L}}\right) \\
&\rightarrow det\left(\mathcal{I}_Z \otimes \widehat{\mathcal{L}}\right) \otimes det\,Tor\mathcal{L} \simeq det\mathcal{L},
\end{aligned}$$

where we have used the facts: (i) $det\mathcal{I}_Z = \mathcal{O}_Y$, (ii) $det\,Tor\mathcal{L} = \mathcal{O}(Div\,Tor\mathcal{L})$ and $Div\,Tor\mathcal{L} \geq 0$, and (iii) there are canonical inclusions $\mathcal{O}_Y \hookrightarrow \mathcal{O}_Y(D) \hookrightarrow \mathcal{K}_Y$ for $D \geq 0$, where $\mathcal{K}_Y$ is the sheaf of total quotient rings of Y. The composition of this sequence of canonical morphisms gives the canonical morphism of $\mathcal{O}_Y$-modules $\mathcal{L} \rightarrow det\mathcal{L}$ claimed. $\square$

Lemma 4.4 [push-pull of weak Δ-collection]. (1) *Let $f : Y \rightarrow Y'$ be a projective birational morphism between schemes of the same uniform dimension. Assume that Y' is irreducible and that f is an isomorphism outside a closed subscheme of codimension ≥ 2 in Y' - in notation, $f|_U : U \xrightarrow{\sim} U'$ -. Let $(L_\rho, u_\rho, c_m)_{\rho,m}$ be a weak Δ-collection on Y and define $L'_\rho := det\,f_*L_\rho$. Then there exists a unique weak Δ-collection (L'_ρ, u'_ρ, c'_m) on Y' that extends the weak Δ-collection $(f|_U)_*(L_\rho, u_\rho, c_m)_{\rho,m}|_U$ on U'.*

(2) *Let $f : Y = Y_0 \cup Y_1 \cup \cdots \rightarrow Y'$ be a projective morphism between schemes of the same uniform dimension that satisfies*

  (i) *Y' is a Noetherian integral (separated) scheme which is regular in codimension-1 (cf. [Ha1: II.6]),*

  (ii) *the restriction $f : Y_0 \rightarrow Y'$ is an isomorphism outside a closed subscheme of codimension ≥ 2 in Y' - in notation, $f|_U : U \xrightarrow{\sim} U'$ -, and*

  (iii) *each Y_i, $i = 1, \ldots$, is mapped to a codmension-1 subscheme D'_i in Y', whose corresponding divisor is also denoted by D'_i, (i.e. $Y_i \rightarrow D'_i$ is a flat family of curves).*

Let $(L_\rho, u_\rho, c_m)_{\rho,m}$ be a weak Δ-collection on Y such that none of $u_\rho|_{Y_i}$ are zero-sections, where $\rho \in \Delta(1)$ and $i = 1, \ldots$, and let $L'_\rho := det\,f_*L_\rho$. Then there exists a canonically constructed weak Δ-collection $(L'_\rho, u'_\rho, c'_m)_{\rho,m}$ on Y' that extends the weak Δ-collection $(f|_{U-Y_1\cup\cdots})_*(L_\rho, u_\rho, c_m)_{\rho,m}|_{U-Y_1\cup\cdots}$ on $U' - D_1 \cup \cdots$.

Proof. For Statement (1), observe that $\otimes_\rho L'_\rho{}^{\otimes\langle m, n_\rho\rangle} \simeq \mathcal{O}_{Y'}$ as abstract $\mathcal{O}_{Y'}$-modules since the former is an invertible $\mathcal{O}_{Y'}$-module that is free outside a locus of codimension ≥ 2 in Y'. The isomorphisms $(f|_U)_*c_m$ extend to unique isomorphisms $\otimes_\rho L'_\rho{}^{\otimes\langle m, n_\rho\rangle} \simeq \mathcal{O}_{Y'}$ by Hartogs extension theorem since, once

fixing a trivialization of the two rank-1 globally free $\mathcal{O}_{Y'}$-modules in question, $(f|_U)_* c_m$ is given by multiplication of a regular function. The sections u'_ρ are given by the canonical morphism $H^0(Y, L_\rho) \to H^0(Y', L'_\rho)$ arising from the combination of the definition of f_* and the canonical morphism $f_* L_\rho \to L'_\rho = \det f_* L_\rho$. The cocycle conditions $c'_{m_1} \otimes c'_{m_2} = c'_{m_1+m_2}$ follow by continuity.

For Statement (2), let us first show that $\otimes_\rho L'_\rho{}^{\otimes \langle m, n_\rho \rangle} \simeq \mathcal{O}_{Y'}$ as abstract $\mathcal{O}_{Y'}$-modules. By the assumption in Statement (2), $Y_i \to D_i$, $i = 1, \ldots,$ are flat families of curves and the relative degree of $rel\text{-}deg_{D_i}(L_\rho|_{Y^i})$ is well-defined. Moreover, recalling the definition of Div, one concludes that

$$L'_\rho \; = \; \det f_* L_\rho \; = \; \det \left((f|_{Y_0})_* (L_\rho|_{Y_0}) \right) \otimes \mathcal{O}_{Y'} \left(\sum_{i=1,\ldots} reldeg_{D_i}(L_\rho|_{Y_i}) \cdot D_i \right)$$

since all $u_\rho|_{Y_i}$ are non-zero sections. By definition, the restriction $(L_\rho|_{Y_i}, u_\rho|_{Y_i}, c_m|_{Y_i})_{\rho,m}$ of $(L_\rho, u_\rho, c_m)_{\rho,m}$ to each component Y_i of Y is a weak Δ-collection on Y_i. In particular, $c_m|_{Y_i} : \otimes_\rho L_\rho|_{Y_i}^{\langle m, n_\rho \rangle} \simeq \mathcal{O}_{Y_i}$ and

$$\sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle \cdot rel\text{-}deg_{D_i}(L_\rho|_{Y_i}) \; = \; 0 \, .$$

Furthermore, the restriction $f|_{Y_0} : Y_0 \to Y'$ is in the situation of Statement (1) and one can define the weak Δ-collection $(L'_{\rho,0}, u'_{\rho,0}, c'_{m,0})_{\rho,m}$ on Y' as in Statement (1) as the push-forward of $(L_\rho, u_\rho, c_m)_{\rho,m}|_{Y_0}$ via $f|_{Y_0}$. It follows that

$$\otimes_{\rho \in \Delta(1)} L'_\rho{}^{\otimes \langle m, n_\rho \rangle} \; = \; \otimes_{\rho \in \Delta(1)} L'_{\rho,0}{}^{\otimes \langle m, n_\rho \rangle} \xrightarrow{c'_{m,0}} \mathcal{O}_{Y'} \, .$$

This defines also the sought-for c'_m. The sections u'_ρ and the cocycle conditions on c'_m follow by the same reasoning as in the case of Statement (1). This concludes the proof. $\qquad \square$

The collapsing morphism.

Proposition 4.5. *Let X be a convex smooth toric variety and $M_d(X)$ be the moduli stack $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1 \times X, (1,d))$ of genus 0 stable map into $\mathbb{P}^1 \times X$ of degree $(1,d)$. Then there exists a natural morphism $\Upsilon : M_d(X) \to \mathcal{AM}_0(X)_d$ of stacks.*

Remark 4.6. Composition of Υ with the morphism $\phi : \mathcal{AM}_0(X)_d \to W_d$ in the proof of Lemma 3.6 gives the morphism $\varphi : M_d(X) \to W_d$ in [L-L-Y : II, Sec. 2.5, Lemma 2.7].

Proof of Proposition. We split the discussions to two cases.

Case (1) : $M_d(X)$ *is a compactification of the component of* $Hom\,(\mathbb{P}^1, \mathbb{P}^1 \times X)$ *that corresponds to genus* 0 *curves in* $\mathbb{P}^1 \times X$ *of degree* $(1, d)$.

Let $(Sch\,/\,\mathbb{C})$ be the category of Noetherian schemes of finite type over $\mathbb{C}$ and ξ be an object in $M_d(X)(S)$ given by

$$F : \mathcal{X} \quad \longrightarrow \quad S \times \mathbb{P}^1 \times X$$
$$\searrow \quad \swarrow$$
$$S \qquad\qquad .$$

Let p_i (resp. p_{ij}) be the composition of F with the projection of $S \times \mathbb{P}^1 \times X$ to its i-th component (resp. the product of its i-th and j-th components).

Assume first that (S, ξ) is an atlas of $M_d(X)$, then S is smooth and outside a divisor D_S of S (i.e. the boundary locus of S) the defining family of stable maps over S parameterizes morphisms from $\mathbb{P}^1$ into $\mathbb{P}^1 \times X$. The map

$$p_{12} : \mathcal{X} \longrightarrow S \times \mathbb{P}^1$$

is a projective birational morphism that is an isomorphism outside a locus of codimension ≥ 2 in $S \times \mathbb{P}^1$. Let $(L_\rho, z_\rho, c_m)_{\rho, m}$ be the universal Δ-collection on X and

$$(L_{\rho, \xi}, u_{\rho, \xi}, c_{m, \xi})_{\rho, m} \;=\; p_3^*(L_\rho, z_\rho, c_m)_{\rho, m}.$$

Then $(L_{\rho, \xi}, u_{\rho, \xi}, c_{m, \xi})_{\rho, m}$ is a Δ-collection on $\mathcal{X}$ over S. The construction satisfies the base change property that, if $f : T \to S$ be a morphism of $\mathbb{C}$-schemes, then there is a canonical isomorphism of Δ-collections

$$(L_{\rho, f^*\xi}, u_{\rho, f^*\xi}, c_{m, f^*\xi})_{\rho, m} \;\simeq\; (f \times Id_{\mathbb{P}^1})^*(L_{\rho, \xi}, u_{\rho, \xi}, c_{m, \xi})_{\rho, m}.$$

In particular, if one equips $M_d(X)$ with the étale topology, then one obtains a Δ-collection on the stack $M_d(X)$ by considering the étale morphisms among atlases.

Let $\mathcal{L}_{\rho, \xi} = p_{12*}L_{\rho, \xi}$. Since p_{12} is an isomorphism over the complement U' of a codimension ≥ 2 locus in $S \times \mathbb{P}^1$, by Lemma 4.4 (1) there exists a unique weak Δ-collection of the form $(det\,\mathcal{L}_{\rho, \xi}, \sigma_{\rho, \xi}, c'_{m, \xi})$ on $S \times \mathbb{P}^1$ (over $Spec\,\mathbb{C}$) such that the restriction of p_{12*} over U' is an isomorphism of weak Δ-collections on U' (over $Spec\,\mathbb{C}$). Since each fiber $\mathbb{P}^1$ of $S \times \mathbb{P}^1$ over S comes from pinching rational subcurves of the corresponding fiber of $\mathcal{X}$ over S and the restriction of $(det\,\mathcal{L}_{\rho, \xi}, \sigma_{\rho, \xi}, c'_{m, \xi})$ to a fiber $\mathbb{P}^1$ defines a morphism from $\mathbb{P}^1$ to X (of possibly lower multi-degrees), $(det\,\mathcal{L}_{\rho, \xi}, \sigma_{\rho, \xi}, c'_{m, \xi})$ must satisfy

the nonvanishing condition of Definition 2.1.3 when restricted to each fiber $\mathbb{P}^1$ of $S \times \mathbb{P}^1$ over S. Consequently, $(det\mathcal{L}_{\rho,\xi}, \sigma_{\rho,\xi}, c'_{m,\xi})$ is a weak Δ-collection on $S \times \mathbb{P}^1$ over S as well and one obtains a map

$$\Omega : \{\,\text{atlases } (S, \xi) \text{ of } M_d(X)\,\} \; \longrightarrow \; \mathcal{AM}_0(X)_d$$

that commutes with the étale base change among atlases of $M_d(X)$.

Fix now an atlas (T, ξ_T) for $M_d(X)$ and let $\xi \in M_d(X)(S)$ for a general $S \in (Sch/\mathbb{C})$. Since $M_d(X)$ is a smooth Deligne-Mumford stack, the pair (ξ_T, ξ) determines a commutative diagram

$$
\begin{array}{ccc}
S' := \mathbf{Isom}\,(\xi_T, \xi) & \stackrel{\alpha}{\longrightarrow} & S \\
\downarrow \beta & & \downarrow \\
T & \longrightarrow & M_d(X)\,,
\end{array}
$$

where α is étale and surjective. The canonical isomorphism $\alpha^*\xi \simeq \beta^*\xi_T$ induces a canonical isomorphism

$$\alpha^*(L_{\rho,\xi}, u_{\rho,\xi}, c_{m,\xi})_{\rho,m} \; \simeq \; \beta^*(L_{\rho,\xi_T}, u_{\rho,\xi_T}, c_{m,\xi_T})_{\rho,m}\,.$$

The weak Δ-collection $\beta^*\Omega(T, \xi_T)$ on $(S' \times \mathbb{P}^1)/S'$ is a descent datum with respect to α and hence descends to a weak Δ-collection on $(S \times \mathbb{P}^1)/S$. One can check that different choices of atlases (T, ξ_T) give rise to the same descent on $(S \times \mathbb{P}^1)/S$, thus one obtains a well-defined morphism of stacks from $M_d(X)$ to $\mathcal{AM}_0(X)_d$. This concludes the proof for Case (1).

Case (2) : *General* $M_d(X)$.

Again, let S be an atlas of $M_d(X)$, which is smooth. Then there is a stratification of S labelled by the dual graphs of the prestable domain curves of stable maps in question. We shall assume that the graph for the maximal stratum is not a point, i.e. we are not in Case (1). Then the projective morphism p_{12} are now in the situation of Statement (2) of Lemma 4.4. Convexity of X implies that the restriction of $u_{\rho,\xi}$, as defined analogous to the discussion in Case (1), to each component of $\mathcal{X}$ is not a zero-section. The proposition now follows from Lemma 4.4 (2) and the same argument as in Case (1) above. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We conclude the notes with three themes along the line for further study.

Theme 1. Further properties and details of the A-twisted moduli stack $\mathcal{AM}_g(X)$.

Theme 2. Construction of natural morphisms between the moduli stack of stable maps and the A-twisted moduli stack that generalize the construction in [L-L-Y].

Theme 3. Generalization of the twisted moduli stack $\mathcal{AM}_g(X)$ to the case of open strings, e.g. [G-J-S].

Appendix. Witten's gauged linear sigma models for mathematicians.

Witten's gauged linear sigma model (GLSM) [Wi1] is one of the universal frameworks or structures that lie behind stringy dualities (e.g. [Gre]). A mathematical review of the related part of [Wi1] (cf. also [M-P]) to the current work is given in this appendix.

• **Introduction to the superland.** [Po2: vol. II. Appendix B] (resp. [Fr]) gives a concise introduction of *spinor representations*, *supersymmetry* (SUSY), *supermultiplets*, and *superfields* and their *component fields* from a string theorist's (resp. mathematician's) aspect. A formulation of *superspaces/manifolds/schemes* that is close in spirit to Grothendieck's formulation of algebraic geometry is given in [Ma: Chapter 4]. This formulation provides a geometry behind the standard text [W-B] on supersymmetry. *Kähler differentials* and *tangent vectors* can be defined as in [Ha1]. *Fermionic integration* is discussed in [We: Sec. 26.6] and [W-B: IX], whose mathematical formulation *Berezin integral* is discussed in [Fr] and [Ma]. *R-symmetry* is discussed in [Fr: Lecture 3] and [We]. *Central extensions* of a supersymmetry algebra and its *BPS representations* are discussed in [Fr], [Po2: vol II], and [We]. *Super linear algebra*, in particular the *parity change functor* $\prod$, is discussed in [Fr: Lectures 1 and 2] and [Ma: Chaper 3]. See also [Arg] and [DEFJKMMW].

• **Supermanifolds and line bundles.** For the purpose of this article we have reduced the role played by the parity change functor $\prod$ in the description as much as possible. All the modules are left modules.

(1) ([Ma: Sec. 4.1]; also [Ha1].) A *supermanifold* $X = (X, \mathcal{O}_X)$ (in smooth, analytic, or algebraic category) is a $\mathbb{Z}/2\mathbb{Z}$-graded ringed topological space $(X, \mathcal{O}_X)$ such that

  (a) The stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at any point $x \in X$ is a local ring.

(b) X is covered by a collection of open sets $\{U_\alpha\}_{\alpha \in I}$ such that each $(U_\alpha, \mathcal{O}_M|_{U_\alpha})$ is isomorphic to $(U_\alpha^0, Sym^\bullet_{\mathcal{O}_{U_\alpha^0}}(\prod \mathcal{E}_\alpha))$, where U_α^0 is an ordinary manifold (in the corresponding category), $\mathcal{E}_\alpha$ is an ordinary locally free coherent $\mathcal{O}_{U_\alpha^0}$-module, and $\prod \mathcal{E}_\alpha$ is the $\mathcal{O}_{U_\alpha^0}$-module $\mathcal{E}_\alpha$ with odd parity.

(c) X_{rd} is a manifold (in the corresponding category), cf. Remark below.

Remark. Let $\mathcal{O}_X = \mathcal{O}_X^{(0)} \oplus \mathcal{O}_X^{(1)}$ be the decomposition of $\mathcal{O}_X$ into the even (i.e. grade 0) and the odd (i.e. grade 1) component and $J_X := \mathcal{O}_X^{(1)} + (\mathcal{O}_X^{(1)})^2$ (the ideal of "superfuzz", cf. [Fr: Lecture 1]). Then X_{rd} is by definition the submanifold of X associated to J_X. Note also that $Sym^\bullet(\prod \mathcal{E}_\alpha) \simeq \bigwedge^\bullet \mathcal{E}_\alpha$ as $\mathcal{O}_{U_\alpha^0}$-modules with all the parities after tensor products erased. X is called *decomposable* if in Condition (b) one can choose $U_\alpha = X$ for some α. In this case there is a surjective affine morphism $X \to X_{\mathrm{rd}}$ such that the composition $X_{\mathrm{rd}} \hookrightarrow X \to X_{\mathrm{rd}}$ is the identity map.

(2) A *line bundle* $\mathcal{L}$ on X is a locally free rank 1 $\mathcal{O}_X$-module. Associated to $\mathcal{L}$ is a finite filtration of $\mathcal{O}_X$-modules: $\mathcal{L} \supset \mathcal{L} \cdot J_X \supset \mathcal{L} \cdot J_X^2 \supset \cdots \supset 0$. Global sections of the associated graded object $Gr\mathcal{L} := \oplus_i (\mathcal{L} \cdot J_X^i / \mathcal{L} \cdot J_X^{i+1})$ are called *component sections* of $\mathcal{L}$. The restriction $\mathcal{L}_{\mathrm{rd}}$ of $\mathcal{L}$ to X_{rd} is a usual line bundle on X_{rd}. When X is decomposable, $Gr\mathcal{L} \simeq \mathcal{O}_M \otimes \mathcal{L}_{\mathrm{rd}}$ as $\mathcal{O}_{X_{\mathrm{rd}}}$-modules with all the parities erased. One may define also the Picard group $Pic(X)$ of X.

• N: **the count of minimal collections.** ([Fr: Lecture 3] and [Po2: vol. II, Appendix B].) The real dimension of a minimal real spin representation at d-dimensional Minkowski space is given by

d	1	2	3	4	5	6	7	8	9	10	11	12
$dim_\mathbb{R}$	1	1	2	4	8	8	16	16	16	16	32	64

In even dimensions, there are two such irreducible representations, distinguished by *left* and *right*. The N that appears in every SUSY literatures counts the number of collections of the odd generators of a SUSY algebra with each collection in a minimal spinor representation of the Lorentz subalgebra of the SUSY algebra.

Example A.1 $[d = 4, N = 2]$. The complexified $d = 4$, $N = 2$ SUSY algebra (as in the Seiberg-Witten theory) contains 8 odd generators in collections

of 4. Each collection of odd generators spans an irreducible representation, $\mathbf{2}$ or $\mathbf{2}'$, of $Spin(1,3) \simeq SL(2,\mathbb{C})$. For $d = 4$, though there are two different minimal spinor representations, they give the same complexification $\mathbf{2} + \mathbf{2}'$. In physics literature SUSY algebras are ususally complexified; thus it is not necessary to distinguish whether it is $\mathbf{2}$ or $\mathbf{2}'$ that appears in the SUSY algebra at $d = 4$. In contrast, at $d = 2$, complexifications of $\mathbf{1}$ and $\mathbf{1}'$ give inequivalent representations of $Spin(1,1)$ and the distinction of left and right is necessary. E.g. $N = (1,1)$ and $N = (0,2)$ label different complexified SUSY algebras at $d = 2$ with 2 odd generators. The distinction is also needed at $d = 10$, cf. the mod-8 periodicity of many properties of spinor representations.

• **Physical supermanifolds.** For simplicity and sufficiency of this paper, we shall assume that the supermanifold X is decomposable, i.e. $\mathcal{O}_X = Sym^{\bullet}_{\mathcal{O}_{X_{\mathrm{rd}}}}(\prod \mathcal{E})$ for some locally free $\mathcal{O}_{X_{\mathrm{rd}}}$-module $\mathcal{E}$. To link X with supersymmetry from physics, it is then required that X_{rd} is a Lorentzian manifold and $\mathcal{E}$ is a spinor bundle on X_{rd}. *Superfields* on X are defined to be global sections of locally free sheaves, e.g. $\mathcal{O}_X = Sym^{\bullet}_{\mathcal{O}_{X_{\mathrm{rd}}}}(\prod \mathcal{E})$, on X.

Example A.2 [$d = 4$, $N = 1$]. (1) *The supergeometry.* $X_{\mathrm{rd}} =$ the Minkowski space (with coordinates $x = (x^0, x^1, x^2, x^3)$) equipped with the standard metric of signature $(-1, 1, 1, 1)$ and $\mathcal{E} = (\mathcal{O}_{X_{\mathrm{rd}}} \otimes \mathbf{2})_{\mathbb{C}} = (\mathcal{O}_{X_{\mathrm{rd}}})_{\mathbb{C}} \otimes_{\mathbb{C}} (\mathbf{2} + \mathbf{2}')$, the complexified spinor bundle on X_{rd}. Fix a set of (anticommuting) generators θ^{α}, $\overline{\theta}^{\dot{\alpha}}$, $\alpha = 1, 2$, for $\prod \mathcal{E}$ as an $(\mathcal{O}_{X_{\mathrm{rd}}})_{\mathbb{C}}$-module. Recall the decomposition $(\mathbf{2} + \mathbf{2}') \bigwedge (\mathbf{2} + \mathbf{2}') = \mathbf{1} + \mathbf{1} + \mathbf{4}$, where $\mathbf{1}$ is the (complexified) 1-dimensional trivial representation and $\mathbf{4}$ is the (complexified) vector representation of $SO(1,3)$, the Pauli matrices σ^m, $m = 0, 1, 2, 3$ and the ε matrices (cf. [W-B: Appendix B] and [Fr: Lecture 3]). Then a superfield from $(\mathcal{O}_X)_{\mathbb{C}}$ can be expressed as (cf. [W-B: Appendix A] for summation conventions)

$$F(x, \theta, \overline{\theta}) = f(x) + \theta\phi(x) + \overline{\theta}\,\overline{\chi}(x) + \theta\theta\, m(x) + \overline{\theta}\,\overline{\theta}\, n(x) + \theta\sigma^m\overline{\theta}\, v_m(x)$$
$$+ \theta\theta\overline{\theta}\,\overline{\lambda}(x) + \overline{\theta}\,\overline{\theta}\,\theta\,\psi(x) + \theta\theta\overline{\theta}\,\overline{\theta}\, d(x)$$

with the component fields from representations of $SO(1,3)$:

1	θ	$\overline{\theta}$	$\theta\theta$	$\overline{\theta}\,\overline{\theta}$	$\theta\sigma^m\overline{\theta}$	$\theta\theta\overline{\theta}$	$\overline{\theta}\,\overline{\theta}\,\theta$	$\theta\theta\overline{\theta}\,\overline{\theta}$
$f(x)$	$\phi(x)$	$\overline{\chi}(x)$	$m(x)$	$n(x)$	$v_m(x)$	$\overline{\lambda}(x)$	$\psi(x)$	$d(x)$
$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}'$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{4}$	$\mathbf{2}'$	$\mathbf{2}$	$\mathbf{1}$

(cf. [W-B: Eq.(4.9)]). The $d = 4$, $N = 1$ SUSY algebra can be realized as an algebra of (differential) operators acting an $F(x, \theta, \overline{\theta})$. In particular the four odd generators are realized as: (This is what physicists call *SUSY generators* of the SUSY algebra.)

$$Q_\alpha \;=\; \frac{\partial}{\partial \theta^\alpha} - \sqrt{-1}\,\sigma^m_{\alpha\dot\alpha}\,\overline{\theta}^{\dot\alpha}\frac{\partial}{\partial x_m}$$

and

$$\overline{Q}_{\dot\alpha} \;=\; -\frac{\partial}{\partial \overline{\theta}^{\dot\alpha}} + \sqrt{-1}\,\sigma^m_{\alpha\dot\alpha}\,\theta^\alpha\frac{\partial}{\partial x_m}\,, \quad \alpha, \dot\alpha = 1,\, 2\,,$$

([Wi1: Eq.(2.1)] and [W-B: Eq.(4.4)]).

(2) *Chiral superfields and chiral multiplets.* A superfield Φ (resp. $\overline{\Phi}$) that satisfies

$$\overline{D}_{\dot\alpha}\Phi \;=\; 0 \quad (\text{ resp. } D_\alpha\overline{\Phi} \;=\; 0)\,,$$

where

$$D_\alpha \;=\; \frac{\partial}{\partial \theta^\alpha} + \sqrt{-1}\,\sigma^m_{\alpha\dot\alpha}\,\overline{\theta}^{\dot\alpha}\frac{\partial}{\partial x_m} \quad \text{and} \quad \overline{D}_{\dot\alpha} \;=\; -\frac{\partial}{\partial \overline{\theta}^{\dot\alpha}} - \sqrt{-1}\,\sigma^m_{\alpha\dot\alpha}\,\theta^\alpha\,\frac{\partial}{\partial x_m}\,,$$

is called *chiral superfield* (resp. *antichiral superfield*). In terms of

$$y^m \;=\; x^m + \sqrt{-1}\theta\sigma^m\overline{\theta}$$

in the coordinate ring of X, such Φ (resp. $\overline{\Phi}$) can be expressed as

$$\Phi(y, \theta) \;=\; \phi(y) + \sqrt{2}\,\theta\psi(y) + \theta\theta F(y)$$

(resp.

$$\overline{\Phi}(\overline{y}, \overline{\theta}) \;=\; \overline{\phi}(\overline{y}) + \sqrt{2}\,\overline{\theta}\overline{\psi}(\overline{y}) + \overline{\theta}\overline{\theta}\overline{F}(\overline{y})\,,$$

) in component fields. The part (ϕ, ψ) is a section of the vector bundle associated to a $d = 4$, $N = 1$ *chiral multiplet* representation (cf. [Fr: Lecture 5, Table 7]). When the Lagrangian for $d = 4, N = 1$ SUSY quantum field theory (SQFT) is considered, the equation of motion for (ϕ, ψ) will involve differential operators while that for F will be purely algebraic. We say that ϕ and ψ are *dynamical* component fields and F an *auxiliary* component field in the chiral multiplet Φ.

(3) *Vector superfields and vector multiplets.* A superfield V that satisfies the reality condition

$$V \;=\; V^\dagger\,,$$

where $V^\dagger$ is the Hermitian conjugate of V ([W-B: Appendix A]), is called a *vector superfield*. In the *Wess-Zumino gauge*, its component field expansion is

$$V \;=\; -\theta\sigma^m\overline{\theta}v_m + \sqrt{-1}\,\theta\theta\overline{\theta}\,\overline{\lambda} - \sqrt{-1}\,\overline{\theta}\,\overline{\theta}\theta\lambda + \frac{1}{2}\,\theta\theta\overline{\theta}\,\overline{\theta}\,D\,,$$

([W-B: Eq.(6.6)] and [Wi1: Eq.(2.11)]). The dynamical components $(\lambda, \overline{\lambda}, v_m)$ is a section of the vector bundle associated to the $d = 4$, $N = 1$ *massless vector multiplet* representation (cf. [Fr: Lecture 5, Table 7]) while D is an auxiliary component, which plays an important role in defining the vacuum manifolds in each phase of a gauged linear sigma model [Wi1].

• **Dimensional reduction** $(d = 4, N = 1) \Rightarrow (d = 2, N = (2, 2))$ **and R-symmetry.** (Cf. [DEFJKMMW], [H-V], [We], and [Wi1].)

(1) The $d = 4$, $N = 1$ *SUSY algebra* is given by generators with (anti-)commutation relations: $((\eta_{mn}) = Diag\,(-1, 1, 1, 1).)$

$$[L_{mn}, L_{m'n'}] = \eta_{nm'}L_{mn'} \;-\; \eta_{mn'}L_{nn'} \;-\; \eta_{n'm}L_{m'n} \;+\; \eta_{n'n}L_{m'm}\,,$$
$$\text{(Lorentz algebra)}$$

$$[L_{mn}, P_{m'}] = \eta_{m'n}P_m - \eta_{m'm}P_n \qquad\qquad \text{(vector representation)}$$

$$[L_{mn}, Q_\alpha] = (\sigma_{mn})_\alpha{}^\beta Q_\beta\,, \quad [L_{mn}, Q_{\dot\alpha}] = Q_{\dot\beta}\,(\overline{\sigma}_{mn})^{\dot\beta}{}_{\dot\alpha}\,,$$
$$\text{(spinor representation)}$$

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2\sigma^m_{\alpha\dot\alpha}P_m \qquad\qquad \text{(Clifford-type algebra)}$$
$$\{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0$$
$$[J, Q_\alpha] = Q_\alpha\,, \quad [J, \overline{Q}_{\dot\alpha}] = -\overline{Q}_{\dot\alpha} \qquad\qquad \text{(R-symmetry } U(1)\,)$$
$$[P_m, P_n] = [J, P_m] = [J, L_{mn}] = 0\,.$$
$$(\,m,\,n,\,m',\,n' = 0, 1, 2, 3\,;\; \alpha,\,\beta = 1, 2\,;\; \dot\alpha,\,\dot\beta = \dot1,\,\dot2,\;\; [\text{W-B: Eq.(A.14)}]\,.)$$

It is customary to call Q_α, $\overline{Q}_{\dot\alpha}$ the *SUSY generators* of the SUSY algebra.

(2) The *dimensional reduction* of the $d = 4$, $N = 1$ SUSY algebra to $d = 2$ is obtained by considering the subalgebra that leaves a specified $\mathbb{R}^{1+1}$ subspace, e.g. the (x^0, x^3)-coordinate plane, in $\mathbb{R}^{1+3}$ invariant. This corresponds to setting the extra conditions

$$P_1 \;=\; P_2 \;=\; L_{01} \;=\; L_{02} \;=\; L_{13} \;=\; L_{23} \;=\; 0$$

to the $d = 4$, $N = 1$ SUSY algebra since these generators generate Lorentz transformations that do not leave the (x^0, x^3)-coordinate plane invariant. The resulting algebra is the $d = 2$, $N = (2, 2)$ *SUSY algebra*. Its generators with renamings are

$$L := L_{03}, \quad H := -P_0, \quad P := P_3,$$
$$Q_- := Q_1, \quad Q_+ := Q_2, \quad \overline{Q}_- := \overline{Q}_{\dot{1}}, \quad \overline{Q}_+ := \overline{Q}_{\dot{2}},$$
$$J_1 := J, \quad J_2 := -2\sqrt{-1}\,L_{12},$$

with commutation relations:

$$[L, H] = -P, \; [L, P] = -H,$$
$$[L, Q_+] = \tfrac{1}{2}Q_+, \; [L, Q_-] = -\tfrac{1}{2}Q_-, \; [L, \overline{Q}_+] = \tfrac{1}{2}\overline{Q}_+, \; [L, \overline{Q}_-] = -\tfrac{1}{2}\overline{Q}_-,$$
$$\{Q_+, \overline{Q}_+\} = 2\,(H - P), \; \{Q_-, \overline{Q}_-\} = 2\,(H + P),$$
$$[J_1, Q_+] = Q_+, \; [J_1, Q_-] = Q_-, \; [J_1, \overline{Q}_+] = -\overline{Q}_+, \; [J_1, \overline{Q}_-] = -\overline{Q}_-$$
$$[J_2, Q_+] = Q_+, \; [J_2, Q_-] = -Q_-, \qquad [J_2, \overline{Q}_+] = -\overline{Q}_+, \; [J_2, \overline{Q}_-] = \overline{Q}_-$$
$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0$$
$$\{Q_-, Q_+\} = \{\overline{Q}_-, \overline{Q}_+\} = \{Q_+, \overline{Q}_-\} = \{Q_-, \overline{Q}_+\} = 0,$$
$$[H, P] = [J_1, H] = [J_1, P] = [J_1, L] = 0$$
$$[J_2, H] = [J_2, P] = [J_2, L] = [J_1, J_2] = 0.$$

Q_- and $\overline{Q}_-$ (resp. Q_+ and $\overline{Q}_+$) are the $d = 2$, $N = (2, 0)$ (resp. $N = (0, 2)$) SUSY generators and the Lorentz generator L_{12} in the original algebra has now become the second R-symmetry generator J_2 of the new SUSY algebra.

Define

$$J_L = \tfrac{1}{2}\,(J_2 - J_1), \qquad (\,\text{left-moving R-symmetry generator}\,)$$
$$J_R = \tfrac{1}{2}\,(J_2 + J_1). \quad (\,\text{right-moving R-symmetry generator}\,)$$

Then

$$[J_L, Q_-] = -Q_-, \quad [J_L, \overline{Q}_-] = \overline{Q}_-, \quad [J_L, Q_+] = 0, \quad [J_L, \overline{Q}_+] = 0$$
$$[J_R, Q_-] = 0, \qquad [J_R, \overline{Q}_-] = 0. \quad [J_R, Q_+] = Q_+, \quad [J_R, \overline{Q}_+] = -\overline{Q}_+,$$

(3) The dimensional reduction of a superfield on $d = 4$ Minkowski space-time to $d = 2$ superfields is obtained by setting two spatial directions, say x^1 and x^2, to be constant and take fields to depend only on x^0 and x^3. Recall Example A.2. Then the rule of conversion of fields from $d = 4$, $N = 1$ to $d = 2$, $N = (2, 2)$ are given by:

$$y^0 := x^0, \qquad y^1 := x^3, \qquad\qquad (\text{for space-time coordinates})$$
$$\sigma := (v_1 - \sqrt{-1}v_2)/\sqrt{2}, \qquad \overline{\sigma} := (v_1 + \sqrt{-1}v_2)/\sqrt{2}$$

$$(\text{reduced vector components in } d = 4$$
$$\Rightarrow \text{complex scalars in } d = 2)$$

$$\begin{cases} (\psi^-, \psi^+) := (\psi^1, \psi^2), \; (\psi_-, \psi_+) := (\psi_1, \psi_2), \\ (\overline{\psi}^-, \overline{\psi}^+) := (\psi^{\dot{1}}, \psi^{\dot{2}}), \; (\overline{\psi}^-, \overline{\psi}^+) := (\psi_{\dot{1}}, \psi_{\dot{2}}). \end{cases} \quad (\text{for spinorial components})$$

The $d = 2\,N = (2,2)$ chiral superfields and vector superfields can be obtained from $d = 4$, $N = 1$ ones via these conversions, cf. Example A.2 and the next item.

• **Gauged linear sigma models.** ([Wi1 : Sec. 2].) Given $d = 2$, $N = (2,2)$ chiral superfields (Φ_i here is the Φ_0 in [Wi1 : Eq.(2.13)])

$$\Phi_i \;=\; \phi_i + \sqrt{2}\theta^+\psi_{i,+} + \sqrt{2}\theta^-\psi_{i,-} + \theta^2 F_i\,, \quad i = 1, \ldots, n\,,$$

in $d = 2$, $N = (2,2)$ chiral coordinates (Φ_i here is the Φ_0 in [Wi1 : Eq.(2.13)])

$$(\,y^0 - \sqrt{-1}(\theta^-\overline{\theta}^- + \theta^+\overline{\theta}^+)\,,\, y^1 + \sqrt{-1}(\theta^-\overline{\theta}^- - \theta^+\overline{\theta}^+)\,)\,,$$

vector superfields, in Wess-Zumino gauge, (cf. connections)

$$
\begin{aligned}
V_a \;=\;& -\sqrt{2}\,(\theta^-\overline{\theta}^+\sigma_a + \theta^+\overline{\theta}^-\overline{\sigma}_a) + (\theta^-\overline{\theta}^- + \theta^+\overline{\theta}^+)v_{a,0} - (\theta^-\overline{\theta}^- - \theta^+\overline{\theta}^+)v_{a,1} \\
&+\sqrt{-2}\,\theta^+\theta^-(\overline{\theta}^+\overline{\lambda}_{a,+} + \overline{\theta}^-\overline{\lambda}_{a,-}) - \sqrt{-2}\,\overline{\theta}^+\overline{\theta}^-(\theta^-\lambda_{a,-} + \theta^+\lambda_{a,+}) \\
&-2\,\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-\,D_a\,, \\
a =\;& 1, \ldots, n-d\,,
\end{aligned}
$$

gauge group $U(1)^{n-d}$ (parameterized by $(t_1, \ldots, t_{n-d})$), and a $U(1)^{n-d}$-action on Φ_i by

$$\Phi_i \;\longrightarrow\; \left(\prod_{a=1}^{n-d} t_a^{Q_{i,a}}\right)\Phi_i\,.$$

Define ([Wi1 : Eq.(2.16)]), (cf. curvatures)

$$
\begin{aligned}
\Sigma_a :=\;& \frac{1}{\sqrt{2}}\overline{D}_+ D_- V_a \\
=\;& \sigma_a - \sqrt{-2}\theta^+\overline{\lambda}_{a,+} - \sqrt{-2}\,\overline{\theta}^-\lambda_{a,-} + \sqrt{2}\theta^+\overline{\theta}^-(D_a - \sqrt{-1}v_{a,01}) \\
&-\sqrt{-1}\overline{\theta}^-\theta^-(\partial_0 - \partial_1)\sigma_a - \sqrt{-1}\theta^+\overline{\theta}^+(\partial_0 + \partial_1)\sigma_a \\
&+\sqrt{2}\overline{\theta}^-\theta^+\theta^-(\partial_0 - \partial_1)\overline{\lambda}_{a,+} + \sqrt{2}\theta^+\overline{\theta}^-\overline{\theta}^+(\partial_0 + \partial_1)\lambda_{a,-} - \theta^+\overline{\theta}^-\theta^-\overline{\theta}^+(\partial_0^2 - \partial_1^2)\sigma_a\,,
\end{aligned}
$$

where $v_{a,01} = \partial_0 v_{a,1} - \partial_1 v_{a,0}$. The associated *gauged linear sigma model* is a 2-dimensional supersymmetric quantum field theory (SQFT) with action

$$L \;=\; L_{\text{kinetic}} + L_W + L_{\text{gauge}} + L_{D,\theta}\,,$$

where

$$L_{\text{kinetic}} = \int d^2y \, d^4\theta \sum_{i=1}^{n} \overline{\Phi}_i \exp\left[2\sum_{a=1}^{n-d} Q_{i,a} V_a\right] \Phi_i \,,$$

$$L_W = -\int d^2y \, d\theta^+ \, d\theta^- \, W(\Phi_i)\big|_{\overline{\theta}^+ = \overline{\theta}^- = 0} - (\text{Hermitian conjugate}) \,,$$

$$L_{\text{gauge}} = -\sum_{a=1}^{n-d} \frac{1}{4\,e_a^2} \int d^2y \, d^4\theta \, \overline{\Sigma}_a \, \Sigma_a \,,$$

and

$$L_{D,\theta} = \sum_{a=1}^{n-d} \int d^2y \, (-r_a D_a + \frac{\theta_a}{2\pi} v_{a,01})$$

$$= \frac{\sqrt{-1}\,t_a}{2\sqrt{2}} \int d^2y \, d\theta^+ \, d\overline{\theta}^- \Sigma\big|_{\theta^- = \overline{\theta}^+ = 0} - \frac{\sqrt{-1}\,\overline{t_a}}{2\sqrt{2}} \int d^2y \, d\theta^- \, d\overline{\theta}^+ \overline{\Sigma}\big|_{\theta^+ = \overline{\theta}^- = 0}$$

with

$$t_a = \sqrt{-1}\,r_a + \frac{\theta_a}{2\pi} \,.$$

The real-valued numerical parameters e_a, r_a, and θ_a are called the *coupling constants* of the theory.

Performing the Fermionic integrations $\int d^4\theta$, $\int d\theta^+ d\theta^-$, and $\int d\overline{\theta}^+ d\overline{\theta}^-$ renders L a complicated expression in terms of component fields on Φ_i and V_a ([Wi1: Eq.(2.19), Eq.(2.21), and Eq.(2.23)]). The $d = 2, N = (2,2)$ SUSY algebra without R-symmetry generators can be realized as an algebra of derivations acting on V_a and on Φ_i with gauge transformations taken into account while the R-symmetry acts on fields via global abelian transformations on the fields. In particular, the SUSY transformations in terms of component fields are given in [Wi1: Eq.(2.12) and Eq.(2.14)] (Cf. Example A.2 (1), [W-B Chapters III-VII], and [We: Sec. 2.7.8]). The action L is invariant under these transformations (and hence supersymmetric).

• **Wick rotation.** Field theories on Riemannian manifolds behave better than those on Lorentzian manifolds. A *Wick rotation* is meant to be an analytic continuation between theories in the two categories (e.g. [P-S]). Some of its geometry is studied in [Liu]. In the current case of flat space-times, such an analytic continuation is realized by setting $y^0 = -iy^2$ and taking (y^1, y^2) as the coordinates of the Wick rotated $d = 2$ space-time. The latter has the Euclidean metric $-(dy^0)^2 + (dy^1)^2 = (dy^2)^2 + (dy^1)^2$ and the tangent bundle group $SO(1,1)$ now becomes $SO(2)$.

• **The A-twist and the B-twist.** ([Wi2], [Wi1], [DEFJKMMW: vol. 2, Witten's lecture, Sec. 14.3], and [F-S: Chapter 7].) Consider the two different twisted embeddings of the $d = 2$ rotation algebra, generated by L, into the $d = 2$, $N = (2,2)$ SUSY algebra:

$$\text{A-twist}: \quad L \mapsto L^A := L - \tfrac{1}{2}J_L + \tfrac{1}{2}J_R$$
$$\text{B-twist}: \quad L \mapsto L^B := L + \tfrac{1}{2}J_L + \tfrac{1}{2}J_R.$$

Since both J_L and J_R commute with all the SUSY algebra generators except $Q_\pm$ and $\overline{Q}_\pm$, the commutation relation of L^A and L^B with SUSY algebra generators are the same as those for L except the following ones:

$$[L^A, Q_+] = Q_+, \quad [L^A, Q_-] = 0, \qquad [L^A, \overline{Q}_+] = 0, \quad [L^A, \overline{Q}_-] = -\overline{Q}_-,$$
$$[L^B, Q_+] = Q_+, \quad [L^B, Q_-] = -Q_-, \quad [L^B, \overline{Q}_+] = 0, \quad [L^B, \overline{Q}_-] = 0.$$

This implies that all the SUSY generators are now of integral spin with respect to either of the twisted tangent bundle groups.

From the commutation relations of the Lorentz generator with SUSY generators, the supermanifolds associated to the Wick-rotated $d = 2$, $N = (2,2)$ SUSY algebra are

$$X = \left(C, \, Sym^\bullet \prod \left((K_C^{\frac{1}{2}})^{\oplus 2} \oplus (K_C^{-\frac{1}{2}})^{\oplus 2} \right) \right),$$

where C is a Riemann surface and K_C is the canonical line bundle of C while the supermanifolds associated to either A-twisted or B-twisted SUSY algebra are

$$X^{\text{twist}} = \left(C, \, Sym^\bullet \prod (\mathcal{O}_C^{\oplus 2} \oplus K_C \oplus K_C^{-1}) \right).$$

When C is the complex plane, a cylinder, or an elliptic curve, $K_C^{\pm\frac{1}{2}} \sim \mathcal{O}_C$ admit nontrivial global sections. Thus both SQFT and its twists can be built on such C. For general C, $K_C^{\pm\frac{1}{2}}$ have no global sections except the zero-section and, hence, only twisted SQFT can be defined on C.

• **Phase structure.** ([Wi1] and [M-P] for GLSM; [Al], [Arg], [Po1], [R-S-Z], and [W-K] for general field-theoretical aspects.) When the coupling constants $(e_a, r_a, \theta_a)_a$ in the action L of the gauged linear model given earlier vary, the nature of the field theories may also vary. Thus $(e_a, r_a, \theta_a)_a$ can be thought of as the coordinates for a space $\mathcal{M}_{GLSM}$ that parameterizes a

family of $d = 2$ field theories. $\mathcal{M}_{GLSM}$ is called the (*Wilson's*) *theory space* of the model. Quantities (e.g. 2-point functions) of the field theories may be turned into defining geometric data (e.g. Zamolodchikov metric) on $\mathcal{M}_{GLSM}$. There exists a stratification of $\mathcal{M}_{GLSM}$ according to the nature of the field theory a point in $\mathcal{M}_{GLSM}$ parameterizes. Each stratum of this stratification is called a *phase* of the GSLM. In general, for quantum considerations of the theory, cutoffs (e.g. of energy) may have to be introduced. These cutoff parameters may also be added in to enlarge the theory space. (Cf. The recent work of Borchard [Bo] enlarges Wilson's theory space by adding also the space of *renormalization prescriptions*. His work should be important to understanding the quantum phase structure on the theory space.)

● **The moduli space of the A-twisted theory in the geometric phase.** Either twist breaks half of the 4 supersymmetries in general. The resulting $d = 2$ SQFT has the same expression as the action L but each of the component fields in the superfield involved lives in a new bundle determined by the twisted spin discussed in Item (The A-twist and the B-twist) above. For the A-twist, the remaining supersymmetries of the twisted gauged linear sigma model are generated by Q_- and $\overline{Q}_+$. These are the SUSY generators that are of A-twisted spin 0. Recall the realization of SUSY algebra with R-symmetry generators removed as an algebra of derivations acting on fields. A field configuration that is annihilated by both Q_- and $\overline{Q}_+$ is called a *supersymmetric field configuration* of the twisted gauged linear sigma model. When the gauge coupling constants e_a are all set equal to some e and the superpotential W is set to zero, the bosonic part of SUSY configurations for the A-twisted theory are given explicitly by the solutions to the following system of equations ([Wi1 : Eq.(3.33), Eq.(3.34), Eq.(3.35)] and [M-P : Eq.(3.54 a-d)])

$$
\begin{aligned}
d\sigma_a &= 0, & a &= 1, \ldots, n-d, \\
\sum_{a=1}^{n-d} Q_i^a \, \sigma_a \, \phi_i &= 0, & i &= 1, \ldots, n, \\
D_{\overline{z}} \, \phi_i &= 0, & i &= 1, \ldots, n, & \quad (*1) \\
D_a + v_{a,12} &= 0, & a &= 1, \ldots, n-d, & \quad (*2)
\end{aligned}
$$

with $D_{\overline{z}}$ a covariant derivative constructed from the $U(1)^{n-d}$ gauge connection $(v_{a,1}, v_{a,2})_a$ and

$$
D_a = -e^2 \left(\sum_{i=1}^{n} Q_i^a \, |\phi_i|^2 - r_a \right)
$$

from the equation of motion for D_a.

When $(r_a)_a$ is in the geometric phase, for which the solution set to the subsystem { Eq.($*1$), Eq.($*2$) } is non-empty, the only solution for σ_a is $\sigma_a = 0$ for all a. Following the study in [Brad], [B-D], and [GP] on vortex-type equations, Witten and Morrison-Plesser thus conclude that the moduli space of the A-twisted theory for $(r_a)_a$ in this phase is given by

$$
\coprod_{\vec{d}} \mathcal{M}_{\vec{d}} = \frac{\left\{ \begin{array}{c} \text{common solutions to} \\ \text{Eq. ($*1$) and Eq. ($*2$)} \end{array} \right\}}{\left\{ \begin{array}{l} \text{unitary abelian gauge transformations} \\ \text{and global complex abelian transforma-} \\ \text{tions} \end{array} \right\}}
$$

$$
= \left\{ \begin{array}{l} \text{solutions to Eq. ($*1$) that satisfy} \\ \text{appropriate stability condition} \end{array} \right\} \bigg/ \left\{ \begin{array}{l} \text{complex abelian gauge} \\ \text{transformations} \end{array} \right\}
$$

$$
= \text{the toric variety} \quad \coprod_{\vec{d}} (Y_{\vec{d}} - F_{\vec{d}}) / G ,
$$

where $Y_{\vec{d}}$, $F_{\vec{d}}$, and G are explained in Explanation/Fact 2.1.2 and Sec. 3 in terms of toric geometry ([M-P : Sec. 3.1]). (Cf. See also [Fr : Lecture 4 and Lecture 5] on the *moduli space of vacua* of a SQFT.)

As already mentioned in [M-P : Sec. 3.7], the above construction, in particular the moduli space $\coprod_{\vec{d}} \mathcal{M}_{\vec{d}}$, has a generalization to higher genus Riemann surfaces as well, following [Cox2]. The main theme of this paper is the study of this generalization of $\coprod_{\vec{d}} \mathcal{M}_{\vec{d}}$ to higher genus.

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