COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 12, Number 1, 183-211, 2004

Global Existence of the m-equivariant Yang-Mills Flow in Four Dimensional Spaces

Min-Chun Hong and Gang Tian

1. Introduction.

The use of non-linear parabolic equations (the heat flow method) to find solutions of corresponding elliptic equations goes back to Eells-Sampson in 1964. In their seminal paper [ES], Eells and Sampson introduced the heat flow for harmonic maps to establish the existence of smooth harmonic maps from a compact Riemmanian manifold into a Riemmanian manifold having non-positive section curvature. In general, the heat flow for harmonic maps even on two dimensional manifolds may develop singularity at finite time (cf. [CDY]). Struwe [St1] established the existence of the unique global weak solution, which is smooth with exception of at most finitely many points, to the heat flow for harmonic maps in two dimensions. The harmonic map flow in two dimensions is very similar to the Yang-Mills flow in four dimensions. It is desirable to have a similar picture for Yang-Mills flow.

In this paper, we consider the Yang-Mills flow in a vector bundle over four dimensional manifolds. Let X be a compact 4-dimensional Riemannian manifold and let $E \to X$ be a vector bundle with a compact Lie group G. Let A be a connection on E. The Yang-Mills functional is

$$\mathrm{YM}(A) = \int_X |F_A|^2 dv_X,$$

where $F_A = d\,A + A \wedge A$ is the curvature of A in E.

A connection A is said to be a solution to the Yang-Mills (heat) flow if it satisfies the following equation:

$$\frac{\partial A}{\partial t} = -D_A^* F_A. \tag{1.1}$$

The study of the Yang-Mills flow (1.1) has been of great interests. Donaldson [D] first introduced the Yang-Mills flow to study the existence of Hermitian Yang-Mills metrics of holomorphic vector bundles and proved the global existence of the regular solution to the Yang-Mills flow over a Hermitian holomorphic vector bundle on a Kähler surface X assuming that the initial unitary connection A_0 has curvature F_{A_0} of type $(1,1)$. Rade [R] established the global existence of the Yang-Mills flow in a vector bundle on three dimension compact Riemannian manifolds. Struwe [St3] (also [Sc1]) proved the global existence of the unique weak solution to the Yang-Mills flow in vector bundles on four dimension manifolds, where the weak solution is regular always from finite singularities. It remains a challenging question whether the Yang-Mills flow in four dimensional manifolds develop singularity at finite time. Schlatter, Struwe and Tahvildar-Zadeh [SST] proved the global existence of the $SO(4)$-equivariant Yang-Mills flow on $\mathbb{R}^4$, which provides some evidences that the Yang-Mills flow in four dimensions may not blow up in finite time. In this paper, we will prove the global existence of the m-equivariant Yang-Mills flow on $\mathbb{R}^4$ (see Theorem 4.1).

The motivation of this paper is based on the fundamental relationship between m-equivariant gauge fields on S^4 and monopoles on the hyperbolic 3-space $\mathbb{H}^3$, described by Atiyah [A]. To study the m-equivariant connections on S^4, the Yang-Mills functional over S^4 can be reduced to the Yang-Mills-Higgs functional in hyperbolic 3-space $\mathbb{H}^3$. Using m-equivariant connections over S^4, L.M. Sibner, R. J. Sibner and K. Uhlenbeck [SSU] proved the existence of non-self dual Yang-Mills connections over S^4. Braam [B] studied the magnetic monopoles on complete three manifolds. Based on these ideas, we also reduce the Yang-Mills flow to the Yang-Mills-Higgs flows in the $\mathbb{H}^3$.

In this paper, we investigate the Yang-Mills flow in a vector bundle over three dimensional complete manifolds, which include hyperbolic 3-space $\mathbb{H}^3$.

Now, let (M,g) be a complete 3-dimensional Riemannian manifold with curvature bounded by $K > 0$ and we assume that

$$\inf_{x\in M} |B_1(x)| > 0,$$

where $|B_1(x)|$ stands for the volume of the unit geodesic ball $B_1(x)$ with respect to g. Let $E \to M$ be a vector bundle with a compact Lie group G. In the sequel we take G to be $SU(2)$ or $SO(3)$ for simplicity. Let A be a connection on E and let Φ be a Higgs field, i.e. a section of $\Omega^0(\mathrm{ad}E)$. The Yang-Mills Higgs functional is

$$\mathrm{YMH}(A,\Phi) = \int_M |F_A|^2 + |D_A\Phi|^2 dv_M,$$

where F_A is the curvature of A, D_A denotes the covariant derivative on sections associate to A and dv_M denotes the volume form of M.

A solution of the Yang-Mills-Higgs flow consists of a connection A and a section Φ satisfying

$$\begin{aligned} \frac{\partial A}{\partial t} &= -\, D_A^* F_A - J; \\ \frac{\partial \Phi}{\partial t} &= -\, D_A^* D_A \Phi, \end{aligned} \tag{1.2}$$

where

$$J = J(D_A, \Phi) = -[D_A\Phi, \Phi].$$

Initial conditions for the flow (1.2) are given by

$$A(x,0) = A_0(x), \quad \Phi(x,0) = \Phi_0(x) \quad \text{for } x \in M, \tag{1.3}$$

where A_0 and Φ_0 are a given connection and a section respectively.

The study of the Yang-Mills-Higgs flow on a complete manifold is of independent interests, as the study of the harmonic map flow from on complete manifolds (cf. [LT]). Not much is known for the Yang-Mills-Higgs flow (1.2) in a general noncompact 3-manifold M. Only in the case of $M = \mathbb{R}^3$, Hassell in [Ha] proved the global existence of the Yang-Mills-Higgs flow assuming that the initial values are sufficiently small.

In this paper, we first establish the global existence of the Yang-Mills-Higgs flow (1.2) over a complete three-manifold (see Theorem 3.7). To prove the local existence for (1.2), we use a trick of De Turck [De]. We consider the gauge equivalent heat flow corresponding to (1.2) through gauge transformations. We also improve the idea of Struwe [St3] to deal with the initial value $(A_0, \Phi_0) \in H^{1,2}$ by considering a background connection B and section Ψ, which are solutions of linear parabolic equations (see (2.3) and (2.9)). We would like to point out that the Weizenböck formula plays an important role. To prove the global existence of (1.2), we use a covering argument to obtain a global estimate (see Lemma 3.5), which is used to extend the smooth solution of (1.2) to any finite time without any singularity. Finally, we apply these results to $M = \mathbb{H}^3$ and obtain the global existence of smooth solutions to the m-equivariant Yang-Mills flow on $\mathbb{R}^4$ (see Theorem 4.1). Our proof also yields non-self dual Yang-Mills connections on S^4 by the heat flow method (see Theorem 4.2). The existence of such connections were previously constructed in [SSU] by the elliptic method.

The paper is organized as follows. In section 2, we prove the local existence of the Yang-Mills-Higgs flow (1.2) in complete three dimensional manifolds. In section 3, we establish the global existence of the Yang-Mills-Higgs flow (1.2) in complete three dimensional manifolds. In section 4, we prove the global existence of smooth solutions to the m-Yang-Mills flow over $\mathbb{R}^4$.

Acknowledgment. Most of the work was carried out when the first author was an ARC fellow at the Australian National University. Partial work was also done when the first author visited the MIT in May 2001 and in Nov. 2002. The first author gratefully acknowledges the hospitality and support of the MIT.

2. Local existence.

In this section, we will establish the local existence of the Yang-Mills-Higgs flow (1.2) with initial value (1.3).

We recall from [He, Chapter 3, Theorem 3.2 and Proposition 3.7] the following property of the Sobolev space in a complete manifold:

Let M be a smooth, complete Riemannian n-manifold with Ricci curvature bounded from below. Assuming that

$$\inf_{x\in M} |B_1(x)| > 0,$$

where $|B_1(x)|$ stands for the volume of $B_1(x)$ with respect to the metric g. Then the Sobolev embedding is true, i.e. for $u \in H^{1,p}(M)$, there exists a constant $A > 0$ such that

$$\left(\int_M |u|^q \, dv_M\right)^{1/q} \le A \left[\left(\int_M |\nabla u|^p \, dv_M\right)^{1/p} + \left(\int_M |u|^p \, dv_M\right)^{1/p}\right],$$

for all $p \le q \le \frac{np}{n-p}$.

Moreover, let (M, g) be a smooth, complete Riemannian n-manifold with positive injective radius, and we assume that the set of smooth functions with compact support in M is dense in $H^{1,p}(M)$ for any $p \ge 1$.

The space of the connection on E is an affine space

$$\mathcal{D} = \{A = A_{ref} + a; \quad a \in \Omega^1(\text{ad}E)\},$$

where A_{ref} is a given smooth connection such that

$$|A_{ref}| \le C, \quad |\nabla A_{ref}| \le C, \quad \int_M |F_{ref}|^2 \, dv_M \le C, \quad \int_M |\nabla F_{ref}|^2 \, dv_M \le C,$$

where C is a fixed constant.

A connection A on E is related to a covariant derivative Δ_A. Let ad E be the adjoint bundle whose sections $s \in \Omega^0(\mathrm{ad}E)$ locally can be represented by the map $s \in U_\alpha \to \mathfrak{g}$, the Lie algebra of G, where $\{U_\alpha\}$ is a cover of M.

The Sobolev space of $H^{l,p}(\Omega^i(\mathrm{ad}E))$ consists of the $\mathfrak{g}$-valued i-forms ϕ in the adjoint bundle ad$E$ with measure coefficients such that

$$\|\phi\|_{H^{l,p}(\Omega^i(E))} = \left(\sum_{k=0}^{l} \|\nabla_{ref}^k \phi\|^p_{L^p(\Omega^i(E))}\right)^{1/p}.$$

We denote by $\langle \cdot, \cdot \rangle$ the pointwise inner product. Since A is compatible with the metric, for any $\phi, \psi \in \Omega^i(\mathrm{ad}E)$ we have

$$d\langle \phi, \psi \rangle = \langle \nabla_A \phi, \psi \rangle + \langle \phi, \nabla_A \psi \rangle .$$

It implies the Kato inequality $|d|\phi|| \le |\nabla_A \phi|$. Then for any $\phi \in H^{1,p}(\Omega^i)$, we have

$$\left(\int_M |\phi|^q \, dv_M\right)^{1/q} \le A\left[\left(\int_M |\nabla_A \phi|^p \, dv_M\right)^{1/p} + \left(\int_M |\phi|^p \, dv_M\right)^{1/p}\right]$$

for $p \le q \le \frac{np}{n-p}$.

From now on, let $n = 3$.

We recall a covering result for a complete manifold (cf. [He, Chapter 1, Lemma 1.1] in the following:

Lemma 2.1. *Let (M, g) be a smooth, complete Riemannian 3-manifold with curvature bounded by some constant $K > 0$, and let $\rho > 0$ be given. There exists a set $\{x_i\}$ of points in M such that for any $r \ge \rho$,*

(i) the family $\{B_r(x_i)\}$ is a uniformly locally finite covering of M, with the property that at any point $x \in M$ at most N of the balls $B_r(x_i)$ meet where $N \le (\frac{8r}{\rho})^3 e^{-4\sqrt{2K}r}$.

(ii) for any $i \ne j$, $B_{\rho/2}(x_i) \cap B_{\rho/2}(x_j) = \emptyset$.

In the sequel, we also assume that the Riemannian curvature Rm of M is bounded, i.e., $|\mathrm{Rm}| \le K$ for some positive constant K.

For each connection D_A, we have the Hodge Laplacian $\triangle_A = D_A^* D_A + D_A D_A^*$ and another crude Laplacian $\nabla_A^* \nabla_A$ on $\Omega^i(\mathrm{ad}E)$. We recall the well-known Weizenböck formula (cf. [La]) as follows: for any $\phi \in \Omega^i(\mathrm{ad}E)$, we have

$$\nabla_A^* \nabla_A \phi = \triangle_A \phi + F_A \# \phi + \mathrm{Rm} \# \phi, \tag{2.1}$$

where we do not need the explicit formula of the two terms $F\#\phi$ and $\mathrm{Rm}\#\phi$, which are multi-linear combination of smooth coefficients satisfying

$$|F\#\phi| \le C|F|\,|\phi|, \quad |\mathrm{Rm}\#\phi| \le C|\mathrm{Rm}||\phi|$$

for some constant $C > 0$.

As a consequence of the Weizenböck formula (2.1), the same proof as in [St3] yields the following:

Lemma 2.2. *Let $D_A = D_{ref} + A$, where $A \in C^1(\Omega^1(adE))$ and $\|A\|_{C^1(M)} \le K_1$ for some positive constant K_1. There exist constants $C_1 = C_1(\eta)$ and $C_2 = C_2(K, K_1)$ such that for any $\phi \in H^{2,2}(\Omega^i(adE))$,*

$$\|\phi\|^2_{H^{2,2}(\Omega^i(adE))} \le C_1 \| \triangle_A \phi\|^2_{L^2(\Omega^i(adE)))} + C_2\|\phi\|^2_{L^2(\Omega^i(adE))}.$$

For any $T > 0$, we consider the space

$$V_T = V_T\left(\Omega^i(\mathrm{ad}E)\right) = L^2\left([0,T]; H^{2,2}(\Omega^i(\mathrm{ad}E)\right) \cap H^{1,2}\left([0,T]; L^2\left(\Omega^i(\mathrm{ad}E)\right)\right)$$

and employ the standard notation of L^p-L^q-norm,

$$\|\phi\|_{L^{p,q}([0,T];\Omega^i(\mathrm{ad}E))} := \left(\int_0^T \|\phi\|^q_{L^p(\Omega^i(\mathrm{ad}E))}\,dt\right)^{1/q}$$

for $1 \le p, q < \infty$. Moreover, we denote by the norm of the space V_T

$$\|\phi\|^2_{V_T} := \|\frac{d}{dt}\phi\|^2_{L^{2,2}([0,T];\Omega^i(\mathrm{ad}E))} + \|\phi\|^2_{L^2([0,t];H^{2,2}(\Omega^i(\mathrm{ad}E)))}.$$

As in [St3], V_T is continuously embedded in

$$L^\infty\left([0,T]; H^{1,2}\left(\Omega^i(ad\,E)\right)\right),$$

and as pointed in [St3], with

$$\sup_{0\le t\le T} \|\phi(\cdot,t)\|^2_{H^{1,2}(\Omega^i(\mathrm{ad}E))} \le \|\phi(0)\|^2_{H^{1,2}(\Omega^i(\mathrm{ad}E))} + 2\|\phi\|^2_{V_T}. \tag{2.2}$$

Moreover, as in [St3], we have

Lemma 2.3. *Let $D_A = D_{ref} + A$, $A \in C^1(\Omega^1(adE))$ with $\|A\|_{C^1(M)} \le K_1$ for some $K_1 > 0$. Then there exist a constant $C_3 = C_3(K, K_1)$, depending on K and K_1, and a finite number $T_1 = T_1(\|A\|_{C^1(M)}, E) > 0$, depending on $\|A\|_{C^1(M)}$, such that for any $\phi \in V_T$*

$$\|\phi\|^2_{V_T} \le C_3\left[\|(\frac{d}{dt} + \triangle_A)\phi\|^2_{L^{2,2}([0,T];\Omega^i(adE))} + \|\phi(0)\|^2_{H^{1,2}(\Omega^i(adE))}\right].$$

Let us assume that in (1.3), A_0 and Φ_0 are smooth, $A_0 \in H^{1,2}(\Omega^1(\mathrm{ad}E))$ and $\Phi_0 \in H^{1,2}(\Omega^0(\mathrm{ad}E))$ is bounded. By the density result of the Sobolev space, there exists a connection A_1, which has compact support in M, such that $\|A_1 - A_0\|_{H^{1,2}(\Omega^1(\mathrm{ad}E))}$ is sufficiently small.

In order to prove the local existence of the heat flow (1.2) with (1.3), we consider the following initial value problem:

$$\begin{aligned} \frac{\partial B}{\partial t} &= - \triangle_{A_1} B, \quad \text{in } M \times (0, \infty) \\ B(0) &= B_0, \quad \text{in } M, \end{aligned} \tag{2.3}$$

where $B_0 = A_0 - A_1$ with $D_{A_1} = D_{ref} + A_1$.

Then we have

Lemma 2.4. *Let $D_1 = D_{ref} + A_1$ be a smooth connection satisfying above conditions and assume that B_0 is smooth and also in $H^{1,2}$. Then the initial problem (2.3) has a unique, global solution $B(x,t)$ where $B(x,t)$ is smooth in $M \times (0,\infty)$, $B(x,t) \in L^2([0,T], H^{2,2}(adE)) \cap C^0([0,T]; H^{1,2}(adE)) \cap H^{1,2}([0,T]; L^2(adE))$ for any $T < \infty$. Moreover, there exits a constant $C(T)$ such that for any $t \in [0,T]$*

$$\|B(\cdot,t)\|_{H^{1,2}(\Omega^1(adE))} \le C(T)\|B_0\|_{H^{1,2}(\Omega^1(adE))}. \tag{2.4}$$

Proof. Let us first consider the problem (2.3) in $M \times [0,T]$ with initial $B(0) = B_0$, where B_0 is smooth, $|B(0)|$ and $|\nabla B_0|$ are bounded by some positive constant. By the Weizenböck formula (2.1), we obtain

$$\begin{aligned} \triangle_{A_1} B &= \nabla^*_{A_1} \nabla_{A_1} B + Rm\# B + F_{D_1}\# B + A_1 \# \nabla B \\ &= \nabla^* \nabla B + f(A_1, \nabla A_1, B, \nabla B), \end{aligned}$$

where the term $f(A_1, \nabla A_1, B, \nabla B) := Rm\#B + \nabla A_1 \# B + A_1 \# \nabla B + A_1 \# A_1 \# B$ is multi-linear. Due to the assumption on A_1 and the curvature Rm, we have

$$|f(A_1, \nabla A_1, B, \nabla B)| \le C(|B| + |\nabla B|).$$

Let $H(x,y,t)$ be the heat kernel of M as in [LT]. Through the heat kernel, equation (2.3) becomes

$$B(x,t) = -\int_0^t \int_M H(x,y,t-s)\left[\triangle_M - \frac{\partial}{\partial s}\right] B(y,s)\, dv_M(y) ds$$
$$= -\int_0^t \int_M H(x,y,t-s) f(A_1, \nabla A_1, B, \nabla B)(y,s)\, dv_M(y) ds,$$

where $\triangle_M$ is the Laplacian-Beltrami operator on M. A similar iteration argument to one in [LT, Theorem 3.3] yields the local existence of the unique smooth solution B of the problem (2.3) with initial value B_0.

Now, we prove the global existence of the unique solution to equation (2.3) with $B(0) = B_0$. Let $T > 0$ be the maximum time such that B is a smooth solution to equation (2.3) in $M \times [0,T)$ with initial value $B(0) = B_0$.

Let $\phi \in C_0^\infty(B_{2R})$ be a cut-off function with $0 \le \phi \le 1$, $|d\phi| \le CR^{-1}$, and $\phi = 1$ in B_R. Using the Weizenböck formula (2.1), we have

$$\frac{d}{dt}\int_M \phi^2 |B|^2\, dv_M = 2\int_M \phi^2 \left\langle \frac{\partial B}{\partial t}, B\right\rangle dv_M = -2\int_M \phi^2 \left\langle \triangle_{A_1} B, B\right\rangle dv_M$$
$$= -2\int_M \phi^2 \left\langle \nabla_{A_1}^* \nabla_{A_1} B, B\right\rangle dv_M + \int_M \phi^2 \left\langle Rm\# B + F_{A_1}\# B, B\right\rangle dv_M$$
$$\le -2\int_M \phi^2 |\nabla_{A_1} B|^2 dv_M + \int_M \left\langle \nabla_{A_1} B, \phi d\phi B\right\rangle dv_M + C\int_M |\phi|^2 |B|^2\, dv_M.$$

for any $t < T$. This implies that for any $t < T$,

$$\int_{B_R} |B(t)|^2\, dv_M \le \int_{B_{2R}} |B_0|^2\, dv_M - \int_0^t \int_{B_R} |\nabla_{A_1} B|^2 dv_M\, d\tau$$
$$+ C\int_0^t \int_{B_{2R}} |B(\tau)|^2\, dv_M\, d\tau.$$

By Lemma 2.1, for any $2R \in (0, R_0]$, there exists a cover $\{B_R(x_i)\}_{i=1}^\infty$ of M with the property that at any point $x \in M$ at most N of the balls $B_{2R}(x_i)$ meet where $R_0 > 0$ is sufficiently small and N is a constant depending on K. By a covering argument, we have

$$\int_M |B(t)|^2\, dv_M \le C\int_M |B_0|^2\, dv_M - \int_0^t \int_M |\nabla_{A_1} B|^2 dv_M\, d\tau$$
$$+ C_1\int_0^t \int_M |B(\tau)|^2\, dv_M\, d\tau.$$

By Gronwall's inequality, we obtain

$$\int_M |B|^2\, dv_M \le Ce^{C_1 t}\int_M |B_0|^2\, dv_M \tag{2.5}$$

for all $t < T$. Let ϕ be again the above cut-off function in B_{2R}. Then we apply the Weizenböck formula (2.1) again to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_M \phi^2|\nabla_{A_1}B|^2\,dv_M = 2\int_M \phi^2\left\langle \nabla_{A_1}\frac{\partial B}{\partial t}, \nabla_{A_1}B\right\rangle dv_M\\
&= -2\int_M \phi^2\left\langle \triangle_{A_1}B, \nabla_{A_1}^*\nabla_{A_1}B\right\rangle dv_M - \int_M \phi\left\langle \triangle_{A_1}B, \nabla\phi\nabla_{A_1}B\right\rangle dv_M\\
&= -2\int_M \phi^2|\nabla_{A_1}^*\nabla_{A_1}B|^2\,dv_M + \int_M \left\langle Rm\#B + F_{A_1}\#B, \nabla_{A_1}^*\nabla_{A_1}B\right\rangle dv_M\\
&\quad - \int_M \phi\left\langle \nabla_{A_1}^*\nabla_{A_1}B + Rm\#B + F_{A_1}\#B, \nabla\phi\nabla_{A_1}B\right\rangle dv_M\\
&\leq -\int_{B_R}|\nabla_{A_1}^*\nabla_{A_1}B|^2\,dv_M + C\int_{B_{2R}}|B|^2\,dv_M + C\int_{B_{2R}}|\nabla_{A_1}B|^2\,dv_M.
\end{aligned}$$

Integrating (2.6) from $[0,t]$ with $t < T$, one obtains from (2.5) that

$$\int_{B_R}|\nabla_{A_1}B(t)|^2\,dv_M \leq \int_{B_{2R}}|\nabla_{A_1}B_0|^2\,dv_M + C\int_0^t\int_{B_{2R}}(|B|^2 + |\nabla_{A_1}B|^2\,dv_M, \tag{2.6}$$

for $t < T$. By a covering argument, we have

$$\int_M|\nabla_{A_1}B(t)|^2\,dv_M \leq C\int_M|\nabla_{A_1}B_0|^2\,dv_M + C\int_0^t\int_M(|B|^2 + |\nabla_{A_1}B|^2)\,dv_M. \tag{2.7}$$

Combining (2.5) with (2.7), we apply Gronwall's inequality again to obtain

$$\|B(t)\|_{H^{1,2}} \leq C\|B_0\|_{H^{1,2}}, \tag{2.8}$$

for any $t < T$. By Lemma 2.3, there exists a constant uniform constant C such that for any $t < T$,

$$\|B\|_{V_t} \leq C\|B_0\|_{H^{1,2}(\Omega^1(\mathrm{ad}E))},$$

so the solution of B can be extended to in the space V_T and B is smooth of (2.3) at $t = T$. By the local existence at $t = T$, there exists a solution of (2.3) in $M \times [T, T+\delta)$. This implies that $T = +\infty$.

Since $B_0 \in H^{1,2}$, there exists a smooth sequence $B_0^{(m)}$ with compact support in M such that $B_0^{(m)}$ converges strongly to B_0 in $H^{1,2}$. By the above result, there exists a global unique smooth solution $B^{(m)}$ of the problem (2.3) with initial values $B_0^{(m)}$. By Lemma 2.3, there exists a uniform $T_1 > 0$ and constant C_2 such that

$$\|B^{(m)}\|_{V_{T_1}} \leq C_2\|B^{(m)}\|_{H^{1,2}(\Omega^1(\mathrm{ad}E))}.$$

Letting $m \to \infty$, there exists a local solution B of (2.3) in the space V_{T_1} for some $T_1 > 0$ with $B(0) = B_0$ in $H^{1,2}$.

This yields the global existence of the unique solution to (2.3) in V_T for any finite time $T > 0$, i.e.

$$B \in C^0\left([0,T); H^{1,2}(\Omega^1(ad\,E))\right) \cap L^2\left([0,T); H^{2,2}(\Omega^1(ad\,E))\right),$$

which are smooth for $t \geq 0$ since B_0 is smooth. (2.4) follows (2.2). This proves our claim. □

Let Ψ be a section in $\Omega^0(\mathrm{ad}E)$ such that

$$\frac{\partial \Psi}{\partial t} + \triangle_1 \Psi = 0 \tag{2.9}$$

with initial value $\Psi(0) = \Phi_0$. For all $T > 0$, the similar proof as in Lemma 3 yields the global existence of a solution $\Psi \in V_T$ of equation (2.9) in $M \times [0,T]$ with initial value $\Psi(0) = \Phi_0$. Using (2.9), we easily obtain

$$\frac{\partial}{\partial t}|\Psi|^2 + d^*d|\Psi|^2 = 2\left\langle \frac{\partial \Psi}{\partial t} + \triangle_1 \Psi, \Psi \right\rangle - 2|D_1\Psi|^2 \leq 0.$$

Then we apply the maximum principle to obtain

$$\sup_M |\Psi(t)| \leq \sup_M |\Psi(0)| = \sup_M |\Phi_0|.$$

Let B be solutions to (2.3) with initial value $B_0 = A_0 - A_1$ and let Ψ be a solution to (2.9) with initial value Φ_0. Following De Turck [De], we make the trick

$$D_A = D_1 + B + a, \Phi = \Psi + v,$$

where $D_1 = D_{ref} + A_1$ and A_1 is a smooth connection with compact support in M in (2.3) so that $\|A_1 - A_0\|_{H^{1,2}(\Omega^1(\mathrm{ad}E))}$ is sufficiently small.

Let us consider a gauge equivalent flow for Yang-Mills-Higgs equations as in [D]:

$$\frac{\partial A}{\partial t} = \frac{\partial (B+a)}{\partial t} = -D_A^* F_A + J + D_A(-D_A^* a), \tag{2.10}$$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial (\Psi + v)}{\partial t} = -D_A^* D_A \Phi + [D_A^* a, \Phi] \tag{2.11}$$

with initial conditions

$$a(0) = 0, \quad v(0) = 0.$$

Then we have

Theorem 2.5. *Let M be a complete noncompact Riemannian 3-manifold with curvature bounded by $K > 0$ and let $E \to M$ be a vector bundle with compact Lie group G. Assume that the initial connection $A_0 \in H^{1,2}(\Omega^1(adE))$ is smooth and the section $\Phi_0 \in H^{1,2}(\Omega^0(adE))$ is smooth and bounded in M. Then there exists $T_0 > 0$ such that the Cauchy problem (2.10)-(2.11) has a unique regular solution on $M \times [0, T_0)$.*

Proof. Let $\tilde{B} = A_1 + B$, where B is the solution of (2.3). Note

$$F_A = F_{\tilde{B}} + D_A a + a\#a.$$

Then

$$D_A^* F_A = D_A^* D_A a + D_{\tilde{B}}^* F_{\tilde{B}}^* + a\#F_b + D_A^*(a\#a).$$

We compute

$$\begin{aligned}\triangle_A a = \triangle_1 a - *(B+a) * (D_1 a + (B+a)\wedge a) + D_1^*[(B+C)\wedge a] \\ + (B+a) * D_1 * a + (B+a)\wedge a) + D_1[*(B+a) * a],\end{aligned} \tag{2.11}$$

where $\triangle_1 a = D_1^* D_1 a + D_1 D_1^* a$. Similarly, we have

$$\triangle_A \Phi = \triangle_1 \Phi + \Delta_1 B\#\Phi + \Delta_1 \Phi\#B + \Delta_1 a\#\Phi + B\#B\#\Phi + B\#a\#\Phi + a\#a\#\Phi.$$

By the above notations, equations (2.10)-(2.11) are equivalent to the following System:

$$\begin{cases} \frac{\partial a}{\partial t} + \triangle_1 a &= f_1 + f_2 \\ \frac{\partial v}{\partial t} + \triangle_1 v &= g_1 + g_2 \end{cases} \tag{2.12}$$

with initial values $a(0) = 0$ and $v(0) = 0$, where

$$f_1 := -\frac{\partial B}{\partial t} - D_1^* F_{\tilde{B}} + B\#F_{\tilde{B}} + D_1\Psi\#\Psi + B\#\Psi\#\Psi,$$

$$g_1 := \Delta_1 B\#\Psi + D_1\Psi\#B + B\#B\#\Psi,$$

$$\begin{aligned} f_2(a, v, \Delta_1 a, \nabla_1 v) := F_{\tilde{B}}\#a + B\#\Delta_1 a + \Delta_1 B\#a + B\#B\#a \\ + \Delta_1 a\#a + B\#a\#a + a\#a\#a + \Delta_1 v\#\Psi + \Delta_1\Psi\#v \\ + \Delta_1 v\#v + a\#\Psi\#\Psi + a\#v\#\Psi + B\#\Psi\#v + a\#v\#v,\end{aligned}$$

and

$$\begin{aligned} g_2(a, v, \Delta_1 a, \nabla_1 v) := \Delta_1 B\#v + D_1 v\#B + D_1 a\#v + B\#B\#v \\ + B\#a\#(\Psi + v) + a\#a\#(v + \Psi) + D_1\#a\#(\Psi + v) + B\#a\#(\Psi + v).\end{aligned}$$

Using Lemma 2.3 with the fact that $a(0) = 0$ and $v(0) = 0$, it follows from (2.2) that

$$\|a\|_{L^\infty([0,T];H^{1,2}(\Omega^1(\text{ad } E))} \leq 2\|a\|_{V_T} \leq C\|\frac{\partial a}{\partial t} + \triangle_1 a\|_{L^{2,2}([0,T];\Omega^1(\text{ad}E))}, \quad (2.13)$$

$$\|v\|_{L^\infty([0,T];H^{1,2}(\Omega^0(\text{ad } E))} \leq 2\|v\|_{V_T} \leq C\|\frac{\partial v}{\partial t} + \triangle_1 v\|_{L^{2,2}([0,T];\Omega^0(\text{ad}E))}, \quad (2.14)$$

for all T with $0 < T \leq T_1$, where T_1 is fixed in Lemma 2.3 depending on $\|A_1\|_{C(M)}$.

Note that $\|B_0\|_{H^{1,2}}$ is sufficiently small. Thus, letting $T > 0$ small enough, we apply Lemma 2.3 to obtain that $\|B\|_{H^{1,2}}$ is sufficiently small.

Now we will find a small T to obtain a-priori bounds of solution $(a, v) \in V_T \times V_T$.

Firstly we estimate terms in f_2 and g_2. By the Sobolev inequality, we have

$$\begin{aligned}\|F_{\tilde{B}}\#a\|_{L^{2,2}(0,T;M)} &\leq C\|F_{\tilde{B}}\|_{H^{1,2}}\|a\|_{L^\infty(H^{1,2})} \\ &\leq C(T\|F_{D_1}\|_{H^{1,2}} + \|B\|_{L^2(H^{2,2})})\|a\|_{V_T} \leq \varepsilon\|a\|_{V_T}\end{aligned}$$

for $T \leq T_2$, where T_2 is a very small constant depending only on D_1 and $\|B_0\|_{H^{1,2}}$ and ε.

Note that $|\Psi|$ is bounded by some constant C. Therefore for a sufficiently small $T_2 > 0$, we apply Sobolev's inequality to obtain

$$\|a\#\Psi\#\Psi\|_{L^{2,2}} \leq C\|a\|_{L^{\infty,4}}\|\Psi\|_{L^2(H^{1,2})} \leq CT\|\Psi_0\|_{H^{1,2}} \leq \varepsilon\|a\|_{V_T}$$

where we use the fact $a(0) = 0$ if $T \leq T_2$.

By a similar argument as in [St3], there exists a very small T_2 depending only ε, D_1 and $\|A_0\|_{H^{1,2}}$

$$\|f_2\|_{L^{2,2}} \leq \varepsilon(\|a\|_{V_T} + \|v\|_{V_T}) \quad \|g_2\|_{L^{2,2}} \leq \varepsilon(\|a\|_{V_T} + \|v\|_{V_T}),$$

if $\|a\|_{V_T} \leq \varepsilon$ and $\|v\|_{V_T} \leq \varepsilon$.

We can select a sufficiently small $T_3 = T_3(\varepsilon) > 0$ depending on T_1 and $\|B_0\|_{H^{1,2}}$ such that for all $T \leq T_3$,

$$\|f_1\|_{L^{2,2}([0,T];\text{ad}E)} + \|f_1\|_{L^{2,2}([0,T];\text{ad}E)} \leq \frac{\varepsilon}{2C}.$$

Therefore, we have

$$\|a\|_{V_T} \leq C\left\|\frac{da}{dt} + \triangle_1 a\right\|_{L^{2,2}} \leq C\|f_1\|_{L^{2,2}} + C\varepsilon\left(\|a\|_{V_T} + \|v\|_{V_T}\right),$$

$$\|v\|_{V_T} \le C \left\| \frac{dv}{dt} + \triangle_1 v \right\|_{L^{2,2}} \le C\|g_1\|_{L^{2,2}} + C\varepsilon \left(\|a\|_{V_T} + \|v\|_{V_T}\right).$$

For a small $T \le T_4 = \min\{T_1, T_2, T_3\}$ and a sufficiently small ε with $C\varepsilon \le 1/2$, we obtain a-priori estimate of a and v in V_T.

Let

$$U_\epsilon^T := \{(a, v) \in V_T \times V_T \mid \|a\|_{V_T} + \|v\|_{V_T} \le \varepsilon\}. \tag{2.15}$$

Given any $(a, v) \in U_\epsilon^T$, there is a unique weak solution (b, w) to the equations

$$\begin{cases} \frac{\partial b}{\partial t} + \triangle_1 b &= f_1 + f_2(a, v, \Delta_1 a, \nabla_1 v), \\ \frac{\partial w}{\partial t} + \triangle_1 w &= g_1 + g_2(a, v, \Delta_1 a, D_1 v), \end{cases}$$

with initial values $b(0) = 0$ and $w(0) = 0$. Write $(b, w) = L(a, v)$. Then (2.13), (2.14), and (2.15) give us that

$$\|b\|_{V_T} + \|w\|_{V_T} = \|L(a, v)\|_{V_T \times V_T} \le \epsilon.$$

Thus $L : U_\epsilon^T \to U_\epsilon^T$.

Moreover, there exists a number θ with $0 < \theta < 1$ such that for two $(a, v), (d, z) \in U_\varepsilon^T$

$$\|L(a, v) - L(d, z)\|_{V_T \times V_T} \le \theta \|(a - d, v - z)\|_{V_T \times V_T}.$$

Using the contraction mapping theorem on $U_\epsilon^T \subset V_T \times V_T$, there exists a unique weak solution $(a, v) \in V_T \times V_T$ of (2.12). By the general theory of quasi-linear parabolic equations, the solution (a, v) is smooth for $t > 0$ since f_1 and g_1 are smooth for $t > 0$, for example, see [LSU]. □

As a consequence of Theorem 2.5, we establish the local existence of the heat flow (1.2) with initial value (1.3):

Theorem 2.6. *Let M be a complete non-compact Riemannian 3-manifold with curvature bounded by $K > 0$ and let $E \to M$ be a vector bundle with compact Lie group G. Assume that the initial connection $A_0 \in H^{1,2}(\Omega^1(adE))$ and the section $\Phi_0 \in H^{1,2}(\Omega^0(adE))$ are smooth. Moreover, $|\Phi_0|$ is bounded in M. Then there exists a number $T_0 > 0$, such that the heat flow (1.2) with initial value (1.3) has the unique regular solution on $M \times [0, T_0)$.*

Proof. By Theorem 2.4, let (A, Φ) be a solution to (2.10)-(2.11) in $M \times [0, T_0)$, i.e.

$$\frac{\partial A}{\partial t} = -D_A^* F_A + J(D_A, \Phi) + D_A(-D_A^* a),$$

$$\frac{\partial \Phi}{\partial t} = -D_A^* D_A \Phi + [D_A^* a, \Phi]$$

with initial value $A(0) = A_0$ and $\Phi(0) = \Phi_0$, where $A = A_1 + B + a$, A_1 is chosen before (2.3) and B is a solution to the Cauchy problem (2.3).

Let $S(t)$ be a family of gauge transformations and consider the initial value problem for $S(t)$:

$$\frac{\partial S}{\partial t} = S \circ (-D_A a), \quad S(0) = I. \tag{2.16}$$

The initial problem (2.16) can easily be solved uniquely for small time $0 \leq t < T_0$. Let $S(t)$ be a family of gauge transformations which is a smooth solution of (2.16), i.e.

$$S^{-1} \circ \frac{\partial S}{\partial t} = -D_A^* a.$$

Then $D_{\bar{A}} = S^{-1^*} D_A$ and $\bar{\Phi} = S^{-1} \circ \Phi$ satisfy the heat flow (1.2), i.e.

$$\frac{\partial \bar{A}}{\partial t} = -\, D_{\bar{A}}^* F_{\bar{A}} - J;$$

$$\frac{\partial \bar{\Phi}}{\partial t} = -\, D_{\bar{A}}^* D_{\bar{A}} \bar{\Phi}.$$

This proves our claim. □

3. Global existence of Yang-Mills-Higgs flow in 3 manifolds.

In this section, we establish the global existence of the Yang-Mills-Higgs flow in three dimensional manifolds M. In the sequel, we denote by $(\cdot, \cdot)$ the L^2-product in E.

We easily have (cf. [Ho])

Lemma 3.1.

$$D_A J = D_A [D_A \Phi, \Phi] = [F_A \Phi, \Phi] - 2 D_A \Phi \wedge D_A \Phi, \tag{3.1}$$

$$D_A^* J = [\triangle_A \Phi, \Phi], \tag{3.2}$$

where $J = J(D_A, \Phi) = -[D_A \Phi, \Phi]$.

Using the heat flow (1.2), we have

$$\frac{dF(D_A(t))}{dt} = \frac{dF(D_A + \epsilon \partial D / \partial t)}{d\epsilon}\Big|_{\epsilon=0} = D_A \frac{\partial A}{\partial t} = -D_A (D_A^* F_A + J). \tag{3.3}$$

Then we have the following energy estimate:

Lemma 3.2. *Let (A, Φ) be a smooth solution to the Yang-Mills-Higgs flow (1.2) on $M \times [0, T]$ and the solution (A, Φ) is gauge-equivalent to $(\bar{A}, \bar{\Phi})$ with $\bar{A} \in V_T$, $\bar{\Phi} \in V_T$. Then for $t_1 \leq T$, we have*

$$\begin{aligned} &YMH(D_A, \Phi)(t_1) + 2\int_0^{t_1}\int_M \left(\left|\frac{\partial \Phi}{\partial t}\right|^2 + |D_A^* F_A + J|^2 \right) dv_M\, dt \\ &= YMH(D_0, \Phi_0). \end{aligned} \tag{3.4}$$

Proof. Since (A, Φ) is gauge-equivalent to $(\bar{A}, \bar{\Phi}) \in V_T$, it follows from (3.3) that

$$\frac{1}{2}\frac{d}{dt}\int_M |F_A|^2\, dv_M = \left(D_A(\frac{\partial A}{\partial t}), F_A \right) = \left(\frac{\partial A}{\partial t}, D_A^* F \right).$$

For any $a \in \Omega^1(\mathrm{ad}E)$, we have

$$D_a\Phi = d\Phi + [a, \Phi].$$

Then we have

$$\langle a(\Phi), D_A\Phi \rangle = \langle a, J \rangle\,. \tag{3.5}$$

Using the heat flow (1.2), one obtains from (3.5) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_M |D_A\Phi|^2\, dv_M &= \left(\frac{\partial (D_A\Phi)}{\partial t},\, D_A\Phi \right) \\ &= \left(\frac{\partial D_A}{\partial t} \circ \Phi + D_A \circ \frac{\partial \Phi}{\partial t},\, D_A\Phi \right) \\ &= -\,(D_A^* F_A + J, J) - \int_M |\frac{\partial \Phi}{\partial t}|^2\, dv_M. \end{aligned}$$

Therefore

$$\frac{1}{2}\int_M |F_A|^2 + |D_A\Phi|^2\, dv_M = -\int_M \left(|D_A^* F_A + J|^2 + |\frac{\partial \Phi}{\partial t}|^2 \right) dv_M.$$

For each $t_1 \leq T$, the desired energy estimate follows from integrating the above identity with respect to t from 0 to t_1. □

Lemma 3.3. *(Maximum Principle) Let (A, Φ) be a smooth solution to (1.2) on $M \times [0, T]$ with initial value (1.3). Then for $t \leq T$*

$$\sup_{x\in M} |\Phi(x, t)| \leq \sup_{x \in M} |\Phi_0(x)|. \tag{3.6}$$

Proof. Since (A, Φ) is a smooth solution to the heat flow (1.2), we easily obtain

$$(\frac{\partial}{\partial t} + \nabla^*\nabla)|\Phi|^2 = -2|D_A\Phi|^2 \leq 0.$$

The desired result follows from the standard maximum principle. □

In order to prove the existence of global regular solution to the heat flow (1.2), we need a local estimate in the following:

Lemma 3.4. *Let $D_A = D_{ref} + A$ with the curvature F_A. Then there exists a small R_0 such that for $a \in \Omega^p(adE)$, a geodesic ball B_R of radius $R < R_0$ with center* 0*, we have*

$$\begin{aligned}\|a\|^2_{L^6(B_{R/2})} + \|\nabla_A a\|^2_{L^2(B_{R/2})} \leq & C(\|D_A a\|^2_{L^2(B_R)} + \|D_A^* a\|^2_{L^2(B_R)}) \\ & + C(1 + R^{-2})\|a\|^2_{L^2(B_R)} \qquad (3.7)\end{aligned}$$

for some constant $C > 0$.

Proof. Let $\phi \in C_0^\infty(B_R)$ be a cut-off function with $0 \leq \phi \leq 1$, $|d\phi| \leq CR^{-1}$, and $\phi = 1$ inside $B_{R/2}$. Then we apply the Weintzenböke formula (2.1) to find

$$\begin{aligned}\|\nabla_A a\|^2_{L^2(B_{R/2})} \leq & \|\nabla_A(\phi a)\|^2_{L^2} = (\nabla_A^*\nabla_A(\phi a), \phi a) \\ = & \|D_A(\phi a)\|^2_{L^2} + \|D_A^*(\phi a)\|^2_{L^2} + (F_A \# \phi a, \phi a) + (Rm \# \phi a, \phi a).\end{aligned}$$

By Hölder's and Sobolev's inequalities, we obtain

$$\begin{aligned}(F_A \# \phi a, \phi a) \leq & C_4 |B_R|^{1/6} \|F_A\|_{L^2(B_R)} \|\phi a\|^2_{L^6} \\ \leq & C_4 R^{1/2} YMH(u_0, D_0)^{1/2} \left(\|\nabla_A(\phi a)\|^2_{L^2} + \|\phi a\|^2_{L^2}\right),\end{aligned}$$

where C_4 is a constant depending on the Sobolev constant and E, the last inequality comes from energy inequality Lemma 3.1. Thus

$$\|\nabla_A(\phi a)\|^2_{L^2(B_{R/2})} \leq C\Big(\|D_A a\|^2_{L^2(B_R)} + \|D_A^* a\|^2_{L^2(B_R)} + (1 + R^{-2})\|a\|^2_{L^2(B_R)}\Big),$$

by choosing a sufficiently small R with $R \leq R_0 = \frac{1}{4C_4^2 \text{YMH } (u_0, D_0)}$. □

Next lemma is a key to establish a global solutions of the heat flow.

Lemma 3.5. *Let (A,Φ) be a smooth solution to (1.2) on $M\times[0,T)$ with initial condition (1.3). Then*

$$\frac{\partial}{\partial t}F_A=-D_A(D_A^*F_A+J)\in L^2\left([\epsilon,T];L^2\left(\Omega^1(adE)\right)\right)$$

and

$$D_A\frac{\partial\Phi}{\partial t}=-D_AD_A^*D_A\Phi\in L^2\left([\epsilon,T];L^2\left(\Omega^0(adE)\right)\right)$$

for any small constant $\epsilon>0$.

Proof. For simplicity, let $D=D_A$, $F=F_A$ and $D\Phi=D_A\Phi$.

Using (1.2), we have

$$\frac{\partial J}{\partial t}=[(D^*F+J)\circ\Phi,\Phi]+[D\frac{\partial\Phi}{\partial t},\Phi]+[D\Phi,\frac{\partial\Phi}{\partial t}].$$

Let $\phi\in C_0^\infty(B_{2R})$ be a cut-off function with $\phi=1$ in B_R.

Using the equation (3.3) and Lemma 3.3, one obtains from (1.2) that

$$\begin{aligned}
\left\|\phi\frac{\partial F}{\partial t}\right\|_{L^2}^2&=-\left(\phi^2(D^*F+J),\,D^*\left(\frac{\partial F}{\partial t}\right)\right)+2\left(\phi d\phi(D^*F+J),\frac{\partial F}{\partial t}\right)\\
&=-\left(\phi^2(D^*F+J),\frac{\partial}{\partial t}(D^*F+J)-\frac{\partial D}{\partial t}\#F-\frac{\partial J}{\partial t}\right)+2\left(\phi d\phi(D^*F+J),\frac{\partial F}{\partial t}\right)\\
&\le-\frac{1}{2}\frac{d}{dt}\|\phi(D^*F+J)\|_{L^2}^2+C\int_M\phi^2(|F|+|D\Phi|)\left(|D^*F+J|^2+\left|\frac{\partial\Phi}{\partial t}\right|^2\right)dv_M\\
&\quad+C\|\phi(D^*F+J)\|_{L^2}^2+\frac{1}{4}\left\|\phi D\frac{\partial\Phi}{\partial t}\right\|_{L^2}^2+\frac{1}{2}\left\|\phi\frac{\partial F}{\partial t}\right\|_{L^2}^2+C\|\,|d\phi|(D^*F+J)\|_{L^2}.
\end{aligned}\tag{3.8}$$

Using the heat flow (1.2), we obtain from (3.9) that

$$\begin{aligned}
\left\|\phi D\frac{\partial\Phi}{\partial t}\right\|_{L^2}^2&=\left(\phi^2D\frac{\partial\Phi}{\partial t},-DD^*D\Phi\right)=\left(\phi^2\left[\frac{\partial(D\Phi)}{\partial t}-\frac{\partial D}{\partial t}\Phi\right],-DD^*D\Phi\right)\\
&\le-\left(\phi^2D^*\frac{\partial(D\Phi)}{\partial t},\,D^*D\Phi\right)+2\int_M|\phi||d\phi||\frac{\partial(D\Phi)}{\partial t}||D^*D\Phi|\,dv_M\\
&\quad+C\int_M\phi^2|D^*F+J|\left|\phi D\frac{\partial\Phi}{\partial t}\right|dv_M\\
&\le-\frac{1}{2}\frac{d}{dt}\|\phi D^*D\Phi\|_{L^2}^2+C\int_M\phi^2|D^*F+J|\left|D\frac{\partial\Phi}{\partial t}\right|dv_M
\end{aligned}$$

$$
\begin{aligned}
&+\left(\phi^2\frac{\partial D}{\partial t}\#D\Phi,\, D^*D\Phi\right)+C\int_M|\phi||d\phi|\left(|D(\frac{\partial\Phi}{\partial t})|+|\frac{\partial A}{\partial t}||\Phi|\right)|D^*D\Phi|\,dv_M\\
\le&-\frac{d}{dt}\|\phi D^*D\Phi\|^2_{L^2}+\frac{1}{4}\left\|\phi D\frac{\partial\Phi}{\partial t}\right\|^2_{L^2}+C\int_M\phi^2|D\Phi|(|D^*F+J|^2+|D^*D\Phi|^2)\,dv_M\\
&+C\|\phi(D^*F+J)\|^2_{L^2}+C\|\phi D^*D\Phi\|^2_{L^2}+C\||d\phi|(D^*F+J)\|^2_{L^2}\\
&+C\||d\phi|D^*D\Phi\|^2_{L^2}. \qquad (3.9)
\end{aligned}
$$

Combining (3.9) with (3.8), we find

$$
\begin{aligned}
&\left\|\phi\frac{\partial}{\partial t}F\right\|^2_{L^2}+\left\|\phi D\frac{\partial\Phi}{\partial t}\right\|^2_{L^2}\\
&\le-\frac{d}{dt}\left(\|\phi D^*D\Phi\|^2_{L^2}+\|\phi(D^*F+J)\|^2_{L^2}\right)\\
&+C\int_M\phi^2(|F|+|D\Phi|)\left(|D^*F+J|^2+|D^*D\Phi|^2+\left|\frac{\partial\Phi}{\partial t}\right|^2\right)dv_M\\
&+C\|\phi(D^*F+J)\|^2_{L^2}+C\|\phi D^*D\Phi\|^2_{L^2}\\
&+C\||d\phi|(D^*F+J)\|^2_{L^2}+C\||d\phi|D^*D\Phi\|^2_{L^2}. \qquad (3.10)
\end{aligned}
$$

By Lemmas 3.1 and 3.3, we have

$$|D^*J|^2\le C|D^*D\Phi|^2|\Phi|^2\le C|D^*D\Phi|.$$

Thus, applying Hölder's inequality, we obtain

$$
\begin{aligned}
&\int_M\phi^2(|F|+|D\Phi|)\left(|D^*F+J|^2+|D^*D\Phi|^2\right)dv_M\\
&\le\int_{B_{2R}}(|F|+|D\Phi|)\left(|D^*F+J|^2+|D^*D\Phi|^2\right)dv_M\\
&\le C YMH(\Phi,D)^{1/2}R^{1/2}\left(\int_{B_{2R}}\left(|D^*F+J|^6+|D^*D\Phi|^6\right)dv_M\right)^{\frac{1}{3}}.
\end{aligned}
$$

Note that $D^*D^*F=0$. Then applying Lemma 3.4 to the above inequality, we obtain

$$
\begin{aligned}
&\int_M\phi^2(|F|+|D\Phi|)\left(|D^*F+J|^2+|D^*D\Phi|^2\right)dv_M\\
&\le CR^{1/2}\int_{B_{4R}}\left(|D(D^*F+J)|^2+|D^*J|^2+|DD^*D\Phi|^2\right)dv_M\\
&\quad+CR^{1/2}\int_{B_{4R}}(1+R^{-2})\left(|D^*F+J|^2+|D^*D\Phi|^2\right)dv_M
\end{aligned}
$$

$$\leq CR^{1/2}\left(\|D(D^*F+J)\|_{L^2}^2+\|DD^*D\Phi\|_{L^2(B_{4R})}^2\right)$$
$$+CR^{1/2}(1+R^{-2})\left(\|D^*F+J\|_{L^2(B_{4R})}+\|D^*D\Phi\|_{L^2(B_{4R})}^2\right). \tag{3.11}$$

Combining (3.10) with (3.11), we have

$$\left(\left\|\frac{\partial}{\partial t}F\right\|_{L^2(B_R)}^2+\left\|\phi D\frac{\partial\Phi}{\partial t}\right\|_{L^2(B_R)}^2\right)$$
$$\leq -\frac{d}{dt}\left(\|\phi D^*D\Phi\|_{L^2}^2+\|\phi(D^*F+J)\|_{L^2}^2\right)$$
$$+CR^{1/2}\left(\|D(D^*F+J)\|_{L^2}^2+\|DD^*D\Phi\|_{L^2(B_{4R})}^2\right)$$
$$+C(1+R^{-2})\left(\|D^*F+J\|_{L^2(B_{4R})}+\|D^*D\Phi\|_{L^2(B_{4R})}^2\right). \tag{3.12}$$

For a given $\tau>0$, we can find $t_0\in(0,\tau]$ such that

$$\|D^*F+J\|_{L^2}^2(t_0)\leq 2\tau^{-1}\int_0^\tau\|D^*F+J\|_{L^2}^2\,dt\leq 2\tau^{-1}\int_0^T\|D^*F+J\|_{L^2}^2\,dt$$

$$\|D^*D\Phi\|_{L^2}^2(t_0)\leq 2\tau^{-1}\int_0^\tau\|D^*D\Phi\|_{L^2}^2\,dt\leq 2\tau^{-1}\int_0^T\int_M\left|\frac{\partial\Phi}{\partial t}\right|^2dv_M\,dt.$$

By Lemma 3.2, we have

$$\int_0^T\left(\|D^*F+J\|_{L^2}^2+\|D\Phi\|_{L^2}^2+\left\|\frac{\partial\Phi}{\partial t}\right\|_{L^2}^2+\|F\|_{L^2}^2\right)dt<\infty.$$

By Lemma 2.1, for any $4R\in(0,R_0]$, there exists a cover $\{B_R(x_i)\}_{i=1}^\infty$ of M with the property that at any point $x\in M$ at most N of the balls $B_{4R}(x_i)$ meet where $R_0>0$ is sufficiently small and N is a constant depending on K.

Integrating both sides of (3.12) from t_0 to T in each ball $B_R(x_i)$ and putting all estimates in all balls $B_R(x_i)$ together, we choose R to be sufficiently small (e.g. $R^{1/2}CN<1/8$) to obtain

$$\int_\tau^T\left(\|D(D^*F+J)\|_{L^2}^2+\left\|D\frac{\partial\Phi}{\partial t}\right\|_{L^2}^2\right)dt\leq C(\tau,T).$$

Since τ is arbitrarily small, together with Lemma 3.2 the last inequality proves our claim. □

Lemma 3.6. *Let (A, Φ) be a smooth solution of (1.2) on $M \times [0, T)$. Then (A, Φ) extends to the space $C([\tau, T]; H^{1,2}(\Omega^1(ad\ E)) \times H^{1,2}(\Omega^0(ad\ E)))$ for any $\tau > 0$.*

Proof. Let $\tau > 0$ be any small constant. By Lemma 3.2, we have

$$\int_0^T \int_M |\frac{\partial A}{\partial t}|^2 \, dx \, dt \le \frac{1}{2} YMH(D_0, \Phi_0),$$

which implies $A \in C^0([0, T]; L^2)$. By Lemmas 3.4 and 3.5, we apply the second Bianchi identity $D_A^* D_A^* F = 0$ to obtain

$$\begin{aligned} &\int_\tau^T \int_M |\frac{\partial A}{\partial t}|^6 dv_M \, dt \\ \le & C \int_\tau^T \int_M (|D_A(D_A^* F_A + J)|^2 + |D_A^*(D_A^* F_A + J)|^2 + |\frac{\partial A}{\partial t}|^2) \, dv_M \\ \le & C \int_\tau^T \int_M |D_A(D_A^* F_A + J)|^2 + | \triangle_A \Phi|^2 + |\frac{\partial A}{\partial t}|^2 \, dv_M \, dt \le C(\tau, T). \end{aligned}$$

Then we have $A \in C^0([\tau, T]; L^q)$ for any $q \in [2, 6]$. Moreover, we have

$$\frac{d}{dt}(D_{ref} A) = \frac{d}{dt} F_A + \frac{d}{dt} A \# A \in L^2([\tau, T]; L^2)$$

which implies $D_{ref} A \in C^0([\tau, T]; L^2)$. Then we employ Lemma 3.6 to find

$$\frac{d}{dt}(D_{ref}^* A) = D_{ref}^* (\frac{d}{dt} A) = A \# D_A^*(F + J) + D_A^* J \in L^2([\tau, T]; L^2),$$

where we use the second Bianchi identity $D_A^* D_A^* F_A = 0$. By Lemmas 3.4-3.5, we have

$$\begin{aligned} \|A(t_1) - A(t_2)\|_{H^{1,2}} \le & C(\|D_{ref}^*[A(t_1) - A(t_2)]\|_{L^2} + \|D_{ref}[A(t_1) - A(t_2)]\|_{L^2} \\ & + \|A(t_1) - A(t_2)\|_{L^2}), \end{aligned}$$

which implies that $A \in C^0([\tau, T]; H^{1,2}(\Omega^1(\mathrm{ad} E))$ for any $\tau > 0$. Similarly, by Lemma 3.6, $\Phi \in C^0([0, T]; L^2)$.

$$\frac{\partial D_A \Phi}{\partial t} = \frac{\partial A}{\partial t} \Phi + D_A \frac{\partial \Phi}{\partial t} \in L^2([\tau, T], L^2)$$

since Φ is bounded. Using Lemmas 3.4-3.5 again, $\Phi \in C^0([\tau, T]; H^{1,2}(\Omega^0(\mathrm{ad} E)))$ for any $\tau > 0$. This proves our claim. □

Theorem 3.7. *Let M be a complete noncompact Riemannian 3-manifold with curvature bounded by $K > 0$ and let $E \to M$ be a vector bundle with compact Lie group G. Assume that the initial connection $A_0 \in H^{1,2}(\Omega^1(adE))$ is smooth, the section $\Phi_0 \in H^{1,2}(\Omega^0(adE))$ is smooth and bounded in M. Then for any finite time $T > 0$, the heat flow (1.2) with initial value (1.3) has a regular solution on $M \times [0, T]$.*

Proof. By the local existence, there exists a finite $T < \infty$ such that the heat flow (1.2) with initial value (1.3) has a smooth solution $D_A(t) = D_{ref} + A(t)$ and $\Phi(t)$ in $M \times [0, T)$. By Lemma 3.7, there exists a connection $D_A(T) = D_{ref} + A(T)$ with $A(T) \in H^{1,2}(\Omega^1(\mathrm{ad}E))$ such that

$$\lim_{t\to T} A(t) = A(T), \quad \lim_{t\to T} \Phi(t) = \Phi(T),$$

in $H^{1,2}$. Letting $t_1 < T$ be sufficiently close to T. By the local existence (Theorem 2.5) at the time $t = t_1$, there exists a T_0 such that the heat flow (1.2) with initial value at $t = t_1$ has a smooth solution in $M \times (t_1, t_1 + T_0)$. We know that T_0 depends only on $\|A(t_1)\|_{H^{1,2}}$ and D_1. As t_1 is closed to T enough, $\|A(t_1) - A(T)\|_{H^{1,2}}$ is sufficiently small. We choose $D_1 = D_{ref} + A_1$ by $A(T)$ such that $\|A_1 - A(T)\|_{H^{1,2}}$ is sufficiently small and find a T_0 depending on D_1 and $A(T)$ with $t_1 + T_0 > T$ such that the heat flow has a smooth solution in $M \times [t_1, t_1 + T_0]$. Therefore we can extend the solution (D_A, ϕ) to any finite $T < \infty$. This proves our claim. □

At the end of this section, we analyze the asymptotical behavior of solutions of the Yang-Mills-Higgs flow (1.2) as $t \to \infty$.

Theorem 3.8. *Let (A, Φ) be a smooth solution of the Yang-Mills-Higgs flow in $M \times [0, \infty)$. Then there exists a suitable sequence $\{t_k\}$ such that as $t_k \to \infty$, $(A(\cdot, t_k), \Phi(\cdot, t_k))$ strongly converges in $H^{1,2}$, up to a gauge transformation, to (A_∞, Φ_∞) which is a solution of the Yang-Mills-Higgs equations*

$$\begin{aligned} -D_A^* F_A &= J, \quad \text{in } M; \\ -D_A^* D_A \Phi &= 0, \quad \text{in } M. \end{aligned} \tag{3.13}$$

Proof. By the energy estimate, there exists a suitable sequence t_k such that as $t_k \to \infty$,

$$\int_M |\frac{\partial A}{\partial t}(\cdot, t_k)|^2 + |\frac{\partial \Phi}{\partial t}(\cdot, t_k)|^2 \, dv_M \to 0.$$

Moreover, we know

$$\int_M |F_{A(t_k)}|^2 + |D_{A(t_k)}\Phi|^2 \, dv_M \leq \mathrm{YMH}(D_{A_0}, \Phi_0), \tag{3.14}$$

$$\int_M |D_A^* F_A(\cdot, t_k)|^2 \, dv_M + \int_M |\triangle_A \Phi|^2 \, dv_M \leq C \tag{3.15}$$

for some constant C independently of k.

Let r be a positive number to be fixed in the sequel. By Lemma 2.1, there exists a sequence $\{x_i\}$ of points of M such that $\{B_r(x_i)\}$ is a uniformly locally covering of M, with property that at any point $x \in M$ at most N of the balls meets, where N is a finite number depending on r and $B_{r/2}(x_i) \cap B_{r/2}(x_j) = \emptyset$ for any $i \neq j$.

Note $D_A F_A = 0$. Using Lemma 3.4, we obtain

$$\int_{B_{r/2}(x_i)} |\nabla_A F_A|^2 \, dv_M \leq C \int_{B_r(x_i)} |D_A^* F_A|^2 + C(1 + r^{-2}) \int_{B_r(x_i)} |F_A|^2 \, dv_M,$$

for $r < R_0$. By a covering argument, it follows from (3.14)-(3.15) that

$$\int_M |\nabla_A F_A|^2(\cdot, t_k) \, dv_M + \int_M |\nabla_A^2 \Phi|^2(\cdot, t_k) \, dv_M \leq C.$$

By Hölder's inequality, we have

$$\int_{B_r(x_i)} |F_{A(t_k)}|^{3/2} \, dv_M \leq C r^{3/4} \mathrm{YMH}(D_{A_0}, \Phi_0)^{3/4} = \varepsilon_0,$$

choosing $r \leq (\frac{\varepsilon_0}{C})^{4/3} \mathrm{YMH}(D_{A_0}, \Phi_0)^{-1}$ for a very small ε_0. It follows from Theorem 3.6 in [U] that there exists a gauge tranformation σ_k such that $\sigma_k(A(t_k))$ weakly converges to A_∞ in $H^{2,2}$ and $\sigma_k(\Phi(t_k))$ weakly converges to Φ_∞ in $H^{2,2}$. Therefore $(\sigma_k(A(t_k)), \sigma(\Phi(t_k))$ converges to (A_∞, Φ_∞) strongly in $H^{1,2}$. Moreover, (A_∞, Φ_∞) is a smooth solution of Yang-Mills-Higgs equations (3.13). □

Remark. *Actually, $(\sigma_k(A(t_k)), \sigma(\Phi(t_k))$ converges to (A_∞, Φ_∞) in C^∞ (cf. [HT]). One can expect to prove uniqueness of the limit solution (A_∞, Φ_∞) (cf. [R], [Si]). The space of solutions of Yang-Mills-Higgs equations (3.13) is path connected.*

4. The m-equivariant Yang-Mills flow on $\mathbb{R}^4$.

In this section, we will discuss the m-equivariant Yang-Mills flow on $\mathbb{R}^4$.

Following the descriptions of Atiyah [A], we introduce m-equivariant connection on S^4 below as in [SSU]. Since Yang-Mills equations in four manifolds are conformally invariant, we can change the conformal structure on $S^4 = \mathbb{R}^4 \cup \{\infty\}$. If we consider $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, introduce poplar coordinates in the first factor, we have

$$\mathbb{R}^4 = \{(r, \alpha, (y_1, y_2)) : \quad r \geq 0, \alpha \in [0, 2\pi), (y_1, y_2) \in \mathbb{R}^2\}.$$

The metric in $\mathbb{R}^4$ is

$$ds^2 = dr^2 + r^2 d\alpha^2 + dy_1^2 + dy_2^2.$$

The action of $U(1)$ is

$$q(t)(r, \alpha, y_1, y_2) = (z, \alpha + t(\text{mod} 2\pi), y_1, y_2).$$

This implies the conformal equivalence

$$U(1) = \mathbb{R}^4 - \{r = 0\} = \mathbb{R}^4 - \mathbb{R}^2 = S^4 - S^2.$$

The S^2 is precisely the fixed point set of the action of $U(1)$ on S^4.

To define a $U(1)$ invariant connection on S^4 with structure group $SU(2)$, a representation $s : U(1) \to G$ gives the gauge transformations of connections, i.e.

$$q(t)^* D_B = s(t)^{-1} \circ D_B \circ s(t).$$

We trivialize on $\mathbb{R}^4$ and assume $s(t) = e^{\hat{i}mt}$ some integer m. Here $\{\hat{i}, \hat{j}, \hat{k}\}$ are a standard basis for $su(2)$. In some gauge, we can write

$$D = d + \hat{B},$$

where

$$\hat{B} = e^{-\hat{i}m\alpha}(\hat{\phi} d\alpha + A) e^{\hat{i}m\alpha}.$$

here $q(t)^* A = A$ and $\hat{\phi} \circ q(t) = \hat{\phi}$. The Higgs filed ϕ must be asymptotic to zero as $r \to 0$ and the integer m describes the representation of class of $U(1) \to SO(3)$ of the symmetry group in the fibre of the gauge group over the fixed point set. We call such connections on S^4 m-equivariant connections.

The usual gauge change by $s = e^{\hat{i}mt}$ takes the connection $d + \hat{B}$ to the connection $d + B$, where $B = \Phi d\alpha + A$, which is singular along the entire plane $r = 0$. Here $\Phi = \hat{\phi} - m\hat{i}$.

The Yang-Mills functional $D_B = d + B$ on $\mathbb{R}^4$ (or S^4) is reduced to the Yang-Mills-Higgs functional on $\mathbb{H}^3$, i.e.

$$\begin{aligned}\mathrm{YM}(B) &= \int_{\mathbb{R}^4} |F_B|^2\, dx_1 dx_2 dy_1 dy_2 \\ &= 2\pi \int_{\mathbb{R}^2} \int_0^\infty [r|F_A|^2 + |D_A\Phi|^2 r^{-1}]\, dr\, dy_1 dy_2 \\ &= 2\pi \int_{\mathbb{H}^3} |D_A\Phi|^2_{\mathbb{H}^3} + |F_A|^2_{\mathbb{H}^3}\, dv_{\mathbb{H}^3} = 2\pi \mathrm{YMH}(A, \Phi). \end{aligned} \tag{4.1}$$

The Yang-Mills equation on $\mathbb{R}^4$

$$-D_B^* F_B = 0 \tag{4.2}$$

is reduced to the following Yang-Mills-Higgs equations on $\mathbb{H}^3$:

$$\begin{aligned} &- D_A^* F_A - J = 0; \\ &- D_A^* D_A \Phi = 0. \end{aligned} \tag{4.3}$$

with $|\Phi(x)| \to m$ as $|r| \to 0$.

The corresponding heat flow (1.1) on $\mathbb{R}^4$ is also reduced to the heat flow on $\mathbb{H}^3$

$$\begin{aligned} \frac{\partial A}{\partial t} &= - D_A^* F_A - J; \\ \frac{\partial \Phi}{\partial t} &= - D_A^* D_A \Phi, \end{aligned} \tag{4.4}$$

with smooth initial value $D_0 = D_{ref} + A_0$ and Φ_0 where $A_0 \in H^{1,2}(\mathbb{H}^3)$ and $\Phi_0 \in H^{1,2}(H^3)$. Let (D_A, Φ) be a global solution to the heat flow (4.4). For the integer m, let $\hat{\Phi} = \Phi + m\hat{\imath}$. Since $D_A\Phi = D_A\hat{\Phi}$, $(D_A, \hat{\Phi})$ is also solution of (4.4) with initial value $(D_0, \Phi_0 + m\hat{\imath})$ where $(A_0, \Phi_0) \in H^{1,2}$.

Since $\mathbb{H}^3$ is a complete three manifold which has constant curvature, by Theorem 3 we have a global existence of the Yang-Mills-Higgs flow (4.4) on $\mathbb{H}^3$, so we have also a global existence of the Yang-Mills flow (1.1).

Theorem 4.1. *Let $B_0 = A_0 + \Phi_0 d\alpha$ be a given m-equivariant connection in $\mathbb{R}^4$ with $(A_0, \Phi_0 - \hat{\imath}m) \in H^{1,2}$ and $\Phi_0 \to \hat{\imath}m$ as $r \to 0$ for some integer m. Then there exists a global solution to the Yang-Mills flow (1.1) on $\mathbb{R}^4$ with initial value B_0.*

Proof. Applying Theorem 3.8 to the case $M = \mathbb{H}^3$, there exists a global smooth solution (A, Φ) of the heat flow (4.4) in $\mathbb{H}^3 \times (0, \infty)$ satisfying

$$(A, \Phi - m\hat{i}) \in C^0([0, \infty); H^{1,2}(\mathbb{H}^3)) \cap H^{1,2}([0, \infty); L^2(\mathbb{H}^3)).$$

At each finite time $t > 0$, we apply Lemma 3.2 for $M = \mathbb{H}^3$ to obtain

$$\int_{\mathbb{R}^4} |F_B|^2 \, dx_1 dx_2 dy_1 dy_2 = 2\pi \int_{\mathbb{H}^3} |D_A \Phi|^2_{\mathbb{H}^3} + |F_A|^2_{\mathbb{H}^3} \, dv_{\mathbb{H}^3} < +\infty.$$

Let $B = A + \Phi d\alpha$. Then $B(t)$ is the solution of (1.1) in $\mathbb{R}^4 \backslash \mathbb{R}^2 \times (0, \infty)$. We will extend B from $\mathbb{R}^4 \backslash \mathbb{R}^2$ to $\mathbb{R}^4$.

For each $\tilde{R} > 0$, we have

$$\int_0^T \int_{\mathbb{R}^4 \cap B_{\tilde{R}}} |\frac{\partial B}{\partial t}|^2 dx \leq 2\pi \int_0^T \int_{\mathbb{R}^2} \int_0^{\tilde{R}} r^2 \left[|\frac{\partial A}{\partial t}|^2_{\mathbb{H}^3} + |\frac{\partial \Phi}{\partial t}|^2_{\mathbb{H}^3} \right] \frac{dr dy_1 dy_2}{r^3} < +\infty,$$

for any $T > 0$. This implies that

$$\frac{\partial B}{\partial t} \in L^2([0, \infty), L^2_{\text{loc}}(\mathbb{R}^4)).$$

Since B satisfies the Yang-Mills flow (1.1) on $\mathbb{R}^4$, a local energy estimate yields

$$\int_0^T \int_{B_R(x)} |\frac{\partial B}{\partial t}|^2 dx \, dt \leq \int_{B_{2R}(x)} |F_{B_0}|^2 dx + C \int_{B_{2R}(x)} |F_{B(T)}|^2 \, dx$$

for every x and a fixed $R > 0$. A simple covering argument on $\mathbb{R}^4$ gives us

$$\int_0^T \int_{\mathbb{R}^4} |\frac{\partial B}{\partial t}|^2 dx \, dt \leq C \int_{\mathbb{R}^4} |F_{B_0}|^2 \, dx + C \int_{\mathbb{R}^4} |F_{B(T)}|^2 \, dx < +\infty.$$

It follows from Theorem 4.1 of [SS] that there exists a limit holonomy of Φ, i.e., $\Phi \to \hat{i} m$ as $r \to 0$, where m is the integer due to the assumption of Φ_0. Using Theorems 5.1-5.2 of [SS], $B(t)$ can be extended from $H^{1,2}(\mathbb{R}^4 \backslash \mathbb{R}^2)$ to $H^{1,2}(\mathbb{R}^4)$ since m is an integer. Therefore we get a solution B of the Yang-Mills flow (1.1) in $\mathbb{R}^4 \times (0, \infty)$ such that B is smooth in $\mathbb{R}^4 \backslash \mathbb{R}^2$ and $B \in C^0([0, \infty), H^{1,2}(\mathbb{R}^4) \cap H^{1,2}([0, \infty), L^2(\mathbb{R}^4))$. We claim the singular set $\mathbb{R}^2 = \{(r, \alpha, (y_1, y_2)) \in \mathbb{R}^4 : r = 0\}$ of the weak solution is removable by the heat flow (1.1) in $\mathbb{R}^4$. Since $B = A + \Phi d\alpha$ is gauge-equivalent to a smooth solution $\tilde{B} = \tilde{A} + \tilde{\Phi} d\alpha$ in $\mathbb{R}^4 \backslash \mathbb{R}^2 \times [0, T]$ such that $(\tilde{A}, \tilde{\Phi}) \in V_T(\mathbb{H}^3)$ satisfies equations (2.10)-(2.11) in $\mathbb{H}^3 \times [0,T]$ for some $T < \infty$. For each $\tilde{R} > 0$, we find

$$\int_0^T \int_{\mathbb{R}^4 \cap B_{\tilde{R}}} (|B|^2 + |\nabla^2 B|^2)\, dx_1 dx_2 dy_1 dy_2$$
$$\leq 2\pi \int_0^T \int_{\mathbb{R}^2} \int_0^{\tilde{R}} r^2 \left[|\tilde{A}|^2_{\mathbb{H}^3} + |\tilde{\Phi}|^2_{\mathbb{H}^3} + |\nabla^2 \tilde{A}|^2_{\mathbb{H}^3} + |\nabla^2 \tilde{\Phi}|^2_{\mathbb{H}^3} \right] \frac{dr dy_1 dy_2}{r^3} < +\infty,$$

This means that for any ball $B_{\tilde{R}} \subset \mathbb{R}^4$ and $T > 0$, $(\tilde{A}, \tilde{\Phi}) \in V_T(B_{\tilde{R}} \times [0,T])$ satisfy equations (2.10)-(2.11). Since equations (2.10)-(2.11) are parabolic, $(\tilde{A}, \tilde{\Phi})$ is smooth across the singular set $\mathbb{R}^2 = \{(r, \alpha, (y_1, y_2)) \in \mathbb{R}^4 : r = 0\}$ (see [LSU] or [St3]). This proves our claim. □

Next, we apply Theorem 4.1 to study the non-self dual Yang-Mills connection. On S^4, the self-dual Yang-Mills (instanton) equations are

$$*F_B = F_B. \tag{4.5}$$

On $\mathbb{H}^3$, the magnetic monopole equations are

$$D_A \Phi = *F_A. \tag{4.6}$$

If (A, Φ) is a magnetic monopole, then

$$\int_{\mathbb{H}^3} |F_A|^2_{\mathbb{H}^3} + |D_A \Phi|^2_{\mathbb{H}^3} dv_{\mathbb{H}^3} = 2 \int_{\mathbb{H}^3} |F_A|^2_{\mathbb{H}^3} dv_{\mathbb{H}^3} = 4km,$$

where k is the winding number of Φ, or the number of zeros of Φ. The integer number m, the mass of the monopole, is the asymptotic value of $|\Phi|^2$ at infinity.

Now we prove the existence of non self-dual connection by obtaining a sequence connections $B(t_j)$ of the Yang-Mills flow (1.2) over $\mathbb{R}^4$ with $t_j \to \infty$ instead of obtaining a sequence B_j by the Ljusternik-Schnirelmann theory.

Theorem 4.2. *(Sibner-Sinber-Uhlenbeck) There exists a nontrivial, non-self dual m-equivariant Yang-Mill connection B in the trivial bundle over S^4.*

Proof. There exists a non-contractible loop in $\mathcal{U}/G$ of $H^{1,\infty}$ connections $B_0^\gamma = A_0^\gamma + \Phi_0^\gamma d\alpha$, $\gamma \in [0, 2\pi]$, on a trivial bundle (see [SSU, Lemma 2]), satisfying

$$\mathrm{YM}(B_0^\gamma) < 8\pi m.$$

Moreover, B_0^γ is smooth in $\mathbb{H}^3$ except the sphere $R=\varepsilon$, and also Lipshitz continuous across the sphere $R=\varepsilon$ where $R=\sqrt{r^2+|y|^2}$, Φ_0^γ is bounded and has no zero point in $\mathbb{H}^3$. By an argument of regularization, we can assume that $B_0^\gamma = A_0^\gamma + \Phi_0^\gamma d\alpha$ are smooth in $\mathbb{H}^3$. We set $(A_0^\gamma, \Phi_0^\gamma)$ as initial values for the heat flow (4.4). Then there exist global smooth solutions (A^γ, Φ^γ) of the heat flow (4.4) in $\mathbb{H}^3 \times [0,\infty)$ with initial values $(A_0^\gamma, \Phi_0^\gamma)$. By the energy inequality, there exists a suitable sequence of $t_k \to \infty$ such that $(A^\gamma, \Phi^\gamma)(\cdot, t_k)$, up to a gauge transformation, strongly converges to $(A_\infty^\gamma, \Phi_\infty^\gamma)$ in $H^{1,2}$, where $(A_\infty^\gamma, \Phi_\infty^\gamma)$ are solutions of (4.3) in $\mathbb{H}^3$. For each γ, we claim that $(A_\infty^\gamma, \Phi_\infty^\gamma)$ must not be any non-trivial monopole, i.e. self-dual solution of (4.6). If $(A_\infty^\gamma, \Phi_\infty^\gamma)$ is a non-trivial monopole, the number k of zeros of the Higgs field Φ_∞^γ must be great than 2 since Φ_0^γ has no zero point. Then $\mathrm{YMH}(A_\infty^\gamma, \Phi_\infty^\gamma) = 4k\pi m$ contradicts to the fact that $\mathrm{YHM}(A_0^\gamma, \Phi_0^\gamma) < 8\pi m$.

We assume that for every γ and for any sequence t_k, $(A^\gamma, \Phi^\gamma)(\cdot, t_k)$ strongly converges in $H^{1,2}$ to the trivial monopole (i.e. $k=0$). Then for every γ, $(A^\gamma, \Phi^\gamma)(\cdot, t)$ strongly converges to the trivial monopole in $H^{1,2}$ as $t \to \infty$. It implies that loops $(A^\gamma, \Phi^\gamma)(\cdot, t)$ shrink to the trivial monopole. This contradicts to the fact that the loops are not contractible in $\mathcal{U}/G$. Therefore there exists a nontrivial non-self-dual solution of (4.3) in $\mathbb{H}^3$. Using the removable codimension two singularity theorem of [SS], $B = A + \Phi d\alpha$ can be extended to a smooth solution, which is neither self-dual nor antiself-dual, of the Yang-Mills equation (4.2). □

References.

[A] M. F. Atiyah, *Magnetic monopoles in hyperbolic space,* Vector Bundles on Algebraic varieties, Tata Institute of Fundamental research, Bombay (1984), 1–33.

[B] P. Braam, *Magnetic monopoles on three-manifolds,* J. Diff. Geom. **30** (1989), 425–464.

[CDY] K.-C. Chang, W.-Y. Ding, and R. Ye, *Finite-time blow up of the heat flow of harmonic maps from surfaces,* J. Diff. Geom. **36** (1992), 507–515.

[CS] Y. Chen and M. Struwe, *Existence and partial regular results for the heat flow for harmonic maps,* Math. Z. **201** (1989), 83–103.

[D] S. K. Donaldson, *Anti-self-dual Yang-Mills connections on complex algebraic surfaces and stable vector bundles,* Proc. Lond. Math. Soc. **50** (1985), 1–26.

[De] D. M. De Turck, *Deforming the metrics in direction of their Ricci tensor*, J. Diff. Goem. **18** (1985), 157–162.

[ES] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[LSU] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type,* Providence, RI: AMS, 1987.

[La] B. Lawson *The theory of gauge fields in four dimensions,* (CBMS Regional Conference Series, vol 58) Providence, RI: AMS, 1987.

[Ha] A. Hassell, *The Yang-Mills-Higgs heat flow on* $\mathbb{R}^3$*,* J. Functional Analysis **111** (1993), 431–448.

[He] E. Hebey, *Nonlinear Analysis on manifolds: Sobolev space and inequalities,* Courant Lecture notes 5, Courant Institute of Mathematical Sciences, New York University, 1999.

[Ho] M.-C. Hong, *Heat flow for Yang-Mills-Higgs field and the Hermitian Yang-Mills-Higgs metric,* Ann. Global Anal. Geom. **20** (2001), 23–46.

[HT] M.-C. Hong and G. Tian, *Asymptotical behaviour of the Yang-Mills flow and singular Yang-Mills connections,* A preprint (2002).

[LT] P. Li and L.-F. Tam, *The heat equation and harmonic maps of complete manifolds,* Invent. Math. **105** (1991), 1–46.

[R] J. Rade, *On the Yang-Mills heat equation in two and three dimensions,* J. Reine Angew. Math. **431** (1992), 123–163.

[Sc] A. Schlatter, *Global existence of the Yang-Mills flow in four dimensions,* J. Reine Angew. Math. **479** (1996), 133–147.

[Sc1] A. Schlatter, *Long-time behaviour of the Yang-Mills flow in four dimensions,* Ann. Global Anal. Geom. **15** (1997), 1–25.

[SST] A. Schlatter, M. Struwe and A. S. Tahvilder-Zadeh, *Global existence of the equivariant Yang-Mills heat flow in four space dimensions,* Amer. J. Math. **120** (1998), 117–128.

[Si] L. Simon, *Asymptotics for a class of non-linear evolution equations with applications to geometric problems,* Annals of Math. **118** (1983), 525–571.

[SS] L. M. Sibner and R. J. Sibner, *Classification of singular Sobolev connections by their holonomy,* Commun. Math. Phys. **144** (1992), 337–350.

[SSU] L. M. Sibner, R. J. Sibner and K. Uhlenbeck, *Solutions to Yang-Mills equations that are not self-dual,* Proc. Natl. Acad. Sci. USA **86** (1989), 8610–8613.

[St1] M. Struwe, *On the evolution of Harmonic maps of Riemannian surfaces,* Communt. Math. Helv. **60** (1985), 558–581.

[St2] M. Struwe, *On the evolution of harmonic maps in higher dimensions,* J. Diff. Geom. **28** (1988), 485–502.

[St3] M. Struwe, *The Yang-Mills flow in four dimensions,* Calc. Var. **2** (1994), 123–150.

[Ti] G. Tian, *Gauge theory and calibrated geometry, I,* Annals of Math. **151** (2000), 193–268.

[U] K. Uhlenbeck, *Connections with L^p bounds on curvature,* Commun. Math. Phys. **83** (1982), 31–42.

Min-Chun Hong
Department of Mathematics, Statistics and Computing Science
University of New England
Armidale, NSW 2351, Australia

Gang Tian
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139, USA