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Brownian Motion on a Submanifold

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Given a submanifold M of a Riemannian manifold N, we give two different constructions of Brownian motion on M: one by "projection" onto M of the Brownian motion on N and the other by a more intrinsic approach. The two procedures lead to very different ways in which vectors are transported along Brownian paths.

Introduction.

Throughout this note N will denote a complete, connected n-dimensional Riemannian manifold and M will be a closed m-dimensional, imbedded submanifold of N which is given the Riemannian structure which it inherits from N. In addition, we will be using ∇^N to denote the Levi-Civita on N, and ∇^M to denote the inherited Levi-Civita on M. Finally, given a piecewise smooth path $p : [0,t] \longrightarrow N$, we will use $\mathcal{T}_p^N \in \mathrm{Hom}\big(T_{p(0)}N; T_{p(t)}\big)$ to denote parallel transport along p. Similarly, if p takes its values in M, then $\mathcal{T}_p^M \in \mathrm{Hom}\big(T_{p(0)}M; T_{p(t)}M\big)$ will be parallel transport along p as a path in M.

Our goal is to examine various relations between the Brownian motion on N and the Brownian motion on M. This sort of analysis was carried out in Chapters 4 and 5 of [3] when $N = \mathbb{R}^n$. However, even in that case, the analysis given there is less complete than the one given here.

1. The Shape Operator.

Given $x \in M$, define the *shape operator* $\mathcal{S}_x \in \mathrm{Hom}\big(T_xM; \mathrm{Hom}(T_xN; T_xN)\big)$ so that if $X_x \in T_xM$, then

$$\mathcal{S}_x(X_x) = \frac{d}{dt}\big(\mathcal{T}^N_{p\restriction[0,t]}\big)^{-1} \circ \Pi_{p(t)} \circ \mathcal{T}^N_{p\restriction[0,t]} X_x \Big|_{t=0}, \tag{1.1}$$

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where $p \in C^1\big([0,\infty);M\big)$ with $p(0)=x$ and $\dot p(0)=X_x$. To see that $\mathcal{S}_x(X_x)$ is well-defined (i.e., independent of the choice of p), observe that if $Y \in TN$ is any vector field on N, then

$$\begin{aligned}\nabla^N_{X_x}(\Pi Y) &= \frac{d}{dt}\Big(\big(\mathcal{T}_{p\restriction[0,t]}\big)^{-1}\circ \Pi_{p(t)}\circ \mathcal{T}_{p\restriction[0,t]}\Big)\big(\mathcal{T}_{p\restriction[0,t]}\big)^{-1}Y_{p(t)}\Big|_{t=0}\\ &= \mathcal{S}_x(X_x)Y_x + \Pi_x\nabla^N_{X_x}Y,\end{aligned}$$

and so

$$\mathcal{S}_x(X_x)Y_x = \nabla^N_{X_x}(\Pi Y) - \Pi_x\nabla^N_{X_x}Y. \tag{1.2}$$

In particular,

$$\begin{aligned}Y\restriction M \in TM \implies \mathcal{S}_x(X_x)Y_x &= \Pi^\perp_x\nabla^N_{X_x}Y\\ &= \nabla^N_{X_x}Y - \nabla^M_{X_x}Y \equiv -H(X_x,Y_x),\end{aligned} \tag{1.3}$$

where H is the second fundumental form; and

$$Y\restriction M \perp TM \implies \mathcal{S}_x(X_x)Y_x = -\Pi_x\nabla^N_{X_x}Y. \tag{1.4}$$

Thus, by choosing $Y \in TN$ so that either $Y\restriction M \in TM$ or $Y\restriction M \perp TM$, we see that

$$\Pi^\perp_x\circ\mathcal{S}_x(X_x) = \mathcal{S}_x(X_x)\circ\Pi_x \text{ and } \Pi_x\circ\mathcal{S}_x(X_x) = \mathcal{S}_x(X_x)\circ\Pi^\perp_x. \tag{1.5}$$

In addition, if $Y_x \in T_xM$, then we can choose $X, Y \in TN$ which agree with X_x and Y_x at x and satisfy $X\restriction M, Y\restriction M \in TM$. Hence, by (1.3),

$$\mathcal{S}_x(X_x)Y_x - \mathcal{S}(Y_x)X_x = \Pi^\perp_x[X,Y]_x = 0,$$

since ∇^N and ∇^M are torsion free and $[X,Y]_x \in T_xM$. In other words,

$$X_x, Y_x \in T_xM \implies \mathcal{S}_x(X_x)Y_x = \mathcal{S}(Y_x)X_x. \tag{1.6}$$

Lemma 1.7. *Given $x \in M$, define $a_x \in \mathrm{Hom}\big(T_xM;\mathrm{Hom}(T_xN;T_xN)\big)$ so that*

$$a_x(X_x) = \mathcal{S}_x(X_x)\circ\big(\Pi_x - \Pi^\perp_x\big). \tag{1.8}$$

Then $a_x(X_x)$ is skew symmetric on T_xN for each $X_x \in T_xM$. Next, for $p \in C^1\big([0,\infty);M\big)$, determine

$$t\in[0,\infty)\longmapsto O_p(t)\in\mathrm{Hom}\big(T_{p(0)}N;T_{p(t)}N\big)$$

by

$$\frac{d}{dt}\big(\mathcal{T}^N_{p\restriction[0,t]}\big)^{-1}\circ O_p(t) = \big(\mathcal{T}^N_{p\restriction[0,t]}\big)^{-1}\circ a_{p(t)}\big(\dot p(t)\big)\circ O_p(t) \quad \text{with } O_p(0)=I.$$

Then, for each $t\in[0,\infty)$, $O_p(t)$ is unitary from $T_{p(0)}N$ onto $T_{p(t)}N$,

$$\Pi_{p(t)}\circ O_p(t) = O_p(t)\circ \Pi_{p(0)} \quad \text{and} \quad \Pi^\perp_{p(t)}\circ O_p(t) = O_p(t)\circ\Pi^\perp_{p(0)}.$$

In fact,

$$O_p(t)\restriction T_{p(0)}M = \mathcal{T}^M_{p\restriction[0,t]}.$$

Proof. Clearly the skew symmetry follows from (1.5).

Next set $x=p(0)$, let $Y_x\in T_N$ be given, and set $Y(t)=O_p(t)Y_x$. Then

$$\frac{D^N}{dt}Y(t) = \mathcal{T}^N_{p\restriction[0,t]}\frac{d}{dt}\big(\mathcal{T}^N_{p\restriction[0,t]}\big)^{-1}Y(t) = a_{p(t)}\big(\dot p(t)\big)Y(t),$$

where $\frac{D^N}{dt}$ denotes N-covariant differentiation along p. Hence, $t\in[0,\infty)\longmapsto Y(t)\in T_{p(t)}N$ is characterized as the solution to

$$\frac{D^N}{dt}Y(t) = a_{p(t)}\big(\dot p(t)\big)Y(t) \quad \text{with } Y(0)=Y_x. \tag{$*$}$$

In particular, because of the skew symmetry of $a_{p(t)}\big(\dot p(t)\big)$,

$$\frac{d}{dt}\big\|Y(t)\big\|^2 = 2\Big\langle \frac{D^N}{dt}Y(t), Y(t)\Big\rangle = 0,$$

and so $O_p(t)$ is unitary. Now set $\tilde Y(t)=\Pi_{p(t)}Y(t)$. Then

$$\frac{D^N}{dt}\tilde Y(t) = \mathcal{S}_{p(t)}\big(\dot p(t)\big)Y(t) + \Pi_{p(t)}a_{p(t)}\big(\dot p(t)\big)Y(t) = a_{p(t)}\big(\dot p(t)\big)\tilde Y(t),$$

where, in the last step, we have again applied (1.5). Thus, by the characterization given in (*), we see that $\tilde Y(t) = O_p(t)\tilde Y(0)$. From this it follows immediately that $\Pi_{p(t)}\circ O_p(t) = O_p(t)\circ\Pi_x$, and, obviously, $\Pi^\perp_{p(t)}\circ O_p(t) = O_p(t)\circ\Pi^\perp_x$ comes along for free. Finally, to prove that $Y(t)=\mathcal{T}^M_{p\restriction[0,t]}Y_x$ when $Y_x\in T_xM$, simple observe that, because $Y(t)\in T_{p(t)}M$,

$$\frac{D^N}{dt}Y(t) = a_{p(t)}\big(\dot p(t)\big)Y(t)\perp T_{p(t)}M$$

follows from (1.5). In other words,

$$\frac{D^M}{dt}Y(t) = \Pi_{p(t)}\frac{D^N}{dt}Y(t) = 0,$$

and this proves that $Y(t) = \mathcal{T}^M_{p\restriction[0,t]}Y_x$. □

Finally, we close this section with the observation that if $\varphi \in C^2(N;\mathbb{R})$ and $x \in M$, then

$$X_x, Y_x \in T_xM \implies \langle X_x, \mathrm{hess}^M_x\varphi Y_x\rangle = \langle X_x, \mathrm{hess}^N_x\varphi Y_x\rangle + \mathcal{S}_x(X_x)Y_x\varphi. \tag{1.9}$$

To see this, simply recall that, for any extension Y of Y_x to N with $Y \restriction M \in TM$,

$$\begin{aligned}\langle X_x, \mathrm{hess}^M_x\varphi Y_x\rangle &= X_xY\varphi - \nabla^M_{X_x}Y\varphi = X_xY\varphi - \nabla^N_{X_x}Y\varphi + \mathcal{S}_x(X_x)Y_x\varphi \\ &= \langle X_x, \mathrm{hess}^N_x\varphi Y_x\rangle + \mathcal{S}_x(X_x)Y_x\varphi.\end{aligned}$$

As a consequence of (1.9) and the representation of Laplacian as the trace of the Hessian, we obtain

$$\Delta^M\varphi = \mathrm{Trace}^M\big(\mathrm{hess}^N_x\varphi\big)\varphi - B\varphi, \tag{1.10}$$

where, for each $x \in M$ and orthonormal basis $\big((E_1)_x, \dots, (E_n)_m\big)$ in T_xN,

$$B_x \equiv -\sum_{i=1}^m \mathcal{S}\big(\Pi_x(E_i)_x\big)\Pi_x(E_i)_x = \sum_{i=1}^m H\big(\Pi_x(E_i)_x, \Pi_x(E_i)_x\big) \tag{1.11}$$

is (apart from normalization) the *mean curvature vector* (cf. page 49 in [2]).

2. Moving to the Orthonormal Frame Bundle.

In this section we will interpret the results of §1 in terms of the orthonormal frame bundle (cf. Chapter 8 of [3] for a treatment using the notation adopted here or [1] for a thorough treatment) $\mathcal{O}(N)$ over N. That is, elements $\mathfrak{f}$ of $\mathcal{O}(N)$ are *frames* $(x, \mathfrak{e}_x)$, where $x \in N$ and $\mathfrak{e}_x = \big((E_1)_x, \dots, (E_n)_x\big)$ is an orthonormal basis in T_xN. We use $\pi : \mathcal{O}(N) \longrightarrow N$ to denote the fiber map $\pi\mathfrak{f} = x$, and, for convenience, we identify $\mathfrak{f}$ with the isometry from $\mathbb{R}^n$ onto T_xN given by

$$\mathfrak{f}\boldsymbol{\xi} = \sum_{i=1}^n \xi_i(E_i)_x \quad \text{for } \boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

Next, recall that $\mathcal{O}(N)$ is a principle bundle over N with fiber the orthogonal group $\mathcal{O}(\mathbb{R}^n)$, and, for $O \in \mathcal{O}(\mathbb{R}^n)$, let $R_O : \mathcal{O}(N) \longrightarrow \mathcal{O}(N)$ be the map defined so that $(R_O\mathfrak{f}) = \mathfrak{f}(O\boldsymbol{\xi})$ for $\mathfrak{f} \in \mathcal{O}(N)$ and $\boldsymbol{\xi} \in \mathbb{R}^n$. Further, given $a \in o(\mathbb{R}^n)$, define the vertical vector field $\lambda(a)$ on $\mathcal{O}(N)$ so that

$$\lambda(a)_{\mathfrak{f}} = \frac{d}{ds} R_{e^{sa}}\mathfrak{f}\Big|_{s=0};$$

and, given $\boldsymbol{\xi} \in \mathbb{R}^n$, define the canonical horizontal vector field $\mathfrak{E}(\boldsymbol{\xi})$ on $\mathcal{O}(N)$ so that $\mathfrak{E}(\boldsymbol{\xi})_{\mathfrak{f}}$ is the horizontal lift to $\mathfrak{f}$ of $\mathfrak{f}\boldsymbol{\xi} \in T_xM$. Finally, the solder form ϕ and connection 1-from ω are defined (cf. page 181 in [3]) on $T_{\mathfrak{f}}\mathcal{O}(N)$ into $\mathbb{R}^n$ and $o(\mathbb{R}^n)$, respectively, so that

$$\mathfrak{X}_{\mathfrak{f}} = \mathfrak{E}\big(\phi(\mathfrak{X}_{\mathfrak{f}})\big)_{\mathfrak{f}} + \lambda\big(\omega(\mathfrak{X}_{\mathfrak{f}})\big)_{\mathfrak{f}}$$

gives the resolution of $\mathfrak{X}_{\mathfrak{f}}$ into its horizontal and vertical components.

Define $\hat{\Pi} : \pi^{-1}(M) \longrightarrow \mathrm{Hom}(\mathbb{R}^n;\mathbb{R}^n)$ so that $\hat{\Pi}_{\mathfrak{f}} = \mathfrak{f}^{-1} \circ \Pi_{\pi(\mathfrak{f})} \circ \mathfrak{f}$. Clearly, for each $\mathfrak{f} \in \pi^{-1}(M)$, $\hat{\Pi}_{\mathfrak{f}}$ is the orthogonal projection onto the subspace of $\boldsymbol{\xi} \in \mathbb{R}^n$ such that $\mathfrak{f}\boldsymbol{\xi} \in T_{\pi(\mathfrak{f})}M$. By using the fact (cf. (8.22) in [3]) that, for any vector field Y on N,

$$\mathfrak{f}^{-1}\big(\nabla^N_{\mathfrak{f}\boldsymbol{\xi}}Y\big) = \mathfrak{E}(\boldsymbol{\xi})_{\mathfrak{f}}\Xi_Y, \quad \text{where } \Xi_Y(\mathfrak{f}) \equiv \mathfrak{f}^{-1}Y_{\pi\mathfrak{f}}, \tag{2.1}$$

we see that, for $\mathfrak{f}\boldsymbol{\xi} \in T_{\pi(\mathfrak{f})}M$,

$$\mathfrak{f}^{-1}\nabla^N_{\mathfrak{f}\boldsymbol{\xi}}(\Pi Y) = \big(\mathfrak{E}(\boldsymbol{\xi})_{\mathfrak{f}}\hat{\Pi}\big)\mathfrak{f}^{-1}Y_{\pi\mathfrak{f}} + \hat{\Pi}_{\mathfrak{f}}\big(\mathfrak{f}^{-1}\nabla^N_{\mathfrak{f}\boldsymbol{\xi}}Y\big).$$

Hence, by (1.2),

$$\hat{\mathcal{S}}_{\mathfrak{f}}(\boldsymbol{\xi}) \equiv \mathfrak{f}^{-1}\circ\mathcal{S}_{\pi\mathfrak{f}}(\mathfrak{f}\boldsymbol{\xi}) \circ \mathfrak{f} = \mathfrak{E}(\boldsymbol{\xi})_{\mathfrak{f}}\hat{\Pi} \quad \text{for } \mathfrak{f} \in \pi^{-1}(M) \text{ and } \boldsymbol{\xi} \in \mathfrak{f}^{-1}(T_{\pi\mathfrak{f}}M). \tag{2.2}$$

Next, observe that $\pi^{-1}(M)$ is a submanifold of $\mathcal{O}(N)$. In fact, for $\mathfrak{f} \in \pi^{-1}(M)$ and $\mathfrak{X}_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}(N)$, $\mathfrak{X}_{\mathfrak{f}} \in T_{\mathfrak{f}}\big(\pi^{-1}(M)\big)$ if and only if $\hat{\Pi}^{\perp}_{\mathfrak{f}}\phi(\mathfrak{X}_{\mathfrak{f}}) = 0$. Thus, for each $\boldsymbol{\xi} \in \mathbb{R}^n$,

$$\mathfrak{f} \in \pi^{-1}(M) \longmapsto \hat{\mathfrak{E}}(\boldsymbol{\xi})_{\mathfrak{f}} \equiv \mathfrak{E}\big(\hat{\Pi}_{\mathfrak{f}}\boldsymbol{\xi}\big)_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}(N)$$

is a vector field on $\pi^{-1}(M)$. Furthermore, if $\varphi \in C^2(N;\mathbb{R})$, then, by (2.2),

$$\hat{\mathfrak{E}}(\boldsymbol{\xi})_{\mathfrak{f}} \circ \hat{\mathfrak{E}}(\boldsymbol{\eta})(\varphi\circ\pi) = \mathfrak{E}(\Pi_{\mathfrak{f}}\boldsymbol{\xi})_{\mathfrak{f}} \circ \mathfrak{E}(\Pi_{\mathfrak{f}}\boldsymbol{\eta})(\varphi\circ\pi) + \mathfrak{E}\big(\hat{\mathcal{S}}_{\mathfrak{f}}(\boldsymbol{\xi})\boldsymbol{\eta}\big)_{\mathfrak{f}}(\varphi\circ\pi).$$

At the same time, because an alternative way to describe $\text{hess}_x^N\varphi$ is to say that

$$\text{hess}_x^N\varphi Y_x = \nabla_{Y_x}\text{grad}^N\varphi,$$

(2.1) leads to

$$\langle \mathfrak{f}\hat{\Pi}_\mathfrak{f}\boldsymbol{\xi}, \text{hess}_{\pi\mathfrak{f}}^N\varphi\mathfrak{f}\hat{\Pi}_\mathfrak{f}\boldsymbol{\eta}\rangle = \mathfrak{E}(\Pi_\mathfrak{f}\boldsymbol{\xi})_\mathfrak{f}\circ\mathfrak{E}(\Pi_\mathfrak{f}\boldsymbol{\eta})(\varphi\circ\pi).$$

Hence, after combining this with the preceding, (1.9) says that

$$\langle \mathfrak{f}\hat{\Pi}_\mathfrak{f}\boldsymbol{\xi}, \text{hess}_{\pi\mathfrak{f}}^M\varphi\mathfrak{f}\hat{\Pi}_\mathfrak{f}\boldsymbol{\eta}\rangle = \hat{\mathfrak{E}}(\boldsymbol{\xi})_\mathfrak{f}\circ\hat{\mathfrak{E}}(\boldsymbol{\eta})(\varphi\circ\pi), \quad \mathfrak{f}\in\pi^{-1}(M) \tag{2.3}$$

In particular, this means that for any orthonormal basis $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ in $\mathbb{R}^n$,

$$\big(\Delta^M\varphi\big)\circ\pi = \sum_{i=1}^n \hat{\mathfrak{E}}(\mathbf{e}_i)^2\varphi\circ\pi \quad \text{on } \pi^{-1}(M). \tag{2.4}$$

3. Brownian Motion, an Extrinsic Approach.

The formula (2.4) provides the basis for a construction of Brownian motion on M via “projection” of the Brownian motion on N.

To see what we have in mind, recall (cf. §8.2 in [3]), one way to construct the Brownian motion on N starting at a point x is to *roll* a Euclidean Brownian motion (i.e., a Wiener process) on T_xN onto N. That is, if $\mathbf{w}$ is a “piecewise smooth” Wiener path in $\mathbb{R}^n$ and $\mathfrak{f}\in\pi^{-1}(x)$, then we determine $\mathfrak{p}^N(\,\cdot\,,\mathfrak{f},\mathbf{w})$ by

$$\dot{\mathfrak{p}}^N(t,\mathfrak{f},\mathbf{w}) = \mathfrak{E}(\dot{\mathbf{w}})_{\mathfrak{p}^N(t,\mathfrak{f},\mathbf{w})} \quad \text{with } \mathfrak{p}^N(0,\mathfrak{f},\mathbf{w}) = \mathfrak{f}$$

and set $p^N(t,\mathfrak{f},\mathbf{w}) = \pi\circ\mathfrak{p}^N(t,\mathfrak{f},\mathbf{w})$. If almost every Wiener path were actually piecewise smooth, the distribution of $\mathbf{w}\rightsquigarrow p^N(\,\cdot\,,\mathfrak{f},\mathbf{w})$ under Wiener measure would be the distribution of Brownian motion on M starting at x. Because almost no Wiener is anywhere smooth, the preceding has to be interpretted by an appropriate limit procedure in which paths $\mathbf{w}$ are first replaced by polygononal approximations. The result of this procedure is equivalent to saying that we want to take $\mathbf{w}\rightsquigarrow\mathfrak{p}^N(\,\cdot\,,\mathfrak{f},\mathbf{w})$ to be the solution to the Stratonovich stochastic differential equation

$$d\mathfrak{p}^N(t,\mathfrak{f},\mathbf{w}) = \sum_{i=1}^n \mathfrak{E}(\mathbf{e}_i)_{\mathfrak{p}^N(t,\mathfrak{f},\mathbf{w})}\circ d\big(\mathbf{e}_i,\mathbf{w}(t)\big) \quad \text{with } \mathfrak{p}^N(0,\mathfrak{f},\mathbf{w}) = \mathfrak{f}.$$

Indeed, if one ignores problems coming from possible explosion, Itô's formula for Stratonovich calculus says that, for any $\Phi \in C^2_c\big(\mathcal{O}(N);\mathbb{R}\big)$,

$$\Phi\big(\mathfrak{p}^N(\,\cdot\,,\mathfrak{f},\mathbf{w})\big) - \int_0^t \left(\frac{1}{2}\sum_1^n \mathfrak{E}(\mathbf{e}_i)^2_{\mathfrak{p}^N(\tau,\mathfrak{f},\mathbf{w})}\Phi\right) d\tau$$

is a martingale under Wiener measure. Thus, since, by another application of (2.1),

$$\big(\Delta^N\varphi\big)\circ\pi = \sum_{i=1}^n \mathfrak{E}(\mathbf{e}_i)^2(\varphi\circ\pi),$$

it is clear that

$$\varphi\big(p^N(t,\mathfrak{f},\mathbf{w})\big) - \int_0^t \big(\tfrac{1}{2}\Delta^N\varphi\big)\big(p^N(\tau,\mathfrak{f},\mathbf{w})\big)\,d\tau$$

is a martingale for each $\varphi \in C^2_c(N;\mathbb{R})$. In other words, $\mathbf{w} \rightsquigarrow p^N(\,\cdot\,,\mathfrak{f},\mathbf{w})$ under Wiener measure has the distribution of a Brownian motion on N starting at x. Moreover, because $\mathfrak{p}^N(\,\cdot\,,\mathfrak{f},\mathbf{w})$ is the horizontal lift of $p^N(\,\cdot\,,\mathfrak{f},\mathbf{w})$ to $\mathfrak{f}$ when $\mathbf{w}$ is piecewise smooth, it is reasonable to say that *horizontal transport along the Brownian curve* $p^N(\,\cdot\,,\mathfrak{f},\mathbf{w})\restriction[0,t]$ is given by $\mathfrak{p}^N(t,\mathfrak{f},\mathbf{w})\circ\mathfrak{f}^{-1}$ even when $\mathbf{w}$ is a generic Wiener path.

With the preceding in mind, we now suppose that $x \in M$ and consider the Stratonovich stochastic differential equation

$$d\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w}) = \sum_{i=1}^n \hat{\mathfrak{E}}(\mathbf{e}_i)_{\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})}\circ d\big(\mathbf{e}_i,\mathbf{w}(t)\big). \tag{3.1}$$

By precisely the same arguement as above, only this time using (2.4), we see that $\mathbf{w} \rightsquigarrow q^M(\,\cdot\,,\mathfrak{f},\mathbf{w}) \equiv \pi\circ\mathfrak{q}^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$ is distributed under Wiener measure like a Brownian motion on M starting at x. Furthermore, it is again reasonable to think of $\mathfrak{q}^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$ as the horizontal lift to $\mathfrak{f}$ of $q^N(\,\cdot\,,\mathfrak{f},\mathbf{w})$. Thus, $\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})\circ\mathfrak{f}^{-1}$ gives parallel transport along $q^M(\,\cdot\,,\mathfrak{f},\mathbf{w})\restriction[0,t]$ as a path in N. However, it does *not* give parallel transport along $q^M(\,\cdot\,,\mathbf{w})$ as a path in M. Indeed, it will seldom even take T_xM into $T_{q^M(t,\mathfrak{f},\mathbf{w})}M$.

A Remark about Explosion: In the preceding discussion, we ignored the question of explosion. Because we are assuming that M is imbedded in N, we can (cf. Theorem 3.64 in [3]) show that explosion never occurs if we can check that, in the sense of distributions, $\Delta^M\rho \le C(1+\rho)$ on M for some $C < \infty$, where $\rho(y) = \mathrm{dist}^N(x,y)^2$ and $\mathrm{dist}^N(x,y)$ denotes the Riemannian distance in N between x and y. In view of (1.10), this is tantamount to testing whether

$$\mathrm{Trace}^M\bigl(\mathrm{hess}^N\rho\bigr) - B\rho \le C(1+\rho)$$

in the sense of distributions. By the arguement in §8.4 of [3], the first term on the left can be handled if there exists an $\alpha > 0$ such that

$$\sum_{i=1}^m \bigl\langle R^N\bigl(Y_y,(E_i)_y\bigr)(E_i)_y, Y_y\bigr\rangle \ge -\alpha\bigl(1+\rho(y)\bigr)\|Y_y\|^2 \qquad \text{for } y\in M \text{ and } Y_y \in T_yN,$$

where $\bigl((E_1)_y,\dots,(E_m)_y\bigr)$ is used to denote an orthonormal basis in T_yM. Thus, if such an α exists, then non-explosion is guaranteed by the existence of a $\beta > 0$ such that $\langle B, \mathrm{grad}^N\rho\rangle \ge -\beta(1+\rho)$.

4. A Second, and More Geometrically Sound, Appraoch.

As we pointed out, although the $q^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$ is indeed Brownian motion on M starting at $\pi\mathfrak{f}$, $\mathfrak{q}^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$ is the *wrong* lift of $q^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$ to $\mathfrak{f}$ if one is interested in parallel transport in M, as opposed to N. In addition, because our construction of the m-dimensional Brownian path $q^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$ used the n-dimensional Wiener path $\mathbf{w}$, one suspects that there should be a tighter construction of Brownian motion on M: a construction which involves only an m-dimensional Wiener path.

Motivated by the preceding comments, we will now take a different tack. To understand the origins of this new approach, set (cf. (2.2))

$$\hat a_{\mathfrak{f}}(\boldsymbol{\xi}) \equiv \hat{\mathcal{S}}_{\mathfrak{f}}\bigl(\hat\Pi_{\mathfrak{f}}\boldsymbol{\xi}\bigr)\circ\bigl(\hat\Pi_{\mathfrak{f}} - \hat\Pi_{\mathfrak{f}}^{\perp}\bigr) = \mathfrak{f}^{-1}\circ a_{\pi\mathfrak{f}}\bigl(\Pi_{\pi\mathfrak{f}}\mathfrak{f}\boldsymbol{\xi}\bigr)\circ\mathfrak{f} \quad \text{for } \mathfrak{f}\in\pi^{-1}(M). \tag{4.1}$$

Using (1.9), it is a straight-forward matter to check that

$$\begin{aligned}&\Bigl(\mathfrak{E}(\boldsymbol{\xi}) + \lambda\bigl(\hat a_{\mathfrak{f}}(\boldsymbol{\xi})\bigr)\Bigr)_{\mathfrak{f}}\circ\Bigl(\mathfrak{E}(\boldsymbol{\eta}) + \lambda\bigl(\hat a(\boldsymbol{\eta})\bigr)\Bigr)(\varphi\circ\pi)\\ &\quad = \langle \mathfrak{f}\boldsymbol{\xi}, \mathrm{hess}^M_{\pi\mathfrak{f}}\varphi\,\mathfrak{f}\boldsymbol{\eta}\rangle \quad \text{for } \mathfrak{f}\in\pi^{-1}(M) \text{ and } \boldsymbol{\xi},\boldsymbol{\eta}\in\mathfrak{f}^{-1}(T_{\pi\mathfrak{f}}M).\end{aligned} \tag{4.2}$$

Indeed, all that one needs to do is remember that vertical vectors kill $\varphi\circ\pi$, observe that

$$\lambda\bigl(\hat a_{\mathfrak{f}}(\boldsymbol{\xi})\bigr)_{\mathfrak{f}}\circ\mathfrak{E}(\boldsymbol{\eta}) = \mathfrak{E}\bigl(\hat a_{\mathfrak{f}}(\boldsymbol{\xi})\boldsymbol{\eta}\bigr)_{\mathfrak{f}} + \mathfrak{E}(\boldsymbol{\xi})_{\mathfrak{f}}\circ\lambda\bigl(\hat a_{\mathfrak{f}}(\boldsymbol{\eta})\bigr),$$

and note that, because $\hat\Pi_{\mathfrak{f}}^{\perp}\boldsymbol{\xi} = 0 = \hat\Pi_{\mathfrak{f}}^{\perp}\boldsymbol{\eta}$, $\hat a_{\mathfrak{f}}(\boldsymbol{\xi})\boldsymbol{\eta} = \hat{\mathcal{S}}(\boldsymbol{\xi})\boldsymbol{\eta}$. In particular, if $(\mathbf{e}_1,\dots,\mathbf{e}_m)$ is an orthonormal basis for $\mathfrak{f}^{-1}(T_{\pi\mathfrak{f}}M)$, then

$$(\Delta_M\varphi)\circ\pi(\mathfrak{f}) = \sum_{i=1}^m \Bigl(\mathfrak{E}(\mathbf{e}_i) + \lambda\bigl(\hat a(\mathbf{e}_i)\bigr)\Bigr)^2_{\mathfrak{f}}(\varphi\circ\pi). \tag{4.3}$$

In order to base a construction of Brownian motion on (4.3), we will make use of the information contained in the following simple lemmas.

Lemma 4.4. *Given a piecewise continuously differentiable, continuous* $\mathfrak{q} : [0,\infty) \longrightarrow \pi^{-1}(M)$*, determine* $t \in [0,\infty) \longmapsto \hat{O}_{\mathfrak{q}}(t) \in \mathrm{Hom}(\mathbb{R}^n;\mathbb{R}^n)$ *by*

$$\frac{d}{dt}\hat{O}_{\mathfrak{q}}(t) = \hat{a}_{\mathfrak{q}(t)}\big(\phi(\dot{\mathfrak{q}}(t)\big)\hat{O}_{\mathfrak{q}}(t) \quad \textit{with } \hat{O}_{\mathfrak{q}}(0) = I. \tag{4.5}$$

Then $\hat{O}_{\mathfrak{q}}(t)$ *is an element of the orthognal group* $\mathcal{O}(\mathbb{R}^n)$ *for each* $t \in [0,\infty)$*. Moreover, if* $\mathfrak{q}$ *is horizontal (i.e.,* $\omega\big(\dot{\mathfrak{q}}(t)\big) \equiv 0$*), then (cf. Lemma 1.7)*

$$\mathfrak{q}(t) \circ \hat{O}_{\mathfrak{q}}(t) \circ \mathfrak{q}(0)^{-1} = O_{\pi\circ\mathfrak{q}}(t), \quad t \in [0,\infty), \tag{4.6}$$

and so

$$\hat{\Pi}_{\mathfrak{q}(t)} \circ \hat{O}_{\mathfrak{q}}(t) = \hat{O}_{\mathfrak{q}}(t) \circ \hat{\Pi}_{\mathfrak{q}(0)}, \quad t \in [0,\infty). \tag{4.7}$$

Proof. Without loss in generality, we will assume that $\mathfrak{q}$ is continuous differentiable everywhere.

Because (cf. the first part of Lemma 1.7) the values of $\hat{a}$ are always in the Lie algebra $o(\mathbb{R}^n)$ of skew symmetric operators, it is trivial to check that $\hat{O}_{\mathfrak{q}}(t) \in \mathcal{O}(\mathbb{R}^n)$ for all $t \geq 0$. To check (4.6) when $\mathfrak{q}$ is horizontal, let $\boldsymbol{\xi} \in \mathbb{R}^n$ be given, and set $X(t) = \mathfrak{q}(t)\hat{O}_{\mathfrak{q}}(t)\boldsymbol{\xi} \in T_{\pi\mathfrak{q}(t)}M$. Then, because $\mathfrak{q}$ is horizontal,

$$\frac{D^N}{dt}X(t) = \mathfrak{q}(t)\frac{d}{dt}\hat{O}_{\mathfrak{q}}(t)\boldsymbol{\xi} = \mathfrak{q}(t)\hat{a}_{\mathfrak{q}(t)}\big(\phi(\dot{\mathfrak{q}}(t)\big)\hat{O}_{\mathfrak{q}}(t)\boldsymbol{\xi} = a\big((\pi\circ\mathfrak{q})^{\cdot}(t)\big)X(t).$$

Since this means that $X(t) = O_{\pi\circ\mathfrak{q}}(t)\mathfrak{q}(0)\boldsymbol{\xi}$, (4.6) follows. Finally, (4.7) is immediate from (4.6) and the corresponding fact (cf. the last part of Lemma 1.7) for $O_{\pi\circ\mathfrak{q}}(t)$. □

Lemma 4.8. *Let* $\mathfrak{p} \in C\big([0,\infty);\pi^{-1}(M)\big)$ *be a piecewise continuously differentiable, set* $p = \pi\circ\mathfrak{p}$*, and let* $\mathfrak{q}$ *be the horizontal lift of* p *to* $\mathfrak{p}(0)$*. Then the following are equivalent:*

(1) $\omega\big(\dot{\mathfrak{p}}(t)\big) = \hat{a}_{\mathfrak{p}(t)}\big(\phi(\dot{\mathfrak{p}}(t))\big)$ *for all* $t \geq 0$ *at which* $\mathfrak{p}$ *is continuously differentiable,*

(2) $\mathfrak{p}(t) = R_{\hat{O}_{\mathfrak{q}}(t)}\mathfrak{q}(t)$*,*

(3) $O_p(t) = \mathfrak{p}(t)\mathfrak{p}(0)^{-1}$*.*

In particular, any one of these implies that $\hat\Pi_{\mathfrak{p}(t)} = \hat\Pi_{\mathfrak{p}(0)}$ *for all* $t \in [0,\infty)$.

Proof. Again we may and will assume that $\mathfrak{p}$ is continuously differentiable everywhere.

The equivalence of (2) and (3) just a restatement of (4.6), and the final conclusion is simply a restatement of (3). To prove the equivalence of (1) and (2), first note that

$$(1) \iff \dot{\mathfrak{p}}(t) = \mathfrak{E}\big(\mathfrak{p}(t)^{-1}\dot p(t)\big)_{\mathfrak{p}(t)} + \lambda\Big(\hat a_{\mathfrak{p}(t)}\big(\mathfrak{p}(t)^{-1}\dot p(t)\big)\Big)_{\mathfrak{p}(t)}.$$

Thus, it suffices to show that if $\mathfrak{r}(t) \equiv R_{\hat O_{\mathfrak{q}}(t)}\mathfrak{q}(t)$, then

$$\dot{\mathfrak{r}}(t) = \mathfrak{E}\big(\mathfrak{r}(t)^{-1}\dot p(t)\big)_{\mathfrak{r}(t)} + \lambda\Big(\hat a_{\mathfrak{r}(t)}\big(\mathfrak{r}(t)^{-1}\dot p(t)\big)\Big)_{\mathfrak{r}(t)}.$$

But $\dot{\mathfrak{r}}(t)$ is equal to

$$\begin{aligned}\big(R_{\hat O_{\mathfrak{q}}(t)}\big)_*\mathfrak{E}\big(\mathfrak{q}(t)^{-1}\dot p(t)\big)_{\mathfrak{q}(t)} &+ \lambda\Big(\hat O_{\mathfrak{q}}(t)^\top \hat a_{\mathfrak{q}(t)}\big(\mathfrak{q}(t)^{-1}\dot p(t)\big)\hat O_{\mathfrak{q}}(t)\Big)_{\mathfrak{r}(t)}\\ &= \mathfrak{E}\big(\mathfrak{r}(t)^{-1}\dot p(t)\big)_{\mathfrak{r}(t)} + \lambda\Big(\hat a_{\mathfrak{r}(t)}\big(\mathfrak{r}(t)^{-1}\dot p(t)\big)\Big)_{\mathfrak{r}(t)}.\end{aligned}$$

□

In order to bring out the goemetric content of the preceding lemmas, we think of $\mathbb{R}^m$ as the subspace of $\boldsymbol{\xi} = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$ such that $\xi_i = 0$ for $m < i \le n$, take $\hat\Pi^0$ to be orthogonal projection from $\mathbb{R}^n$ onto $\mathbb{R}^m$, and introduce the space

$$\mathcal{O}^N(M) \equiv \big\{\mathfrak{f} \in \pi^{-1}(M) : \hat\Pi_{\mathfrak{f}} = \hat\Pi^0\big\}.$$

It should be clear that $\mathcal{O}^N(M)$ is submanifold of $\mathcal{O}(N)$. In fact, it is subbundle whose base is M and fiber is

$$\mathcal{O}^{\mathbb{R}^n}(\mathbb{R}^m) \equiv \big\{O \in \mathcal{O}(\mathbb{R}^n) : \hat\Pi^0 \circ O = O \circ \hat\Pi^0\big\}.$$

To see this, let $x \in M$ be given and note that there exists an open neighborhood U of x in N on which there exist vector fields E_i, $1 \le i \le n$, such that $\big((E_1)_y,\dots,(E_n)_y\big)$ is an orthonormal basis in $T_y(N)$ for all $y \in U$ and $(E_i)_y \in T_yM$ for all $y \in U \cap M$ and $1 \le i \le m$. Now set

$$\mathfrak{f}_y = \Big(y, \big((E_1)_y,\dots,(E_n)_y\big)\Big),$$

and observe that

$$\big(y, O\big) \in U \times \mathcal{O}(\mathbb{R}^n) \longmapsto \big(y, R_O \mathfrak{f}_y\big) \in \pi^{-1}(U)$$
$$\big(y, O\big) \in (U \cap M) \times \mathcal{O}^{\mathbb{R}^n}(\mathbb{R}^m) \longmapsto \big(y, R_O \mathfrak{f}_y\big) \in \pi^{-1}(U) \cap \mathcal{O}^N(M)$$

are homeomorphic. Finally, let $o^{\mathbb{R}^n}(\mathbb{R}^m)$ denote the Lie algebra of $a \in o(\mathbb{R}^n)$ such that $\hat{\Pi}^0 a = a\hat{\Pi}^0$, and note that $o^{\mathbb{R}^n}(\mathbb{R}^m)$ is the Lie algebra for the Lie group $\mathcal{O}^{\mathbb{R}^n}(\mathbb{R}^m)$.

Lemma 4.9. *If $\mathfrak{f} \in \mathcal{O}^N(M)$ and $a \in o(\mathbb{R}^n)$, then $\lambda(a)_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}^N(M)$ if and only if $a \in o^{\mathbb{R}^n}(\mathbb{R}^m)$. Moreover, if*

$$\mathfrak{E}^M(\boldsymbol{\xi})_{\mathfrak{f}} = \mathfrak{E}(\boldsymbol{\xi})_{\mathfrak{f}} + \lambda\big(\hat{a}_{\mathfrak{f}}(\boldsymbol{\xi})\big)_{\mathfrak{f}} \quad \textit{for } \mathfrak{f} \in \mathcal{O}^N(M) \textit{ and } \boldsymbol{\xi} \in \mathbb{R}^m,$$

then $\mathfrak{E}^M(\boldsymbol{\xi})_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}^N(M)$. Finally, for $\mathfrak{f} \in \mathcal{O}^N(M)$ and $\mathfrak{X}_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}(N)$, $\mathfrak{X}_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}^N(M)$ if and only if

$$\phi(\mathfrak{X}_{\mathfrak{f}}) \in \mathbb{R}^m \quad \textit{and} \quad \omega^M(\mathfrak{X}_f) \equiv \omega(\mathfrak{X}_{\mathfrak{f}}) - \hat{a}_{\mathfrak{f}}\big(\phi(\mathfrak{X}_{\mathfrak{f}})\big) \in o^{\mathbb{R}^n}(\mathbb{R}^m),$$

in which case $\boldsymbol{\xi} = \phi(\mathfrak{X}_{\mathfrak{f}})$ and $a = \omega^M(\mathfrak{X}_{\mathfrak{f}})$ are the unique elements of $\mathbb{R}^m$ and $o^{\mathbb{R}^n}(\mathbb{R}^m)$, respectively, such that $\mathfrak{X}_{\mathfrak{f}} = \mathfrak{E}^M(\boldsymbol{\xi})_{\mathfrak{f}} + \lambda(a)_{\mathfrak{f}}$.

Proof. Let $\mathfrak{f} \in \mathcal{O}^N(M)$. The fact that $a \in o(\mathbb{R}^n)$ is an element of $o^{\mathbb{R}^n}(\mathbb{R}^m)$ if and only if $\lambda(a)_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}^N(M)$ is just a restatement of the fact that $o^{\mathbb{R}^n}(\mathbb{R}^m)$ is the Lie algebra for $\mathcal{O}^{\mathbb{R}^n}(\mathbb{R}^m)$. Next let $\boldsymbol{\xi} \in \mathbb{R}^m$. To see that $\mathfrak{E}^M(\boldsymbol{\xi})_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}^N(M)$, we need only find a continuously differentiable $\mathfrak{p} : [0,\infty) \longrightarrow \mathcal{O}^N(M)$ such that $\mathfrak{p}(0) = \mathfrak{f}$ and $\dot{\mathfrak{p}}(0) = \mathfrak{E}^M(\boldsymbol{\xi})_{\mathfrak{f}}$. To this end, determine $\mathfrak{p} : [0,\infty) \longrightarrow \pi^{-1}(M)$ by

$$\dot{\mathfrak{p}}(t) = \hat{\mathfrak{E}}(\boldsymbol{\xi})_{\mathfrak{p}(t)} + \lambda\big(\hat{a}_{\mathfrak{p}(t)}(\boldsymbol{\xi})\big)_{\mathfrak{p}(t)} \quad \text{with } \mathfrak{p}(0) = \mathfrak{f}.$$

Clearly $\dot{\mathfrak{p}}(0) = \mathfrak{E}^M(\boldsymbol{\xi})_{\mathfrak{f}}$. In addition, $\omega\big(\dot{\mathfrak{p}}(t)\big) = \hat{a}_{\mathfrak{p}(t)}\big(\phi(\dot{\mathfrak{p}}(t)\big)$ for all $t \geq 0$, and so, by the last part of Lemma 4.8, $\hat{\Pi}_{\mathfrak{p}(t)} = \hat{\Pi}_{\mathfrak{p}(0)} = \hat{\Pi}^0$ for all $t \geq 0$. Thus $\mathfrak{p}(t) \in \mathcal{O}^N(M)$ for all $t \geq 0$.

To complete the proof from here, let $\mathfrak{X}_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}(N)$ be given. Then $\boldsymbol{\xi} = \phi(\mathfrak{X}_{\mathfrak{f}})$ and $b = \omega(\mathfrak{X}_{\mathfrak{f}})$ are the unique elements of $\mathbb{R}^n$ and $o(\mathbb{R}^n)$ such that $\mathfrak{X}_{\mathfrak{f}} = \mathfrak{E}(\boldsymbol{\xi})_{\mathfrak{f}} + \lambda(a)_{\mathfrak{f}}$. Obviously, $\mathfrak{X}_{\mathfrak{f}} \in T_{\mathfrak{f}}\pi^{-1}(M)$ if and only if $\boldsymbol{\xi} \in \mathbb{R}^m$. Hence, if $\mathfrak{X}_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}^N(M)$, and we write

$$\mathfrak{X}_{\mathfrak{f}} = \mathfrak{E}^M(\boldsymbol{\xi})_{\mathfrak{f}} + \lambda(b)_{\mathfrak{f}},$$

then $a = b - \hat{a}_{\mathfrak{f}}(\boldsymbol{\xi}) = \omega^M(\mathfrak{X}_{\mathfrak{f}})$, and, because $\lambda(a)_{\mathfrak{f}} = \mathfrak{X}_{\mathfrak{f}} - \mathfrak{E}^M(\boldsymbol{\xi})_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}^N(M)$, $a \in o^{\mathbb{R}^n}(\mathbb{R}^m)$. Conversely, if $\mathfrak{X}_{\mathfrak{f}} = \mathfrak{E}^M(\boldsymbol{\xi})_{\mathfrak{f}} + \lambda(a)_{\mathfrak{f}}$ for some $\boldsymbol{\xi} \in \mathbb{R}^m$ and $a \in o^{\mathbb{R}^n}(\mathbb{R}^m)$, then $\mathfrak{X}_{\mathfrak{f}} \in T_{\mathfrak{f}}\mathcal{O}^N(M)$, $\boldsymbol{\xi} = \phi(\mathfrak{X}_{\mathfrak{f}})$, and $a = \omega^M(\mathfrak{X}_{\mathfrak{f}})$. □

We are now in a position to achieve both the goals set at the opening of the section. Namely, let $\mathbf{w}(\,\cdot\,)$ be an $\mathbb{R}^n$-valued Wiener process, and choose $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ to be an orthonormal basis in $\mathbb{R}^n$ with $\mathbf{e}_i\in\mathbb{R}^m$ for $1\le i\le m$. Given $\mathfrak{f}\in\mathcal{O}^N(M)$, determine $\mathfrak{p}^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$ by the Stratonovich stochastic differential equation

$$d\mathfrak{p}^M(t,\mathfrak{f},\mathbf{w})=\sum_{i=1}^m \mathfrak{E}^M(\mathbf{e}_i)_{\mathfrak{p}^M(t,\mathfrak{f},\mathbf{w})}\circ d(\mathbf{e}_i,\mathbf{w}(t))\quad \text{with } \mathfrak{p}^M(0,\mathfrak{f},\mathbf{w})=\mathfrak{f}. \tag{4.10}$$

(Notice that, although $\mathbf{w}$ is $\mathbb{R}^n$-valued, only the $\mathbb{R}^m$-component $\hat{\Pi}^0\mathbf{w}$ enters (4.10).) By the first part of Lemma 4.9, so long as it has not exploded, $t\rightsquigarrow\mathfrak{p}^M(t,\mathfrak{f},\mathbf{w})$ takes its values in $\mathcal{O}^N(M)$. Furthermore, if $p^M(\,\cdot\,,\mathfrak{f},\mathbf{w})\equiv \pi\circ\mathfrak{p}^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$, then, by (4.3), $\mathbf{w}\rightsquigarrow p^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$ is a Brownian motion on M starting at $x\equiv\pi\mathfrak{f}$; and the fact that $\mathfrak{p}^M(t,\mathfrak{f},\mathbf{w})\in\mathcal{O}^N(M)$ becomes the statement that

$$\Pi_{p^M(t,\mathfrak{f},\mathbf{w})}\circ\mathfrak{p}^M(t,\mathfrak{f},\mathbf{w})=\mathfrak{p}^M(t,\mathfrak{f},\mathbf{w})\circ\hat{\Pi}_{\mathfrak{f}}.$$

In fact, by (3) in Lemma 4.8, we know that

$$\mathfrak{p}^M(t,\mathfrak{f},\mathbf{w})\mathfrak{f}^{-1}=O_{p^M(\,\cdot\,,\mathfrak{f},\mathbf{w})}(t) \tag{4.11}$$

for piecewise smooth $\mathbf{w}$'s, and so, even when $\mathbf{w}$ is a generic Wiener path, we can use (4.11) to define $O_{p^M(\,\cdot\,,\mathfrak{f},\mathbf{w})}(t)$, in which case it is clear that $O_{p^M(\,\cdot\,,\mathfrak{f},\mathbf{w})}(t)\restriction T_{\pi\mathfrak{f}}M$ gives us a notion of parallel transport along $p^M(\,\cdot\,,\mathfrak{f},\mathbf{w})\restriction[0,t]$ as a path in M.

5. The Relationship between the Constructions in §3 and §4.

It may be useful to point out how $\mathfrak{p}^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$ and $\mathfrak{q}^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$ are related. The idea is that because (cf. Lemma 4.8) $t\rightsquigarrow R_{\hat{O}_{\mathfrak{q}}(t)}$ converts the horizontal lift $\mathfrak{q}$ to $\mathcal{O}(N)$ of a path $p:[0,\infty)\longrightarrow M$ into its horizontal lift to $\mathcal{O}^N(M)$, the same ought to be true for Brownian paths.

Thus, choose an orthonormal basis $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ in $\mathbb{R}^n$ so that $\mathbf{e}_i\in\mathbb{R}^m$ when $1\le i\le m$. Next, given $\mathfrak{f}\in\mathcal{O}^N(M)$, determine $\mathfrak{q}^M(\,\cdot\,,\mathfrak{f},\mathbf{w})$ relative to $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ by (3.1). At the same time, consider the Stratonovich stochastic differential equation

$$d\hat{O}(t,\mathfrak{f},\mathbf{w})=\sum_{i=1}^n \hat{a}_{\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})}(\mathbf{e}_i)\hat{O}(t,\mathfrak{f},\mathbf{w})\circ d\big(\mathbf{e}_i,\mathbf{w}(t)\big)$$
$$\text{with } \hat{O}(0,\mathfrak{f},\mathbf{w})=I.$$

Obviously, $\hat{O}(t, \mathfrak{f}, \mathbf{w})$ plays the role of $\hat{O}_{\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})}$ for generic Wiener paths $\mathbf{w}$. Thus, by Lemma 4.8, we should expect that

$$\mathbf{w} \rightsquigarrow \mathfrak{r}(\,\cdot\,, \mathfrak{f}, \mathbf{w}) \equiv R_{\hat{O}(\,\cdot\,,\mathfrak{f},\mathbf{w})}\mathfrak{q}^M(\,\cdot\,, \mathfrak{f}, \mathbf{w}) \tag{5.1}$$

should have the same distribution as $\mathbf{w} \rightsquigarrow \mathfrak{p}^M(\,\cdot\,, \mathfrak{f}, \mathbf{w})$. (Indeed, at first, one might have guessed that $\mathfrak{r}(\,\cdot\,, \mathfrak{f}, \mathbf{w})$ would be even equal to $\mathfrak{p}^M(\,\cdot\,, \mathfrak{f}, \mathbf{w})$. However, as we are about to see, that guess ignores a required rotation of $\mathbf{w}$.) To test this expectation, first note that, as a consequence of (2) $\iff$ (3) in Lemma 4.8, we know that $\mathfrak{r}(\,\cdot\,, \mathfrak{f}, \mathbf{w})$ is a path in $\mathcal{O}^N(M)$. Second, by Itô's formula for Stratonovich calculus,

$$\begin{aligned} d\mathfrak{r}(t, \mathfrak{f}, \mathbf{w}) = \sum_{i=1}^{n} \Big(&\hat{\mathfrak{E}}\big(\hat{O}(t, \mathfrak{f}, \mathbf{w})^\top \mathbf{e}_i\big)_{\mathfrak{r}(t,\mathfrak{f},\mathbf{w})} \\ &+ \lambda\big(\hat{O}(t, \mathfrak{f}, \mathbf{w})^\top \hat{a}_{\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})}(\mathbf{e}_i)\hat{O}(t, \mathfrak{f}, \mathbf{w})\big)\Big)_{\mathfrak{r}(t,\mathfrak{f},\mathbf{w})} \circ d\big(\mathbf{e}_i, \mathbf{w}(t)\big). \end{aligned}$$

Next observe that

$$\hat{O}(t, \mathfrak{f}, \mathbf{w})^\top \hat{a}_{\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})}(\mathbf{e}_i)\hat{O}(t, \mathfrak{f}, \mathbf{w}) = \hat{a}_{\mathfrak{r}(t,\mathfrak{f},\mathbf{w})}\big(\hat{O}(t, \mathfrak{f}, \mathbf{w})^\top \mathbf{e}_i\big).$$

Further, observe that $\hat{\Pi}_{\mathfrak{r}(t,\mathfrak{f},\mathbf{w})} = \hat{\Pi}^0$ is equivalent to

$$\hat{\Pi}_{\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})}\hat{O}(t, \mathfrak{f}, \mathbf{w}) = \hat{O}(t, \mathfrak{f}, \mathbf{w})\hat{\Pi}^0. \tag{5.2}$$

Hence, we have now shown that

$$\begin{aligned} d\mathfrak{r}(t, \mathfrak{f}, \mathbf{w}) &= \sum_{i=1}^{m} \mathfrak{E}^M\left(\hat{O}(t, \mathfrak{f}, \mathbf{w})^\top \mathbf{e}_i\right)_{\mathfrak{r}(t,\mathfrak{f},\mathbf{w})} \circ d\big(\mathbf{e}_i, \mathbf{w}(t)\big) \\ &= \sum_{i,j=1}^{m} \mathfrak{E}^M(\mathbf{e}_j)_{\mathfrak{r}(t,\mathfrak{f},\mathbf{w})} \circ d\big(\mathbf{e}_j, \mathbf{W}(t)\big), \end{aligned}$$

where

$$\mathbf{W}(t) \equiv \sum_{i=1}^{n} \int_0^t \left(\hat{O}(\tau, \mathfrak{f}, \mathbf{w})^\top \mathbf{e}_i, \mathbf{e}_j\right) \circ d\big(\mathbf{e}_i, \mathbf{w}(\tau)\big).$$

Hence, if we can check that $\hat{\Pi}^0\mathbf{W} = \hat{\Pi}^0\tilde{\mathbf{w}}$ where $\tilde{\mathbf{w}}$ is an $\mathbb{R}^n$-valued Wiener process, we will know that

$$R_{O(\,\cdot\,,\mathfrak{f},\mathbf{w})}\mathfrak{q}^M(\,\cdot\,, \mathfrak{f}, \mathbf{w}) = \mathfrak{p}^M\big(\,\cdot\,, \mathfrak{f}, \tilde{\mathbf{w}}\big), \tag{5.3}$$

and therefore that $\mathbf{w} \rightsquigarrow R_{O(\cdot,\mathfrak{f},\mathbf{w})}\mathfrak{q}^M(\cdot,\mathfrak{f},\mathbf{w})$ does indeed have the same distribution as $\mathbf{w} \rightsquigarrow \mathfrak{p}^M(\cdot,\mathfrak{f},\mathbf{w})$.

To see that $\hat{\Pi}^0\mathbf{W}$ is an $\mathbb{R}^m$-valued Wiener path, note that

$$\begin{aligned} d\big(\hat{\Pi}^0\mathbf{W}\big) =& \hat{\Pi}^0\hat{O}(t,\mathfrak{f},\mathbf{w})^\top d\mathbf{w}(t) \\ & - \frac{1}{2}\left(\sum_{i=1}^n \hat{\Pi}^0\hat{O}(t,\mathfrak{f},\mathbf{w})^\top \hat{a}_{\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})}(\mathbf{e}_i)\mathbf{e}_i\right) dt, \end{aligned}$$

where the first term on the left is an Itô differential. Second, by (5.2),

$$\begin{aligned} & -\sum_{i=1}^n \hat{\Pi}^0\hat{O}(t,\mathfrak{f},\mathbf{w})^\top \hat{a}_{\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})}(\mathbf{e}_i)\mathbf{e}_i \\ & = \sum_{i=1}^n \hat{O}(t,\mathfrak{f},\mathbf{w})^\top \hat{a}_{\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})}(\mathbf{e}_i)\hat{\Pi}^\perp_{\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})}\mathbf{e}_i. \end{aligned}$$

Finally, because, for each t and $\mathbf{w}$, the expressions in the preceding are independent of the choice of the orthonormal basis $(\mathbf{e}_1,\dots,\mathbf{e}_n)$, by choosing the basis so that each element is either orthogonal to or in $\mathfrak{q}^M(t,\mathfrak{f},\mathbf{w})^{-1}T_{q^M(t,\mathfrak{f},\mathbf{w})}M$, we see that the right hand side must vanishes identically. In other words,

$$\hat{\Pi}^0\mathbf{W} = \hat{\Pi}^0\tilde{\mathbf{w}} \quad \text{when } \tilde{\mathbf{w}}(t) \equiv \int_0^t \hat{O}(\tau,\mathfrak{f},\mathbf{w})\,d\mathbf{w}(\tau). \tag{5.4}$$

Because $\hat{O}(\cdot,\mathfrak{f},\mathbf{w})$ takes its values in $\mathcal{O}(\mathbb{R}^n)$ and the preceding integral is taken in the sense of Itô, $\tilde{\mathbf{w}}$ is an $\mathbb{R}^n$-valued Wiener process.

Remark: It is amusing to recognize that the difference between $\tilde{\mathbf{w}}(t)$ and $\mathbf{W}(t)$ is precisely

$$\begin{aligned} & \frac{1}{2}\int_0^t \hat{O}(\tau,\mathfrak{f},\mathbf{w})^\top \mathfrak{q}^M(\tau,\mathfrak{f},\mathbf{w})^{-1}B_{q^M(\tau,\mathfrak{f},\mathbf{w})}\,d\tau \\ & \qquad = \frac{1}{2}\int_0^t \mathfrak{r}(\tau,\mathfrak{f},\mathbf{w})^{-1}B_{q^M(\tau,\mathfrak{f},\mathbf{w})}\,d\tau, \end{aligned}$$

where (cf. (1.11)) B is the mean curvature vector.

6. A Technical Addendum about Cartan's Structural Equations and Gauss's Formula.

Use $\mathcal{O}(M)$ to denote the bundle of orthonormal frames over M, and define the solder form ϕ and connection 1-form ω accordingly (cf. page 181 of

[3]). Next, let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be an orthonormal basis in $\mathbb{R}^n$ with $\mathbf{e}_i \in \mathbb{R}^m$ for $1 \le i \le m$. It should be apparent that the map $F : \mathcal{O}^N(M) \longrightarrow \mathcal{O}(M)$ which takes $\mathfrak{f} \in \mathcal{O}^N(M)$ into $\big(\pi\mathfrak{f}, (\mathfrak{f}\mathbf{e}_1, \dots, \mathfrak{f}\mathbf{e}_m)\big) \in \mathcal{O}(M)$ is a smooth surjection which preserves the bundle structure. In addition, one sees that $\phi^M \equiv \phi \restriction \mathcal{O}^N(M)$ and $\omega^M \equiv \hat{\Pi}^0 \circ \omega \circ \hat{\Pi}$ are, respectively, the pullbacks under F of the solder form ϕ and connection 1-form ω on $\mathcal{O}(M)$. Similarly, for each $\boldsymbol{\xi} \in \mathbb{R}^m$, $F_*\mathfrak{E}^M(\boldsymbol{\xi})_\mathfrak{f}$ is the horizontal lift to $F(\mathfrak{f}) \in \mathcal{O}(M)$ of $F(\mathfrak{f})\boldsymbol{\xi} \in T_{\pi\mathfrak{f}}M$. In other words, $F_*\mathfrak{E}^M(\boldsymbol{\xi})$ is not only well-defined, it is the canonical horizontal vector field on $\mathcal{O}(M)$ corresponding to $(\xi_1, \dots, \xi_m)$. In particular, $-\omega\big(F_*\big[\mathfrak{E}^M(\boldsymbol{\xi}), \mathfrak{E}^M(\boldsymbol{\eta})\big]\big)$ is the curvature 2-form (cf. (8.44) in [3]) on $\mathcal{O}(M)$ at $F(\mathfrak{f})$.

All the above considerations should make one suspect that ϕ^M and ω^M might satisfy the Cartan Structural equations (cf. page 194 in [3]), and that the computation of $\big[\mathfrak{E}^M(\boldsymbol{\xi}), \mathfrak{E}^M(\boldsymbol{\eta})\big]$ ought to lead to an interesting form of Gauss's formula (cf. (3.27) in [2]). The key to verifying these suspicions is contained in the following lemma.

Lemma 6.1. *There is a map*

$$\mathfrak{f} \in \mathcal{O}^N(M) \longmapsto \Omega^M_\mathfrak{f} \in \mathrm{Hom}\big(\mathbb{R}^m \times \mathbb{R}^m; o^{\mathbb{R}^n}(\mathbb{R}^m)\big)$$

such that, for each $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^m \times \mathbb{R}^m$

$$\big[\mathfrak{E}^M(\boldsymbol{\xi}), \mathfrak{E}^M(\boldsymbol{\eta})\big]_\mathfrak{f} = -\lambda\big(\Omega^M_\mathfrak{f}(\boldsymbol{\xi}, \boldsymbol{\eta})\big)_\mathfrak{f}, \quad \mathfrak{f} \in \mathcal{O}^N(M). \tag{6.2}$$

Moreover, for each $\boldsymbol{\xi} \in \mathbb{R}^m$ *and* $a \in o(\mathbb{R}^n)$,

$$\lambda(a)_\mathfrak{f}\hat{a}(\boldsymbol{\xi}) = \big[\hat{a}_\mathfrak{f}(\boldsymbol{\xi}), a\big] + \hat{a}_\mathfrak{f}(a\boldsymbol{\xi}), \quad \mathfrak{f} \in \mathcal{O}^N(M) \tag{6.3}$$

Finally, if $\boldsymbol{\xi} \in \mathbb{R}^m$ *and* $a \in o^{\mathbb{R}^n}(\mathbb{R}^m)$, *then*

$$\big[\lambda(a), \mathfrak{E}^M(\boldsymbol{\xi})\big]_\mathfrak{f} = \mathfrak{E}^M(a\boldsymbol{\xi})_\mathfrak{f}. \tag{6.4}$$

Proof. Proving the existence of Ω^M with the required properties is equivalent to checking that $\big[\mathfrak{E}^M(\boldsymbol{\xi}), \mathfrak{E}^M(\boldsymbol{\eta})\big]_\mathfrak{f}$ is vertical. But, by (4.2), we know that

$$\mathfrak{E}^M(\boldsymbol{\xi})_\mathfrak{f} \circ \mathfrak{E}^M(\boldsymbol{\eta})(\varphi \circ \pi) = \langle \mathfrak{f}\boldsymbol{\xi}, \mathrm{hess}^M_{\pi\mathfrak{f}}\varphi\mathfrak{f}\boldsymbol{\eta}\rangle,$$

which, because the connection on M is Levi-Civita's, means that $\pi_*\big[\mathfrak{E}^M(\boldsymbol{\xi}), \mathfrak{E}^M(\boldsymbol{\eta})\big]_\mathfrak{f} = 0$.

To check (6.3), simply note that

$$\lambda(a)_{\mathfrak{f}}\hat{a}(\boldsymbol{\xi}) = \frac{d}{ds}e^{-sa}\hat{a}_{\mathfrak{f}}\big(e^{sa}\boldsymbol{\xi}\big)_{\mathfrak{f}}e^{sa}\Big|_{s=0} = \big[\hat{a}_{\mathfrak{f}}(\boldsymbol{\xi}), a\big] + \hat{a}_{\mathfrak{f}}(a\boldsymbol{\xi}).$$

To prove (6.4), let $a \in o^{\mathbb{R}^n}(\mathbb{R}^m)$ be given, and use (8.5) and (8.15) in [3] together with (6.2) above to justify

$$\begin{aligned}&\big[\lambda(a), \mathfrak{E}^M(\boldsymbol{\xi})\big]_{\mathfrak{f}} = \mathfrak{E}(a\boldsymbol{\xi})_{\mathfrak{f}} + \big[\lambda(a), \lambda\big(\hat{a}(\boldsymbol{\xi})\big)\big]_{\mathfrak{f}}\\ &= \mathfrak{E}^M(a\boldsymbol{\xi})_{\mathfrak{f}} + \lambda\Big(\big[a, \hat{a}_{\mathfrak{f}}(\boldsymbol{\xi})\big] + \lambda(a)\hat{a}(\boldsymbol{\xi}) - \hat{a}_{\mathfrak{f}}(a\boldsymbol{\xi})\Big)_{\mathfrak{f}} = \mathfrak{E}^M(a\boldsymbol{\xi})_{\mathfrak{f}}.\end{aligned}$$

□

To see that Cartan's structural equations hold for ϕ^M and ω^M, one can now use exactly the same procedure as was used on page 194 of [3]. That is, one calculates $d\phi^M(\mathfrak{X}, \mathfrak{Y})$ and $d\omega^M(\mathfrak{X}, \mathfrak{Y})$ at $\mathfrak{f} \in \mathcal{O}^N(M)$ by considering what happens when $\mathfrak{X}$ and $\mathfrak{Y}$ are either $\mathfrak{E}^M(\boldsymbol{\xi})$, for some $\boldsymbol{\xi} \in \mathbb{R}^m$. or $\lambda(a)$, for some $a \in o^{\mathbb{R}^n}(\mathbb{R}^m)$. By using (8.5) in [3] together with (6.2) and (6.4) above, these computations lead immediately to the Cartan structural equations:

$$d\phi^M = -\omega^M \wedge \phi^M \quad \text{and} \quad d\omega^M = \omega^M \wedge \omega^M + \Omega^M \circ \phi^M, \tag{6.5}$$

where $\Omega^M \circ \phi^M(\mathfrak{X}, \mathfrak{Y}) \equiv \Omega^M\big(\phi^M(\mathfrak{X}), \phi^M(\mathfrak{Y})\big)$.

Finally, we want to find an expression for Ω^M in terms of the curvature 2-form Ω for $\mathcal{O}(N)$. To this end, observe that

$$\begin{aligned}\big[\mathfrak{E}^M(\boldsymbol{\xi}), \mathfrak{E}^M(\boldsymbol{\eta})\big] =& \big[\mathfrak{E}(\boldsymbol{\xi}), \mathfrak{E}(\boldsymbol{\eta})\big] + \big[\mathfrak{E}(\boldsymbol{\xi}), \lambda\big(\hat{a}(\boldsymbol{\eta})\big)\big]\\ &+ \big[\lambda\big(\hat{a}(\boldsymbol{\xi})\big), \mathfrak{E}(\boldsymbol{\eta})\big] + \big[\lambda\big(\hat{a}(\boldsymbol{\xi})\big), \lambda\big(\hat{a}(\boldsymbol{\eta})\big)\big].\end{aligned} \tag{*}$$

By definition,

$$\big[\mathfrak{E}(\boldsymbol{\xi}), \mathfrak{E}(\boldsymbol{\eta})\big] = -\lambda\big(\Omega(\boldsymbol{\xi}, \boldsymbol{\eta})\big). \tag{a}$$

By (8.5) in [3],

$$\begin{aligned}&\big[\mathfrak{E}(\boldsymbol{\xi}), \lambda\big(\hat{a}(\boldsymbol{\eta})\big)\big] + \big[\lambda\big(\hat{a}(\boldsymbol{\xi})\big), \mathfrak{E}(\boldsymbol{\eta})\big]\\ &= \mathfrak{E}\big(\hat{a}(\boldsymbol{\eta})\boldsymbol{\xi} - \hat{a}(\boldsymbol{\xi})\boldsymbol{\eta}\big) + \lambda\big(\mathfrak{E}(\boldsymbol{\xi})\hat{a}(\boldsymbol{\eta}) - \mathfrak{E}(\boldsymbol{\eta})\boldsymbol{\xi}\big).\end{aligned}$$

Notice that, because $\mathfrak{f}\boldsymbol{\xi}, \mathfrak{f}\boldsymbol{\eta} \in T_{\pi\mathfrak{f}}M$, (1.6) implies that

$$\hat{a}_{\mathfrak{f}}(\boldsymbol{\xi})\boldsymbol{\eta} - \hat{a}_{\mathfrak{f}}(\boldsymbol{\eta})\boldsymbol{\xi} = \mathfrak{f}^{-1}\big(\mathcal{S}(\mathfrak{f}\boldsymbol{\xi})\mathfrak{f}\boldsymbol{\eta} - \mathcal{S}(\mathfrak{f}\boldsymbol{\eta})\mathfrak{f}\boldsymbol{\xi}\big) = 0.$$

At the same time, by (2.2),

$$\mathfrak{E}(\boldsymbol{\xi})\hat{a}(\boldsymbol{\eta}) - \mathfrak{E}(\boldsymbol{\eta})\hat{a}(\boldsymbol{\xi}) = \Big(\big[\mathfrak{E}(\boldsymbol{\xi}), \mathfrak{E}(\boldsymbol{\eta})\big]\hat{\Pi}\Big) \circ \big(\hat{\Pi} - \hat{\Pi}^{\perp}\big) + 2\big[\hat{\mathcal{S}}(\boldsymbol{\eta}), \hat{\mathcal{S}}(\boldsymbol{\xi})\big]$$
$$= -\big[\hat{\Pi}, \Omega(\boldsymbol{\xi}, \boldsymbol{\eta})\big] \circ \big(\hat{\Pi} - \hat{\Pi}^{\perp}\big) - 2\big[\hat{\mathcal{S}}(\boldsymbol{\xi}), \hat{\mathcal{S}}(\boldsymbol{\eta})\big].$$

Thus,

$$\big[\mathfrak{E}(\boldsymbol{\xi}), \lambda\big(\hat{a}(\boldsymbol{\eta})\big)\big] + \big[\lambda\big(\hat{a}(\boldsymbol{\xi})\big), \mathfrak{E}(\boldsymbol{\eta})\big]$$
$$= -\lambda\Big(\big[\hat{\Pi}, \Omega(\boldsymbol{\xi}, \boldsymbol{\eta}) \circ (\hat{\Pi} - \hat{\Pi}^{\perp})\big] + 2\big[\hat{\mathcal{S}}(\boldsymbol{\xi}), \hat{\mathcal{S}}(\boldsymbol{\eta})\big]\Big). \tag{b}$$

Finally, by (6.3),

$$\big[\lambda\big(\hat{a}(\boldsymbol{\xi})\big), \lambda(\boldsymbol{\eta})\big] = \lambda\Big(\big[\hat{a}(\boldsymbol{\xi}), \hat{a}(\boldsymbol{\eta})\big] + \lambda\big(\hat{a}(\boldsymbol{\xi})\big)\hat{a}(\boldsymbol{\eta}) - \lambda\big(\hat{a}(\boldsymbol{\eta})\big)\hat{a}(\boldsymbol{\xi})\Big)$$
$$= -\lambda\big(\big[\hat{a}(\boldsymbol{\xi}), \hat{a}(\boldsymbol{\eta})\big]\big) = \lambda\big(\big[\hat{\mathcal{S}}(\boldsymbol{\xi}), \hat{\mathcal{S}}(\boldsymbol{\eta})\big]\big).$$

Hence, when we put this together with (a) and (b) and plug them into (*), we conclude that

$$\big[\mathfrak{E}^M(\boldsymbol{\xi}), \mathfrak{E}^M(\boldsymbol{\eta})\big]$$
$$= -\lambda\Big(\Omega(\boldsymbol{\xi}, \boldsymbol{\eta}) + \big[\hat{\Pi}, \Omega(\boldsymbol{\xi}, \boldsymbol{\eta})\big] \circ (\hat{\Pi} - \hat{\Pi}^{\perp}) + \big[\hat{\mathcal{S}}(\boldsymbol{\xi}), \hat{\mathcal{S}}(\boldsymbol{\eta})\big]\Big)$$
$$= -\lambda\Big(\hat{\Pi} \circ \Omega(\boldsymbol{\xi}, \boldsymbol{\eta}) \circ \hat{\Pi} + \hat{\Pi}^{\perp} \circ \Omega(\boldsymbol{\xi}, \boldsymbol{\eta}) \circ \hat{\Pi}^{\perp} + \big[\hat{\mathcal{S}}(\boldsymbol{\xi}), \hat{\mathcal{S}}(\boldsymbol{\eta})\big]\Big).$$

This not only give another proof that $\big[\mathfrak{E}^M(\boldsymbol{\xi}), \mathfrak{E}^M(\boldsymbol{\eta})\big]$ is vertical, it shows that

$$\Omega^M(\boldsymbol{\xi}, \boldsymbol{\eta}) = \hat{\Pi} \circ \Omega(\boldsymbol{\xi}, \boldsymbol{\eta}) \circ \hat{\Pi} + \hat{\Pi}^{\perp} \circ \Omega(\boldsymbol{\xi}, \boldsymbol{\eta}) \circ \hat{\Pi}^{\perp} + \big[\hat{\mathcal{S}}(\boldsymbol{\xi}), \hat{\mathcal{S}}(\boldsymbol{\eta})\big]. \tag{6.6}$$

To see that (6.6) gives the Gauss formula relating the Riemann curvature R^M on M to the Riemann curvature R^N on N, recall (cf. (8.54) in [3]) that the Cartan structural equations lead to

$$R^N(X_{\pi\mathfrak{f}}, Y_{\pi\mathfrak{f}}) = \mathfrak{f} \circ \Omega\big(\mathfrak{f}^{-1}X_{\pi\mathfrak{f}}, \mathfrak{f}^{-1}Y_{\pi\mathfrak{f}}\big) \circ \mathfrak{f}^{-1}$$

for all $\mathfrak{f} \in \mathcal{O}(N)$ and $X_{\pi\mathfrak{f}}, Y_{\pi\mathfrak{f}} \in T_{\pi\mathfrak{f}}N$. In the same way, (6.5) shows that

$$R^M(X_{\pi\mathfrak{f}}, Y_{\pi\mathfrak{f}}) = \mathfrak{f} \circ \hat{\Pi}_{\mathfrak{f}} \circ \Omega^M\big(\mathfrak{f}^{-1}X_{\pi\mathfrak{f}}, \mathfrak{f}^{-1}Y_{\pi\mathfrak{f}}\big) \circ \hat{\Pi}_{\mathfrak{f}} \circ \mathfrak{f}^{-1}$$

for all $\mathfrak{f} \in \mathcal{O}^N(M)$ and $X_{\pi\mathfrak{f}}, Y_{\pi\mathfrak{f}} \in T_{\pi\mathfrak{f}}M$. Hence, (6.6) leads to Gauss's formula

$$R^M\big(X_x, Y_x\big) = \Pi_x \circ \Big(R^N\big(X_x, Y_x\big) + \big[\mathcal{S}_x(X_x), \mathcal{S}(Y_x)\big]\Big) \restriction T_xM \tag{6.7}$$

for all $x \in M$ and $X_x, Y_x \in T_xM$.

References.

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