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Geometric Estimates for the Logarithmic Fast Diffusion Equation

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1. Introduction.

We consider solutions u of the logarithmic fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta \log u \tag{1.1}$$

on the plane $\mathbb{R}^2$, with initial data $f \geq 0$ of finite mass. Δ denotes the Euclidean Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

with respect to the standard metric $ds^2 = dx^2 + dy^2$. It has been observed by S. Angenent and L. Wu [17], [18] that equation (1.1) represents the evolution of the conformally equivalent metric g with

$$ds^2 = u\,(dx^2 + dy^2)$$

under the *Ricci Flow*, which evolves a metric $ds^2 = g_{ij}\,dx^i dx^j$ by its Ricci curvature R_{ij} with

$$\frac{\partial g_{ij}}{\partial t} = -2\,R_{ij}. \tag{1.2}$$

The equivalence follows easily from the observation that the conformal metric $g_{ij} = u\,I_{ij}$ has scalar curvature $R = -(\Delta \log u)/u$ and in two dimensions $R_{ij} = \frac{1}{2}\,R\,g_{ij}$. We use this equivalence to deduce geometric estimates on the solution u near its vanishing time T.

Equation (1.1) arises also in physical applications, as a model for long Van-der-Wals interactions in thin films of a fluid spreading on a solid surface, if certain nonlinear fourth order effects are neglected [6], [2], [3]. In that

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framework $u(x,t)$ denotes the height of the liquid film at the point x and the instant t.

Equation (1.1) can be also understood as the formal limit, as $m \to 0$, of the fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta \left(\frac{u^m}{m} \right) \tag{1.3}$$

as shown in [12] and [16]. Notice that the exponent $m = 0$ in dimension $n = 2$ is critical for equation (1.3), since $m = (n-2)/n$ defines the critical exponent for (1.3) in the sense of [11].

We will consider solutions with finite total mass

$$A = \int_{\mathbb{R}^2} u \, dxdy < \infty.$$

Geometrically A is the area of the plane in the conformal metric g. Since u goes to zero when (x,y) tends to infinity, the equation is not uniformly parabolic. The equation becomes singular, when u tends to zero. This results in many interesting phenomena, in particular solutions are not unique [7]. It is shown in [7] that given an initial data $f \geq 0$ with finite mass and a constant $\lambda \geq 0$, there exists a solution u_λ of equation (1.1) with initial data $u_\lambda(\cdot, 0) = f$, satisfying

$$\frac{d}{dt} \int_{\mathbb{R}^2} u_\lambda(x,t) \, dx = -2\pi(2+\lambda).$$

The solution u_λ exists up to the exact time T_λ, which is determined in terms of the initial mass and λ by

$$T_\lambda = \frac{1}{2\pi(2+\lambda)} \int_{\mathbb{R}^2} f(x) \, dx.$$

We restrict our attention to the *maximal solutions* u of (1.1), which vanish at the exact time

$$T = \frac{1}{4\pi} \int_{\mathbb{R}^2} f(x) \, dx.$$

Geometrically this corresponds to the condition that the conformal metric is complete.

Our results consist of upper and lower bounds on the geometric *width* w of the solution and on the *maximum curvature* $\mathbb{R}$. Precise pointwise bounds on the solution u near its vanishing time were derived in the rotationally symmetric case by King [13].

Before we state our main results, let us define the width w of a solution u. Let $F : \mathbb{R}^2 \to [0, \infty)$ denote a proper function on the plane, i.e., a closed function F such that $F^{-1}(a)$ is compact for every $a \in [0, \infty)$. We define the width of F to be the supremum of the lengths of the level curves of F

$$w(F) = \sup_c L\,\{F = c\}.$$

Then, we define the width w of a metric on the plane to be the infimum

$$w = \inf_F \; w(F)$$

over all smooth proper functions F. Note that for a solution u the length of a curve Γ in the conformal metric is just

$$L(\Gamma) = \int_\Gamma \sqrt{u}\, d\sigma$$

where $d\sigma$ is the standard Euclidean length in the plane. Thus the width w of the metric $ds^2 = u\,(dx^2 + dy^2)$ on the plane is given by

$$w = \inf_F \; \sup_c \int_\Gamma \sqrt{u}\, d\sigma.$$

As we noted already, the maximal solution will exist only up to some time $T < \infty$ when the area A goes to zero. We recall [7] that for the maximal solution

$$\frac{dA}{dt} = -\int R\, da = -4\pi$$

so that $T = A_0/4\pi$, where A_0 is the initial area, and at each time

$$A = 4\pi\,(T - t).$$

Our estimates will only depend on the time to collapse $T - t$. However, they do not scale in the usual way. We will first show that the width w is proportional to $T - t$.

Theorem 1.1. *There exist constants $\gamma > 0$ and $C < \infty$ for which*

$$\gamma\,(T - t) \le w \le C\,(T - t) \tag{1.4}$$

on $0 < t \le T$.

Let us note that in the case where the solution $u = u(r,t)$ is radially symmetric the width w of the metric is given by $2\pi r\sqrt{u}$, and hence (1.4) implies the pointwise bound

$$\gamma\,(T-t) \le \max_{r\ge 0} r\,\sqrt{u}(r,t) \le C\,(T-t)$$

on the solution u.

We will also show that the maximum curvature $\mathbb{R}$ is proportional to $1/(T-t)^2$. This implies that the blow up of the curvature is of type II.

Theorem 1.2. *There exist constants $\gamma > 0$ and $C < \infty$ with*

$$\frac{\gamma}{(T-t)^2} \le R_{\max} \le \frac{C}{(T-t)^2} \tag{1.5}$$

on $0 < t \le T$.

Lets us note that since $R = -(\Delta \log u)/u$ the bound on $\mathbb{R}$ also provides information about the pointwise behavior of u near its vanishing time T.

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2. The Upper Bound on the width w.

In this section we will prove the upper bound

$$w \le C\,(T-t) \tag{2.1}$$

on the width w. The proof involves the construction of a Busemann function F with

$$w(F) \le C\,A \tag{2.2}$$

where A denotes the area of the plane under a conformal metric g. This is just a statement about the geometry of metrics on the plane with finite area and scalar curvature bounded below, and uses no other fact about the flow.

Let g be a metric on the plane with finite area and sectional curvature $K \ge -1$. Schoen and Yau [15] prove that for any complete metric g on a manifold M of dimension n with Ricci curvature bounded below by

$$\mathrm{Ric} \ge -(n-1)\,k^2$$

for a constant k, if $s(Q) = d(P,Q)$ is the geodesic distance to a fixed point P, then at any point where s is smooth

$$\Delta_g s \leq (n-1)\,[\,k + \frac{1}{s}\,].$$

In our case, where $n = 2$, $k = 1$, we obtain the inequality

$$\Delta_g s \leq 1 + \frac{1}{s}. \tag{2.3}$$

However (2.3) does not make sense at points in the *cut locus* set $\mathrm{Cut}(P)$ of P where s is not smooth. Nevertheless, one can follow the argument in [15] (Chapter I, Proposition 1.1) to show that (2.3) holds globally in the distributional sense. For the convenience of the reader, we will next present an outline of this argument.

For any point P on our surface Σ (equivalent to $\mathbb{R}^2$ equipped with the metric g) we define the map $s(Q) = d(P,Q)$ for all $Q \in \Sigma$, where $d(\cdot,\cdot)$ denotes the geodesic distance with respect to the metric g. Then $s(Q)$ is a Lipschitz continuous function on Σ, and hence differentiable almost everywhere.

Consider the exponential map $\,exp_P\, : T_P\Sigma \to \Sigma$. For a vector $X \in T_P\Sigma$ with $\|X\| =< X, X >^{\frac{1}{2}} = 1$, let $\gamma(t)$ be the unique geodesic starting from P along the direction X (i.e., $\gamma(0) = P$, $\gamma'(0) = X$). Then we have $exp_P(tX) = \gamma(t)$, for $t > 0$. When t is small, γ is the unique minimal geodesic joining P and $exp_P(tX)$, also $d\,exp_P|_{tX}\, : T_{tX}(T_P\Sigma) \to T_{\gamma(t)}(\Sigma)$ is an isomorphism. However, as t increases these properties may be violated. Let

$$t_0 = \sup\{\, t > 0 : \gamma \text{ is the unique minimal geodesic joining } P \text{ and } \gamma(t)\,\}.$$

If $t_0 < \infty$, then $\gamma(t_0)$ is called a *cut point* of P. The set of all cut points of P is called the *cut locus* and denoted by $\mathrm{Cut}(P)$. If we denote by $S_P = \{\, X \in T_P(\Sigma) : \|X\| = 1\,\}$, it is clear that for any $X \in S_P$ there can be at most one cut point on the geodesic $\,exp_P(tX)$, $t > 0$. If $exp_P(t_0\,X) = Q$ is a cut point of P then we set $\zeta(X) = d(P,Q)$; if there is no cut point we set $\zeta(X) = \infty$. Since Σ is 2-dimensional, we can identify $X \in S_P$ with the angle $\theta \in [0, 2\pi)$ of X from a fixed direction. Hence ζ can be identified as a 2π-periodic function $\bar{\zeta}(\theta)$, $\theta \in \mathbb{R}$. Define

$$E_P = \{\, t\,X : 0 \leq t < \zeta(X),\, X \in S_P\,\} = \{\, (\rho,\theta) : \rho < \bar{\zeta}(\theta)\,\}.$$

Then it can be shown that $\,exp_P\, : E_P \to exp_P(E_P)$ is a diffeomorphism. Clearly, $\mathrm{Cut}(P) = \partial exp_P(E_P)$. Also, $\mathrm{Cut}(P)$ has 2-dimensional measure zero. In addition, it can be shown that the function $\zeta(X) = \bar{\zeta}(\theta)$ is smooth and corresponds to points of the cut locus with exactly two distinct minimizing geodesics, except of a set Λ' (corresponding to the *singular part* of the cut locus Λ) which has Hausdorff measure $H^1(exp_P(\Lambda')) = H^1(\Lambda) = 0$.

We are now ready to show that (2.3) holds in the sense of distributions. Let

$$\Omega = exp_P(E_P). \tag{2.4}$$

Then $\Sigma = \mathrm{Cut}(P) \cup \Omega$ and the exponential map $exp_P : E_P \to \Sigma \setminus Cut(P)$ provides a maximal normal coordinate chart at P. Let $\phi \in C_0^\infty(\Sigma)$, with $\phi \geq 0$. Since $Cut(P)$ has measure zero we have

$$\int_\Sigma s\,\Delta\phi = \int_\Omega s\,\Delta\phi.$$

Set $C = \mathrm{Cut}(P)\backslash\Lambda$ (the regular part of the cut locus). The set C corresponds to an open set C' in θ where $\bar{\zeta}(\theta)$ is smooth.

Approximate the function $\bar{\zeta}(\theta)$ by a sequence of smooth functions $\bar{\zeta}_\varepsilon(\theta)$, such that $\bar{\zeta}_\varepsilon \uparrow \bar{\zeta}$. Since $\bar{\zeta}$ is smooth on C', we have that $\bar{\zeta}_\varepsilon \to \bar{\zeta}$ in C^∞ on compacts of C'. Setting

$$\Omega_\varepsilon = exp_P(E_P^\varepsilon); \quad E_P^\varepsilon = \{\, (\rho,\theta) :\ \rho < \bar{\zeta}_\varepsilon(\theta)\,\}$$

it is clear that

$$\int_{\Omega_\varepsilon} s\,\Delta_g\phi \to \int_\Omega s\,\Delta_g\phi, \qquad \text{as } \varepsilon \to 0.$$

On the other hand, since s is smooth on Ω_ε, we may apply Green's formula to get

$$\int_{\Omega_\varepsilon} s\,\Delta_g\phi = \int_{\Omega_\varepsilon} \phi\,\Delta_g s + \int_{\partial\Omega_\varepsilon} s\,\frac{\partial\phi}{\partial\nu_\varepsilon} - \int_{\partial\Omega_\varepsilon} \phi\,\frac{\partial s}{\partial\nu_\varepsilon}. \tag{2.5}$$

The last term in the above identity is nonnegative since the star-shapeness of E_P^ε implies that $\partial s/\partial\nu_\varepsilon > 0$ on $\partial\Omega_\varepsilon$. To control the other boundary term, we write $\partial\Omega_\varepsilon = C_\varepsilon \cup \Lambda_\varepsilon$, with C_ε, Λ_ε approximating the regular part C and singular part Λ of the cut locus respectively, and $H^1(\Lambda_\varepsilon) \to 0$ as $\varepsilon \to 0$ (since $H^1(\Lambda) = 0$). Then

$$\int_{\Lambda_\varepsilon} s\,\frac{\partial\phi}{\partial\nu_\varepsilon} \to 0, \qquad \text{as } \varepsilon \to 0$$

since $H^1(\Lambda_\varepsilon) \to 0$, and

$$\int_{C_\varepsilon} s\,\frac{\partial\phi}{\partial\nu_\varepsilon} \to \int_C s\,\frac{\partial\phi}{\partial\nu_\varepsilon} = 0, \qquad \text{as } \varepsilon \to 0$$

because of cancellation in pairs (since points in C have exactly two distinct minimizing geodesics). Hence, by taking the limit $\varepsilon \to 0$ in (2.5) and using the a.e pointwise bound (2.3) we conclude that

$$\int_\Sigma s\,\Delta_g \phi = \int_\Omega s\,\Delta_g \phi = \int_{\Omega_\varepsilon} \Delta_g s\,\phi \leq \int_\Omega (1+\frac{1}{s})\,\phi$$

for all $\phi \in C_0^\infty(\Sigma)$, $\phi \geq 0$, showing that (2.3) holds globally on Σ in the distributional sense.

We will now follow a similar approximation as above to show that if $s(Q) = d(P,Q)$ is the distance to a fixed point P, then for any constant $c > 0$

$$\int_{s\leq c} \Delta_g s = L(\{\, s = c\}) \tag{2.6}$$

with $L(\{\, s = c\})$ denoting the geodesic length of the set $\{\, s = c\}$ in the metric g. Let Ω be defined by (2.4) and set $\Omega(c) = \Omega \cap \{s \leq c\}$. Then

$$\Omega(c) = exp_P(\{\,(\rho,\theta);\ \rho < \eta(\theta)\,\})$$

with $\bar\eta(\theta) = \min(\bar\zeta(\theta), c)$, $\bar\zeta$ defined as above. In addition

$$\int_{s\leq c} \Delta_g s = \int_{\Omega(c)} \Delta_g s.$$

Consider next the sets:

$A = \{$ the points where $s = c$ in $\Sigma \setminus \mathrm{Cut}(P)\,\}$
$B = \{$ the points where $s = c$ on the regular part of $\mathrm{Cut}(P)\,\}$
$C = \{$ the points where $s < c$ on the regular part of $\mathrm{Cut}(P)\,\}$
$D = \{$ the points where $s \leq c$ on the singular part of $\mathrm{Cut}(P)\,\}$.

Notice that the sets A and C correspond (via the exponential map exp_P) to open sets A' and C' in θ such that $\bar\eta = c$ is constant in A' and smooth in C'. Hence, we can approximate the function $\bar\eta(\theta)$ by a sequence of smooth functions $\bar\eta_\varepsilon \uparrow \bar\eta$ such that $\bar\eta_\varepsilon = c$ in A' and $\bar\eta_\varepsilon \to \bar\eta$ in C^∞ on compacts of C'. Set

$$\Omega_\varepsilon(c) = exp_P(\{\,(\rho,\theta);\ \rho < \bar\eta_\varepsilon(\theta)\,\}).$$

Since s is smooth on $\Omega_\varepsilon(c)$, we may apply Green's formula to get

$$\int_{\Omega_\varepsilon(c)} \Delta_g s = \int_{\partial\Omega_\varepsilon(c)} \nabla_g s \cdot N. \tag{2.7}$$

By the previous discussion, we may express $\partial\Omega_\varepsilon(c)$ as $\partial\Omega_\varepsilon(c) = A_\varepsilon \cup B_\varepsilon \cup C_\varepsilon \cup D_\varepsilon$ such that $H^1(B_\varepsilon \cup D_\varepsilon) \to 0$, as $\varepsilon \to 0$ and

$$\int_{A_\varepsilon} \nabla_g s \cdot N \to \int_A \nabla_g s \cdot N = \int_{\{s=c\}} \nabla_g s \cdot N = L(\{s=c\})$$

(since $\nabla_g s \cdot N = 1$) while

$$\int_{C_\varepsilon} \nabla_g s \cdot N \to \int_C \nabla_g s \cdot N = 0$$

due to cancellation in pairs. Hence we conclude (2.6).

Next choose an origin 0 and a sequence of points $P_j \to \infty$, and for any point Q define the Busemann function

$$F(Q) = \lim_{j\to\infty} (s_j(Q) - s_j(0))$$

where $s_j(Q) = d(P_j, Q)$ is the distance to the point P_j. By the triangular inequality

$$|d(P_j, Q) - d(P_j, 0)| \le d(0, Q).$$

Hence, the Busemann function $F(Q)$ is well defined. Let us denote by $F_j(Q)$ the function

$$F_j(Q) = s_j(Q) - s_j(0).$$

Then by (2.3) the estimate

$$\Delta_g F_j \le 1 + \frac{1}{s_j(Q)}$$

holds in the distributional sense. Hence, we may apply (2.6) to get that

$$L(\{\, F_j = c \,\}) = \iint_{\{\, F_j \le c \,\}} \Delta_g F_j \le \iint_{\{\, F_j \le c \,\}} \left(1 + \frac{1}{s_j(Q)}\right) \, da.$$

Taking the limit $j \to \infty$, and using that on the complete metric g, the limit $\lim_{j\to\infty} s_j(X) = \infty$, we finally obtain the bound

$$L\{\, F = c \,\} \le \iint_{\{\, F_j \le c \,\}} 1 \; da = A\{\, F \le c \,\}$$

where $L\{\, F = c \,\}$ is the length of the level curve of F and $A\{\, F \le c \,\}$ is the area inside. If the sectional curvature K satisfies $K \ge -\kappa$, instead of $K \ge -1$, we obtain the bound

$$L\{\, F = c \,\} \le \sqrt{\kappa}\, A\{\, F \le c \,\} \tag{2.8}$$

since the sectional curvature K scales like $1/L^2$.

It is shown in [8] that for any solution u to the Ricci flow on the plane, the scalar curvature $R = 2K$ evolves by

$$\frac{\partial R}{\partial t} = \Delta_g R + R^2.$$

It follows by the maximum principle that the minimum scalar curvature satisfies

$$R \geq -\frac{1}{t}.$$

Observing that $R = -(\Delta \log u)/u$, the above bound is nothing but the well known Aronson-Bénilan inequality [1] $u_t \leq 1/t$. If we are at least half-way to the time of collapse $t \geq T/2$, then $R \geq -2/T$ and $K \geq -1/T$. Hence, by (2.8)

$$w(F) = \sup_c L\{\, F = c \,\} \leq \frac{1}{\sqrt{T}} \sup_c A\{\, F \leq c \,\} \leq \frac{4\,\pi}{\sqrt{T}}\,(T-t)$$

which gives the desired bound

$$w = \inf_F w(F) \leq C\,(T-t).$$

3. The lower bound on the width.

The lower bound $w \geq \gamma\,(T-t)$ on the width w will follow from the isoperimetric estimate similar to the one in [9], but with different scaling to reflect the behavior at infinity. Any simple curve Γ with length $L(\Gamma)$, divides the plane into two regions, one inside Γ with area $A_{\text{in}}(\Gamma)$ and one outside with area $A_{\text{out}}(\Gamma)$. We define the isoperimetric ratio

$$I(\Gamma) = L(\Gamma)\,\left(\frac{1}{A_{\text{in}}(\Gamma)} + \frac{1}{A_{\text{out}}(\Gamma)}\right)$$

and let

$$I = \inf_\Gamma I(\Gamma)$$

be the smallest isoperimetric ratio over all curves Γ. In the case of S^2 we used in [9] the natural scaling $L(\Gamma)^2$, since area scales like length squared; but here we see that $L(\Gamma)$ is proportional to A_{out}.

Lemma 3.1. *For any maximal solution u to the Ricci flow on the plane satisfying $u \leq C_0/(r^2 \log^2 r)$, at $t = 0$, the isoperimetric ratio I is bounded* *below, up to the collapsing time T, i.e., there exists a constant $\alpha > 0$ for which*

$$I(t) := \inf_{\Gamma} L(\Gamma) \left(\frac{1}{A_{in}(\Gamma)} + \frac{1}{A_{out}(\Gamma)} \right) \geq \alpha > 0 \qquad (3.1)$$

on $0 \leq t \leq T$.

Before we proceed with the proof of the Lemma, let us state, for the convenience of the reader, an a'priori pointwise estimate on the solution u, from above an below, shown in [16].

Proposition 3.2 ([16]). *For any maximal solution to equation* (1.1) *on the plane, satisfying the initial bound*

$$u \leq \frac{C_0}{r^2 \log^2 r}, \qquad \text{at } t = 0, \qquad (3.2)$$

there exists uniform positive constants C and c and constants $R(t)$, depending on t, such that the pointwise bounds

$$u \leq \frac{C}{r^2 \log^2 r}, \qquad r > 1 \qquad (3.3)$$

and

$$u \geq \frac{c}{r^2 \log^2 (\frac{r}{R(t)})} \qquad r > R(t) \qquad (3.4)$$

hold for all $0 \leq t \leq T$, up to the vanishing time T of u.

Proof of Lemma 3.1. The bound from above (3.3) shows that the area outside of radius $r > 1$ is bounded above by

$$A_{\text{out}} \leq \frac{C}{\log r}$$

while the lower bound (3.4) shows that any curve enclosing the origin at radius no more than r, with $r >> R(t)$, has length

$$L(\Gamma) \geq \frac{c}{\log r + \log R(t)}.$$

Hence, for each time $0 < t < T$, as $r \to \infty$, the isoperimetric ratio remains bounded below away from zero, uniformly in time. Moreover, I is bounded below away from zero at any time $t < T$. We only need to show that $I(t)$ does not decay to zero, as $t \to T$, assuming that at each time $t < T$, the minimum of the isoperimetric ratio $I(t)$ is achieved at a curve Γ.

To this end, we will prove any $0 < t < T$, $I(t)$ satisfies the differential inequality

$$\frac{dI(t)}{dt} \geq -C\, I^3 \tag{3.5}$$

for some constant $C < \infty$. This will prevent $I(t)$ from going down to zero at the time T.

Let Γ be an optimal curve at time t, with $0 < t < T$ and let $\Gamma(s)$ be the parallel curve to Γ at distance s, outside for $s > 0$ and inside for $s < 0$. Then, as shown in [9], the length L of Γ satisfies the parabolic equation

$$\frac{\partial L}{\partial t} = \frac{\partial^2 L}{\partial s^2} \tag{3.6}$$

under the Ricci Flow. We also have

$$\frac{\partial A_{\text{in}}}{\partial s} = L \qquad \text{and} \qquad \frac{\partial A_{\text{out}}}{\partial s} = -L.$$

Moreover, for the Ricci Flow (with $R = 2K$)

$$\frac{\partial A_{\text{in}}}{\partial t} = -2 \iint_{\text{in}} K\, da \qquad \text{and} \qquad \frac{\partial A_{\text{out}}}{\partial t} = -2 \iint_{out} K\, da.$$

By the Gauss-Bonnet Theorem, we have

$$\int_{\text{in}} K\, da + \int_{\Gamma} k\, ds = 2\pi$$

and

$$\int_{\text{out}} K\, da - \int_{\Gamma} k\, ds = 0.$$

since the inside disc has Euler class 1 and the outside annulus has Euler class 0. Hence, from the above formulas, we conclude

$$-\frac{1}{2}\,\frac{\partial A_{\text{in}}}{\partial t} + \int_{\Gamma} k\, ds = 2\pi$$

while

$$\frac{1}{2}\,\frac{\partial A_{\text{out}}}{\partial t} + \int_{\Gamma} k\, ds = 0.$$

Moreover, the first variation formula for arc-length gives

$$\frac{\partial L}{\partial s} = \int_{\Gamma} k\, ds.$$

Combining these results gives

$$\frac{\partial A_{\rm in}}{\partial t} = 2\,\frac{\partial L}{\partial s} - 4\pi \qquad \text{and} \qquad \frac{\partial A_{\rm out}}{\partial t} = -2\,\frac{\partial L}{\partial s}.$$

Now we can compute the evolution of I. Since

$$\log I = \log L + \log(A_{\rm in} + A_{\rm out}) - \log A_{\rm in} - \log A_{\rm out}$$

we find that

$$\frac{1}{I}\,\frac{\partial I}{\partial t} = \frac{1}{L}\,\frac{\partial L}{\partial t} - \frac{4\pi}{A_{\rm in} + A_{\rm out}} - \frac{2\,\frac{\partial L}{\partial s} - 4\pi}{A_{\rm in}} + \frac{2\,\frac{\partial L}{\partial s}}{A_{\rm out}} \tag{3.7}$$

while

$$\frac{1}{I}\,\frac{\partial I}{\partial s} = \frac{1}{L}\,\frac{\partial L}{\partial s} + 0 - \frac{1}{A_{\rm in}}\,L + \frac{1}{A_{\rm out}}\,L \tag{3.8}$$

and

$$\begin{aligned} \frac{1}{I}\,\frac{\partial^2 I}{\partial s^2} - \frac{1}{I^2}\left(\frac{\partial I}{\partial s}\right)^2 &= \frac{1}{L}\,\frac{\partial^2 L}{\partial s^2} - \frac{1}{L^2}\left(\frac{\partial L}{\partial s}\right)^2 \\ &\quad + \frac{L^2}{A_{\rm in}^2} + \frac{L^2}{A_{\rm out}^2} - \frac{1}{A_{\rm in}}\,\frac{\partial L}{\partial s} + \frac{1}{A_{\rm out}}\,\frac{\partial L}{\partial s}. \end{aligned} \tag{3.9}$$

At the maximum of I we have $\partial I/\partial s = 0$. Hence, (3.8) gives

$$\frac{\partial L}{\partial s} = L^2\left(\frac{1}{A_{\rm in}} - \frac{1}{A_{\rm out}}\right) \tag{3.10}$$

Combining (3.7), (3.9) and (3.10) we obtain

$$\begin{aligned} \frac{1}{I}\left(\frac{\partial I}{\partial t} - \frac{\partial^2 I}{\partial s^2}\right) &\geq \frac{1}{L}\left(\frac{\partial L}{\partial t} - \frac{\partial^2 L}{\partial s^2}\right) \\ &\quad + L^2\left[\left(\frac{1}{A_{\rm in}} - \frac{1}{A_{\rm out}}\right)^2 - \left(\frac{1}{A_{\rm in}^2} + \frac{1}{A_{\rm out}^2}\right)\right]. \end{aligned} \tag{3.11}$$

Equations (3.6) and (3.11) imply

$$\frac{1}{I}\left(\frac{\partial I}{\partial t} - \frac{\partial^2 I}{\partial s^2}\right) \geq L^2\left[\left(\frac{1}{A_{\rm in}} - \frac{1}{A_{\rm out}}\right)^2 - \left(\frac{1}{A_{\rm in}^2} + \frac{1}{A_{\rm out}^2}\right)\right]. \tag{3.12}$$

We can easily estimate

$$\left(\frac{1}{A_{\rm in}} - \frac{1}{A_{\rm out}}\right)^2 - \left(\frac{1}{A_{\rm in}^2} + \frac{1}{A_{\rm out}^2}\right) \geq -C\left(\frac{1}{A_{\rm in}} + \frac{1}{A_{\rm out}}\right)^2$$

and finally obtain the inequality

$$\frac{1}{I}\left(\frac{\partial I}{\partial t}-\frac{\partial^2 I}{\partial s^2}\right)\geq -C\,L^2\left(\frac{1}{A_{\text{in}}}+\frac{1}{A_{\text{out}}}\right)^2=-C\,I^2$$

or

$$\frac{\partial I}{\partial t}\geq\frac{\partial^2 I}{\partial s^2}-C\,I^3. \tag{3.13}$$

Since, $\partial^2 I/\partial s^2\geq 0$ at the minimum curve Γ, we finally obtain the desired inequality (3.5) which is equivalent to

$$\frac{d}{dt}\left(\frac{1}{I^2}\right)\leq 2C.$$

Integrating in time, and using that $I\geq\alpha_0$, at $t=0$, we find that

$$I(t)\geq\frac{I_0}{\sqrt{1+2CI_0^2\,t}}$$

with $I_0=I(0)$. Since I_0 is bounded from above and below by a uniform constant the bound (3.1) readily follows. This finishes the proof of Lemma 3.1.

To show the lower bound on the width $w\geq\gamma\,(T-t)$, notice first that for any proper function F the area inside the level set $\{F=c\}$ is monotone increasing in c going from 0 to A, while the area outside is monotone decreasing from A to 0. So from some value of c they will be equal

$$A_{\text{in}}\{\,F=c\,\}=A_{\text{out}}\{\,F=c\,\}=\frac{A}{2}.$$

But

$$L\{\,F=c\,\}\left[\frac{1}{A_{\text{in}}\{\,F=c\,\}}+\frac{1}{A_{\text{out}}\{\,F=c\,\}}\right]\geq I.$$

Thus, for some c, we have

$$L\{\,F=c\,\}\geq I\,\frac{A}{4}.$$

Since $I\geq\alpha$ and $A=4\pi\,(T-t)$, we finally obtain

$$w\geq\alpha\,\pi\,(T-t)=\gamma\,(T-t)$$

with $\gamma=\alpha\,\pi>0$. This completes the proof of Theorem 1.

4. The upper bound on the curvature.

The upper bound $R_{\max} \le C/(T-t)^2$ follows from the estimate on a potential function introduced by H.-D. Cao [4]. Let

$$r = \frac{\int R\, da}{\int 1\, da}$$

be the average scalar curvature. Since

$$\int R\, da = 2 \int K\, da = 4\,\pi$$

and

$$\frac{d}{dt} \int 1\, da = - \int R\, da$$

we find

$$\frac{dr}{dt} = r^2.$$

Definition 4.1. The potential $\bar f$ is the solution of the equation

$$\Delta_g \bar f = R - r \tag{4.1}$$

with mean value zero.

The existence and uniqueness of the potential function $\bar f$ is shown in the next Proposition.

Proposition 4.2. *Assume that the metric $g_{ij} = u\, dx_i\, dx_j$ satisfies the pointwise bounds on $\mathbb{R}^2$*

$$c\, \min(1, \frac{1}{|x|^2 \log^2 |x|}) \le u \le C\, \min(1, \frac{1}{|x|^2 \log^2 |x|}). \tag{4.2}$$

for some positive constants c, C. Then, equation (4.1) admits a solution f which is unique up to a constant. Moreover, each solution f satisfies the gradient estimate

$$|D_g f| \le M \tag{4.3}$$

for some constant M, depending only on c and C.

We notice that equation (4.1) is equivalent to equation

$$\Delta f = (R - r)\, u \qquad \text{on} \quad \mathbb{R}^2 \tag{4.4}$$

with Δ denoting the Euclidean Laplace operator. The forcing term $h := (R - r)\, u$ has finite mass and zero mean. Moreover, since R is bounded and u satisfies (4.2), h satisfies the bound

$$|h| \le C\, \min(1, \frac{1}{|x|^2 \log^2 |x|}). \tag{4.5}$$

Proposition 4.3. *Under the above assumptions on h, there exists a unique up to a constant solution of equation*

$$\Delta f = h \qquad \text{on} \quad \mathbb{R}^2.$$

Moreover, each solution f satisfies the gradient estimate

$$|Df| \le M\, \min(1, \frac{1}{|x| \log |x|}) \tag{4.6}$$

for some constant M depending on C and c.

Since

$$|D_g f| = \frac{1}{\sqrt{u}}\, |Df|$$

and u satisfies the bounds (4.2), Proposition 4.2 readily follows from Proposition 4.3.

Proof of Proposition 4.3. The uniqueness of f, up to a constant, is a direct consequence of Liouville's Theorem.

For existence, we define f as the Newtonian Potential of h, namely

$$f(x) = \int_{\mathbb{R}^2} \log |x - y|\, h(y)\, dy. \tag{4.7}$$

Then, f is well defined since h is locally bounded and satisfies the bound (4.5). To obtain the gradient estimate we differentiate (4.7) with respect to x_i and use the fact that h has mean zero to show that

$$D_i f(x) = \int_{\mathbb{R}^2} \left(\frac{x_i - y_i}{|x - y|^2} - \frac{x_i}{|x|^2} \right) h(y)\, dy.$$

It is clear that $D_i f$ is locally bounded, since h is locally bounded and integrable. We will next show that

$$|D_i f| \le \frac{M}{|x|\, \log |x|}, \qquad |x| >> 1.$$

Set

$$k(x,y) = \frac{x_i - y_i}{|x-y|^2} - \frac{x_i}{|x|^2}.$$

We then express

$$\int_{\mathbb{R}^2} k(x,y)\, h(y)\, dy = \left(\int_{|x-y| \le \frac{|x|}{2}} + \int_{\frac{|x|}{2} < |x-y| < 2|x|} + \int_{|x-y| \ge 2|x|} \right) k(x,y)\, h(y)\, dy$$

and we estimate the three integrals separately. For the first and the last integral we simply use the estimate

$$|k(x,y)| \le \frac{1}{|x-y|} + \frac{1}{|x|}.$$

When $|x-y| \le |x|/2$, then $|y| \ge |x|/2$, and hence

$$|h(y)| \le \frac{C}{|x|^2 \log^2 |x|}.$$

Hence

$$I_1 := |\int_{|x-y| \le \frac{|x|}{2}} k(x,y)\, h(y)\, dy| \le \frac{C}{|x|^2 \log^2 |x|} \int_{|x-y| \le \frac{|x|}{2}} \left(\frac{1}{|x-y|} + \frac{1}{|x|} \right) dy$$

showing that

$$I_1 \le \frac{C}{|x| \log^2 |x|}.$$

When $|x-y| \ge 2|x|$, then $|y| \ge |x|$ and $|k(x,y)| \le 2/|x|$. Hence

$$I_3 := |\int_{|x-y| \ge 2|x|} k(x,y)\, h(y)\, dy| \le \frac{2}{|x|} \int_{|y| \ge |x|} \frac{C}{|y|^2 \log^2 |y|}\, dy$$

showing that

$$I_3 \le \frac{C}{|x| \log |x|}.$$

Finally, to estimate I_2 we set

$$\phi(w) = \frac{w_i}{|w|^2}$$

so that

$$k(x,y) = \phi(x-y) - \phi(x).$$

Hence, for y in the set $\frac{|x|}{2} < |x-y| < 2|x|$, we have

$$|k(x,y)| \leq \max_{|x|/2 \leq |w| \leq 2|x|} |\nabla \phi(w)| \cdot |y|.$$

Since $|\nabla \phi(w)| \leq 16/|w|^2$, we obtain the estimate

$$|k(x,y)| \leq \frac{C}{|x|^2} \cdot |y|.$$

This gives

$$I_2 := | \int_{\frac{|x|}{2} \leq |x-y| \leq 2|x|} k(x,y)\, h(y)\, dy \,| \leq \frac{C}{|x|^2} \int_{|y| \leq 3|x|} |y|\, h(y)\, dy.$$

Using once more the bound (4.5) we conclude that

$$I_2 \leq \frac{C}{|x|\, \log^2 |x|}.$$

Combining all three estimates, we finally obtain (4.6).

We will next compute the evolution of the potential $\bar{f}$ with zero mean.

Proposition 4.4. *The potential $\bar{f}$ evolves by*

$$\frac{\partial \widetilde{f}}{\partial t} = \Delta_g \widetilde{f} + r\, \widetilde{f} - b$$

where

$$b = \frac{\int |D_g \widetilde{f}|^2 \, d\mu}{\int 1 \, d\mu} \tag{4.8}$$

with $d\mu = u\, dx_i dx_j$.

Proof. Since $\Delta_g \widetilde{f} = R - r$, differentiating in time and using the evolution equations of the metric g and the curvature R, we compute

$$\Delta_g \left(\frac{\partial \widetilde{f}}{\partial t} \right) = \Delta_g \left(\Delta_g \widetilde{f} + r\, \widetilde{f} \right).$$

Hence

$$\frac{\partial \widetilde{f}}{\partial t} = \Delta_g \widetilde{f} + r\, \widetilde{f} - b$$

for some number b which is constant in space and depends only on time. It is easy to compute that b is given by (4.8).

Proposition 4.5. *The potential*

$$f = \bar{f} + \alpha(t)$$

with

$$\alpha(t) = \frac{1}{q} \int_0^t b(t)\, q(t)\, dt, \qquad q(t) = e^{-\int_0^t r(t)\, dt} \tag{4.9}$$

evolves by

$$\frac{\partial f}{\partial t} = \Delta_g f + r\, f \tag{4.10}$$

and satisfies the gradient estimate (4.3)*.*

Proof. By (4.9) we easily compute that

$$\alpha' = -\frac{q'}{q}\, \alpha + b$$

with

$$\frac{q'}{q} = -r.$$

Hence

$$\alpha' = r\, \alpha + b$$

implying that

$$\frac{\partial f}{\partial t} = \frac{\partial \widetilde{f}}{\partial t} + \alpha' = (\Delta_g \widetilde{f} + r\, \widetilde{f} - b) + (ra + b) = \Delta_g f + r\, f$$

as desired. Since α depends only on time, the gradient bound (4.3) still holds.

Now, using the evolution equation (4.10) we find that

$$\frac{\partial}{\partial t} |D_g f|^2 = \Delta_g |D_g f|^2 - 2\, |D_g^2 f|^2 + 2r\, |D_g f|^2$$

and use

$$|D_g^2 f|^2 = |\overset{\circ}{D_g^2} f|^2 + \frac{1}{2}\, (\Delta_g f)^2$$

where Δ_g is the trace of $D_g^2 f$ and $|\overset{\circ}{D_g^2} f|^2$ is the trace free part of the Hessian, to conclude that

$$\frac{\partial}{\partial t} (|D_g f|^2 + R - r) = \Delta_g (|D_g f|^2 - R + r) - 2\, |\overset{\circ}{D_g^2} f|^2 + 2r\, (|D_g f|^2 + R - r).$$

Since $|\overset{\circ}{D^2} f|^2 \geq 0$, the maximum principle shows that

$$\frac{d}{dt} \log \left(|D_g f|^2 + R - r\right)_{\max} \leq 2r.$$

Since $dr/dt = r^2$ and $r \to \infty$, as $t \to T$, we have $r = 1/(T-t)$ exactly. Thus

$$r = \frac{d}{dt} \log \frac{1}{T-t}$$

and hence

$$\log \left(|D_g f|^2 + R - r\right)_{\max} - 2 \log \frac{1}{T-t}$$

is decreasing in time. This proves that

$$|D_g f|^2 + R - r \leq \frac{C}{(T-t)^2}$$

implying the curvature bound

$$R \leq r + \frac{C}{(T-t)^2}.$$

Using that $r = 1/(T-t) \leq C/(T-t)^2$, we finally obtain the bound

$$R \leq \frac{C}{(T-t)^2}$$

as desired.

5. The lower bound on the curvature.

The lower bound $\mathbb{R} \geq c/(T-t)^2$ will be obtained by combining the upper bound on the width $w \leq C\,(T-t)$, shown in section 2, with a result on formation of singularities on the Ricci flow in [8]. We know that $\mathbb{R} \to \infty$ as $t \to T < \infty$, since otherwise by the existence of W.S. Shi [14] on complete solutions of the Ricci flow, the solution would continue past T. Moreover, from [10] we can form a limit of dilations of the solution to obtain a blow-up of the singularity which is either of type I ($\mathbb{R}\,(T-t) \leq C$) or of type II ($\mathbb{R}\,(T-t) \to \infty$), as $t \to T$.

Now type I cannot occur; for if $\mathbb{R}\,(T-t) \leq C$ and $A \leq C\,(T-t)$, the type I limit should be a positive solution for $-\infty < t < T$ with positive curvature $R > 0$ and finite area, which would necessarily be compact, and hence a sphere S^2 or projective space RP^2 instead of the plane $\mathbb{R}^2$.

Suppose then that we obtain a limit solution of type II. Thus is a complete solution with $R > 0$ and $R \leq 1$ everywhere for $-\infty < t < \infty$, and $R = 1$ at the origin and at time $t = 0$. Since the maximum curvature is attained, by [10] this solution is a Ricci soliton, i.e. a solution g_{ij} satisfying

$$M_{ij} := D_i D_j f - \frac{1}{2}\,\Delta f \cdot g_{ji} = 0.$$

It moves only by diffeomorphism, so its shape remains unchanged. It is shown in [9] that in dimension 2 the only soliton solution is a cigar, which at time $t = 0$ looks like

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

and flows by conformal dilation. It is asymptotic to a flat cylinder at infinity, with maximum curvature at the origin. Since the cigar occurs as a limit of blow-ups of the original solution, this means that for time t near T the solution is as close as we wish to a scaling of the cigar by a constant factor over as large a compact set around 0 as we wish. This easily implies that the width w is as close as we wish to $2\pi/\sqrt{\mathbb{R}}$ or else greater, since that is the width of a sequence of cigars of maximum curvature $\mathbb{R}$ at the origin. But this show that

$$\frac{2\pi - \epsilon}{\sqrt{\mathbb{R}}} \leq w \leq C\,(T - t)$$

from which we get

$$\mathbb{R} \geq \frac{c}{(T - t)^2}.$$

This completes the proof of Theorem 1.2.

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