On the Asymptotic Scalar Curvature Ratio of Complete Type I-like Ancient Solutions to the Ricci Flow on Noncompact 3-manifolds

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1. Introduction.

Complete noncompact Riemannian manifolds with nonnegative sectional curvature arise naturally in the Ricci flow when one takes the limits of dilations about a singularity of a solution of the Ricci flow on a compact 3-manifold [H-95a]. To analyze the singularities in the Ricci flow one needs to understand these manifolds in depth. There are three invariants, asymptotic scalar curvature ratio, asymptotic volume ratio and aperture, that have been used to study the geometry of these manifolds at infinity.

Let $(\mathcal{M}^n, g)$ be a complete noncompact Riemannian manifold with nonnegative sectional curvature and let $O \in \mathcal{M}$ be some point which we call the origin. The *asymptotic scalar curvature ratio* (ASCR) is defined by

$$\text{ASCR}(\mathcal{M}, g) = \limsup_{d(x,O) \to \infty} R(x) \cdot d(x, O)^2, \tag{1}$$

where R is the scalar curvature. ASCR is a measure of how non-flat the manifold is at infinity. Since the sectional curvature is nonnegative, there is a positive constant c depending only on n such that

$$c^{-1} \cdot \limsup_{d(x,O) \to \infty} |Rm(x)| \cdot d(x, O)^2 \leq \text{ASCR}(\mathcal{M}, g)$$

$$\leq c \cdot \limsup_{d(x,O) \to \infty} |Rm(x)| \cdot d(x, O)^2$$

where Rm is the Riemann curvature tensor. In the literature sometimes

$$\limsup_{d(x,O) \to \infty} |Rm(x)| \cdot d(x, O)^2$$

is used as the definition of ASCR. It is clear that ASCR is independent of the choice of origins and is invariant under scaling.

ASCR has been used to study gap theorems. In particular, in [ESS-89] Eschenburg, Schroeder and Strake proved that if $\left(\mathcal{M}^{2k+1}, g\right)$ is a complete noncompact odd-dimensional Riemannian manifold with positive sectional curvature, then $\mathrm{ASCR}(\mathcal{M}, g) > 0$. These types of results are generally referred to as gap theorems since they show the existence of a gap between flat $\mathbb{R}^n$ and metrics of positive curvature on $\mathbb{R}^n$. Gap theorems have been proved by Greene-Wu [GW-82], Kasue and Sugahara [KS-87], and Drees [D-94]. For a survey of the history of gap theorems related to the notion of ASCR, see [G-97]. ASCR has also been used to study the structure of manifolds at infinity [PT-01].

Below we give a few examples about how ASCR is used in singularity analysis in the Ricci flow.

1. In [H-95a] Hamilton showed that for a solution to the Ricci flow on a compact 3-manifold forming a Type II singularity and satisfying an injectivity radius estimate, there exists a sequence of dilations converging to a complete solution $\left(\mathcal{M}_\infty^3, g_\infty\left(t\right)\right)$ defined for all $t \in (-\infty, \infty)$ with nonnegative bounded sectional curvature and attaining its maximum of the scalar curvature on space and time. In [H-93b] it is shown using the differential Harnack inequality of Li-Yau-Hamilton type [H-93a] that the universal covering solution $\left(\widetilde{\mathcal{M}}_\infty^3, \tilde{g}_\infty\left(t\right)\right)$ of such a solution must be a stationary solution (also called *Ricci solitons*) of the Ricci flow in the space of metrics modulo diffeomorphisms flowing along a gradient vector field, that is, there exists a 1-parameter family of diffeomorphisms $\varphi_t : \widetilde{\mathcal{M}}_\infty^3 \to \widetilde{\mathcal{M}}_\infty^3$ such that $\tilde{g}_\infty\left(t\right) = \varphi_t^*\left(\tilde{g}_\infty\left(0\right)\right)$ and the 1-parameter family of vector fields $X\left(t\right)$ generated by φ_t are the gradients of functions $f\left(t\right)$. In §20 of [H-95a] it is shown that such a Ricci soliton must have $\mathrm{ASCR}(g\left(t\right)) = \infty$ for all $t \in (-\infty, \infty)$.

2. In §22 of [H-95a] Hamilton showed using a geometric result about bumps of curvature from §21 of [H-95a] that if a complete solution to the Ricci flow with bounded curvature has $\mathrm{ASCR}(g\left(t\right)) = \infty$, then one can perform dimension reduction. In particular, provided there is a local injectivity radius estimate, there exists a sequence of points $\{y_\alpha\}_{\alpha \in \mathbb{N}}$ in $\mathcal{M}^3$ with $\lim_{\alpha \to \infty} d_{g(0)}\left(y_\alpha, O\right) = \infty$ such that the sequence of dilated solutions on balls $\left\{\left(B_{g_\alpha(0)}\left(y_\alpha, r_\alpha\right), g_\alpha\left(t\right)\right), y_\alpha\right\}_{\alpha \in \mathbb{N}}$, where

$$g_\alpha\left(t\right) = R\left(g_\alpha\right)\left(y_\alpha, 0\right) g\left(t \cdot R\left(g_\alpha\right)^{-1}\left(y_\alpha, 0\right)\right)$$

and $\lim_{\alpha \to \infty} r_\alpha^2 R\left(g_\alpha\right)\left(y_\alpha, 0\right) = \infty$, converges to complete solution to the Ricci flow $\left(\mathcal{P}_\infty, k_\infty\left(t\right), y_\infty\right)$ with bounded nonnegative sectional

curvature which splits metrically as the product of $\mathbb{R}$ and a solution on a surface with positive curvature. This is why it is important to know when a noncompact ancient solution has infinite asymptotic scalar curvature ratio, when this is true one should be able to perform dimension reduction.

3. In §5.3 of [H-97] it is shown that for an ancient solution $(\mathcal{M}, g(t))$ to the Ricci flow which is complete with bounded positive curvature operator and satisfies certain pinching conditions, then ASCR$(\mathcal{M}, g(t)) = \infty$ if and only if for some fixed time the solution metric has an arbitrarily necklike end (see §2 below for a definition).

The *asymptotic volume ratio* is defined by

$$AVR(g) \doteqdot \lim_{r \to \infty} \frac{\text{Vol}\,[B\,(O, r)]}{r^n},$$

where $B(O, r) = \{x : d(O, x) < r\}$. It is also used in singularity analysis in the Ricci flow. In particular, in §19 of [H-95a] Hamilton showed that a complete ancient Type I-like solution to the Ricci flow with bounded positive curvature operator and finite ASCR must have AVR$(g(t)) > 0$ for all t. He also showed there that the scalar curvature decays exactly quadratically:

$$0 < c \le R\,(x, t)\,d_{g(t)}\,(x, O)^2 \le C$$

where c and C depend on O and t.

This paper arises from exploring the following idea inspired by the work of Hamilton [H-97]. *If a piece of a positively curved complete noncompact manifold is sufficiently close to a long standard cylinder, then its asymptotic scalar curvature ratio is large.*

2. Main results.

To state the main result of this paper we need to define the so-called (ε, k, L)-necks. First we recall a few basic definitions concerning necks from §3.2 of [H-97]. A topological *neck* in a differentiable manifold $\mathcal{M}^n$ is a local diffeomorphism

$$N : S^{n-1} \times [a, b] \to \mathcal{M}^n,$$

for some $a < b$.

Let g be a Riemannian metric on $\mathcal{M}$ and $g \doteqdot N^*\,(g)$ be the pulled-back metric on on $S^{n-1} \times [a, b]$. A neck is called *normal* if it satisfies the following five conditions.

1. (**good slices**) Each slice $S^{n-1} \times \{z\} \subset S^{n-1} \times [a, b]$, $z \in [a, b]$, has constant mean curvature with respect to g.

2. (**good parametrizations of the slices**) The identity map

$$\iota : \left(S^{n-1} \times \{z\}, \bar{g} \right) \to \left(S^{n-1} \times \{z\}, g \right)$$

   is harmonic for all $z \in [a, b]$, where $\bar{g}$ is the standard metric.

3. (**taking in account that conformal maps of S^2 are harmonic**) When $n = 3$ the center of mass of $S^2 \times \{z\} \subset \mathbb{R}^3 \times \{z\}$ with respect to g is the origin:

$$\int_{S^2 \times \{z\}} \vec{x} \, dA_g \left(\vec{x} \right) = \vec{0} \in \mathbb{R}^3$$

   for all $z \in [a, b]$, where dA_g is the area form.

4. (**good spacing of the slices**) The spacing of the slices are normalized using volume by

$$\mathrm{Vol} \left(S^{n-1} \times [z, w], g \right) = \omega_{n-1} \int_z^w r(y)^n \, dy,$$

   where

$$r(y) \doteqdot \left(\frac{\mathrm{Area} \left(S^{n-1} \times \{y\}, g \right)}{\omega_{n-1}} \right)^{\frac{1}{n-1}}$$

   is called the *mean radius*.

5. (**aligning the parametrizations**) If $\bar{V}$ is a Killing vector field on a slice $\left(S^{n-1} \times \{z\}, \bar{g} \right)$, then the unit normal vector field ν of $S^2 \times \{z\} \subset \mathbb{R}^3 \times \{z\}$ with respect to g satisfies

$$\int_{S^{n-1} \times \{z\}} \bar{g}(\bar{V}, \nu) dA_{\bar{g}} = 0$$

   for all $z \in [a, b]$.

Note that in condition 4 it is important that the power of $r(y)$ is n, not $n - 1$.

A neck is called (ε, k)-*cylindrical* if

1. (**conformally close to cylindrical in C^0**) the metric

$$\widehat{g}(\theta, z) \doteq r(z)^{-2} g(\theta, z)$$

satisfies

$$|\widehat{g} - \bar{g}|_{\bar{g}} \leq \varepsilon,$$

2. (**conformally close to cylindrical up to C^k**)

$$\left| \bar{\nabla}^j \widehat{g} \right|_{\bar{g}} \leq \varepsilon$$

for all $1 \leq j \leq k$, where $\bar{\nabla}$ is the covariant derivative with respect to $\bar{g}$,

3. (**mean radius changes slowly**)

$$\left| \frac{d^j \log r(z)}{dz^j} \right| \leq \varepsilon$$

for all $1 \leq j \leq k$.

A neck N in a manifold $\mathcal{M}^n$ is called an *embedded neck* if $N : S^{n-1} \times [a, b] \to \mathcal{M}^n$ is an embedding.

Definition 1. *(i) A map $N : S^{n-1} \times [a, b] \to \mathcal{M}$ is called an (ε, k, L)-neck if it is a normal (ε, k)-cylindrical neck and $b - a \geq 2L$.*

(ii) A complete Riemannian manifold $(\mathcal{M}, g)$ with exactly one topological end is called to have an arbitrarily necklike end if for every (ε, k, L) there exists an (ε, k, L)-neck in $(\mathcal{M}, g)$.

Definition (ii) is from §5.3 of [H-97] (p. 71). We write $(\varepsilon_1, k_1, L_1) \preccurlyeq (\varepsilon_2, k_2, L_2)$ if $\varepsilon_1 \leq \varepsilon_2$, $k_1 \geq k_2$ and $L_1 \geq L_2$. In the next lemma we collect a few simple properties of (ε, k, L)-necks.

Lemma 2. *(i) (ε, k, L)-necks are scale-invariant, i.e., if $N : S^{n-1} \times [a, b] \to (M^n, g)$ is an (ε, k, L)-neck then $N : S^{n-1} \times [a, b] \to (M^n, \lambda^2 \cdot g)$ is also an (ε, k, L)-neck for any $\lambda > 0$.*

(ii) If $N : S^{n-1} \times [a, b] \to \mathcal{M}$ is an $(\varepsilon_1, k_1, L_1)$-neck and $(\varepsilon_1, k_1, L_1) \preccurlyeq (\varepsilon_2, k_2, L_2)$, then N is an $(\varepsilon_2, k_2, L_2)$-neck.

(iii) If S_r^{n-1} is the sphere of radius r, then for all $\varepsilon > 0$ and $k \in \mathbb{N}$ the product manifold $\mathcal{M} = S_r^{n-1} \times [-rL, rL]$ has an (ε, k, L)-neck given by the obvious map.

The main result of this paper is the following theorem which will be proved in §4.

Theorem 3. *For every odd integer $n \geq 3$ and every positive $C_0 < \infty$ there exists $(\varepsilon_0, k_0, L_0)$ such that if $(\mathcal{M}^n, g)$ is a complete, orientable, noncompact Riemannian manifold with bounded positive sectional curvature and $\mathcal{M}$ contains an embedded (ε, k, L)-neck with $(\varepsilon, k, L) \preccurlyeq (\varepsilon_0, k_0, L_0)$, then*

$$ASCR\left(g\right) \geq C_0.$$

The main result may be thought of as a odd-dimensional *quantitative* version of the "only if" part of Theorem 3.1 in §5.3 of [H-97], which states that a complete noncompact ancient solution to the Ricci flow on a four-manifold with bounded positive curvature operator satisfying certain pinching conditions has an arbitrarily necklike end at sometime if and only if the asymptotic scalar curvature ratio is infinite. In particular, we obtain a weaker characterization of $\text{ASCR}(g\left(t\right)) = \infty$, which is one of the ingredients used in the proof of Theorem 4 below.

The main result has the following useful consequence in dimension three. Recall that an ancient solution is called *Type-I like* if

$$\sup_{\mathcal{M}^n \times (-\infty, 0]} |t| \cdot |Rm\left(x, t\right)| < \infty.$$

Theorem 4. *If $\left(\mathcal{M}^3, g\left(t\right)\right)$, $-\infty < t < \omega$, is a complete noncompact ancient Type I-like solution to the Ricci flow with bounded positive sectional curvature on an orientable 3-manifold, then $ASCR(g\left(t\right)) = \infty$ for all $t \in (-\infty, \omega)$.*

We shall prove this theorem in §5. This answers a conjecture of Hamilton when $n = 3$. In §22 of [H-95a] (p. 93) he writes:

> We do not know any examples of complete noncompact ancient solutions of positive curvature operator with $Rs^2 < \infty$ and $R|t| < \infty$, and we conjecture none exist, since the curvature has had plenty of space and time to dissipate.

Conjecture 5. *There does not exist complete noncompact ancient Type I-like solutions to the Ricci flow with bounded positive curvature operator.*

By the theorem above such solutions, if they were to exist, must necessarily have infinite asymptotic scalar curvature ratio. When $n = 2$ it is proved in Theorem 26.1 of [H-95a] that there does not exist complete noncompact ancient Type I-like solutions to the Ricci flow with positive sectional curvature.

For the mean curvature flow, which in general has very similar properties as the Ricci flow, there are no complete noncompact strictly convex Type I-like ancient solutions by Huisken's classification using his monotonicity formula [Hu-90].

3. Relative volumes and necks.

The main result of this section, Proposition 10 below, is to combine the relative volume comparison theorem (Lemma 12) and the existence of an embedded (ε, k, L)-neck to show that there are small relative volumes. Note that this result holds in both odd and even dimensions.

3.1. Busemann functions.

Let $(\mathcal{M}^n, g)$ be a complete noncompact Riemannian manifold. Given a point $Q \in \mathcal{M}$ and a ray γ emanating from Q, the Busemann function $b_\gamma : M \to \mathbb{R}$ associated to γ is defined by

$$b_\gamma(x) \doteqdot \lim_{t \to +\infty} [t - d(\gamma(t), x)].$$

Let $\mathcal{R}$ be the set of all rays emanating from Q. The Busemann function $b_Q : M \to \mathbb{R}$ with the base point Q is defined by

$$b_Q(x) = \sup_{\gamma \in \mathcal{R}} b_\gamma(x)$$

We collect some well-known properties of Busemann functions on complete noncompact Riemannian manifolds with nonnegative sectional curvature in the following lemma (for a proof see, for example, [CG-72] and [LT-87]).

Lemma 6. Let $(\mathcal{M}^n, g)$ be a complete noncompact Riemannian manifold with nonnegative sectional curvature. Then
    (i) The Busemann function b_Q is proper, Lipschitz with Lipschitz constant 1 and bounded from below.
    (ii) The sublevel sets $C_r \doteqdot \{x \in \mathcal{M} : b_Q(x) \le r\}$ are compact and totally convex.
    (iii) Let $S_r \doteqdot b_Q^{-1}(r)$ denote level set of the Busemann function. Then for any $r_1 \le r_2$

$$S_{r_1} = \{x \in C_{r_2} : d(x, S_{r_2}) = r_2 - r_1\}.$$

    (iv) $|b_Q(x)| \le d(x, Q)$, which implies $B(Q, r) \subset C_r$.

The following lemma says that any sufficiently long minimal geodesic segment emanating from Q can be well approximated by a ray emanating from Q and that the Busemann function is similar at infinity to the distance function to the base point Q (see Li-Tam [LT-87] Theorem 2.3 on p. 177, Kasue [K-88] Lemma 1.4 on p. 598, Drees [D-94] Lemma 1 on p. 80 and for an exposition see [CKE]). For the convenience of the reader we give a proof here.

Lemma 7. *Let $(\mathcal{M}^n, g)$ be a complete noncompact Riemannian manifold with nonnegative sectional curvature. Define $\theta : [0, \infty) \to [0, \pi]$ by*

$$\theta(r) = \sup_{\sigma \in \mathcal{S}(r)} \inf_{\gamma \in \mathcal{R}} \angle_Q(\sigma'(0), \gamma'(0)),$$

where $\mathcal{S}(r)$ is the set of all minimal geodesic segments σ of length $L(\sigma) \geq r$ emanating from Q. Then
   (i) $\theta(r)$ is a nonincreasing function of r and

$$\lim_{r \to +\infty} \theta(r) = 0.$$

   (ii) The function b_Q and $d(\cdot, Q)$ are asymptotically equal, more precisely

$$(1 - \theta[d(x, Q)]) \cdot d(x, Q) \leq b_Q(x) \leq d(x, Q) \tag{2}$$

for all $x \in \mathcal{M}$.

Proof. (i) From the definition it is clear that $\theta(r)$ is a nonincreasing function of r. If $\lim_{r \to \infty} \theta(r) \neq 0$, there exists $\epsilon > 0$, a sequence of points $p_i \in \mathcal{M}$ with $d(p_i, Q) \nearrow +\infty$, and minimal geodesic segments σ_i joining Q and p_i parametrized by arc length such that

$$\angle_Q(\sigma_i'(0), \gamma'(0)) \geq \epsilon$$

for each i and all rays $\gamma \in \mathcal{R}$. By the compactness of the unit sphere in $T_Q\mathcal{M}$, there is a subsequence such that $\lim_{j \to +\infty} \sigma_{i_j}'(0) \doteq V_\infty$ exists. Let $\sigma_\infty : [0, +\infty) \to \mathcal{M}$ be the unique geodesic with $\sigma_\infty(0) = Q$ and $\sigma_\infty'(0) = V_\infty$. It is clear that $\sigma_\infty \in \mathcal{R}$. In particular the condition

$$\angle_Q(\sigma_{i_j}'(0), \sigma_\infty'(0)) \geq \epsilon$$

is impossible for large enough j. (i) is proved

(ii) For any $x \in \mathcal{M}$ let σ be a minimal geodesic from Q to x and $r = d(x, Q)$. Since $\mathcal{R}$ is a closed set, there is a ray $\gamma \in \mathcal{R}$ such that $\angle_Q(\sigma'(0), \gamma'(0)) \leq \theta(r)$. Then it follows from sectional curvature $K_{\mathcal{M}} \geq 0$ and Toponogov's comparison theorem that $d(x, \gamma(r)) \leq \theta(r) \cdot r$. Hence

$$b_Q(x) \geq b_\gamma(x) \geq r - d(x, \gamma(r)) \geq (1 - \theta(r)) \cdot r.$$

The lemma is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3.2. Necks in manifolds with positive sectional curvature.

We call a neck $N : S^{n-1} \times [a, b] \to \mathcal{M}$ *absolute* (ε, k)-*cylindrical* if it satisfies the following two inequalities.

1.

$$\left| \frac{1}{r^2 \left(\frac{a+b}{2} \right)} \cdot g - \bar{g} \right|_{\bar{g}} \leq \varepsilon \quad \text{on } S^{n-1} \times [a, b] \tag{3}$$

   where $r \left(\frac{a+b}{2} \right)$ is the mean radius of $S^{n-1} \times \{ \frac{a+b}{2} \}$.

2.

$$\left| \bar{\nabla}^j \left(\frac{1}{r^2 \left(\frac{a+b}{2} \right)} \cdot g \right) \right|_{\bar{g}} \leq \varepsilon \quad \text{on } S^{n-1} \times [a, b] \tag{4}$$

   for all $1 \leq j \leq k$, where $\bar{\nabla}$ is the covariant derivative with respect to $\bar{g}$.

We call a topological neck an *absolute* (ε, k, L)-*neck* if it is a normal absolute (ε, k)-cylindrical neck and $b - a \geq 2L$. When $(\varepsilon_1, k_1, L_1) \preccurlyeq (\varepsilon_2, k_2, L_2)$, an absolute $(\varepsilon_1, k_1, L_1)$-neck is an absolute $(\varepsilon_2, k_2, L_2)$-neck. The following result, whose proof is given in Appendix A, holds for necks in arbitrary Riemannian manifolds.

Lemma 8. *Given* (ε, k, L), *there exists* $\varepsilon' = \varepsilon'(\varepsilon, k, L) \leq \varepsilon$ *such that if* $N : S^{n-1} \times [a, b] \to \mathcal{M}$ *is an* (ε', k, L)-*neck, then* N *is an absolute* (ε, k, L)-*neck.*

Let $N : S^{n-1} \times [-L, L] \to \mathcal{M}^n$ be an embedded neck in a complete noncompact Riemannian manifold $(\mathcal{M}^n, g)$ with positive sectional curvature. By Gromoll and Meyer [GM-69], $\mathcal{M}^n$ is diffeomorphic to $\mathbb{R}^n$. Thus it follows from the solution of the Schoenflies Conjecture in dimension $n \neq 4$ [M-59], [B-60] that the center sphere $N \left(S^{n-1} \times \{0\} \right)$ bounds a differentiable ball in

$\mathcal{M}^n$ when $n \neq 4$. When $n = 4$ Hamilton proved the following lemma (see Theorem 1.1 in §7 of [H-97], the proof there works for all $n \geq 2$).

Lemma 9. *There exists $(\varepsilon_a, k_a, L_a)$ having the following property. For any $(\varepsilon, k, L) \preccurlyeq (\varepsilon_a, k_a, L_a)$ and any complete noncompact Riemannian manifold $(\mathcal{M}^n, g)$ with positive sectional curvature which has an embedded (ε, k, L)-neck $N : S^{n-1} \times [-L, L] \to \mathcal{M}^n$, the center sphere $N\left(S^{n-1} \times \{0\}\right)$ bounds a differentiable ball in $\mathcal{M}^n$.*

3.3. Necks and relative volumes.

Let ω_{n-1} be the volume of the sphere S^{n-1} of radius 1, and $\varepsilon_b \doteqdot \varepsilon'(\frac{1}{10}, 1, L_b)$ be a function of L_b as in Lemma 8.

Proposition 10. *For any $\delta > 0$ there is $L_b \geq \max(L_a, 16)$ having the following property. For any $(\varepsilon, k, L) \preccurlyeq (\min(\varepsilon_a, \varepsilon_b), k_a, L_b)$, if there is an embedded (ε, k, L)-neck N in a complete noncompact Riemannian manifold $(\mathcal{M}^n, g)$ with positive sectional curvature*

$$N : S^{n-1} \times [-L, L] \to (\mathcal{M}^n, g),$$

then without any loss of generality we may assume that the component of $\mathcal{M} - N\left(S^{n-1} \times [-L_b, L_b]\right)$ bounded by $N(S^{n-1} \times \{-L_b\})$ is diffeomorphic to a ball and for any $Q \in N(S^{n-1} \times \{-L_b\})$ and there exists $r_0 > 0$ such that for any $R_2 \geq R_1 \geq r_0$ the following relative volume estimate holds:

$$\frac{Vol\left[\, \overline{B}(Q, R_2) \backslash B(Q, R_1) \right]}{\frac{\omega_{n-1}}{n} \cdot (R_2^n - R_1^n)} \leq \delta. \tag{5}$$

Remark 11. *The $r_0 > 0$ we shall choose has the property that for all $R_2 \geq R_1 \geq r_0$, $\overline{B}(Q, R_2) \backslash B(Q, R_1)$ is contained in the component (diffeomorphic to $S^{n-1} \times \mathbb{R}$) of $\mathcal{M} - N\left(S^{n-1} \times [-L_b, L_b]\right)$ bounded by $N(S^{n-1} \times \{L_b\})$.*

Recall that a geodesic is called normal if it is parametrized by arc length. To prove this proposition we need the following form of the relative volume comparison theorem (see Theorem 3.1, p. 226 in Zhu [Z-97]). Let Γ be any measurable subset of the unit sphere $S_p^{n-1} \subset T_p\mathcal{M}$. Given $r \leq R$, let

$$A_{r,R}^{\Gamma}(p) \doteqdot \left\{ x \in \mathcal{M} : \begin{array}{c} r \leq d(x, p) \leq R \text{ and there exists a} \\ \text{normal minimal geodesic } \gamma \text{ from} \\ \gamma(0) = p \text{ to } x \text{ satisfying } \gamma'(0) \in \Gamma \end{array} \right\}.$$

Fix any point p_H in the simply-connected space form of dimension n and of constant sectional curvature H, let $A^\Gamma_{r,R}(p_H)$ be the corresponding set in the space form. Clearly $A^\Gamma_{r,R}(p) \subset \overline{B}(p,R) \backslash B(p,r)$, and if $\Gamma = S^{n-1}_p$ then $A^\Gamma_{r,R}(p) = \overline{B}(p,R) \backslash B(p,r)$.

Lemma 12. (Bishop-Gromov relative volume comparison theorem) *Let* $(\mathcal{M}^n, g)$ *be a complete Riemannian manifold with* $Ric_\mathcal{M} \geq (n-1)H$. *If* $r \leq R$, $s \leq S$, $r \leq s$, $R \leq S$ *and* Γ *is as above, then*

$$\frac{\text{Vol}(A^\Gamma_{s,S}(p))}{\text{Vol}^H(A^\Gamma_{s,S}(p_H))} \leq \frac{\text{Vol}(A^\Gamma_{r,R}(p))}{\text{Vol}^H(A^\Gamma_{r,R}(p_H))},$$

where Vol^H *is the volume in the space form.*

Proof of Proposition 10. By multiplying the metric g by a positive constant if necessary, we may assume that the center sphere $N(S^{n-1} \times \{0\})$ has mean radius $r(0) = 1$. Note that N remains an (ε, k, L)-neck after the scaling and the desired estimate (5) does not change after the scaling. Let $N^b = N|_{S^{n-1} \times [-L_b, L_b]} : S^{n-1} \times [-L_b, L_b] \to \mathcal{M}$. It is clear that N^b is a (ε, k, L_b)-neck.

It follows from Lemma 9 that $\mathcal{M} - N^b \left(S^{n-1} \times [-L_b, L_b] \right)$ has two components $\mathcal{U}_1$ and $\mathcal{U}_2$, where $\mathcal{U}_1$ is diffeomorphic to an open ball B^n and without loss of generality we may assume bounds $N^b \left(S^{n-1} \times \{-L_b\} \right)$, and $\mathcal{U}_2$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$ and bounds $N^b \left(S^{n-1} \times \{L_b\} \right)$. Let $Q \in N^b(S^{n-1} \times \{-L_b\})$. Define for $R_2 > R_1 > 0$

$$\Gamma = \left\{ \gamma'(0) \in S^{n-1}_Q : \begin{array}{c} \text{there is } x \text{ such that } R_1 \leq d(Q,x) \leq R_2 \\ \text{and there is exactly one normal minimal} \\ \text{geodesic } \gamma \text{ from } \gamma(0) = Q \text{ to } x. \end{array} \right\}.$$

Then $A^\Gamma_{R_1,R_2}(Q)$ is a subset of $\overline{B}(Q,R_2) \backslash B(Q,R_1)$ and

$$\text{Vol}\left[A^\Gamma_{R_1,R_2}(Q) \right] = \text{Vol}\left[\overline{B}(Q,R_2) \backslash B(Q,R_1) \right] \tag{6}$$

since $\left[\overline{B}(Q,R_2) \backslash B(Q,R_1) \right] \backslash A^\Gamma_{R_1,R_2}(Q)$ is contained in the set of cut locus points of Q, which has measure zero.

Let $H = 0$. The corresponding space form is Euclidean space, and

$$\text{Vol}^0(A^\Gamma_{R_1,R_2}(0)) = \frac{m(\Gamma)}{n} \cdot (R_2^n - R_1^n)$$

where $m(\Gamma)$ is the measure of Γ in S^{n-1}_Q. We will apply Lemma 12 to (6) to prove (5). Thus we need to find for comparison another relative volume

which is less or equal to δ. We will find this other relative volume by using the embedded neck.

Choose r_0 large enough (depending on the neck N^b and the manifold $\mathcal{M}$), so that $B(Q, r_0) \supset \mathcal{U}_1 \cup N^b\left(S^{n-1} \times [-L_b, L_b]\right)$. This implies that if $R_2 > R_1 > r_0$, then $A^{\Gamma}_{R_1, R_2}(Q) \subset \mathcal{U}_2$.

For any normal minimal geodesic $\gamma_0 : [0, \ell_0] \to \mathcal{M}$ with $\gamma_0(0) = Q$, $\gamma_0(\ell_0) \in A^{\Gamma}_{R_1, R_2}(Q)$ and $\gamma_0'(0) \in \Gamma$, then γ_0 will intersect $N^b\left(S^{n-1} \times \{0\}\right)$ at some (exactly one) point, say $\gamma_0(w_0)$, and we claim that

$$w_0 \geq \frac{9}{10}L_b \ . \tag{7}$$

To see the claim, since $\varepsilon_b \doteqdot \varepsilon'(\frac{1}{10}, 1, L_b)$, by Lemma 8 N^b is an absolute $\left(\frac{1}{10}, 1, L_b\right)$-neck and hence (here we use the assumption that mean radius $r(0) = 1$)

$$\frac{9}{10} | \bullet |_{N^b_* \overline{g}} \leq | \bullet |_g \leq \frac{11}{10} | \bullet |_{N^b_* \overline{g}} \tag{8}$$

where $N^b_* \overline{g}$ is the push-forward metric. From $\gamma_0(w_0) \in N^b\left(S^{n-1} \times \{0\}\right)$ and $Q = \gamma_0(0) \in N^b\left(S^{n-1} \times \{-L_b\}\right)$ we have

$$d_{N^b_* \overline{g}}\left(\gamma_0(0), \gamma_0(w_0)\right) \geq L_b.$$

Hence

$$w_0 = d_g\left(\gamma_0(0), \gamma_0(w_0)\right) \geq \frac{9}{10} d_{N_* \overline{g}}\left(\gamma_0(0), \gamma_0(w_0)\right) = \frac{9}{10}L_b.$$

We now claim that (this is only a rough estimate)

$$w_0 + 10 \leq r_0. \tag{9}$$

Since $\mathcal{M} \backslash B(Q, r_0) \subset \mathcal{U}_2$, we have

$$r_0 \geq d_g(Q, \mathcal{U}_2) \geq d_g(Q, N^b(S^{n-1} \times \{L_b\}).$$

Since any minimal geodesic from Q to any $p \in N^b(S^{n-1} \times \{L_b\})$ must intersect $N^b(S^{n-1} \times \{0\})$, we have

$$d_g(Q, N^b(S^{n-1} \times \{L_b\}) \geq d_g(Q, N^b(S^{n-1} \times \{0\})$$
$$+ d_g(N^b(S^{n-1} \times \{0\}, N^b(S^{n-1} \times \{L_b\}).$$

By (8) the diameter

$$diam_g(S^{n-1} \times \{0\}) \leq \frac{11}{10} \cdot diam_{N^b_* \overline{g}}(S^{n-1} \times \{0\}) = \frac{11\pi}{10},$$

it follows

$$d_g(Q, N^b(S^{n-1} \times \{0\}) \geq d_g(Q, \gamma(w_0)) - \frac{11\pi}{10} = w_0 - \frac{11\pi}{10}.$$

From (8) we have

$$d_g(N^b(S^{n-1} \times \{0\}, N^b(S^{n-1} \times \{L_b\})$$
$$\geq \frac{9}{10} \cdot d_{N^b_*\overline{g}}(N^b(S^{n-1} \times \{0\}, N^b(S^{n-1} \times \{L_b\}) = \frac{9}{10} \cdot L_b$$

Hence
$$w_0 - \frac{11\pi}{10} + \frac{9}{10} \cdot L_b \leq r_0.$$

The claim follows since $L_b \geq 16$.

Now we choose the r, R, s and S in Lemma 12 as

$$r \doteq w_0 - 2, \quad R \doteq w_0 + 2, \quad s \doteq R_1, \quad S \doteq R_2.$$

It is clear from the choice of $L_b \geq 16$ and $r_0 < R_1$ that $r \leq R \leq s \leq S$. We claim

Sublemma

$$A^\Gamma_{w_0-2,w_0+2}(Q) \subset \left\{ x \in \mathcal{M} : d_g\left(x, N^b\left(S^{n-1} \times \{0\}\right)\right) \leq 6 \right\},$$

where Γ is the set defined above.

Remark 13. *Intuitively, the set* $\{x \in \mathcal{M} : d_g\left(x, N^b\left(S^{n-1} \times \{0\}\right)\right) \leq 6\}$ *is close to a standard cylinder of length* 12 *and radius* 1.

Problem 14. *Given* R_1 *and* R_2, *let* Γ *be defined as above.*

Proof of sublemma. Let R_1, R_2, Γ and w_0 be as above. Note that $\gamma_0(w_0) \in N^b\left(S^{n-1} \times \{0\}\right)$ and Γ corresponds to R_1 and R_2. Given any point $x \in A^\Gamma_{w_0-2,w_0+2}(Q)$, let $\gamma : [0, \ell_1] \to \mathcal{M}$ be a normal minimal geodesic with $\gamma(0) = Q$, $\gamma'(0) \in \Gamma$, $\gamma(\ell_1) \in \overline{B}(Q, R_2) \backslash B(Q, R_1)$ and $\gamma(\ell_x) = x$ for some $\ell_x \in [w_0 - 2, w_0 + 2]$. The geodesic γ exists since $x \in A^\Gamma_{w_0-2,w_0+2}(Q)$ implies there exists a normal minimal geodesic $\bar{\gamma} : [0, \ell_x] \to \mathcal{M}$ with $\bar{\gamma}'(0) \in \Gamma$, $\bar{\gamma}(\ell_x) = x$ and $\ell_x \in [w_0 - 2, w_0 + 2]$. Since $\bar{\gamma}'(0) \in \Gamma$, $\bar{\gamma}$ extends to a normal minimal geodesic γ as above. From $\gamma(\ell_1) \notin B(Q, R_1)$, we have $\ell_1 \geq R_1 > r_0$. Then $\gamma(\ell_1) \notin \mathcal{U}_1 \cup N^b\left(S^{n-1} \times [-L_b, L_b]\right)$ and hence γ will intersect

$N^b \left(S^{n-1} \times \{0\} \right)$ at some point $\gamma(w_x)$ (for the same reason as before as applied to γ_0). That is,

$$\gamma(w_x) \in N^b \left(S^{n-1} \times \{0\} \right).$$

Since the mean radius of $N^b \left(S^{n-1} \times \{0\} \right)$ is 1, we have by (8)

$$d_g(\gamma(w_x), \gamma_0(w_0)) \leq \frac{11\pi}{10}.$$

Hence we get

$$|w_x - w_0| \leq \frac{11\pi}{10} \leq 4. \tag{10}$$

From the triangle inequality

$$|d_g(Q, \gamma(w_x)) - d_g(Q, \gamma_0(w_0))| \leq d_g(\gamma(w_x), \gamma_0(w_0)).$$

On the other hand

$$w_0 - 2 \leq d_g(Q, x) = \ell_x \leq w_0 + 2. \tag{11}$$

Combining (10) and (11) we get

$$|l_x - w_x| \leq 6, \tag{12}$$

which implies

$$d_g(\gamma(w_x), x) = d_g \left(\gamma(w_x), \gamma(\ell_x) \right) \leq 6.$$

Since $\gamma(w_x) \in N \left(S^{n-1} \times \{0\} \right)$,

$$d_g(N^b \left(S^{n-1} \times \{0\} \right), x) \leq 6.$$

This completes the proof of the sublemma. $\square$

It follows from the sublemma and (8) that

$$\mathrm{Vol} \left[A_{w_0-2,w_0+2}^{\Gamma}(Q) \right]$$
$$\leq \mathrm{Vol} \left(\{x \in \mathcal{M} : d_g(x, N^b \left(S^{n-1} \times \{0\} \right)) \leq 6\} \right)$$
$$\leq \mathrm{Vol} \left(\{x \in \mathcal{M} : d_{N^b_* \overline{g}}(x, N^b \left(S^{n-1} \times \{0\} \right)) \leq \frac{10}{9} \cdot 6\} \right)$$
$$\leq \left(\frac{11}{10} \right)^n \cdot \mathrm{Vol}_{N^b_* \overline{g}} \left(\{x \in \mathcal{M} : d_{N^b_* \overline{g}}(x, N^b \left(S^{n-1} \times \{0\} \right)) \leq \frac{10}{9} \cdot 6\} \right)$$
$$= \left(\frac{11}{10} \right)^n \cdot \omega_{n-1} \cdot 1^n \cdot \frac{10}{9} \cdot 12.$$

We now finish the proof of the proposition. Applying Lemma 12 we get

$$\frac{\text{Vol}\left[B(Q,R_2)\backslash B(Q,R_1)\right]}{\frac{m(\Gamma)}{n}\cdot(R_2^n-R_1^n)} \leq \frac{\text{Vol}\left(A_{w_0-2,w_0+2}^{\Gamma}(Q)\right)}{\frac{m(\Gamma)}{n}\cdot((w_0+2)^n-(w_0-2)^n)}$$

since $w_0-2 \leq w_0+2 \leq r_0 \leq R_1 \leq R_2$. Replacing the common factor $m(\Gamma)$ by ω_{n-1} and applying the estimate above, we get

$$\frac{\text{Vol}\left[B(Q,R_2)\backslash B(Q,R_1)\right]}{\frac{\omega_{n-1}}{n}\cdot(R_2^n-R_1^n)} \leq \frac{\text{Vol}\left(A_{w_0-2,w_0+2}^{\Gamma}(Q)\right)}{\frac{\omega_{n-1}}{n}\cdot((w_0+2)^n-(w_0-2)^n)}$$

$$\leq \frac{\left(\frac{11}{10}\right)^n \cdot \omega_{n-1}\cdot\frac{10}{9}\cdot 12}{\frac{\omega_{n-1}}{n}\cdot((w_0+2)^n-(w_0-2)^n)}.$$

We have proved by using (7)

$$\frac{\text{Vol}\left[B(Q,R_2)\backslash B(Q,R_1)\right]}{\frac{\omega_{n-1}}{n}\cdot(R_2^n-R_1^n)} \leq \frac{\left(\frac{11}{10}\right)^n \cdot \omega_{n-1}\cdot\frac{10}{9}\cdot 12}{\frac{\omega_{n-1}}{n}\cdot((\frac{9}{10}L_b+2)^n-(\frac{9}{10}L_b-2)^n)}. \qquad (13)$$

If we choose $L_b \geq \max(L_a, 16)$ satisfying

$$\frac{\left(\frac{11}{10}\right)^n \cdot \omega_{n-1}\cdot\frac{10}{9}\cdot 12}{\frac{\omega_{n-1}}{n}\cdot((\frac{9}{10}L_b+2)^n-(\frac{9}{10}L_b-2)^n)} \leq \delta, \qquad (14)$$

and choose $\varepsilon_b = \varepsilon'(\frac{1}{10}, 1, L_b)$, then the proposition follows from (13) and (14).

4. Proof of the main result.

The main part of this section is devoted to estimate the relative volume in (5) from below by ASCR when R_1 and R_2 is large and dimension n is odd; see Proposition 21 below. The main result Theorem 3 is proved at the very end of this section.

4.1. Asymptotic scalar curvature ratio.

Let $(\mathcal{M}^n, g)$ be a complete, noncompact Riemannian manifold with positive sectional curvature and $Q \in \mathcal{M}$. Define a function $a: \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+$ by

$$a(r)^2 = \sup_{x\in\mathcal{M}\backslash B(Q,r)} R\left(x\right)d\left(x,Q\right)^2, \qquad (15)$$

a function $\kappa(r) : \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+$ by

$$\kappa(r) = \sup_{x \in \mathcal{M} \setminus B(Q,r)} R(x),$$

and a function $\rho(r) : \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+$ by

$$\rho(r) \doteqdot \frac{\pi r}{4a(r)}.$$

The following lemma is clear.

Lemma 15. *(i) $a(r)$ is positive and monotone nonincreasing.*
(ii) $ASCR(g) = \lim_{r \to \infty} a(r)^2$.
(iii) $\kappa(r) \cdot r^2 \le a(r)^2$.

To prove Proposition 21, we assume $ASCR(g) < +\infty$ since otherwise the proposition is clearly true. For any $\eta_1 \in (0,1)$ there is $r_1 = r_1(\eta_1, \mathcal{M})$ such that

$$a(r) \le \sqrt{ASCR(g)} + \eta_1, \text{ for all } r \ge r_1.$$

Clearly we have

$$\frac{a(r)}{r} \to 0^+ \quad \text{as } r \to +\infty,$$

$$\rho(r) \to +\infty \quad \text{as } r \to +\infty$$

$$\frac{r}{\rho(r)} \le \frac{4(\sqrt{ASCR(g)} + \eta_1)}{\pi} \quad \text{for } r \ge r_1. \tag{16}$$

4.2. The hypersurfaces $\widehat{S}_r(\rho)$.

Let S_r be the level set of the Busemann function b_Q (defined in §3.1) and C_r be the sublevel set of the Busemann function b_Q. If S_r is smooth, we define $\widehat{b}_Q \doteqdot b_Q$, $\widehat{C}_r \doteqdot C_r$ and $\widehat{S}_r \doteqdot S_r$. If S_r is not smooth, since $\mathcal{M}$ has positive sectional curvature so b_Q is strictly convex [GW-74], we can smooth b_Q (see, for example, p. 158 of [ESS-89] or [GW-76]). For any positive $\eta_2 \le 1$ there is a smooth and strictly convex function $\widehat{b}_Q$ such that

$$\left| \widehat{b}_Q(x) - b_Q(x) \right| < \eta_2 \text{ for all } x \in \mathcal{M}. \tag{17}$$

We define $\widehat{C}_r \doteqdot \widehat{b}_Q^{-1}(-\infty, r)$ and $\widehat{S}_r \doteqdot \widehat{b}_Q^{-1}(r)$. So in any case we have a smooth and strictly convex hypersurface $\widehat{S}_r$ and a strictly convex set $\widehat{C}_r$. It is clear that $\widehat{S}_r \doteqdot \partial \widehat{C}_r$.

We define the hypersurfaces $\widehat{S}_r(\rho)$ parallel to $\widehat{S}_r$ by

$$\widehat{S}_r(\rho) \doteq \left\{ x \in \mathcal{M} : d\left(x, \widehat{C}_r\right) = \rho \right\}.$$

The following, which is Lemma 2 on p. 157 of [ESS-89], gives an estimate for the second fundamental form of these parallel hypersurfaces.

Lemma 16. (bounds for the 2nd fundamental form of parallel hypersurfaces) *(i) If sectional curvature $K \leq \varepsilon^2$ on $\mathcal{M} - \widehat{C}_r$, then the parallel hypersurfaces are smooth embedded hypersurfaces for $0 < \rho < \pi/(2\varepsilon)$.*

(ii) Let g_r^ρ and $h_r^\rho \in C^\infty\left(S^2 T^* \widehat{S}_r(\rho)\right)$ denote the first and second fundamental forms of $\widehat{S}_r(\rho)$, respectively. Then

$$-\varepsilon \tan(\varepsilon\rho) g_r^\rho \leq h_r^\rho \leq \frac{1}{\rho} g_r^\rho.$$

(iii) Taking $\rho = \pi/(4\varepsilon)$ in (ii) we have

$$-\varepsilon g_r^{\pi/(4\varepsilon)} \leq h_r^{\pi/(4\varepsilon)} \leq \frac{4\varepsilon}{\pi} g_r^{\pi/(4\varepsilon)}.$$

(iv) The Weingarten map $L_r^{\pi/(4\varepsilon)} : T\widehat{S}_r(\pi/(4\varepsilon)) \to T\widehat{S}_r(\pi/(4\varepsilon))$ satisfies

$$\left\| L_r^{\pi/(4\varepsilon)} \right\| \leq \frac{4\varepsilon}{\pi},$$

where $\|L\| \doteq \max_{|v|=1} |L(v)|$.

Remark 17. *Note that if ε is small, then $\pi/(2\varepsilon)$ is large. That is, the parallel hypersurfaces $\widehat{S}_r(\rho)$ are smooth for large ρ. Our conclusion is that, assuming $K \leq \varepsilon^2$ on $\mathcal{M} - \widehat{C}_r$, the second fundamental form of $\widehat{S}_r(\pi/(4\varepsilon))$ is small. The reason we have the weaker $1/\rho$ upper bound on the second fundamental form is that $\widehat{S}_r$ could be close to a point (like a small sphere). In particular, the $1/\rho$ upper bound is sharp for $\widehat{S}_0$ a point in euclidean space.*

To apply Lemma 16 we need to estimate ε in terms of $\kappa(r)$ and hence $a(r)$ in regards to the condition that the sectional curvature $K \leq \varepsilon^2$ on $\mathcal{M} - \widehat{C}_r$. We will use the following elementary result whose proof is given in Appendix B.

Lemma 18. *(i) $\widehat{S}_r \subset b_Q^{-1}((r - \eta_2, r + \eta_2))$ and $\widehat{S}_r \subset \mathcal{M} \backslash \overline{B}(Q, r - \eta_2)$.*
    (ii) $B(Q, r - \eta_2) \subset b_Q^{-1}((-\infty, r - \eta_2)) \subset \widehat{C}_r \subset b_Q^{-1}((-\infty, r + \eta_2))$.
    (iii) $\widehat{S}_r(\rho) \subset \mathcal{M} \backslash \overline{B}(Q, r + \rho - \eta_2)$.
    (iv) If $\widehat{S}_r \subset B(Q, \eta)$, then $\widehat{S}_r(\rho) \subset B(Q, \eta + \rho)$.

4.3. Estimate of the area of $\widehat{S}_r(\rho)$ by Gauss-Bonnet formula.

Note that sectional curvatures are less or equal to scalar curvatures pointwise on a manifold with nonnegative sectional curvature, i.e., $K_x \leq R(x)$. By Lemma 18 (ii) and Lemma 15 (iii)

$$\sup_{\mathcal{M}\backslash\widehat{C}_r} K_x \leq \sup_{\mathcal{M}\backslash B(Q,r-\eta_2)} R(x) = \kappa(r - \eta_2) \leq \frac{a(r - \eta_2)^2}{(r - \eta_2)^2} = \left(\frac{\pi}{4\rho\,(r - \eta_2)}\right)^2 .$$

Hence by Lemma 16 (i), (ii) the hypersurface $\widehat{S}_r(\rho)$ is smooth for $\rho \leq \rho(r-\eta_2)$ and its Weingarten map $L_r(\rho)$ is bounded by

$$\|L_r(\rho)\| \leq \frac{1}{\rho} \text{ for any } \rho \leq \rho(r - \eta_2). \tag{18}$$

Indeed, since $\tan\theta \leq \frac{1}{\theta}$ for $0 < \theta \leq \pi/4$, we have $\frac{\pi}{4\rho(r-\eta_2)} \tan\left(\frac{\pi}{4\rho(r-\eta_2)}\rho\right) \leq \frac{1}{\rho}$ if $\rho \leq \rho\,(r - \eta_2)$.

For the remainder of this section we assume that n is odd and consider only $\rho \in (0, \rho\,(r - \eta_2))$. Let $m \doteq (n - 1)/2$. We shall apply the Gauss-Bonnet formula to the hypersurface $\widehat{S}_r(\rho)$ for $r \geq r_1$ as defined in subsection 4.1. Following [ESS-89] [GW-82], let $G_r(\rho)$ be the Gauss-Bonnet integrand of $\widehat{S}_r(\rho)$ with the induced metric. There are many instances of the formula for $G_r(\rho)$ in the literature (see for example p. 749 of [C-44] or p. 740 of [GW-82]), in general $G_r(\rho)$ is defined by

$$\frac{2}{\omega_{n-1}} G_r(\rho)\, dV = \frac{1}{2^{2m}\pi^m m!} \sum \epsilon_{i_1,\ldots,i_{2m}} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2m-1} i_{2m}},$$

where Ω_{ij} are the curvature 2-forms of the induced metric on $\widehat{S}_r(\rho)$. Recall that $\omega_{n-1} = \mathrm{Vol}\big(S^{2m}\big) = 2^{2m+1}\pi^m m!/\,(2m)!$.Hence

$$G_r(\rho)\, dV = \frac{1}{(2m)!} \sum \epsilon_{i_1,\ldots,i_{2m}} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2m-1} i_{2m}}.$$

By the Gauss-Bonnet formula

$$\frac{2}{\omega_{n-1}} \int_{\widehat{S}_r(\rho)} G_r(\rho)\, dV_{\widehat{S}_r(\rho)} = \chi\left(\widehat{S}_r(\rho)\right) = 2. \tag{19}$$

We need to estimate the Gauss-Bonnet integrand (see [GW-82] p. 740ff or [ESS-89] p. 160ff). Define

$$Q_r(\rho) \doteq G_r(\rho) - \det[L_r(\rho)].$$

From Lemma 18 (iii) we have the following estimate of the sectional curvature K_x of $\mathcal{M}$ at points in $\widehat{S}_r(\rho)$

$$\sup_{x \in \widehat{S}_r(\rho)} K_x \leq \kappa(r + \rho - \eta_2)$$

We estimate $Q_r(\rho)$ as on p. 160 of [ESS-89]. Below $c(n)$ is a constant depending only on n. For $r \geq 1 \geq \eta_2$

$$|Q_r(\rho)| \leq c(n) \cdot \sum_{p=1}^{m} \kappa(r + \rho - \eta_2)^p \cdot \left(|L_r(\rho)|^2\right)^{m-p}$$

$$\leq c(n) \cdot \sum_{p=1}^{m} \frac{a(r + \rho - \eta_2)^{2p}}{(r + \rho - \eta_2)^{2p}} \cdot \left(\frac{1}{\rho^2}\right)^{m-p}$$

$$\leq c(n) \cdot \sum_{p=1}^{m} a(r + \rho - \eta_2)^{2p} \cdot \frac{1}{\rho^{n-1}}.$$

It follows from the monotonicity of $a(r)$ that for $r \geq 1$

$$|Q_r(\rho)| \leq c(n) \cdot \frac{1}{\rho^{n-1}} \cdot \sum_{p=1}^{m} a(r - \eta_2)^{2p}. \tag{20}$$

From (18), (19) and (20) we get for $r \geq 1$

$$\omega_{n-1} = \int_{\widehat{S}_r(\rho)} G_r(\rho) dV_{\widehat{S}_r(\rho)}$$

$$\leq \int_{\widehat{S}_r(\rho)} \left(|Q_r(\rho)| + |\det[L_r(\rho)]|\right) dV_{\widehat{S}_r(\rho)}$$

$$\leq \int_{\widehat{S}_r(\rho)} \left(c(n) \cdot \frac{1}{\rho^{n-1}} \cdot \sum_{p=1}^{m} a(r - \eta_2)^{2p} + \frac{1}{\rho^{n-1}}\right) dV_{\widehat{S}_r(\rho)}.$$

Thus we have obtained the following lower bound for the areas of the hypersurfaces

Lemma 19. *There is a constant $c(n)$ depending only on n such that for any $r \geq 1$ and $\rho \leq \rho(r - \eta_2) = \frac{\pi(r-\eta_2)}{4a(r-\eta_2)}$*

$$\text{Area}\left[\widehat{S}_r(\rho)\right] \geq \frac{\omega_{n-1}}{c(n) \cdot \sum_{p=1}^{m} a(r - \eta_2)^{2p} + 1} \cdot \rho^{n-1}. \tag{21}$$

4.4. The final argument.

Lemma 20. *(i) For $r \geq 1$*

$$\widehat{S}_r \subset B\left(Q, \frac{r + \eta_2}{1 - \theta\,(r - \eta_2)}\right) \Big\backslash\, B\,(Q, r - \eta_2).$$

(ii) For $r \geq 1$ and $\rho \leq \rho(r - \eta_2)$

$$\widehat{S}_r(\rho) \subset B\left(Q, \frac{r + \eta_2}{1 - \theta\,(r - \eta_2)} + \rho\right) \Big\backslash\, B\,(Q, r - \eta_2).$$

Proof. (i) For any $x \in \widehat{S}_r$ it follows from Lemma 18 (i) that $d(Q, x) \geq r - \eta_2$ and $b_Q(x) \leq r + \eta_2$. By Lemma 7 (ii), (i) we have for

$$b_Q(x) \geq (1 - \theta\,(d(Q, x))) \cdot d(Q, x) \geq (1 - \theta(r - \eta_2)) \cdot d(Q, x),$$

Hence

$$d(Q, x) \leq \frac{r + \eta_2}{1 - \theta\,(r - \eta_2)}.$$

(ii) This follows from (i) and Lemma 18 (iv). $\qquad\square$

By Lemma 7 (i) there is a $r_2 = r_2(\mathcal{M})$ such that $\theta(r) < 1/2$ for all $r \geq r_2$.

Proposition 21. *Let $(\mathcal{M}^n, g)$ be a complete, noncompact Riemannian manifold with positive sectional curvature and $Q \in \mathcal{M}$. For any $\eta_1, \eta_2 > 0$ there is $r_1 = r_1(\eta_1, \mathcal{M})$ and $r_2 = r_2(\mathcal{M})$ such that for any $r \geq \max\{r_1, r_2\} + 5$*

$$\mathrm{Vol}\left[B\left(Q, \frac{r + \eta_2}{1 - \theta\,(r - \eta_2)} + \rho(r - \eta_2)\right) \Big\backslash\, B(Q, r - \eta_2)\right]$$

$$\geq \frac{\omega_{n-1}}{c(n) \cdot \sum_{p=1}^{m}(\sqrt{ASCR(g)} + \eta_1)^{2p} + 1} \cdot \frac{\rho(r - \eta_2)^n}{n}.$$

Proof. From Lemma 20 (ii)

$$\cup_{0 \leq \rho \leq \rho(r-\eta_2)}\widehat{S}_r(\rho) \subset B\left(Q, \frac{r + \eta_2}{1 - \theta\,(r - \eta_2)} + \rho(r - \eta_2)\right) \Big\backslash\, B(Q, r - \eta_2)$$

we get by the area estimate (21)

$$
\begin{aligned}
\mathrm{Vol}&\left[B\left(Q, \frac{r+\eta_2}{1-\theta\,(r-\eta_2)} + \rho(r-\eta_2)\right) \setminus B(Q, r-\eta_2)\right] \\
&\geq \mathrm{Vol}(\cup_{0\leq\rho\leq\rho(r-\eta_2)}\widehat{S}_r(\rho)) \\
&\geq \int_0^{\rho(r-\eta_2)} \frac{\omega_{n-1}}{c(n)\cdot\sum_{p=1}^m a(r-\eta_2)^{2p}+1}\cdot\rho^{n-1}d\rho \\
&= \frac{\omega_{n-1}}{c(n)\cdot\sum_{p=1}^m a(r-\eta_2)^{2p}+1}\cdot\frac{\rho(r-\eta_2)^n}{n}.
\end{aligned}
\tag{22}
$$

The proposition follows from $\theta\,(r-\eta_2)\leq\frac{1}{2}$ and $a(r-\eta_2)\leq\sqrt{ASCR(g)}+\eta_1$ when $r\geq r\geq\max\{r_1,r_2\}+5$. $\qquad\square$

Now we finish the proof of Theorem 3. Let $(\varepsilon_0,k_0,L_0) = (\min(\varepsilon_a,\varepsilon_b),k_a,L_b)$ given in Proposition 10. Let $N:S^{n-1}\times[-L,L]\to\mathcal{M}$ be a (ε,k,L)-neck with $(\varepsilon,k,L)\preccurlyeq(\varepsilon_0,k_0,L_0)$ and $Q\in N(S^{n-1}\times\{-L_0\})$; this will give us a constant r_0. Choose positive $\eta_1<1$ and $\eta_2<1$, this will give two constants r_1 and r_2. Choosing an $r\geq\max\{r_0,r_1,r_2\}+5$ and letting $R_2\doteq\frac{r+\eta_2}{1-\theta(r-\eta_2)}+\rho(r-\eta_2)$ and $R_1\doteq r-\eta_2$ in Proposition 10, we get from (13) and (22)

$$
\begin{aligned}
&\frac{\left(\frac{11}{10}\right)^n\cdot\omega_{n-1}\cdot\frac{10}{9}\cdot12}{\frac{\omega_{n-1}}{n}\cdot\left(\left(\frac{9}{10}L_0+2\right)^n-\left(\frac{9}{10}L_0-2\right)^n\right)} \\
&\geq \frac{\frac{\omega_{n-1}}{c(n)\cdot\sum_{p=1}^m a(r-\eta_2)^{2p}+1}\cdot\frac{\rho(r-\eta_2)^n}{n}}{\frac{\omega_{n-1}}{n}\cdot\left(\left[\frac{r+\eta_2}{1-\theta(r-\eta_2)}+\rho(r-\eta_2)\right]^n-(r-\eta_2)^n\right)} \\
&\geq \frac{1}{c(n)\cdot\sum_{p=1}^m a(r-\eta_2)^{2p}+1}\cdot\frac{1}{\left(\frac{r+\eta_2}{[1-\theta(r-\eta_2)]\cdot\rho(r-\eta_2)}+1\right)^n} \\
&\geq \frac{1}{c(n)\cdot\sum_{p=1}^m a(r-\eta_2)^{2p}+1}\cdot\frac{1}{\left(\frac{12}{\pi}a(r-\eta_2)+1\right)^n},
\end{aligned}
$$

where in the last inequality we have used $\frac{r+\eta_2}{\rho(r-\eta_2)}\leq\frac{6}{\pi}a(r-\eta_2)$ and $\theta\,(r-\eta_2)\leq1/2$. This is because $r\geq5$ and $\frac{r-\eta_2}{\rho(r-\eta_2)}=\frac{4}{\pi}a(r-\eta_2)$. Using

$a(r - \eta_2) \leq \sqrt{ASCR(g)} + \eta_1$ we get

$$
\frac{\left(\frac{11}{10}\right)^n \cdot \omega_{n-1} \cdot 1^n \cdot \frac{10}{9} \cdot 12}{\frac{\omega_{n-1}}{n}\left(\left(\frac{9}{10}L_0 + 2\right)^n - \left(\frac{9}{10}L_0 - 2\right)^n\right)}
$$
$$
\geq \frac{1}{c(n) \cdot \sum_{p=1}^{m}\left(\sqrt{ASCR(g)} + \eta_1\right)^{2p} + 1} \cdot \frac{1}{\left(\frac{12}{\pi}\left(\sqrt{ASCR(g)} + \eta_1\right) + 1\right)^n}.
$$

Now it is clear that we can make $ASCR(g) \geq C_0$ if we choose L_0 large enough. Theorem 3 is proved.

5. Existence of necklike points in ancient solutions.

We first recall the following result of Hamilton which is based on an estimate of Hamilton (Theorem 24.4 in [H-95a]) and Ivey [I-93].

Lemma 22. If $\left(\mathcal{M}^3, g\left(t\right)\right)$, $t \in (-\infty, \omega)$, is a complete ancient solution of the Ricci flow with bounded curvature, then $g\left(t\right)$ has nonnegative sectional curvature for all $t \in (-\infty, \omega)$.

Proof. See [H-95a] or for a more detailed proof [CKE].   □

We also note that in a lemma in §19 of [H-95a], Hamilton also proved that if $(\mathcal{M}^n, g\left(t\right))$ is a non-Ricci flat ancient solution with nonnegative Ricci curvature, then there exists a constant $c_0 > 0$ depending only on n such that

$$
\liminf_{t \to -\infty} |t| \sup_{x \in \mathcal{M}} R\left(x, t\right) \geq c_0.
$$

(See also Lemma 19.4 of [H-95a] for a related result for Type I singularities.) In some sense this may be considered as an elementary gap-type result for ancient solutions to the Ricci flow regarding the geometry at $t = -\infty$ of the space-time manifold $\mathcal{M}^n \times (-\infty, \omega)$.

Following the terminology of [CKE] we shall say that (x, t) is an *ancient Type I-like c-essential point* if

$$
|\mathrm{Rm}\left(x, t\right)| \cdot |t| \geq c > 0.
$$

We say that (x, t) is a δ-*necklike point* if there exists a unit 2-form θ at (x, t) such that

$$
|\mathrm{Rm} - R\left(\theta \otimes \theta\right)| \leq \delta \left|\mathrm{Rm}\right|.
$$

We have the following result due to the first named author which we believe to have first appeared in the unpublished notes [CKE]. Similar results were proved earlier by Hamilton in Theorem 24.6 of [H-95a] on the existence of necklike points in Type I singular solutions to the Ricci flow on compact 3-manifolds on finite time intervals $[0, T)$ (see also Theorem 3.3 in §2.3 of [H-97] for a similar result in dimension four) and Corollary 3.5 of §2.3 of [H-97] on the existence of necklike points in Type I-like ancient solutions with bounded positive isotropic curvature.

Theorem 23. *Let $\left(\mathcal{M}^3, g\left(t\right)\right), t \in \left(-\infty, \omega\right)$, be a complete ancient solution of the Ricci flow with bounded positive sectional curvature. Suppose that*

$$\sup_{\mathcal{M} \times (-\infty, 0]} |t|^\gamma R\left(x, t\right) < \infty$$

for some $\gamma > 0$. Then either

1. *$\left(\mathcal{M}, g\left(t\right)\right)$ is isometric to a shrinking spherical space form, or*

2. *there exists a constant $c > 0$ such that for all $\tau \in \left(-\infty, 0\right]$ and $\delta > 0$, there exist $x \in \mathcal{M}$ and $t \in \left(-\infty, \tau\right)$ such that (x, t) is an ancient Type I-like c-essential point and a δ-necklike point.*

Proof. We shall show that if for every $c > 0$ there exist $\tau \in \left(-\infty, 0\right]$ and $\delta > 0$ such that there are no ancient Type I-like c-essential δ-necklike points before time τ, then $(\mathcal{M}, g(t))$ is isometric to a shrinking spherical space form.

By the hypothesis, there exists $\gamma > 0$ such that

$$K \doteq \sup_{\mathcal{M} \times (-\infty, 0]} |t|^\gamma R\left(x, t\right) < \infty.$$

(When $\gamma = 1$, this is the definition of an ancient Type I-like solution.) Since the scalar curvature of $(\mathcal{M}, g(t))$ is positive, the function

$$G \doteq |t|^{\gamma \varepsilon / 2} \frac{|\overset{\circ}{\mathrm{Rm}}|^2}{R^{2-\varepsilon}}$$

is well-defined. Since the sectional curvatures are positive, $|\overset{\circ}{\mathrm{Rm}}| \le |\mathrm{Rm}| \le R$, and we have the estimates

$$G \le R^\varepsilon |t|^{\gamma \varepsilon / 2} \le K^\varepsilon |t|^{-\gamma \varepsilon / 2},$$

which show that G is bounded for all times $-\infty < t \le 0$ and satisfies

$$\lim_{t \to -\infty} \sup_{x \in \mathcal{M}} G\left(x, t\right) = 0. \tag{23}$$

If there are no ancient c-essential δ-necklike points on the time interval $(-\infty, \tau]$, then for every $x \in \mathcal{M}$ and $t \in (-\infty, \tau)$ either

$$|\mathrm{Rm}\left(x, t\right)| \cdot |t| < c \tag{24}$$

or we have

$$|\mathrm{Rm} - R\left(\theta \otimes \theta\right)| > \delta\, |\mathrm{Rm}| \tag{25}$$

for every unit 2-form θ at (x, t).

A straightforward computation yields that if ϕ is a nonnegative function and ψ is a positive function, both defined on space and time, then

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{\phi^\alpha}{\psi^\beta}\right) = \alpha\frac{\phi^{\alpha-1}}{\psi^\beta}\left(\frac{\partial}{\partial t} - \Delta\right)\phi - \beta\frac{\phi^\alpha}{\psi^{\beta+1}}\left(\frac{\partial}{\partial t} - \Delta\right)\psi$$

$$- \alpha\left(\alpha - 1\right)\frac{\phi^{\alpha-2}}{\psi^\beta}\left|\nabla\phi\right|^2 - \beta\left(\beta + 1\right)\frac{\phi^\alpha}{\psi^{\beta+2}}\left|\nabla\psi\right|^2$$

$$+ 2\alpha\beta\frac{\phi^{\alpha-1}}{\psi^{\beta+1}}\left\langle\nabla\phi, \nabla\psi\right\rangle.$$

Taking $\phi = (-t)^{\gamma\varepsilon/2}\,|\overset{\circ}{\mathrm{Rm}}|^2$, $\psi = R$, $\alpha = 1$, and $\beta = 2 - \varepsilon$, a computation yields

$$\frac{\partial}{\partial t}G \le \Delta G + \frac{2\left(1 - \varepsilon\right)}{R}\left\langle\nabla G, \nabla R\right\rangle + 2J,$$

where

$$J \doteq \frac{|t|^{\gamma\varepsilon/2}}{R^{3-\varepsilon}}\left[\varepsilon|\overset{\circ}{\mathrm{Rm}}|^2\left(|\mathrm{Rm}|^2 - \frac{\gamma R}{4\,|t|}\right) - P\right]$$

and

$$P \doteq \lambda^2\left(\mu - \nu\right)^2 + \mu^2\left(\lambda - \nu\right)^2 + \nu^2\left(\lambda - \mu\right)^2 \ge 0$$

(here λ, μ, ν are the eigenvalues of Rm.) Fix any (x, t) with $t < \tau \le 0$. If the first alternative (24) holds there with $c \le \gamma/8$, then we may estimate the term

$$|\mathrm{Rm}|^2 - \frac{\gamma R}{4\,|t|} \le R\left(|\mathrm{Rm}| - \frac{\gamma}{4\,|t|}\right) < R\left(\frac{c}{|t|} - \frac{\gamma}{4\,|t|}\right) \le -\frac{\gamma R}{8\,|t|}$$

and hence dropping the $-P \le 0$ term yields

$$J \le -\frac{\gamma\varepsilon}{8\,|t|}G.$$

On the other hand if the second alternative (25) holds, then Lemma 27 in Appendix C implies there exists $\eta(\delta) > 0$ such that $P \geq \eta |\mathrm{Rm}|^2 |\overset{\circ}{\mathrm{Rm}}|^2$ whence it follows that taking $\varepsilon \leq \eta$ gives

$$J \leq -\frac{\gamma \varepsilon}{4|t|} G.$$

Thus in either case, G is a subsolution of the heat equation for all times $-\infty < t < \tau$, because

$$\frac{\partial}{\partial t} G \leq \Delta G + \frac{2(1-\varepsilon)}{R} \langle \nabla G, \nabla R \rangle - \frac{\gamma \varepsilon}{2|t|} G.$$

By the weak maximum principle, which applies even when $\mathcal{M}$ is noncompact since the both the curvatures and G are bounded, $\sup_{x \in \mathcal{M}} G(x,t)$ is a non-increasing function of time. We may then use (23) to conclude that $G \equiv 0$, hence that $(\mathcal{M}^3, g(t))$ locally isometric to a round $\mathcal{S}^3$. Since $(\mathcal{M}^3, g(t))$ is complete, we conclude that it is compact and globally isometric to a spherical space form $\mathcal{S}^3 / \Gamma$. $\qquad \square$

Now we shall assume that $\mathcal{M}^3$ is noncompact. Since $g(t)$ has positive sectional curvature, by a result of Gromoll and Meyer [GM-69], there is an injectivity radius estimate:

Proposition 24. $\mathcal{M}^3$ is diffeomorphic to $\mathbb{R}^3$ and

$$inj\left(\mathcal{M}^3, g(t)\right) \geq \frac{\pi}{\sqrt{K_{\sup}(t)}}$$

where $K_{\sup}(t)$ is the supremum of the sectional curvatures of $g(t)$.

We may apply this estimate and a standard compactness theorem to obtain a cylinder limit solution. In particular, we have:

Theorem 25. If $\left(\mathcal{M}^3, g(t)\right)$, $t \in (-\infty, \omega)$, is a complete noncompact Type I-like ancient solution of the Ricci flow with bounded positive sectional curvature on an orientable 3-manifold, then there exists a sequence of points and times $(x_i, t_i) \in \mathcal{M}^3 \times (-\infty, \omega)$ such that the dilated and translated solutions $\left(\mathcal{M}^3, g_i(t), x_i\right)$, $t \in (-\infty, \omega_i)$, where

$$g_i(t) = R(x_i, t_i) \cdot g\left(t_i + \frac{t}{R(x_i, t_i)}\right)$$

and $\omega_i = R(x_i, t_i)(T - t_i)$, limit to a solution $\left(\mathcal{M}_\infty^3, g_\infty(t), x_\infty\right)$, $t \in (-\infty, \omega_\infty)$, to the Ricci flow isometric to the standard shrinking cylinder $S^2 \times \mathbb{R}$.

Remark 26. *The conclusion also holds with the additional condition on the sequence of times:* $\lim_{i\to\infty} t_i = -\infty$.

Proof. Choose any sequence $\{\delta_i\}_{i\in\mathbb{N}}$ with $\lim_{i\to\infty}\delta_i = 0$. By Theorem 23 (with $\gamma = 1$,) since $\mathcal{M}$ is noncompact (and hence $(\mathcal{M}, g(t))$ is not isometric to a shrinking spherical space form), there exists a constant $c > 0$ such that there exists points and times (x_i, t_i) such that

$$|\mathrm{Rm}\,(x_i, t_i)| \cdot |t_i| \geq c > 0.$$

and

$$|\mathrm{Rm} - R\,(\theta \otimes \theta)|\,(x_i, t_i) \leq \delta_i\,|\mathrm{Rm}\,(x_i, t_i)|\,.$$

for some unit 2-forms θ_i. By Gromoll and Meyer's injectivity radius estimate, we may apply Hamilton's Gromov-type compactness theorem for solutions of the Ricci flow [H-95b] to the sequence of pointed solutions $\left(\mathcal{M}^3, g_i(t), x_i\right)$, $t \in (-\infty, \omega_i)$ where

$$g_i(t) = R(x_i, t_i) \cdot g\left(t_i + \frac{t}{R(x_i, t_i)}\right)$$

and $\omega_i = R(x_i, t_i)(T - t_i)$. We obtain a complete limit ancient solution to the Ricci flow $\left(\mathcal{M}^3_\infty, g_\infty(t), x_\infty\right)$, $t \in (-\infty, \omega_\infty)$, on a noncompact[1] orientable 3-manifold with bounded nonnegative sectional curvature and

$$\mathrm{Rm}\,(g_\infty(x_\infty, 0)) = R(g_\infty(x_\infty, 0))(\theta_\infty \otimes \theta_\infty)$$

for some unit 2-form θ_∞. By the strong maximum principle we conclude that the universal covering solution $\left(\widetilde{\mathcal{M}}^3_\infty, \tilde{g}_\infty(t)\right)$, $t \in (-\infty, \omega_\infty)$, is isometric to the product of $\mathbb{R}$ and a complete ancient Type I-like solution $\left(\mathcal{N}^2, h_\infty(t)\right)$, $t \in (-\infty, \omega_\infty)$, to the Ricci flow on a surface with bounded positive curvature. By Theorem 26.1 of [H-95a], $\left(\mathcal{N}^2, h_\infty(t)\right)$ is isometric to a shrinking round sphere (and in particular, $\mathcal{N}^2$ is compact.) Now there are only two noncompact orientable quotients of $S^2 \times \mathbb{R}$: $S^2 \times \mathbb{R}$ itself and $\mathbb{R}\tilde{\times}\mathbb{R}P^2$, the nontrivial $\mathbb{R}$-bundle over $\mathbb{R}P^2$. Topologically, $\mathbb{R}\tilde{\times}\mathbb{R}P^2 \cong S^2\times[0,\infty)/\sim$ where $(x,0) \sim (-x,0)$, which is diffeomorphic to $\mathbb{R}P^3 - \bar{B}^3$. If $\mathcal{M}^3_\infty \cong \mathbb{R}\tilde{\times}\mathbb{R}P^2$, then $\mathcal{M}^3$ admits an embedded (one-sided) $\mathbb{R}P^2$ (since $\mathcal{M}^3_\infty$ does). However, by the work of Gromoll and Meyer [GM-69], we know that $\mathcal{M}^3$ is diffeomorphic to $\mathbb{R}^3$. This yields a contradiction. Hence $\left(\mathcal{M}^3_\infty, g_\infty(t), x_\infty\right)$ is isometric to $\mathbb{R} \times \left(\mathcal{N}^2, h_\infty(t)\right)$, where $\left(\mathcal{N}^2, h_\infty(t)\right)$ is a shrinking round 2-sphere. $\qquad\square$

---

[1]If $\mathcal{M}^3_\infty$ were compact, then $\mathcal{M}^3$ would be diffeomorphic to $\mathcal{M}^3_\infty$, which contradicts the asumption that $\mathcal{M}^3$ is noncompact.

We are now in a position to prove Theorem 4 assuming Theorem 3.

Proof of Theorem 4. Let $\left(\mathcal{M}^3, g\left(t\right)\right), -\infty < t < \omega$, be a complete orientable noncompact ancient Type I-like solution to the Ricci flow with bounded positive sectional curvature. By Theorem 25, there exists a sequence of points and times $(x_i, t_i) \in \mathcal{M}^3 \times (-\infty, \omega)$ such that the dilated and translated solutions $\left(\mathcal{M}^3, g_i\left(t\right), x_i\right), t \in (-\infty, \omega_i)$, limit to the standard shrinking cylinder $S^2 \times \mathbb{R}$. Now, by Theorem 3, given any $A_0 < \infty$, there exists $(\varepsilon_0, k_0, L_0)$ such that if $\left(\mathcal{M}^3, g\right)$ is a complete, noncompact Riemannian manifold with bounded positive sectional curvature and containing an (ε, k, L)-neck with $(\varepsilon, k, L) \preccurlyeq (\varepsilon_0, k_0, L_0)$, then $\mathrm{ASCR}(g) \geq A_0$. On the other hand, since $\left(\mathcal{M}^3, g_i\left(0\right), x_i\right)$ limits to a standard cylinder $S^2 \times \mathbb{R}$, for i large enough, there exists an $(\varepsilon_0, k_0, L_0)$-neck in $\left(\mathcal{M}^3, g_i\left(0\right), x_i\right)$. This implies

$$\mathrm{ASCR}\left(g\left(t_i\right)\right) = \mathrm{ASCR}\left(g_i\left(0\right)\right) \geq A_0.$$

In §19 of [H-95a] Hamilton has proved that the asymptotic scalar curvature ratio of a complete ancient solution to the Ricci flow with bounded nonnegative curvature operator (when $n = 3$, this is the same as nonnegative sectional curvature) is constant in time. This implies that for all $t \in (-\infty, \omega)$ we have $\mathrm{ASCR}(g\left(t\right)) \geq A_0$. Since $A_0 < \infty$ is arbitrary, we conclude that

$$ASCR\left(g\left(t\right)\right) \equiv \infty$$

for all $t \in (-\infty, \omega)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

6. Appendices.

6.1. A.

Proof of Lemma 8. It suffices to show that there is ε' such that if $N : S^{n-1} \times [a, b] \to \mathcal{M}$ is an (ε', k, L)-neck then inequalities (3) and (4) are satisfied on $z \in [\frac{a+b}{2} - L, \frac{a+b}{2} + L]$.

Assume $\varepsilon' \leq \min\left\{1, \frac{\ln 2}{2L}\right\}$ so that $\left|\frac{d \ln r(z)}{dz}\right| \leq \varepsilon'$. Using the inequality $|e^x - 1| \leq 2|x|$ for $|x| \leq \ln 2$, we get for $z \in [\frac{a+b}{2} - L, \frac{a+b}{2} + L]$,

$$\left| \ln \frac{r^2(z)}{r^2\left(\frac{a+b}{2}\right)} \right| \leq 2 \left| \int_{(a+b)/2}^{z} \frac{d}{d\zeta} \left(\ln r\left(\zeta\right)\right) d\zeta \right| \leq 2L\varepsilon' \leq \ln 2,$$

which implies

$$\left| \frac{r^2(z)}{r^2(\frac{a+b}{2})} - 1 \right| \le 2 \left| \ln \frac{r^2(z)}{r^2(\frac{a+b}{2})} \right| \le 4\varepsilon' L.$$

From $\left| \frac{1}{r^2(z)} g - \bar{g} \right|_{\bar{g}} \le \varepsilon'$ we get

$$\left| \frac{1}{r^2(z)} \cdot g \right|_{\bar{g}} \le \sqrt{n} + \varepsilon'$$

since $|\bar{g}|_{\bar{g}}^2 = \bar{g}^{ij}\bar{g}_{ij} = n$. Hence for $z \in [\frac{a+b}{2} - L, \frac{a+b}{2} + L]$

$$\left| \frac{1}{r^2(\frac{a+b}{2})} \cdot g - \bar{g} \right|_{\bar{g}} \le \left| \frac{1}{r^2(z)} \cdot g - \bar{g} \right|_{\bar{g}} + \left| \left(\frac{r^2(z)}{r^2(\frac{a+b}{2})} - 1 \right) \cdot \left(\frac{1}{r^2(z)} \cdot g \right) \right|_{\bar{g}}$$

$$\le \varepsilon' + 4\varepsilon' L \cdot (\sqrt{n} + \varepsilon').$$

If we choose

$$\varepsilon' \le \min \left\{ 1, \frac{\ln 2}{2L}, \frac{\varepsilon}{1 + 8L\sqrt{n}} \right\},$$

inequality (3) will hold for $z \in [\frac{a+b}{2} - L, \frac{a+b}{2} + L]$.

Let

$$R(z) = \ln \frac{r^2(z)}{r^2(\frac{a+b}{2})}.$$

Assume now $\varepsilon' \le \min \left\{ \frac{1}{2}, \frac{\ln 2}{2L} \right\}$. From $\left| \frac{d^j \log r(z)}{dz^j} \right| \le \varepsilon'$ for $1 \le j \le k$ and $e^{R(z)} \le 1 + 4\varepsilon' L$ we get for $1 \le j \le k$ and $z \in [\frac{a+b}{2} - L, \frac{a+b}{2} + L]$

$$\left| \frac{d^j R(z)}{dz^j} \right| \le 2\varepsilon'$$

and

$$\left| \frac{d^j}{dz^j} \frac{r^2(z)}{r^2(\frac{a+b}{2})} \right| = \left| \frac{d^j}{dz^j} e^{R(z)} \right| \le c(j) \cdot e^{R(z)} \cdot \max_{i_1 + \cdots + i_p = j} \left\{ \left| \frac{d^{i_1} R(z)}{dz^{i_1}} \bullet \cdots \bullet \frac{d^{i_p} R(z)}{dz^{i_p}} \right| \right\}$$

$$\le c(j) \cdot (1 + 4\varepsilon' L) \cdot 2\varepsilon'$$

where $c(j)$ is a constant depending only on j. Hence for $1 \le j \le k$ and

$$z \in [\tfrac{a+b}{2} - L, \tfrac{a+b}{2} + L]$$

$$\left| \bar{\nabla}^j \left(\frac{1}{r^2\left(\frac{a+b}{2}\right)} \cdot g \right) \right|_{\bar{g}} = \left| \bar{\nabla}^j \left(e^{R(z)} \cdot \frac{1}{r^2\left(z\right)} \cdot g \right) \right|_{\bar{g}}$$

$$\leq \sum_{i=0}^{j} \binom{j}{i} \cdot \left| \frac{d^i}{dz^i} e^{R(z)} \right| \cdot \left| \bar{\nabla}^{j-i} \left(\frac{1}{r^2\left(z\right)} \cdot g \right) \right|_{\bar{g}}$$

$$\leq (1 + 4\varepsilon' L)\varepsilon' + \sum_{i=1}^{j-1} \binom{j}{i} \cdot c(i) \cdot (1 + 4\varepsilon' L) \cdot 2\varepsilon' \cdot \varepsilon' + c(j) \cdot (1 + 4\varepsilon' L) \cdot 2\varepsilon'.$$

Note that in the last inequality above we have used $\left| \bar{\nabla}^{j-i} \left(\frac{1}{r^2(z)} \cdot g \right) \right|_{\bar{g}} \leq \varepsilon'$
for $1 \leq j - i \leq k$. So there is a constant C_k depending only on k, such that
if we choose

$$\varepsilon' \leq \min\left\{ \frac{1}{2}, \frac{\ln 2}{2L}, \frac{\varepsilon}{C_k \cdot (1 + 4L)} \right\},$$

then inequality (4) will hold for $z \in [\tfrac{a+b}{2} - L, \tfrac{a+b}{2} + L]$. The lemma is proved.
$\square$

6.2. B.

Proof of Lemma 18. We shall use the properties of Busemann functions in
Lemma 6.

(i) For any $x \in \widehat{S}_r$, $\widehat{b}_Q(x) = r$. It follows from (17) that $|b_Q(x) - r| < \eta_2$.
Hence $\widehat{S}_r \subset b_Q^{-1}((r - \eta_2, r + \eta_2))$.

For any $x \in \widehat{S}_r$, $b_Q(x) > r - \eta_2$. Since $b_Q(x) \leq d(Q, x)$, $d(Q, x) > r - \eta_2$
and $\widehat{S}_r \subset \mathcal{M} \backslash \overline{B}(Q, r - \eta_2)$.

(ii) For any $x \in B(Q, r - \eta_2)$, $d(Q, x) < r - \eta_2$. Since $b_Q(x) \leq d(Q, x)$,
$b_Q(x) < r - \eta_2$ and $B(Q, r - \eta_2) \subset b_Q^{-1}((-\infty, r - \eta_2))$.

For any $x \in b_Q^{-1}((-\infty, r - \eta_2))$, $b_Q(x) < r - \eta_2$. By (17), $\widehat{b}_Q(x) <$
$b_Q(x) + \eta_2 < r$. Hence $b_Q^{-1}((-\infty, r - \eta_2)) \subset \widehat{C}_r$.

For any $x \in \widehat{C}_r$, $\widehat{b}_Q(x) < r$. By (17), $b_Q(x) < \widehat{b}_Q(x) + \eta_2 < r + \eta_2$. Hence
$\widehat{C}_r \subset b_Q^{-1}((-\infty, r + \eta_2))$.

(iii) For any $x \in \widehat{S}_r(\rho)$, $d(x, \widehat{S}_r) = d(x, \widehat{C}_r) = \rho$. Let γ be a minimal
geodesic from Q to x. It is clear that γ intersects $\widehat{S}_r$ at some point $\gamma(l)$, thus

$$d(Q, x) = d(Q, \gamma(l)) + d(\gamma(l), x) \geq d(Q, \widehat{S}_r) + d(\widehat{S}_r, x) > r - \eta_2 + \rho,$$

and hence $\widehat{S}_r(\rho) \subset \mathcal{M} \backslash \overline{B}(Q, r + \rho - \eta_2)$.

(iv) For any $x \in \widehat{S}_r(\rho)$ let $y \in \widehat{S}_r$ be a point satisfying $d(x, y) = d(x, \widehat{S}_r) = \rho$. Then from $\widehat{S}_r \subset B(Q, \eta)$

$$d(Q, x) \leq d(Q, y) + d(y, x) < \eta + \rho,$$

so that $\widehat{S}_r(\rho) \subset B(Q, \eta + \rho)$. $\square$

6.3. C.

Here we give the proof of an estimate of Hamilton used in the proof of Theorem 23.[2]

Lemma 27. *If for some $\delta \in (0, 1)$ we have $|\mathrm{Rm} - R\,(\theta \otimes \theta)|^2 \geq \delta \, |\mathrm{Rm}|^2$ for every unit 2-form θ, then*

$$P \geq \frac{\delta}{96\,(3 - \delta)} \, |\mathrm{Rm}|^2 \, |\overset{\circ}{\mathrm{Rm}}|^2.$$

Proof. We may assume without loss of generality that $|\lambda| \geq |\mu| \geq |\nu|$. The hypothesis implies that

$$\mu^2 + \nu^2 + \mu\nu \geq \frac{\delta}{2} \left(\lambda^2 + \mu^2 + \nu^2 \right),$$

and hence that $\mu^2 + \nu^2 \geq \frac{\delta}{3-\delta}\lambda^2$. Since $|\mu| \geq |\nu|$ by assumption, we have

$$P = \lambda^2 \left(\mu - \nu \right)^2 + \mu^2 \left(\lambda - \nu \right)^2 + \nu^2 \left(\lambda - \mu \right)^2$$

$$\geq \lambda^2 \left(\mu - \nu \right)^2 + \frac{\delta}{2\,(3 - \delta)} \lambda^2 \left(\lambda - \nu \right)^2 + \nu^2 \left(\lambda - \mu \right)^2.$$

Now notice that

$$|\mathrm{Rm}|^2 = \lambda^2 + \mu^2 + \nu^2 \leq 3\lambda^2$$

and

$$|\overset{\circ}{\mathrm{Rm}}|^2 = \frac{1}{3} \left[(\lambda - \mu)^2 + (\lambda - \nu)^2 + (\mu - \nu)^2 \right] \leq \frac{4}{3} \left(\lambda^2 + \mu^2 + \nu^2 \right) \leq 4\lambda^2.$$

So if $|\nu| \leq |\lambda|/2$, we have

$$P \geq \frac{\delta}{2\,(3 - \delta)} \lambda^2 \left(\lambda - \nu \right)^2 \geq \frac{\delta}{8\,(3 - \delta)} \lambda^4 \geq \frac{\delta}{96\,(3 - \delta)} \, |\mathrm{Rm}|^2 \, |\overset{\circ}{\mathrm{Rm}}|^2,$$

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[2]This is a version of an estimate in the proof of Theorem 24.6 of [H-95a].

while if $|\nu| > |\lambda|/2$, we get

$$P > \lambda^2 (\mu - \nu)^2 + \frac{\delta}{2(3-\delta)} \lambda^2 (\lambda - \nu)^2 + \frac{1}{4} \lambda^2 (\lambda - \mu)^2$$

$$\geq \frac{\delta}{2(3-\delta)} \lambda^2 \left[(\mu - \nu)^2 + (\lambda - \nu)^2 + (\lambda - \mu)^2 \right] \geq \frac{\delta}{2(3-\delta)} |\text{Rm}|^2 \, |\overset{\circ}{\text{Rm}}|^2,$$

because $\frac{1}{4} > \frac{\delta}{2(3-\delta)}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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