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Regular Hypersurfaces, Intrinsic Perimeter and Implicit Function Theorem in Carnot Groups

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1. Introduction.

In the last few years, a systematic attempt to develop geometric measure theory in metric spaces has become the object of many studies. Such a program, already suggested in Federer's book [17], has been explicitly formulated and carried on by several authors. We only mention some of them: De Giorgi [14], [15], [16], Gromov [28], [29], Preiss and Tisěr [44], Kirchheim [33], David & Semmes [11], Cheeger [9] and Ambrosio and Kirchheim [3], [4].

In this paper we study, inside a special class of metric spaces i.e. the Carnot groups, a classical problem in Geometric Measure Theory that is the problem of defining regular hypersurfaces and different reasonable surface measures on them, and of understanding their relationships (here hypersurface means simply codimension 1 surface).

First of all a few words on the ambient space: Carnot groups, each one endowed with its Carnot-Carathéodory distance d_c (hereafter abbreviated as cc-distance), are particularly interesting metric spaces not only because they appear in many different mathematical theories (e.g. Several Complex Variables, Partial Differential Equations, Control Theory) but also because they provide examples of spaces that are non Euclidean at any scale yet have a rich geometric structure as families of natural translations and dilations.

We recall briefly the definition: a Carnot group is a connected, simply connected, nilpotent Lie group $\mathbb{G} \equiv (\mathbb{R}^n, \cdot)$ with graded Lie algebra $\mathfrak{g}$ that is

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Lie generated by its first layer V_1, the so called horizontal layer,

$$\mathfrak{g} = V_1 \oplus ... \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \text{ if } i > k. \tag{1}$$

Assume that $X_1, \dots, X_m$ is a family of left invariant vector fields that is also an orthonormal basis of $V_1 \equiv \mathbb{R}^m$ at the origin, that is $X_1(0) = \partial_{x_1}, \dots, X_m(0) = \partial_{x_m}$. The Lie algebra of the group $\mathbb{G}$ can be canonically endowed with a family of dilations, so that $\mathbb{G}$ is also a homogeneous group with homogeneous dimension $Q = \sum_{i=1}^{k} i \dim V_i$ and k is called step of the group (see [19]).

We say that an absolutely continuous curve $\gamma : [0,T] \to \mathbb{G}$ is a sub-unit curve with respect to $X_1, \dots, X_m$ if there exist real measurable functions $c_1(s), \dots, c_m(s)$, $s \in [0,T]$ such that $\sum_j c_j^2 \leqslant 1$ and

$$\dot{\gamma}(s) = \sum_{j=1}^{m} c_j(s) X_j(\gamma(s)), \qquad \text{for a.e. } s \in [0,T].$$

If $p, q \in \mathbb{G}$, their cc-distance $d_c(p,q)$ is

$$d_c(p,q) = \inf \{T > 0 : \ \gamma : [0,T] \to \mathbb{G} \text{ is subunit}, \gamma(0) = p, \gamma(T) = q\}. \tag{2}$$

The fact, that under assumption (1), $d_c(p,q)$ is finite for any p, q is the content of Chow theorem (see e.g. [6] or [28]). We recall that the topology induced on $\mathbb{R}^n$ by d_c is the Euclidean topology, but from a metric point of view $\mathbb{G}$ and Euclidean $\mathbb{R}^n$ can be dramatically different: indeed there are no (even local) bilipschitz maps from a general non commutative group $\mathbb{G}$ to Euclidean spaces. In particular d_c is not locally equivalent to a Riemannian distance. This fact, proved by Semmes (see [45]), relies on a Rademacher's type theorem due to Pansu ([42] see also Vodop'yanov [47]) and on algebraic and metric properties related to the non-commutativity of $\mathbb{G}$. Moreover observe that the intrinsic Hausdorff dimension of a Carnot group $\mathbb{G}$ (i.e. with respect to the cc-distance d_c) agrees with its homogeneous dimension Q. Notice that Q is always strictly larger than n, the topological dimension of $\mathbb{G}$ (see [40]).

Coming now to the problem of surfaces and their measures, observe that the notion of regular surface in a group is not a completely obvious one and that its full comprehension is certainly a starting point for a geometrical understanding of the group. A rich and stimulating discussion on this topic can be found in Section 2 of [28].

There is a classical definition of 'good' surface in a metric space that goes back at least to Federer (see [17] 3.2.14). According to it, a 'good' surface in a metric space is the image of an open subset of an Euclidean space via a Lipschitz map. Such a notion has been successfully used recently by Ambrosio and Kirchheim (see [3], [4]) to develop a theory of currents in metric spaces. Unfortunately it does not fit the geometry of Carnot groups: indeed, as proved by the same authors in [3], in the Heisenberg group $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$, for example, any Lipschitz image of an open subset of $\mathbb{R}^{2n}$ would be purely unrectifiable. This fact simply means that open subsets of $\mathbb{R}^d$ are not appropriate as parameter spaces of surfaces inside a group $\mathbb{G}$; it is necessary to use open subsets of metric spaces more strictly related to $\mathbb{G}$ (about this, see also the definition of rectifiable sets in Carnot groups given in [43]).

On the other hand there is a way of circumventing this difficulty when dealing with codimension 1 surfaces. In any Euclidean space $\mathbb{R}^d$, a C^1 hypersurface can be equivalently viewed (locally) as the zero set of a function $f : \mathbb{R}^d \to \mathbb{R}$ with non-vanishing gradient and in Carnot groups it is natural to follow the same approach.

If U is an open subset of $\mathbb{G}$ and $f : \mathrm{U} \to \mathbb{R}$ we say that f belongs to $C^1_{\mathbb{G}}(\mathrm{U})$ when f and $Xf := (X_1 f, ..., X_m f)$ are continuous functions in U. We say that $S \subset \mathbb{R}^n$ is a $\mathbb{G}$-*regular hypersurface* if for any $p \in S$ there is an open $\mathrm{U} \ni p$ and $f \in C^1_{\mathbb{G}}(\mathrm{U})$ such that

$$S \cap \mathrm{U} = \{q \in \mathrm{U} : \quad f(q) = 0 \ \text{ and } Xf(q) \neq 0\}. \tag{3}$$

In [28], Gromov proved that a topological $(n-1)$-dimensional surface in $\mathbb{G}$ has intrinsic Hausdorff dimension larger than $Q-1$. Here we prove that regular hypersurfaces have precisely intrinsic Hausdorff dimension $Q-1$ and topological dimension $n-1$ (but they might have Euclidean Hausdorff dimension larger than $n-1$).

We recall also that in [22] when studying regular surfaces and surface measures in the special case of the Heisenberg groups $\mathbb{H}^n$, we defined S to be an $\mathbb{H}$-regular surfaces if (3) holds. This definition came out to be a good one, since we were able to prove there an implicit function theorem, yielding a local continuous parametrization of S, an integral representation of the intrinsic Hausdorff measure, and an area type formula. These results made us able to extend, to the setting of $\mathbb{H}^n$, De Giorgi's theory on the rectifiability of the boundary of finite perimeter sets as well as De Giorgi's generalized Gauss-Green formula.

In this paper we prove in a general Carnot group $\mathbb{G}$ a corresponding implicit function theorem. It might be surprising that the statement reads as in $\mathbb{H}^n$, because it is known that, usually, passing from Heisenberg groups to general Carnot groups could make things quite different because a Carnot group can be strongly different from another one, once more due to the possible different stratifications of their Lie algebras that make their geometries not (even locally) comparable.

By the way, as for De Giorgi's rectifiability theory in general Carnot groups, the problem is far from be fully settled, even if a positive answer is given in [23] for a large class of Carnot groups containing all step 2 groups.

Our implicit function theorem states (see Theorem 2.1) that, locally, $S \cap \mathrm{U}$ is the graph in the directions of the integral lines of an appropriate vector field X_i, where $X_i f \neq 0$, of a continuous function ϕ. A more precise statement is as follows:

Assume, without loss of generality, that in (3) we have $0 \in S$, $X_1 f(0) \neq 0$ and that $X_1(0) = \partial_{x_1}$ (see also (18)), then there are $\tilde{\mathrm{U}}$, I_δ and ϕ, with $0 \in \tilde{\mathrm{U}} \subset \mathrm{U}$, $I_\delta = \{\xi \in \mathbb{R}^{n-1} : \ |\xi_i| \leq \delta\}$ and $\phi : I_\delta \to \mathbb{R}$ such that

$$S \cap \tilde{\mathrm{U}} = \{p \in \tilde{\mathrm{U}} : \ p = \exp\left(\phi(\xi) X_1\right)(0, \xi) := \Phi(\xi), \ \xi \in I_\delta\}. \tag{4}$$

In general ϕ is not as regular as one might wish (see Example 3 and Theorem 6.5, vi in[22]), nevertheless through ϕ we can write explicitly the surface measure of S in local coordinates

$$|\partial E|_{\mathbb{G}}(\tilde{\mathrm{U}}) = \int_{I_\delta} \frac{\sqrt{\sum_{i=1}^m |X_i f(\Phi)|^2}}{X_1 f(\Phi)}\, d\mathcal{L}^{n-1} \tag{5}$$

The perimeter measure $|\partial E|_{\mathbb{G}}$ comes from considering S as the topological boundary of the set $E = \{p \in \mathbb{G} : \ p = \exp(tX_1)(\xi), \ t < \phi(\xi), \xi \in I_\delta\}$. Then $S \cap \tilde{\mathrm{U}} = \partial E \cap \tilde{\mathrm{U}}$ and the perimeter measure $|\partial E|_{\mathbb{G}}$ is defined as the total variation of the characteristic function of E. The measure $|\partial E|_{\mathbb{G}}$ is supported on S. The subscript $\mathbb{G}$ is somehow incorrect because the perimeter as well as the cc-distance d_c depend on the choice of the orthonormal family $X = X_1, \ldots, X_m$: changing the base in the horizontal fiber V_1 actually changes the perimeter, but the two remain comparable. Observe that this is the same as in a Riemannian manifold: changing the metric tensor changes the perimeter and the new one is comparable with the old one. Later on we shall fix the family X and this amounts to the choice of a privileged coordinate system in $\mathfrak{g}$.

On the other hand also different surface measures can be considered, all of them depending on the cc-distance. We want to compare the following ones: the $(Q-1)$-Hausdorff measure $\mathcal{H}_{\mathbb{G}}^{Q-1}$,the $(Q-1)$-spherical Hausdorff measure $\mathcal{S}_{\mathbb{G}}^{Q-1}$ and the Minkowski content $\mathcal{M}$. We recall briefly the definitions.

$$\mathcal{H}_{\mathbb{G}}^{Q-1}(S) = \lim_{\delta \to 0^+} \mathcal{H}_{\mathbb{G},\delta}^{Q-1}(S); \qquad \mathcal{S}_{\mathbb{G}}^{Q-1}(S) = \lim_{\delta \to 0^+} \mathcal{S}_{\mathbb{G},\delta}^{Q-1}(S)$$

where respectively

$$\mathcal{H}^{Q-1}_{\mathbb{G},\delta}(S) = \inf\left\{\sum_i \left(\frac{\mathrm{diam} C_i}{2}\right)^{Q-1} :\ S \subset \bigcup_i C_i;\ \mathrm{diam} C_i < \delta\right\},$$

$$\mathcal{S}^{Q-1}_{\mathbb{G},\delta}(S) = \inf\left\{\sum_i \left(\frac{\mathrm{diam} B_i}{2}\right)^{Q-1} :\ S \subset \bigcup_i B_i;\ \mathrm{diam} B_i < \delta\right\}$$

where the infimum is taken with respect to closed sets C_i in the first line or closed cc-balls B_i in the second one.

For any open $\mathrm{U} \subset \mathbb{R}^n$ the Minkowski content $\mathcal{M}(S)(\mathrm{U})$ is

$$\mathcal{M}(S)(\mathrm{U}) = \lim_{\delta \to 0^+} \frac{1}{2\delta} \mathcal{L}^n(S_\delta \cap \mathrm{U})$$

provided the limit exists, where $S_\delta = \{p \in \mathbb{G} :\ d_c(p, S) < \delta\}$ and $\mathcal{L}^n$ is the n-dimensional Lebesgue measure in $\mathbb{R}^n$, which is, by the way, the Haar measure of $\mathbb{G}$.

In the same hypotheses of the implicit function theorem, with the notations of (3) and (5), we prove that (see Theorem 3.3) the perimeter and the $\mathcal{S}^{Q-1}_{\mathbb{G}}$ (and hence $\mathcal{H}^{Q-1}_{\mathbb{G}}$) are comparable measures, i.e. there is $\alpha > 1$ and a Borel function $\mathfrak{s} : S \cap \mathrm{U} \to \mathbb{R}$ such that for all $\tilde{\mathrm{U}} \subset \mathrm{U}$ we have $\frac{1}{\alpha} \le \mathfrak{s} \le \alpha$ a.e. on $S \cap \tilde{\mathrm{U}}$ and

$$\int_{S \cap \tilde{\mathrm{U}}} \mathfrak{s}\, d\mathcal{S}^{Q-1}_{\mathbb{G}} = |\partial E|_{\mathbb{G}}(\tilde{\mathrm{U}}) \tag{6}$$

Moreover in [41], if S is also an Euclidean C^∞ surface, it is proved that

$$|\partial E|_{\mathbb{G}}(\tilde{\mathrm{U}}) = \mathcal{M}(S)(\tilde{\mathrm{U}}). \tag{7}$$

In some special Carnot groups, as the Heisenberg groups, the function $\mathfrak{s}$ in (6) is a constant, and one gets the following stronger version of (6)

$$|\partial E|_{\mathbb{G}}(\tilde{\mathrm{U}}) = c\, \mathcal{S}^{Q-1}_{\mathbb{G}}(S \cap \tilde{\mathrm{U}}). \tag{8}$$

It may be interesting to compare the notion of $\mathbb{G}$-regular hypersurface and that of Euclidean C^1 hypersurface. Strictly speaking, the two classes are different. However, it is enough to remove a "negligeable" closed set from an Euclidean C^1 hypersurface to obtain a $\mathbb{G}$-regular hypersurface, whereas $\mathbb{G}$-regular hypersurfaces can be drammatically irregular from an Euclidean point of view: they may be $(n-1)$-topological submanifold with Euclidean dimension larger than $n-1$. For more precise statements, see Subsection 3.2.

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1.1. Definitions and Notations.

Consider a family X of vector fields $X = (X_1, ..., X_m) \in \mathrm{Lip}\,(\mathbb{R}^n; \mathbb{R}^n)^m$. As usual we identify vector fields and differential operators. If

$$X_j(x) = \sum_{i=1}^{n} c_i^j(x)\partial_i, \quad j = 1, ..., m,$$

define the $m \times n$ matrix

$$C(x) = [c_i^j(x)]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}. \tag{9}$$

Given the family X of Lipschitz continuous vector fields it is well known that subunit curves can be defined as we do in the Introduction for Carnot groups and conseqently the Carnot-Carathéodory distance d_c is well defined provided that there is a subunit curve joining each couple of points. Through the paper, whenever the Carnot-Carathéodory distance is mentioned, we are assuming implicitely that this connectivity property holds and that the distance d_c is continuous with respect to the Euclidean topology. We shall denote $U_c(p, r)$ the open balls associated with d_c.

X_j^* is the operator formally adjoint to X_j in $L^2(\mathbb{R}^n)$, that is the operator which for all $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \phi(x) X_j \psi(x)\, dx = \int_{\mathbb{R}^n} \psi(x) X_j^* \phi(x)\, dx.$$

Moreover, if $f \in L^1_{\mathrm{loc}}$ is a scalar function and $\phi \in (L^1_{\mathrm{loc}})^m$ is a m-vector valued function, we define the X-*gradient* and X-*divergence* as the following distributions:

$$Xf := (X_1 f, ..., X_m f), \quad \mathrm{div}_X\,(\phi) := -\sum_{j=1}^{m} X_j^* \phi_j.$$

Let us remind now the notion of functions of bounded X-variation and recall some of their properties (see [8], [20] and [25]). Let $\Omega \subset \mathbb{R}^n$ be an open set and let

$$F(\Omega; \mathbb{R}^m) := \{\phi \in C_0^1(\Omega; \mathbb{R}^m) : |\phi(x)| \le 1 \ \ \forall x \in \Omega\}. \tag{10}$$

The space $BV_X(\Omega)$ is the set of functions $f \in L^1(\Omega)$ such that

$$\|Xf\|(\Omega) := \sup_{\phi \in F(\Omega;\mathbb{R}^m)} \int_\Omega f(x)\mathrm{div}_X\,(\phi)(x)\,dx < \infty. \tag{11}$$

The space $BV_{X,\mathrm{loc}}(\Omega)$ is the set of functions belonging to $BV_X(U)$ for each open set $U \subset\subset \Omega$. From Riesz representation theorem it follows that if $f \in BV_{X,\mathrm{loc}}(\Omega)$ then the total variation $\|Xf\|$ is a Radon measures on Ω (see [17], 2.2.5).

Theorem 1.1. (Structure of BV_X functions) *Let $BV_{X,\mathrm{loc}}(\Omega)$, then there exists a $\|Xf\|$-measurable function $\sigma_f : \Omega \to \mathbb{R}^m$ such that $|\sigma_f(x)|_{\mathbb{R}^m} = 1$, for $\|Xf\|$-almost every x, and*

$$\int_\Omega f(x)\mathrm{div}_X\,\phi(x)\,dx = \int_\Omega \langle \phi(x), \sigma_f(x)\rangle_{\mathbb{R}^m} d\|Xf\|,$$

for all $\phi \in F(\Omega;\mathbb{R}^m)$.

In perfect analogy with the Euclidean setting, the total variation $\|Xf\|$ is lower semicontinuous with respect to L^1 convergence, (see [8], [20] and [25]), that is we have

Proposition 1.2. (Lower semicontinuity) *Let $f, f_k \in L^1(\Omega)$, $k \in \mathbb{N}$. If $f_k \to f$ in $L^1(\Omega)$, then*

$$\liminf_{k\to\infty} \|Xf_k\|(\Omega) \geq \|Xf\|(\Omega).$$

Definition 1.3. (X-Caccioppoli sets) A measurable set $E \subset \mathbb{R}^n$ is a set with locally finite X-perimeter in Ω (or is a X-Caccioppoli set) if the indicatrix function $\mathbf{1}_E \in BV_{X,\mathrm{loc}}(\Omega)$. In this case the total variation measure $\|X\mathbf{1}_E\|_\infty$ is called *perimeter measure of* E and is indicated as $|\partial E|_X$; hence we have

$$|\partial E|_X(U) := \|X\mathbf{1}_E\|(U) < \infty \tag{12}$$

for every open set $U \subset\subset \Omega$. Moreover, the vector function $\sigma_{\mathbf{1}_E}$ appearing in Theorem 1.1, is called X-*generalized inner normal* of E and we set

$$\nu_E(x) := -\sigma_{\chi_E}(x). \tag{13}$$

It is important to observe that when a Caccioppoli set F has a topological boundary ∂F that is an Euclidean C^1 submanifold of $\mathbb{R}^n$, then the perimeter measure can be represented by integration with respect to the Euclidean $n-1$ Hausdorff measure. Precisely we have (see [8])

Proposition 1.4. *If F is a X-Caccioppoli set with C^1 boundary, then the X-perimeter has the following representation*

$$|\partial F|_X(\Omega) = \int_{\partial F \cap \Omega} |C(x) n_F(x)| d\mathcal{H}^{n-1}, \tag{14}$$

here $n_F(x)$ is the Euclidean unit outward normal to F, C is the coefficient matrix of the vector fields (see (9)), and $\mathcal{H}^{n-1}$ is the Euclidean $(n-1)$-dimensional Hausdorff measure.

It is also important to notice that the domain of applicability of formula (14) is restricted to Euclidean regular hypersurfaces. On the other hand, even in simple Carnot groups (see the definition below) the boundary of finite perimeter sets is, in general, a highly irregular set from an Euclidean point of view. Indeed, not only the Euclidean normal can fail to exist almost everywere, but even the Euclidean metric dimension of the boundary could exceed $n-1$, so making the right hand side of (14) divergent for all Ω ([34]). The perimeter representation proved in (vi) of Theorem 2.1 can be viewed as a generalization of (14) to the boundary of finite perimeter sets (see Remark 3.10).

Finally, we recall the definition of *Carnot group* and some of its structures (see [18], [31], [41], [32] and [42]). Let $\mathbb{G} = (\mathbb{R}^n, \cdot)$ be a Lie group whose Lie algebra $\mathfrak{g}$ admits a *stratification*, i.e. there exist linear subspaces $V_1, ..., V_k$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \text{ if } i > k, \tag{15}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the elements $[X, Y]$ with $X \in V_1$ and $Y \in V_i$.

Via the exponential map, it is possible to induce on $\mathbb{G}$, in a canonical way, a family of automorphisms of the group, called *dilations*, $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ $(\lambda > 0)$ such that

$$\delta_\lambda x \equiv \delta_\lambda(x_1, ..., x_n) = (\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n), \tag{16}$$

where $1 = \alpha_1 = ... = \alpha_m < \alpha_{m+1} \leq ... \leq \alpha_n$ are integers and $m = \dim(V_1)$ (see [19] Chapter 1).

The n-dimensional Lebesgue measure in $\mathbb{R}^n$, denoted by $\mathcal{L}^n$, is the Haar measure of the group $\mathbb{G}$. This means that if $E \subset \mathbb{R}^n$ is measurable, then $\mathcal{L}^n(x \cdot E) = \mathcal{L}^n(E)$ for all $x \in \mathbb{G}$. Moreover, if $\lambda > 0$ then $\mathcal{L}^n(\delta_\lambda E) = \lambda^Q \mathcal{L}^n(E)$.

The group law can be written in the form

$$x \cdot y = x + y + \mathcal{Q}(x, y), \quad x, y \in \mathbb{R}^n \tag{17}$$

where $\mathcal{Q} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ has polynomial components and $\mathcal{Q}_1 = ... = \mathcal{Q}_m = 0$ (see [46], Chapter 12, Section 5). Note that the inverse x^{-1} of an element $x \in \mathbb{G}$ has the form $x^{-1} = -x = (-x_1, \dots, -x_n)$.

Choose now a family $X = (X_1, \dots, X_m)$ of left invariant vector fields that is also an orthonormal basis of $V_1 \equiv R^m$ at the origin, in particular we choose X so that $X_1(0) = \partial_{x_1}, \dots, X_m(0) = \partial_{x_m}$. It happens that also these vector fields X_j have polynomial coefficients; more precisely they have the form

$$X_j(x) = \partial_j + \sum_{i=m+1}^{n} a_i^j(x)\partial_i, \quad X_j(0) = \partial_j, \quad j = 1, ..., m; \tag{18}$$

moreover each polynomial a_i^j is homogeneous with respect to the dilations of the group, that is $a_i^j(\delta_\lambda(x)) = \lambda^{\alpha_i - \alpha_j} a_i^j(x)$ (see [19], Proposition 1.26). From this homogeneity it follows that if j belongs to the l-layer, that is if $h_{l-1} := \sum_{i=1}^{l-1} \dim V_i < j \le h_l := \sum_{i=1}^{l} \dim V_i$, then

$$a_i^j(x) = a_i^j(x_1, \dots, x_{h_{l-1}}). \tag{19}$$

We refer to $\{X_1, \dots, X_m\}$ as *canonical generating vector fields* of the group. The subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$ with fibers

$$H\mathbb{G}_x = \text{span}\ \{X_1(x), \dots, X_m(x)\}, \qquad x \in \mathbb{G}$$

is called *horizontal bundle*. We endow each fiber $H\mathbb{G}_x$ with a scalar product $\langle \cdot, \cdot \rangle_x$ and a norm $|\cdot|_x$ that make the moving frame $\{X_1(x), \dots, X_m(x)\}$ to be orthonormal. We shall drop the index x in the scalar product or in the norm, writing $\langle \psi, \phi \rangle$ for $\langle \psi(x), \phi(x) \rangle_x$, if there is no ambiguity.

We shall identify each section of $H\mathbb{G}$ with its canonical coordinates with respect to this moving frame. This way, a section ϕ is identified with a function $\phi = (\phi_1, \dots, \phi_m) : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$. The spaces of smooth sections of the horizontal bundle are denoted respectively by $\mathbf{C}^k(\mathcal{U}, H\mathbb{G})$, $\mathbf{C}_0^k(\mathcal{U}, H\mathbb{G})$, $\mathbf{C}^\infty(\mathcal{U}, H\mathbb{G})$ or $\mathbf{C}_0^\infty(\mathcal{U}, H\mathbb{G})$.

A metric d_c on $\mathbb{G}$ is defined, via Carnot-Carathéodory construction from the vector fields $X_1, \dots, X_m$. Notice that the set of subunit curves joining any two points in $\mathbb{G}$ is never empty, by Chow's theorem, because the rank of the Lie algebra generated by $X_1, \dots, X_m$ is n; hence d_c is a distance on $\mathbb{G}$ inducing on $\mathbb{R}^n$ the Euclidean topology. Notice also that, being defined through left invariant vector fields, d_c enjoys the further property of being well behaved with respect to left translations and dilations. We mean that

$$d_c(z \cdot x, z \cdot y) = d_c(x, y) \quad \text{and} \quad d_c(\delta_\lambda x, \delta_\lambda y) = \lambda d_c(x, y) \tag{20}$$

for $x, y, z \in \mathbb{G}$ and $\lambda > 0$.

The integer $Q = \sum_{j=1}^n \alpha_j = \sum_{i=1}^k i \dim V_i$ is called the homogeneous dimension of the group; Q turns out to be the Hausdorff dimension of $\mathbb{R}^n$ with respect the cc-distance d_c (see [40]).

Inside a group $\mathbb{G}$, we depart slightly from our previous notations. Once a canonical generating family of vector fields X for $\mathbb{G}$ is fixed, we write $\nabla_{\mathbb{G}}$ for X, $\mathrm{div}_{\mathbb{G}}$ for div_X, $|\partial E|_{\mathbb{G}}$ for $|\partial E|_X$, and so on. In particular, we say that a continuous function f belongs to $\mathbf{C}^1_{\mathbb{G}}(\mathcal{U})$ if $\nabla_{\mathbb{G}} f$ is a continuous vector–valued function in $\mathcal{U}$.

As observed for the distance, also the perimeter measure $|\partial E|_{\mathbb{G}}$ enjoys the further properties of being invariant under group translations and $(Q-1)$-homogeneous with respect to group dilations, that is, for all open $\mathcal{O} \subset \mathbb{R}^n$, $x \in \mathbb{G}$ and $\lambda > 0$, we have

$$|\partial E|_{\mathbb{G}}(\mathcal{O}) = |\partial (x \cdot E)|_{\mathbb{G}}(x \cdot \mathcal{O}) \quad \text{and} \quad |\partial E|_{\mathbb{G}}(\mathcal{O}) = \lambda^{1-Q} |\partial(\delta_\lambda E)|_{\mathbb{G}}(\delta_\lambda \mathcal{O}). \tag{21}$$

A remarkable property of the $\mathbb{G}$-perimeter is provided by the following Q-dimensional isoperimetric inequality ([25]).

Proposition 1.5. (Isoperimetric inequality) *There is a positive constant $c_I > 0$ such that for any $\mathbb{G}$-Caccioppoli set E, for all $x \in \mathbb{G}$ and $r > 0$,*

$$\min\{\mathcal{L}^n(E \cap U_c(x, r)), \mathcal{L}^n(E^c \cap U_c(x, r))\}^{\frac{Q-1}{Q}} \le c_I |\partial E|_{\mathbb{G}}(U_c(x, r)) \tag{22}$$

and

$$\min\{\mathcal{L}^n(E), \mathcal{L}^n(E^c)\}^{\frac{Q-1}{Q}} \le c_I |\partial E|_{\mathbb{G}}(\mathbb{R}^n). \tag{23}$$

We define now $\mathbb{G}$-regular hypersurfaces in a Carnot group $\mathbb{G}$, mimicking Definition 6.1 in [22], as non critical level sets of functions in $\mathbf{C}^1_{\mathbb{G}}(\mathbb{R}^\setminus)$.

Definition 1.6. ($\mathbb{G}$-regular hypersurfaces) Let $\mathbb{G}$ be a Carnot group. We shall say that $S \subset \mathbb{G}$ is a $\mathbb{G}$-*regular hypersurface* if for every $x \in S$ there exist a neighborhood $\mathcal{U}$ of x and a function $f \in \mathbf{C}^1_{\mathbb{G}}(\mathcal{U})$ such that

$$S \cap \mathcal{U} = \{y \in \mathcal{U} : f(y) = 0\}; \tag{i}$$

$$\nabla_{\mathbb{G}} f(y) \neq 0 \quad \text{for } y \in \mathcal{U}. \tag{ii}$$

$\mathbb{G}$-regular surfaces have a unique tangent plane at each point. This follows from a Taylor formula for functions in $\mathbf{C}^1_{\mathbb{G}}(\mathbb{R}^n)$ that is basically proved in [42].

Proposition 1.7. *If* $f \in \mathbf{C}^1_{\mathbb{G}}(U_c(p,r))$, *then*

$$f(x) = f(p) + \sum_{j=1}^{m} (X_j f)(p)(x_j - p_j) + o(d_c(x,p)), \qquad \textit{as} \ \ x \to p. \tag{24}$$

If $S = \{x : f(x) = 0\} \subset \mathbb{G}$ is a $\mathbb{G}$-regular hypersurface, the *tangent group* $T^g_{\mathbb{G}} S(x)$ to S at x is

$$T^g_{\mathbb{G}}\ S(x) := \{v = (v_1, \dots, v_n) \in \mathbb{G} : \ \sum_{j=1}^{m} X_j f(x) v_j = 0\}.$$

By (17), $T^g_{\mathbb{G}} S(x)$ is a subgroup of $\mathbb{G}$, that is proper, by (ii) of Definition 1.6. We can define the *tangent plane* to S at x as

$$T_{\mathbb{G}} S(x) := x \cdot T^g_{\mathbb{G}} S(x).$$

We stress that this is a good definition. Indeed the tangent plane does not depend on the particular function f defining the surface S. This is a consequence of points (i) and (iii) of implicit function theorem below that yields

$$T^g_{\mathbb{G}} S(x) = \{v \in \mathbb{G} : \ \langle \nu_E(x), \pi_x v \rangle_x = 0\}$$

where ν_E is the generalized inward unit normal defined in (13) and $\pi_x(v) = \sum_{j=1}^m v_j X_j(x)$. Notice that the map $v \mapsto \pi_x(v)$, for $x \in \mathbb{G}$ fixed,

$$\pi_x(v) = \sum_{j=1}^{m} v_j X_j(x). \tag{25}$$

is a smooth section of $H\mathbb{G}$.

Notice also that, once more from *(iii)* of Theorem 2.1, it follows that ν_E is a continuous function.

If $v^0 = \sum_{i=1}^m v_i X_i(0) \in H\mathbb{G}_0$ we define the halfspaces $S^{\pm}_{\mathbb{G}}(0, v^0)$ as

$$S^+_{\mathbb{G}}(0, v^0) := \{x \in \mathbb{G} : \sum_{i=1}^{m} x_i v_i > 0\} \text{ and } S^-_{\mathbb{G}}(0, v^0) := \{x \in \mathbb{G} : \sum_{i=1}^{m} x_i v_i < 0\}.$$

Their common boundary is the vertical plane

$$\Pi(0, v^0) := \{x : \sum_{i=1}^{m} x_i v_i = 0\}.$$

If $v = \sum_{i=1}^{m} v_i X_i(y) \in H\mathbb{G}_y$, $S_{\mathbb{G}}^{\pm}(y, v)$ and $\Pi(y, v)$ are the translated sets

$$S_{\mathbb{G}}^{\pm}(y, v) := y \cdot S_{\mathbb{G}}^{\pm}(0, v^0) \quad \text{and} \quad \Pi(y, v) = y \cdot \Pi(0, v^0)$$

where v and v^0 have the same components v_i with respect to the left invariant basis X_i. Hence

$$S_{\mathbb{G}}^{\pm}(y, v) = \{x \in \mathbb{G} : \sum_{i=1}^{m} (x_i - y_i) v_i > 0 (< 0)\}. \tag{26}$$

Clearly, $T_{\mathbb{G}} S(x) = \Pi(x, \nu_E(x))$.

2. The Implicit Function Theorem.

Our main result states that a $\mathbb{G}$-regular hypersurface $S = \{f(y) = 0\}$, that is boundary of the set $E = \{f(y) < 0\}$, can be locally parameterized through a function $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ so that the $\mathbb{G}$-perimeter of E can be written explicitly in terms of $\nabla_{\mathbb{G}} f$ and Φ.

Theorem 2.1. (Implicit Function Theorem) *Let Ω be an open set in $\mathbb{R}^n$ identified with a Carnot group $\mathbb{G}$, $0 \in \Omega$, and let $f \in \mathbf{C}_{\mathbb{G}}^1(\Omega)$ be such that $f(0) = 0$ and $X_1 f(0) > 0$. Define*

$$E = \{x \in \Omega : f(x) < 0\}, \quad S = \{x \in \Omega : f(x) = 0\},$$

and, for $\delta > 0$, $h > 0$

$$I_\delta = \{\xi = (\xi_2, \ldots, \xi_n) \in \mathbb{R}^{n-1}, |\xi_j| \leqslant \delta\}, \qquad J_h = [-h, h].$$

If $\xi = (\xi_2, \ldots, \xi_n) \in \mathbb{R}^{n-1}$ and $t \in J_h$, denote now by $\gamma(t, \xi)$ the integral curve of the vector field X_1 at the time t issued from $(0, \xi) = (0, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, i.e.

$$\gamma(t, \xi) = \exp(tX_1)(0, \xi).$$

Then there exist $\delta, h > 0$ such that the map $(t, \xi) \to \gamma(t, \xi)$ is a diffeomorphism of a neighborhood of $J_h \times I_\delta$ onto an open subset of $\mathbb{R}^n$, and, if we *denote by $\mathcal{U} \subset\subset \Omega$ the image of* $\mathrm{Int}(J_h \times I_\delta)$ *through this map, we have that $E \cap \mathcal{U}$ is connected and*

$$E \text{ has finite } \mathbb{G}\text{-perimeter in } \mathcal{U}; \tag{i}$$

$$\partial E \cap \mathcal{U} = S \cap \mathcal{U}; \tag{ii}$$

$$\nu_E(x) = -\frac{\nabla_{\mathbb{G}} f(x)}{|\nabla_{\mathbb{G}} f(x)|_x} \text{ for all } x \in S \cap \mathcal{U}, \tag{iii}$$

*where ν_E is the generalized inner unit normal defined by (*13*), that can be identified with a section of $H\mathbb{G}$ with $|\nu(x)|_x = 1$ for $|\partial E|_{\mathbb{G}}$-a.e. $x \in \mathcal{U}$. In particular, ν_E can be identified with a continuous function and $|\nu| \equiv 1$. Moreover, there exists a unique function*

$$\phi = \phi(\xi) : I_\delta \to J_h$$

such that the following parameterization holds: if $\xi \in I_\delta$, put $\Phi(\xi) = \gamma(\phi(\xi), \xi)$, then

$$S \cap \tilde{\mathcal{U}} = \{x \in \tilde{\mathcal{U}} : x = \Phi(\xi), \xi \in I_\delta\}; \tag{iv}$$

$$\phi \text{ is continuous;} \tag{v}$$

the $\mathbb{G}$-perimeter has an integral representation:

$$|\partial E|_{\mathbb{G}}(\tilde{\mathcal{U}}) = \int_{I_\delta} \frac{\sqrt{\sum_{j=1}^m |X_j f(\Phi(\xi))|^2}}{X_1 f(\Phi(\xi))}\, d\mathcal{L}_\xi^{n-1}. \tag{vi}$$

Proof. The proof will be divided in several steps.

Step 1. Construction of the continuous function Φ. Clearly, the map $(t, \xi) \to \gamma(t, \xi)$ is continuously differentiable, and then it is a diffeomorphism of a neighborhood of $J_h \times I_\delta$ onto an open subset of $\mathbb{R}^n$ since its Jacobian determinant at $(0, 0)$ is $= 1$.

Let us choose now $\delta, h > 0$ so that $X_1 f > 0$ in $\tilde{\mathcal{U}}$, and let ρ_ϵ be a Friedrichs' mollifier. If we put $f_\epsilon = f * \rho_\epsilon$ ($0 < \epsilon < \mathrm{dist}(\tilde{\mathcal{U}}, \mathbb{R}^n \setminus \Omega)$), then $f_\epsilon \to f$ as $\epsilon \to 0$ uniformly on $\tilde{\mathcal{U}}$, because of the continuity of f. Analogously, $(X_j f) * \rho_\epsilon \to X_j f$ as $\epsilon \to 0$ uniformly on $\bar{\mathcal{U}}$ for $j = 1, \dots, m$. We can use now a regularization argument that goes back to Friedrichs ([24]) and has been used recently for cc-metrics in [26] and [21]. Notice that we did not purpusely use group mollifiers ([19]), for the group structure does not really play a role in this proof (see Theorem 2.4 below). We have

$$X_j f_\epsilon = (X_j f) * \rho_\epsilon - ((X_j f) * \rho_\epsilon - X_j f_\epsilon)$$

for $j = 1, \ldots, m$. Let us prove now that $(X_j f) * \rho_\epsilon - X_j f_\epsilon \to 0$ as $\epsilon \to 0$ uniformly on $\tilde{\mathcal{U}}$. In fact, if $x \in \tilde{\mathcal{U}}$, denoting by $\partial_\ell f$ and $\partial_\ell \rho$ the partial derivatives respectively of f and ρ with respect to their ℓ-th argument, we have

$$
\begin{aligned}
&((X_j f) * \rho_\epsilon)(x) - X_j f_\epsilon(x) \\
&= \int_{|x-y|<\epsilon} \sum_{\ell=m+1}^{n} a_\ell^j(y) \partial_\ell f(y) \rho_\epsilon(x-y) dy \\
&\qquad - \int_{|x-y|<\epsilon} \sum_{\ell=m+1}^{n} a_\ell^j(x) \frac{\partial}{\partial x_\ell}\big(f(y)\rho_\epsilon(x-y)\big)\, dy \\
&= \int_{|x-y|<\epsilon} \sum_{\ell=m+1}^{n} a_\ell^j(y) \partial_\ell f(y) \rho_\epsilon(x-y) dy \\
&\qquad - \int_{|x-y|<\epsilon} \sum_{\ell=m+1}^{n} a_\ell^j(x) f(y) (\partial_\ell \rho_\epsilon)(x-y) dy \\
&= \sum_{\ell=m+1}^{n} \int_{|x-y|<\epsilon} \big(a_\ell^j(y) - a_\ell^j(x)\big) f(y) \partial_\ell \rho_\epsilon(x-y) dy \\
&\quad \text{(since } a_\ell^j \text{ does not depend on the } \ell\text{-th variable)} \\
&= \sum_{\ell=m+1}^{n} \int_{|x-y|<\epsilon} \big(a_\ell^j(y) - a_\ell^j(x)\big)\big(f(y) - f(x)\big) \partial_\ell \rho_\epsilon(x-y) dy \\
&\qquad + f(x) \sum_{\ell=m+1}^{n} \int_{|x-y|<\epsilon} \big(a_\ell^j(y) - a_\ell^j(x)\big) \partial_\ell \rho_\epsilon(x-y) dy \\
&= I_1(x) + f(x) I_2(x).
\end{aligned}
$$

Again since $a_\ell^j(y)$ does not depend on the ℓ-th variable, we have

$$
\begin{aligned}
I_2(x) &= -\sum_{\ell=m+1}^{n} \int_{|x-y|<\epsilon} \big(a_\ell^j(y) - a_\ell^j(x)\big) \frac{\partial}{\partial y_\ell} \rho_\epsilon(x-y) dy \\
&= -\sum_{\ell=m+1}^{n} \int_{|x-y|<\epsilon} \frac{\partial}{\partial y_\ell}\Big(\big(a_\ell^j(y) - a_\ell^j(x)\big) \rho_\epsilon(x-y) \Big) dy = 0,
\end{aligned}
$$

since the function $y \to \big(a_\ell^j(y) - a_\ell^j(x)\big)\rho_\epsilon(x-y)$ is supported in $\{y : |x-y| < \epsilon\}$.

On the other hand, if we denote by ω the modulus of continuity of f, then

$$|I_1(x)| \leqslant c\epsilon^{n+1}\omega(\epsilon) \sum_{\ell=m+1}^{n} \max |\partial_\ell \rho_\epsilon| \to 0 \qquad \text{for any } x \in \tilde{\mathcal{U}}$$

as $\epsilon \to 0$. This proves that

$$X_j f_\epsilon \to X_j f \;\text{ as }\; \epsilon \to 0 \;\text{ uniformly on }\; \bar{\mathcal{U}} \;\text{ for }\; j = 1, \ldots, m. \tag{27}$$

Notice now that for any $\xi \in I_\delta$

$$\frac{\partial}{\partial s} f_\epsilon(\gamma(s,\xi)) = (X_1 f_\epsilon)(\gamma(s,\xi))$$

converges uniformly with respect to $s \in J_h$ to $(X_1 f)(\gamma(s,\xi))$, and hence the map

$$s \mapsto f(\gamma(s,\xi))$$

is differentiable for all s, $|s| \leqslant h$ and

$$\frac{\partial}{\partial s} f(\gamma(s,\xi)) = (X_1 f)(\gamma(s,\xi)) > 0 \tag{28}$$

when $\xi \in I_\delta$ and $s \in J_h$.

Since $f(0) = 0$ and $\gamma(0,0) = 0$, then $f(\gamma(h,0)) > 0 > f(\gamma(-h,0))$ and, by continuity,

$$f(\gamma(h,\xi)) > 0 > f(\gamma(-h,\xi))$$

if $\xi \in I_\delta$, provided δ is small enough, so that the existence of ϕ such that (iv) holds can be proved by an usual continuity argument.

Let us prove now that ϕ is continuous. To this end, it will be enough to show that, if $\xi_k \in I_\delta$ for $k \in \mathbb{N}$, and $\xi_k \to \xi$, then there exists a subsequence $(\xi_{k_j})_{j\in\mathbb{N}}$ such that $\phi(\xi_{k_j}) \to \phi(\xi)$. In fact, by the compactness of J_h, we can extract a subsequence $\phi(\xi_{k_j})$ converging to $\phi_0 \in J_h$. By the continuity of f

$$0 = f(\gamma(\phi(\xi_{k_j}), \xi_{k_j})) \to f(\gamma(\phi_0, \xi)),$$

and then $\phi_0 = \phi(\xi)$, by (iv). This proves (v).

Step 2. $\mathcal{U} \cap \partial E = \mathcal{U} \cap S$**.** By the continuity of f, clearly $\partial E \subseteq S$. On the other hand, let $x \in S \cap \mathcal{U}$ be given; then $x = \gamma(t,\xi)$ for some $t \in J_h$ and $\xi \in I_\delta$. Notice that the first component of $\gamma(t,\xi)$ equals t, since the first component of X_1 is 1; thus necessarily $t = x_1$. As we proved above, the function $s \to f(\gamma(s,\xi))$ is strictly increasing and vanishes for $s = x_1$.

Then there exists a sequence of points $\gamma(s_k, \xi) \in E$ converging to x, so that $S \subseteq \partial E$. This proves (ii).

Later on, we shall need to use the fact that $\mathcal{L}^n(\partial E \cap \mathcal{U}) = 0$. To this end, put

$$\tilde{S}_r = \{x \in \mathcal{U}, x = \gamma(s, \xi), \xi \in I_\delta, s \in J_h, |s - \phi(\xi)| < \frac{1}{r}\};$$

we have

$$\partial E \cap \mathcal{U} = \bigcap_{r=1}^{\infty} \tilde{S}_r.$$

Indeed, if $x \in \bigcap_{r=1}^{\infty} \tilde{S}_r$, then $x = \gamma(\phi(\xi), \xi)$, so that $f(x) = 0$ and hence $x \in S$. If now, as usual, $\mathcal{J}_\gamma$ denotes the Jacobian matrix of γ, we have

$$|\tilde{S}_r| = \int_{\tilde{S}_r} dx \leqslant \int_{I_\delta} d\xi \int_{|s-\phi(\xi)|<1/r} |\det \mathcal{J}_\gamma(s, \xi)| \, ds \leqslant C_\delta \frac{1}{r},$$

since

$$|\det \mathcal{J}_\gamma(s, \xi)| = 1 + o(1) \qquad \text{as} \qquad (s, \xi) \to (0, 0).$$

Thus $\mathcal{L}^n(\partial E \cap \mathcal{U}) = 0$, since $\tilde{S}_{r+1} \subseteq \tilde{S}_r$ for all $r \in \mathbb{N}$.

Step 3. E has finite $\mathbb{G}$-perimeter in $\mathcal{U}$. The proof of this step is based on the construction of a family of functions $\{h_\epsilon\}$ bounded in $BV_{\mathbb{G}}(\mathcal{U})$ converging to $\mathbf{1}_E$ in $L^1(\mathcal{U})$. Let us consider again the approximations f_ϵ we introduced at the beginning of the proof, and consider the functions $g_\epsilon, g : J_h \times I_\delta \to \mathbb{R}$ as

$$g_\epsilon(\xi_1, \xi) := f_\epsilon(\gamma(\xi_1, \xi)) \;, \; g(\xi_1, \xi) := f(\gamma(\xi_1, \xi))$$

As we showed above, $\frac{\partial g_\epsilon}{\partial \xi_1}(\xi_1, \xi)$ converges uniformly to $X_1 f(\gamma(\xi_1, \xi))$ on $J_h \times I_\delta$. Since $\gamma(\xi_1, \xi) \in \bar{\mathcal{U}}$, we can assume

$$(X_1 f_\epsilon)(\gamma(\xi_1, \xi)) = \frac{\partial g_\epsilon}{\partial \xi_1}(\xi_1, \xi) \geqslant \mu > 0 \text{ on } J_h \times I_\delta \quad \text{ for } 0 < \epsilon < \epsilon_0. \tag{29}$$

Thus we can apply the classical implicit function theorem in $J_h \times I_\delta$ to obtain the existence of a smooth function $\phi_\epsilon : I_\delta \to J_h$ such that

$$f_\epsilon(\gamma(\phi_\epsilon(\xi), \xi)) \equiv g_\epsilon(\phi_\epsilon(\xi), \xi) \equiv 0, \tag{30}$$

provided δ is small enough. We stress the fact that the choice of δ and h defining I_δ and J_h can depend on μ but it is independent of ϵ: indeed first we notice that $g_\epsilon(h, 0) \geq h\mu > 0$ and $g_\epsilon(-h, 0) \leq -h\mu < 0$ for all $\epsilon \in (0, \epsilon_1)$. Since $g_\epsilon \to g$ uniformly, choose now $\epsilon_1 = \epsilon_1(h, \mu)$ such that

$\sup|g_\epsilon - g| < h\mu/3$ for $\epsilon < \epsilon_1$, and $|g(h,\xi) - g(h,0)| < h\mu/3$ for $\xi \in I_\delta$, provided $\delta < \delta_0 = \delta_0(h,\mu)$. Then, if $\xi \in I_\delta$, $0 < \epsilon < \epsilon_1$, and $0 < \delta < \delta_0$, we have

$$\begin{aligned} g_\epsilon(h,\xi) &\geqslant g_\epsilon(h,0) - |g_\epsilon(h,\xi) - g_\epsilon(h,0)| \\ &\geqslant g_\epsilon(h,0) - |g_\epsilon(h,\xi) - g(h,\xi)| - |g(h,\xi) - g(h,0)| \\ &\quad - |g(h,0) - g_\epsilon(h,0)| > 0. \end{aligned}$$

Analogously $g_\epsilon(-h,\xi) < 0$ for $\xi \in I_\delta$, $0 < \epsilon < \epsilon_1$, and $0 < \delta < \delta_0$. Thus, the function $\phi_\epsilon(\xi)$ is well defined for $\xi \in I_\delta$. Let us prove now that $\phi_\epsilon \to \phi$ uniformly in I_δ: by contradiction, suppose there exist $\sigma > 0$, $\epsilon_k \to 0$, $(\xi_k)_{k\in\mathbb{N}}$ in I_δ such that

$$|\phi_{\epsilon_k}(\xi_k) - \phi(\xi_k)| \geqslant \sigma.$$

Without loss of generality we may assume that

$$\xi_k \to \xi \in I_\delta \text{ and } \phi_{\epsilon_k}(\xi_k) \to \phi_0 \in J_h$$

as $k \to \infty$, so that $|\phi_0 - \phi(\xi)| \geqslant \sigma$. On the other hand

$$0 = g_{\epsilon_k}(\phi_{\epsilon_k}(\xi_k), \xi_k) = f_{\epsilon_k}(\gamma(\phi_{\epsilon_k}(\xi_k)), \xi_k) \to f(\gamma(\phi_0, \xi)),$$

since $f_{\epsilon_k} \to f$ uniformly on $\bar{\mathcal{U}}$. By the uniqueness of ϕ, this implies $\phi_0 = \phi(\xi)$, a contradiction.

Denote by $\gamma^{-1} : \mathcal{U} \to \operatorname{Int}(J_h \times I_\delta)$ the inverse map of γ; because of the structure of X_1, we can write $\gamma^{-1}(x) = (x_1, \theta(x))$, where $\theta : \mathcal{U} \to \operatorname{Int}(I_\delta)$ is a smooth map.

If now $H(s) = 1$ for $s > 0$ and $H(s) = 0$ for $s \leqslant 0$, put

$$h_\epsilon(x) = H(\phi_\epsilon(\theta(x)) - x_1).$$

and

$$E_\epsilon = \{x \in \mathcal{U} \,:\, x_1 < \phi_\epsilon(\theta(x))\}.$$

We want to show that

$$h_\epsilon = \mathbf{1}_{E_\epsilon} \to \mathbf{1}_E \text{ in } L^1(\mathcal{U}).$$

By dominated convergence theorem, we need only to prove that $h_\epsilon \to \mathbf{1}_E$ a.e. in $\mathcal{U}$. This follows because $\phi_\epsilon \to \phi$ uniformly on I_δ.

Take now $x \in E \cap \mathcal{U}$, $x = \gamma(x_1, \xi)$, $\xi \in I_\delta$; by definition

$$f(\gamma(\phi(\xi), \xi)) = 0 > f(x) = f(\gamma(x_1, \xi)),$$

and hence $\phi(\xi) > x_1$, since the map

$$s \mapsto f(\gamma(s,\xi))$$

is strictly increasing on $[-h,h]$. Since $\phi_\epsilon(\xi) \to \phi(\xi)$, then $\phi_\epsilon(\xi) > x_1$ if ϵ is close to zero, that implies that $h_\epsilon(x) = 1 \to 1 = \mathbf{1}_E(x)$ as $\epsilon \to 0$. The same argument can be carried out if $f(x) > 0$, and hence a.e. since $|S \cap \mathcal{U}| = 0$.

We can prove now that E has finite $\mathbb{G}$-perimeter in $\mathcal{U}$; to this end we need only to show that the h_ϵ's have equibounded $\mathbb{G}$-variations, since the $\mathbb{G}$-variation is L^1-lsc (see Proposition 1.2).

To this end, take $\psi \in \mathbf{C}_0^\infty(\mathcal{U}, H\mathbb{G})$, $|\psi(x)|_x \leqslant 1$ for all $x \in \mathcal{U}$. We have:

$$\begin{aligned}\int_{\mathcal{U}} h_\epsilon \mathrm{div}_{\mathbb{G}}\, \psi\, dx &= \int_{\mathcal{U}\cap E_\epsilon} \mathrm{div}_{\mathbb{G}}\, \psi\, dx = \int_{\mathcal{U}\cap E_\epsilon} \mathrm{div}\, ({}^tC\psi)\, dx \\ &= \int_{\mathcal{U}\cap \partial E_\epsilon} \langle \psi, Cn\rangle_{\mathbb{R}^m}\, d\mathcal{H}^{n-1} \leqslant \int_{\mathcal{U}\cap \partial E_\epsilon} |Cn|_{\mathbb{R}^m}\, d\mathcal{H}^{n-1}.\end{aligned}$$

On the other hand, a parameterization of $\mathcal{U} \cap \partial E_\epsilon$ is given by

$$\Phi_\epsilon = \Phi_\epsilon(\xi) := \gamma(\phi_\epsilon(\xi), \xi),$$

with $\xi \in I_\delta$. Indeed, let us prove that Φ_ϵ is injective: suppose $\gamma(\phi_\epsilon(\xi),\xi) = \gamma(\phi_\epsilon(\xi'),\xi')$; since the first component of $\gamma(t,\xi)$ is t, this implies that $\phi_\epsilon(\xi) = \phi_\epsilon(\xi')$, and then that $\xi = \xi'$, by the uniqueness of the solution of the Cauchy problem.

Thus, from standard area formula,

$$\int_{\mathcal{U}} h_\epsilon \mathrm{div}_{\mathbb{G}}\, \psi\, dx \leqslant \int_{I_\delta} \left| C(\Phi_\epsilon)\, \frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \xi_n} \right| d\xi. \tag{31}$$

Notice now that the j-th component of $C(\Phi_\epsilon(\xi))\, \dfrac{\partial \Phi_\epsilon}{\partial \xi_2}(\xi) \wedge \cdots \wedge \dfrac{\partial \Phi_\epsilon}{\partial \xi_n}(\xi)$ is

$$\langle X_j(\Phi_\epsilon(\xi)), \frac{\partial \Phi_\epsilon}{\partial \xi_2}(\xi) \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \xi_n}\rangle(\xi).$$

To achieve our proof, the following identity will be crucial:

$$\left| \langle X_k, \frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \xi_n}\rangle \right| = \left| \frac{X_k f_\epsilon}{X_1 f_\epsilon} \right|, \tag{32}$$

for $k = 1, \ldots, m$, where $\frac{\partial \Phi_\epsilon}{\partial \xi_j} = \frac{\partial \Phi_\epsilon}{\partial \xi_j}(\xi)$, $X_k = X_k(\Phi_\epsilon(\xi))$, $X_1 f_\epsilon = X_1 f_\epsilon(\Phi_\epsilon(\xi)) > 0$ and $X_k f_\epsilon = X_k f_\epsilon(\Phi_\epsilon(\xi))$, $\xi \in I_\delta$.

To prove (32), let us first notice that, if we write $\gamma = (\gamma_1, \dots, \gamma_n)$, then, because of the structure of the a_i^j, we get

$$\gamma_\ell(t,\xi) = \xi_\ell + \int_0^t g_\ell(s, \xi_2, \dots, \xi_{\ell-1})\, ds,$$

for suitable functions $g_\ell : \mathbb{R}^\ell \to \mathbb{R}$, $\ell = 2, \dots, n$. Hence

$$\frac{\partial \gamma_\ell}{\partial \xi_j} = 0 \quad \text{if } j > \ell, \quad \frac{\partial \gamma_\ell}{\partial \xi_\ell} = 1. \tag{33}$$

Moreover, if we write $\phi_\epsilon = \phi_\epsilon(\xi)$, $\Phi_\epsilon = (\Phi_{\epsilon,1}, \dots, \Phi_{\epsilon,n})$,

$$\frac{\partial}{\partial \xi_j}\Phi_{\epsilon,\ell}(\xi) = \frac{\partial}{\partial \xi_j}\gamma_\ell(\phi_\epsilon,\xi) = \frac{\partial \gamma_\ell}{\partial \xi_j}(\phi_\epsilon,\xi) + \gamma_\ell'(\phi_\epsilon,\xi)\frac{\partial \phi_\epsilon}{\partial \xi_j} = \frac{\partial \gamma_\ell}{\partial \xi_j}(\phi_\epsilon,\xi) + a_\ell^1 \frac{\partial \phi_\epsilon}{\partial \xi_j}, \tag{34}$$

where $a_\ell^1 = a_\ell^1(\gamma(\phi_\epsilon,\xi))$. Differentiating now (30) with respect to ξ_ℓ, we get

$$0 \equiv \sum_{\ell=1}^n \frac{\partial f_\epsilon}{\partial x_\ell}\left(\frac{\partial \gamma_\ell}{\partial \xi_j} + a_\ell^1 \frac{\partial \phi_\epsilon}{\partial \xi_j}\right), \tag{35}$$

and hence

$$\sum_{\ell=1}^n \frac{\partial f_\epsilon}{\partial x_\ell}\frac{\partial \gamma_\ell}{\partial \xi_j} = -(X_1 f_\epsilon)\cdot \frac{\partial \phi_\epsilon}{\partial \xi_j} \quad , \quad j = 2, \dots, n, \tag{36}$$

i.e.

$$\sum_{\ell=1}^n \frac{\partial f_\epsilon}{\partial x_\ell}\nabla_\xi \gamma_\ell = -(X_1 f_\epsilon)\cdot \nabla \phi_\epsilon. \tag{37}$$

Suppose now $2 \leqslant k \leqslant m$; to prove (32), we want to calculate

$$\begin{aligned} D \ &:= \ \left|\langle X_k, \frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \xi_n}\rangle\right| = \left|\det\left[X_k, \frac{\partial \Phi_\epsilon}{\partial \xi_2}, \dots, \frac{\partial \Phi_\epsilon}{\partial \xi_n}\right]\right| \\ &= \ \left|\det\left[(a_1^k, \nabla_\xi \gamma_1(\phi_\epsilon,\xi)), \dots, (a_n^k, \nabla_\xi \gamma_n(\phi_\epsilon,\xi))\right]\right|, \end{aligned}$$

where, in the first determinant, $X_k, \frac{\partial \Phi_\epsilon}{\partial \xi_2}, \dots, \frac{\partial \Phi_\epsilon}{\partial \xi_n}$ are columns, whereas in the last determinant we must see the n-dimensional vectors $(a_1^k, \nabla_\xi \gamma_1(\phi_\epsilon,\xi)), \dots, (a_n^k, \nabla_\xi \gamma_n(\phi_\epsilon,\xi))$ as the rows of a matrix. By (33), $\gamma_1(\phi_\epsilon,\xi) \equiv \phi_\epsilon$, so that $\nabla_\xi \gamma_1(\phi_\epsilon,\xi) = \nabla_\xi \phi_\epsilon$, and hence, keeping in mind that $a_1^k = 0$ since $k > 1$, by (34) and chain rule, we get

$$D = \left|\det\left[(0, \nabla_\xi \phi_\epsilon), (a_2^k, (\nabla_\xi \gamma_2) + a_2^1 \nabla_\xi \phi_\epsilon), \dots, (a_n^k, (\nabla_\xi \gamma_n) + a_n^1 \nabla_\xi \phi_\epsilon)\right]\right|, \tag{38}$$

where $(\nabla_\xi\gamma_j)$ stands for $(\nabla_\xi\gamma_j)(\phi_\epsilon(\xi),\xi)$.

By standard properties of the determinant, i.e. by subtracting a_i^1 times the first row from the i-th row, it follows that

$$D=\left|\det\Big[(0,\nabla_\xi\phi_\epsilon),(a_2^k,\nabla_\xi\gamma_2),\dots,(a_n^k,\nabla_\xi\gamma_n)\Big]\right|.$$

Subtract now the first column from the k-th column; we get

$$D=\left|\det\Big[(0,\nabla_\xi\phi_\epsilon),(a_2^k,\nabla_\xi\gamma_2-a_2^ke_k^{n-1}),\dots,(a_n^k,\nabla_\xi\gamma_n-a_n^ke_k^{n-1})\Big]\right|,$$

where $\{e_2^{n-1},\dots,e_n^{n-1}\}$ is the canonical orthonormal basis of $\mathbb{R}^{n-1}$, so that $e_2^{n-1}=(1,0,\dots,0)$, and so on.

Multiply now the i-th row by $\dfrac{\partial f_\epsilon}{\partial x_i}(X_1f_\epsilon)^{-1}$, sum up when i runs from 2 to n, and then add to the first row. By (37), denoting by $*$ a suitable real number that will turn out to be irrelevant, we get

$$D=\left|\det\Big[(*,-\frac{X_kf_\epsilon}{X_1f_\epsilon}\,e_k^{n-1}),(a_2^k,\nabla_\xi\gamma_2-a_2^ke_k^{n-1}),\dots,(a_n^k,\nabla_\xi\gamma_n-a_n^ke_k^{n-1})\Big]\right|.$$

Notice now that $\gamma_k(t,\xi)=\xi_k$, since $2\leqslant k\leqslant m$ and X_1 has zero k-component so that $\nabla_\xi\gamma_k=e_k^{n-1}$, and that $a_k^k=1$; hence $(a_k^k,\nabla_\xi\gamma_k-a_k^ke_k^{n-1})=(1,0)$. Thus, if we develop the determinant with respect to the first column, all but the k-th cofactor vanish since they contain a zero row. Again since $a_k^k=1$ we obtain

$$\begin{aligned}D=\Big|\det\Big[&-\frac{X_kf_\epsilon}{X_1f_\epsilon}e_k^{n-1},\nabla_\xi\gamma_2-a_2^ke_k^{n-1},\dots,\nabla_\xi\gamma_{k-1}-a_{k-1}^ke_k^{n-1},\\&\nabla_\xi\gamma_{k+1}-a_{k-1}^ke_k^{n+1},\dots,\nabla_\xi\gamma_n-a_n^ke_k^{n-1}\Big]\Big|.\end{aligned}$$

Remember now that $a_i^k=0$ if $i\leqslant m$ and $i\neq k$, and that $\nabla_\xi\gamma_i=e_i^{n-1}$ for $2\leqslant i\leqslant m$. We have

$$\begin{aligned}D&=\left|\frac{X_kf_\epsilon}{X_1f_\epsilon}\right|\Big|\det[e_k^{n-1},e_2^{n-1},\dots,e_{k-1}^{n-1},e_{k+1}^{n-1},\dots,e_m^{n-1},\\&\qquad\nabla_\xi\gamma_{m+1}-a_{m+1}^ke_k^{n-1},\dots]\Big|\\&=\left|\frac{X_kf_\epsilon}{X_1f_\epsilon}\right|\Big|\det[e_2^{n-1},\dots,e_m^{n-1},\nabla_\xi\gamma_{m+1}-a_{m+1}^ke_k^{n-1},\dots]\Big|.\end{aligned}$$

By (33), keeping in mind that $k\leqslant m$ and that hence all components of e_k^{n-1} vanish when their index is greater than m, the matrix

$[e_2^{n-1}, \dots, e_m^{n-1}, \nabla_\xi \gamma_{m+1} - a_{m+1}^k e_k^{n-1}, \dots]$ is a lower triangular matrix, with all diagonal terms equal to 1. Hence

$$D = \left| \frac{X_k f_\epsilon}{X_1 f_\epsilon} \right|.$$

Consider now the case $k = 1$. Arguing as in the proof of (38), in this case we have to evaluate

$$\begin{aligned} D &:= \left| \langle X_1, \frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \xi_n} \rangle \right| = \left| \det \left[X_1, \frac{\partial \Phi_\epsilon}{\partial \xi_2}, \dots, \frac{\partial \Phi_\epsilon}{\partial \xi_n} \right] \right| \\ &= \left| \det \left[(a_1^1, \nabla_\xi \gamma_1(\phi_\epsilon, \xi)), \dots, (a_n^1, \nabla_\xi \gamma_n(\phi_\epsilon, \xi)) \right] \right| \\ &= \left| \det \left[(1, \nabla_\xi \phi_\epsilon), (a_2^1, (\nabla_\xi \gamma_2) + a_2^1 \nabla_\xi \phi_\epsilon), \dots, (a_n^1, (\nabla_\xi \gamma_n) + a_n^1 \nabla_\xi \phi_\epsilon) \right] \right|, \end{aligned}$$

since $a_1^1 = 1$. Again as above, think of the vectors

$$(1, \nabla_\xi \phi_\epsilon), (a_2^1, (\nabla_\xi \gamma_2) + a_2^1 \nabla_\xi \phi_\epsilon), \dots, (a_n^1, (\nabla_\xi \gamma_n) + a_n^1 \nabla_\xi \phi_\epsilon)$$

as of the row of a matrix, and subtract from the i-th row ($i \geqslant 2$) the first row multiplied by a_i^1. In such a way, the first column turns out to have all zero entries but the first one which is 1, and we get

$$D = |\det [\nabla_\xi \gamma_2, \dots, \nabla_\xi \gamma_n]| = 1,$$

since the above matrix is a lower triangular matrix with all entries in the principal diagonal equal 1, by (33).

Thus (32) is proved, and hence it follows from (31) that

$$\begin{aligned} \int_{\mathcal{U}} h_\epsilon \operatorname{div}_{\mathbb{G}} \psi \, dx &\leqslant \int_{I_\delta} \left| C(\Phi_\epsilon) \frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \xi_n} \right| d\xi \\ &\leqslant \int_{I_\delta} \left(\sum_j |X_j f_\epsilon(\Phi_\epsilon)|^2 \right)^{1/2} |X_1 f_\epsilon(\Phi_\epsilon)|^{-1} d\xi \leqslant \text{Const.} \, |I_\delta|, \end{aligned} \tag{39}$$

since the X_j's and X_1^{-1} are equibounded, by (27) and (29).

Thus, the functions h_ϵ have equibounded variations, and hence, as we pointed out above, $\mathbf{1}_E \in BV_{\mathbb{G}}(\mathcal{U})$.

Step 4. The area formula for the $\mathbb{G}$-perimeter of E in U. We want now to prove the explicit formula for $|\partial E|_{\mathbb{G}}(\mathcal{U})$ given by (vi). Again as above, if $\psi \in \mathbf{C}_0^\infty(\mathcal{U}, H\mathbb{G})$, $|\psi(x)|_x \leqslant 1$, by dominated convergence theorem, we have:

$$\begin{aligned}\int_{\mathcal{U}} \mathbf{1}_E \operatorname{div}_{\mathbb{G}}\psi\, dx &= \lim_{\epsilon\to 0} \int_{\mathcal{U}} h_\epsilon \operatorname{div}_{\mathbb{G}}\psi\, dx \\ &= \lim_{\epsilon\to 0} \int_{I_\delta} \langle \psi(\Phi_\epsilon(\xi)), C(\Phi_\epsilon(\xi))\, \frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \tau}\rangle_{\mathbb{R}^m} d\xi \qquad (40)\\ &= \int_{I_\delta} \langle \psi(\Phi(\xi)), \nabla_{\mathbb{G}} f(\Phi(\xi))\rangle_{\Phi(\xi)} |X_1 f(\Phi(\xi))|^{-1} d\xi,\end{aligned}$$

again by dominated convergence theorem. Indeed, we showed above that $\Phi_\epsilon(\xi) \to \Phi(\xi)$ pointwise for $\xi \in I_\delta$, $X_j f_\epsilon \to X_j f$ uniformly on $\tilde{\mathcal{U}}$ as $\epsilon \to 0$, and $\Phi_\epsilon(\xi) \in \tilde{\mathcal{U}}$ when $\xi \in I_\delta$, since, as we pointed out above, $|\phi_\epsilon(\xi)| \leqslant h$ uniformly with respect to $\epsilon \in (0,1)$.

Taking now the supremum with respect to ψ in (40), the proof of (vi) is complete.

Let us prove now (iii). Since $\mathbf{1}_E \in BV_{\mathbb{G}}(\mathcal{U})$, then, by [20], p. 880, $\nabla_{\mathbb{G}} \mathbf{1}_E = (X_1, \dots, X_m)\mathbf{1}_E$ is a Radon measure. Moreover, for all $\psi \in \mathbf{C}_0^0(\mathcal{U}, H\mathbb{G})$, arguing as above, we get

$$-\langle \nabla_{\mathbb{G}} \mathbf{1}_E, \psi\rangle_{\mathbb{R}^m} = \int_I \langle \psi(\Phi(\xi)), \nabla_{\mathbb{G}} f(\Phi(\xi))\rangle_{\Phi(\xi)} |X_1 f(\Phi(\xi))|^{-1} d\xi.$$

Taking the supremum on all $\psi \in \mathbf{C}_0^0(\mathcal{U}, H\mathbb{G})$, $|\psi(x)|_x \leqslant 1$, supp $\psi \subseteq \mathcal{O} \subseteq \mathcal{U}$, where $\mathcal{O}$ is an open set, we obtain

$$|\partial E|_{\mathbb{G}}(\mathcal{O}) = \int_{\Phi^{-1}(\mathcal{O})} \frac{|\nabla_{\mathbb{G}} f|}{X_1 f} \circ \Phi\, d\xi. \qquad (41)$$

Thus, with the notations of [38], Definition 1.17, $|\partial E|_{\mathbb{G}}$ is the image of the measure

$$d\mu = \frac{|\nabla_{\mathbb{G}} f|}{X_1 f} \circ \Phi\, d\xi$$

in $\mathbb{R}^{n-1}$ under the map Φ, given that equality (41) still holds for any Borel set $\mathcal{O}$ because both measures are Radon measures ([38], Theorem 1.18). Hence, by [38], Theorem 1.19, for all $\psi \in \mathbf{C}_0^0(\mathcal{U}, H\mathbb{G})$ we have

$$\int_{\mathcal{U}} \langle \psi(x), \nu_E(x)\rangle_x d|\partial E|_{\mathbb{G}} = \int_I \langle \psi \circ \Phi, \nu_E \circ \Phi\rangle_\Phi \frac{|\nabla_{\mathbb{G}} f|}{X_1 f} \circ \Phi\, d\xi.$$

We notice that Theorem 1.19 in [38] would require $\langle \psi(x), \nu_E(x)\rangle_x$ to be a Borel function. Nevertheless, the result is still true since $\langle \psi(x), \nu_E(x)\rangle_x$ is

$|\partial E|_{\mathbb{G}}$-measurable, thanks to [17], 2.3.6. Thus, keeping in mind that

$$\int_{\mathcal{U}} \langle \psi(x), \nu_E(x) \rangle_x d|\partial E|_{\mathbb{G}} = -\int_{\mathcal{U}} \mathbf{1}_E \operatorname{div}_{\mathbb{G}} \psi \, dx,$$

by (40) and by the arbitrariness of the choice of ψ, we obtain

$$\nu_E \circ \Phi = -\frac{\nabla_{\mathbb{G}} f}{|\nabla_{\mathbb{G}} f|} \circ \Phi$$

a.e. in I_δ with respect to Lebesgue measure in $\mathbb{R}^{n-1}$, and hence

$$\nu_E = -\frac{\nabla_{\mathbb{G}} f}{|\nabla_{\mathbb{G}} f|}$$

$|\partial E|_{\mathbb{G}}$-a.e. on $S \cap \mathcal{U}$, because of (41). Thus (iii) is proved. □

Remark 2.2. If the assumption $X_1 f(0) > 0$ is replaced by $X_j f(0) > 0$ for some $j = 2, \ldots, m$, then we can always reduce ourselves to the case previously considered provided we renumber the first m variables. Hence the above Theorem still holds under this new assumption, provided we replace everywhere the X_1 by X_j.

It is clear from the proof of Theorem 2.1 that in fact the group structure does not play any role in the result but for the structure of the vector fields, even if some statement throughout the proof itself must be slightly modified when the fiber bundle structure fails to exist.

Thus the following definition is quite natural.

Definition 2.3. Let $X = \{X_1, \ldots, X_m\}$ be a family of Lipschitz continuous vector fields. We say that X is a family *of Carnot type* if

$$X_j(x) = \partial_j + \sum_{i=m+1}^{n} a_i^j(x)\partial_i, \quad X_j(0) = \partial_j, \quad j = 1, ..., m, \tag{42}$$

with $a_i^j(x) = a_i^j(x_1, ..., x_{i-1})$, $j = 1, \ldots, m$.

Then the previous Theorem can be stated as follows.

Theorem 2.4. *Let $X = \{X_1, \ldots, X_m\}$ be a family of Lipschitz continuous vector fields of Carnot type in $\mathbb{R}^n$, let Ω be an open subset of $\mathbb{R}^n$, $0 \in \Omega$, and let f be a continuous real-valued function in Ω such that $X_1 f, \ldots, X_m f$ are continuous functions. Then the conclusions of Theorem* 2.1 *hold.*

3. Surface Measures on Hypersurfaces in $\mathbb{G}$.

3.1. $\mathbb{G}$-regular hypersurfaces.

We want to study $\mathbb{G}$-regular hypersurfaces from an intrinsic point of view. In particular we want to compare the perimeter measure, on a $\mathbb{G}$-regular hypersurface S, and the intrinsic $(Q-1)$-Hausdorff measure of S. Observe that it makes sense to speak of the perimeter measure of S given that S is locally the boundary of a finite $\mathbb{G}$-perimeter set (as proved in Theorem 2.1).

We begin with an easy proposition, more or less explicitely contained in Theorem 2.1, showing that S is locally the homeomorphic image of a (vertical) hyperplane in $\mathbb{R}^n$. In particular this fact implies that the topological dimension of S is $n-1$.

Proposition 3.1. *Assume that S is a $\mathbb{G}$-regular hypersurface in a Carnot group $\mathbb{G}$. Then for any $y \in S$ there exists an open, connected neighborhood $\mathcal{U}_y$ of y such that*
(i) there are $\sigma > 0$, $\delta > 0$ and an homeomorphism $F : (-\sigma, \sigma) \times \operatorname{Int} I_\delta \to \mathcal{U}_y$, such that $F(0,0) = y$ and

$$F(\{0\} \times \operatorname{Int} I_\delta) = S \cap \mathcal{U}_y;$$

(ii) the set $E_1 := F((-\sigma, 0) \times \operatorname{Int} I_\delta)$ has finite perimeter in $\mathcal{U}_y$ and

$$S \cap \mathcal{U}_y = \partial E_1 \cap \mathcal{U}_y.$$

Proof. By definition of $\mathbb{G}$-regular hypersurface, if $r > 0$ is sufficiently small, there are $U_c(y,r)$ and $f \in \mathbf{C}^1_{\mathbb{G}}(U_c(y,r))$ such that

$$S \cap U_c(y,r) = \{x \in U_c(y,r) \, : \, f(x) = 0\}$$

and $Xf \neq 0$ in $U_c(y,r)$. Without loss of generality we may assume $X_1 f > 0$ in $U_c(y,r)$. We keep the notations of Theorem 2.1. For fixed $\sigma > 0$ and $\delta > 0$ to be chosen sufficiently small, we consider the map $F = F(t,\xi)$: $(-\sigma,\sigma) \times \mathrm{Int} I_\delta \to \mathbb{R}^n$ given by

$$F(t,\xi) = y \cdot \gamma(\phi(\xi) - t, \xi) := y \cdot \exp((\phi(\xi) - t)X_1)(0,\xi).$$

It is clear that F is an homeomorphism from $(-\sigma,\sigma) \times \mathrm{Int} I_\delta \to \mathcal{U}_y :=$ $F((-\sigma,\sigma) \times \mathrm{Int} I_\delta) \subset U_c(y,r)$. Indeed F is the composition of the homeomorphism $(t,\xi) \to (\phi(\xi) - t, \xi)$ and of the diffeomorphism $(s,\eta) \to$ $y \cdot \gamma(s,\eta) := y \cdot \exp(sX_1)(0,\eta)$.

By Theorem 2.1, (iv), locally $S = \{\gamma(\phi(\xi), \xi)\}$, so that $F(\{0\} \times \operatorname{Int} I_\delta) = S \cap \mathcal{U}_y$, and

$$\mathcal{U}_y \setminus S = F\left((-\sigma, 0) \times \operatorname{Int} I_\delta\right) \cup F\left((0, \sigma) \times \operatorname{Int} I_\delta\right) := E_1 \cup E_2.$$

Each E_i is a connected open set. By (28), the map $s \mapsto f(\gamma(s, \xi))$ is strictly increasing and vanishes at $s = \phi(\xi)$. Since $\phi(\xi) - t > \phi(\xi)$ when $t < 0$, we conclude that $f > 0$ in E_1 and that $f < 0$ in E_2. Hence $E_1 = \{x \in \mathcal{U}_y : f(x) > 0\}$ and from (i) of Theorem 2.1 the thesis follows. □

Remark 3.2. Notice that Proposition 3.1 does not follow from an Euclidean local invertibility theorem, because, in general, the Euclidean C^1-regularity of the local chart F fails to hold in Carnot groups. Indeed, if F were continuously differentiable in the usual sense, then also the map ϕ would be continuously differentiable, since the first component of $F(\xi, t)$ is precisely $\phi(\xi) - t$. But in [22], Example 2 after Theorem 6.5, in the setting of the Heisenberg group $\mathbb{G} = \mathbb{H}^1$, the authors provided an example of a $\mathbb{G}$-regular hypersurface such that the parameterization ϕ is not even Euclidean Lipschitz continuous.

Our next Theorem is a mild regularity result; in it we observe that $\mathbb{G}$-regular hypersurfaces do not have cusps or spikes if they are studied with respect to the intrinsic cc-distance, while they can be very irregular as Euclidean submanifolds. To make precise the former statement we recall the notion of *essential boundary* (or *measure theoretic boundary*) $\partial_* F$ of a set $F \subset \mathbb{G}$

$$\partial_* F := \left\{ x \in \mathbb{G} : \limsup_{r \to 0^+} \min \left\{ \frac{\mathcal{L}^n(F \cap U_c(x, r))}{\mathcal{L}^n(U_c(x, r))}, \frac{\mathcal{L}^n(F^c \cap U_c(x, r))}{\mathcal{L}^n(U_c(x, r))} \right\} > 0 \right\}, \tag{43}$$

where we remind that $U_c(x, r)$ denotes the open cc-ball. Notice that the definition above makes sense in any metric measure space and that the essential boundary does not change if, in Definition (43), the distance d_c is substituted by an equivalent distance d.

Theorem 3.3. *Let $\Omega \subset \mathbb{G}$ be a fixed open set, and let E be such that $\partial E \cap \Omega = S \cap \Omega$, where S is a $\mathbb{G}$-regular hypersurface. Then*

$$\partial E \cap \Omega = \partial_* E \cap \Omega. \tag{44}$$

Proof. Clearly, to prove (44), we need only to show that $\partial E \cap \Omega \subseteq \partial_* E \cap \Omega$. Fix $y \in \partial E \cap \Omega$. By definition of $\mathbb{G}$-regular surface and from (i) of Theorem

2.1, there exists an open ball $U_c(r) := U_c(y,r)$ and $f \in \mathbf{C}^1_{\mathbb{G}}(U_c(r))$ such that

$$\begin{gathered}\partial E \cap U_c(r) = \{x \in U_c(r) \,:\, f(x) = 0\};\\ |\nabla_{\mathbb{G}} f|_{\mathbb{R}^m} > 0 \text{ in } U_c(r);\\ E \cap U_c(r) = \{x \in U_c(r) \,:\, f(x) < 0\}.\end{gathered}$$

If $r > 0$ we define the dilated sets

$$E_{y,r} := y \cdot \delta_r(y^{-1} \cdot E) = \{x \in \mathbb{G} \,:\, y \cdot \delta_{1/r}(y^{-1} \cdot x) \in E\},$$

then we have

$$\mathbf{1}_{E_{y,r}} \to \mathbf{1}_{S^-_{\mathbb{G}}(y,\nu_E(y))} \quad \mathcal{L}^n\text{-a.e. in } \mathbb{G} \text{ as } r \to 0^+. \tag{45}$$

Indeed, for all $p \in S^-_{\mathbb{G}}(y,\nu_E(y))$, (iii) of Theorem 2.1 implies that

$$\sum_{i=1}^m X_i f(y)(p_i - y_i) < 0.$$

and, recalling that $f(y) = 0$, by Proposition 1.7, we have $f(y \cdot \delta_s(y^{-1} \cdot p)) = s \sum_{i=1}^m X_i f(y)(p_i - y_i) + o(s) < 0$ for $s \to 0$, hence $y \cdot \delta_s(y^{-1} \cdot p) \in E$ and in turn, $p \in E_{y,s}$, for s small.
Reversing the inequalities, if $p \in S^+_{\mathbb{G}}(y,\nu_E(y))$ we get $p \in E^c_{y,s}$ for s small. Since $\mathcal{L}^n(\mathbb{G} \setminus (S^+_{\mathbb{G}}(y,v) \cup S^-_{\mathbb{G}}(y,v))) = 0$ we have proved (45).

Now put $S^\pm(y) := S^\pm_{\mathbb{G}}(y,\nu_E(y))$, then, from (45) it follows that

$$\begin{aligned}\liminf_{r\to 0^+} \min &\left\{ \frac{\mathcal{L}^n(E \cap U_c(r))}{\mathcal{L}^n(U_c(r))}, \frac{\mathcal{L}^n(E^c \cap U_c(r))}{\mathcal{L}^n(U_c(r))} \right\}\\ &\geq \min \left\{ \frac{\mathcal{L}^n(U_c(1) \cap S^+(y))}{\mathcal{L}^n(U_c(1))}, \frac{\mathcal{L}^n(U_c(1) \cap S^-(y))}{\mathcal{L}^n(U_c(1))} \right\}.\end{aligned} \tag{46}$$

Indeed, from (45) it follows that $\mathbf{1}_{E_r} \to \mathbf{1}_{S^-(y)}$ in $L^1_{\mathrm{loc}}(\mathbb{G})$, whence

$$\mathcal{L}^n(E_{y,r} \cap U_c(1)) \to \mathcal{L}^n(S^-(y) \cap U_c(1)) \qquad \text{as } \ r \to 0^+. \tag{47}$$

On the other hand

$$\mathcal{L}^n(E_{y,r} \cap U_c(1)) = \frac{\mathcal{L}^n(E \cap U_c(r))}{r^Q} = \mathcal{L}^n(U_c(1)) \frac{\mathcal{L}^n(E \cap U_c(r))}{\mathcal{L}^n(U_c(r))},$$

and hence from (47) we get that

$$\lim_{r\to 0^+} \frac{\mathcal{L}^n(E \cap U_c(r))}{\mathcal{L}^n(U_c(r))} = \frac{\mathcal{L}^n(S^-(y) \cap U_c(1))}{\mathcal{L}^n(U_c(1))}.$$

Analogously, since $\partial E \cap \Omega = \partial E^c \cap \Omega$ and $\nu_{E^c}(0) = -\nu_E(0)$, we get also

$$\lim_{r \to 0^+} \frac{\mathcal{L}^n(E^c \cap U_c(r))}{\mathcal{L}^n(U_c(r))} = \frac{\mathcal{L}^n(S^+(y) \cap U_c(1))}{\mathcal{L}^n(U_c(1))}.$$

Thus (46) holds.

Now we notice explicitly that

$$\min\left\{\frac{\mathcal{L}^n(U_c(1) \cap S^+(y))}{\mathcal{L}^n(U_c(1))}, \frac{\mathcal{L}^n(U_c(1) \cap S^-(y))}{\mathcal{L}^n(U_c(1))}\right\} > 0. \tag{48}$$

Indeed, both $U_c(0,1) \cap \mathrm{Int} S^+_{\mathbb{G}}(y, \nu_E(y))$ and $U_c(0,1) \cap \mathrm{Int} S^-_{\mathbb{G}}(y, \nu_E(y))$ are open and not empty, since the ball $U_c(0,1)$ is symmetric with respect to the group inversion. This concludes the proof of (44). □

Our next results are about the relations between perimeter and intrinsic Hausdorff measures. In the setting of the Heisenberg group, in [10] it is proved that the perimeter of an Euclidean $C^{1,1}$-hypersurface is equivalent to its $(Q-1)$-dimensional intrinsic Hausdorff measure, whereas in [22] it is proved that on the boundary of sets of finite intrinsic perimeter the $(Q-1)$-dimensional intrinsic spherical Hausdorff measure coincides – after a suitable normalization – with the perimeter measure. In the setting of general Carnot groups the problem is essentially open.

Our results here are more clearly stated in a situation slightly more general than the one considered up to now. From now on d will be a metric on $\mathbb{G}$, translation invariant, homogeneous and comparable with d_c. That is we assume that, for all $x, y, z \in \mathbb{G}$ and $\lambda > 0$

$$d(x \cdot y, x \cdot z) = d(y, z) \quad \text{and} \quad d(\delta_\lambda y, \delta_\lambda z) = \lambda d(y, z), \tag{49}$$

and that there is $c_d > 1$ such that

$$\frac{1}{c_d} d(y, z) \le d_c(y, z) \le c_d d(y, z), \quad \text{for all} \quad y, z \in \mathbb{G}. \tag{50}$$

We indicate as $U_d(x, r)$, $\mathcal{S}_d^{Q-1}$ and $\mathcal{H}_d^{Q-1}$ respectively the open balls and the Hausdorff measures with respect to the new distance d, keeeping the notation $U_c(x, r)$, $\mathcal{S}_{\mathbb{G}}^{Q-1}$ and $\mathcal{H}_{\mathbb{G}}^{Q-1}$ when working with the cc-distance.

We need the following differentiation Theorem whose proof can be found in Federer's book (see [17], Theorems 2.10.17 and 2.10.18). Notice that Federer states this result in a much more general context, i.e. for regular measures in metric spaces.

Theorem 3.4. (Federer's differentiation Theorem) *If*

$$\lim_{r\to 0}\frac{|\partial E|_{\mathbb{G}}(U_d(x,r))}{2^{-Q+1}(\operatorname{diam} U_d(x,r))^{Q-1}} = \mathfrak{s}(x), \qquad \textit{for} \quad |\partial E|_{\mathbb{G}}\text{-}a.e.\ x\in\mathbb{G}, \tag{51}$$

then

$$|\partial E|_{\mathbb{G}} = \mathfrak{s}(x)\mathcal{S}_d^{Q-1}\llcorner\partial E \tag{52}$$

Theorem 3.5. *If d is a distance on $\mathbb{G}$ satisfying (49) and (50), and if $\mathfrak{s}_d : H\mathbb{G}_0\setminus\{0\}\to\mathbb{R}$, is the 0-homogeneous function defined as*

$$\mathfrak{s}_d(v) := \mathcal{L}^{n-1}\left(U_d(0,1)\cap\Pi(0,v)\right),$$

then

$$\begin{aligned}|\partial E|_{\mathbb{G}}\llcorner\Omega &= \mathfrak{s}_d\circ\nu_E\ \mathcal{S}_{\mathbb{G}}^{Q-1}\llcorner(S\cap\Omega)\\ &= \mathcal{L}^{n-1}\left(U_d(0,1)\cap T_{\mathbb{G}}^g S(x)\right)\ \mathcal{S}_{\mathbb{G}}^{Q-1}\llcorner(S\cap\Omega).\end{aligned} \tag{53}$$

Moreover, there is a constant $\alpha_d>1$, depending only on the distance d, such that

$$0<\frac{1}{\alpha_d}\leqslant\mathfrak{s}_d(v)\leqslant\alpha_d<\infty.$$

Remark 3.6. If the distance d under consideration is invariant with respect to rotations of $H\mathbb{G}_0\simeq\mathbb{R}^m$, then the function $\mathfrak{s}_d$ is constant and, with an appropriate choice of the normalization constant in the definition of the Hausdorff measure, (53) takes the particularly neat form

$$|\partial E|_{\mathbb{G}} = \mathcal{S}_d^{Q-1}\llcorner S. \tag{54}$$

We do not know how large is the class of groups whose cc-distance enjoys this property. It certainly comprises the Heisenberg groups. For the groups in this class we have

$$|\partial E|_{\mathbb{G}} = \mathcal{S}_c^{Q-1}\llcorner S. \tag{55}$$

Nevertheless, even if d_c were not rotationally invariant, it always exists another *true metric* invariant, homogeneous, comparable with d_c that is also invariant by rotations of $H\mathbb{G}_0$ (for an example see the Appendix in [23]). If one computes the Hausdorff measure with respect to it, then (54) holds.

Proof of Theorem 3.5. Observe that under our hypotheses on the distance d it follows that $\operatorname{diam} U_d(x,r)=2r$, then, from Theorem 3.4, to prove (53) it is enough to show that

$$\lim_{r\to 0} r^{1-Q}|\partial E|_{\mathbb{G}}(U_c(x,r)) = \mathcal{L}^{n-1}\left(U_c(0,1)\cap T_{\mathbb{G}}^g S(x)\right). \tag{56}$$

We are going to prove (56) by a direct computation exploiting formula (vi) of Theorem 2.1. Notice also that, without loss of generality, we can assume that $x = 0$.

From the homogeneity of the perimeter (see (21))

$$r^{1-Q}|\partial E|_{\mathbb{G}}(U_c(0,r)) = |\partial E_{0,r}|_{\mathbb{G}}(U_c(0,1)). \tag{57}$$

For all $r > 0$ that are so small that $\partial E \cap U_c(0,r) = \{x : f(x) = 0\}$, with $f \in \mathbf{C}^1_{\mathbb{G}}(U_c(0,r))$ and $X_1 f(y) > c_1 > 0$ in $U_c(0,r)$, we have

$$\partial E_{0,r} \cap U_c(0,1) = \{x \in U_c(0,1) : \frac{1}{r} f \circ \delta_r(x) = 0\}.$$

Notice that $f_{1/r} := \frac{1}{r} f \circ \delta_r \in \mathbf{C}^1_{\mathbb{G}}(U_c(0,1))$ and that $X_1 f_{1/r}(y) > c_1 > 0$ in $U_c(0,1)$.

Given that, as shown in Theorem 2.1, the equation $f(x) = 0$ defines implicitely a function $\phi : I_\delta \to \mathbb{R}$, it follows that the equation $f_{1/r}(x) = 0$ defines implicitely the function $\phi_{1/r} : I_{\delta/r} \to \mathbb{R}$ acting as $\phi_{1/r}(\xi) = \frac{1}{r}\phi \circ \delta_r(\xi)$ and a map $\Phi_{1/r} : I_{\delta/r} \to \mathbb{G}$ acting as

$$\Phi_{1/r}(\xi) = \exp\left(\phi_{1/r} X_1\right)(0,\xi).$$

Hence by formula (vi) we get

$$|\partial E_{0,r}|_{\mathbb{G}}(U_c(0,1)) = \int\limits_{\Phi^{-1}_{1/r}(U_c(0,1))} \frac{\sqrt{\sum_{j=1}^m (X_j f_{1/r})^2}}{X_1 f_{1/r}} \circ \Phi_{1/r} \, d\mathcal{L}^{n-1}. \tag{58}$$

In order to pass to the limit as $r \to 0$ in (58), notice that, from Taylor's formula (24),

$$\begin{aligned} f_{1/r}(x) &\to f_\infty(x) := \langle Xf(0), \pi_0 x\rangle_0 \\ X_j f_{1/r}(x) &\to X_j f_\infty(x) = X_j f(0) \quad \text{for} \quad j = 1,\dots,m, \end{aligned} \tag{59}$$

as $r \to 0$, uniformly on $U_d(0,1)$. Hence, if we put $\nu = (\nu_1,\dots,\nu_m) := \nu_E(0) = \frac{Xf(0)}{|Xf(0)|_0}$, then from the assumption $X_1 f(y) > c_1 > 0$ it follows $\nu_1 > 0$ and we get

$$\frac{\sqrt{\sum_{j=1}^m (X_j f_{1/r})^2}}{X_1 f_{1/r}} \to \frac{1}{\nu_1} \tag{60}$$

as $r \to 0$, uniformly on $U_d(0,1)$. Finally, for all $\xi \in \Pi(0,e_1) \equiv \mathbb{R}^{n-1}$, since $f_\infty(\Phi_\infty(0,\xi)) = 0$, we get

$$\begin{aligned} &\lim_{r\to 0} \phi_{1/r}(\xi) = \phi_\infty(\xi) = -\frac{\sum_{j=2}^m \nu_j \xi_j}{\nu_1} \\ &\lim_{r\to 0} \Phi_{1/r}(\xi) = \Phi_\infty(\xi) = \exp\left(\phi_\infty(\xi) X_1\right)(0,\xi). \end{aligned} \tag{61}$$

Notice that Φ_∞ is a map from $\Pi(0,e_1)$ to $T^g_{\mathbb{G}}S(0) \subset \mathbb{G}$ and the norm of its Jacobian is precisely $\frac{1}{\nu_1}$. Indeed, writing explicitely $\Phi_\infty(\xi) = (\Phi_{\infty,1}(\xi), \dots, \Phi_{\infty,n}(\xi))$, we have

$$\begin{aligned} \Phi_\infty(\xi) &:= \exp\left(\phi_\infty(\xi) X_1\right)(0,\xi) \\ &= \left(-\frac{\sum_{j=2}^m \nu_j \xi_j}{\nu_1}, \xi_2, \dots, \xi_m, \Phi_{\infty,m+1}(\xi), \dots, \Phi_{\infty,n}(\xi)\right) \end{aligned} \tag{62}$$

and from(19) we know that the last $n-m$ components $\Phi_{\infty,j}$ do not depend on all the variables ξ_j, but, precisely, if $h_{l-1} < j \le h_l$, then $\Phi_{\infty,j}(\xi) = \Phi_{\infty,j}(\xi_2, \dots, \xi_{h_{l-1}})$. Hence the Jacobian matrix $\mathcal{J}_{\Phi_\infty}$ is the $n \times (n-1)$ matrix

$$\mathcal{J}_{\Phi_\infty}(\xi) = \begin{bmatrix} -\frac{\nu_2}{\nu_1} & -\frac{\nu_3}{\nu_1} & \dots & -\frac{\nu_m}{\nu_1} & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ \frac{\partial \Phi_{\infty,m+1}}{\partial \xi_2} & \frac{\partial \Phi_{\infty,m+1}}{\partial \xi_3} & \dots & \frac{\partial \Phi_{\infty,m+1}}{\partial \xi_m} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Phi_{\infty,n}}{\partial \xi_2} & \frac{\partial \Phi_{\infty,n}}{\partial \xi_3} & \dots & \frac{\partial \Phi_{\infty,n}}{\partial \xi_m} & \frac{\partial \Phi_{\infty,n}}{\partial \xi_{m+1}} & \dots & 1 \end{bmatrix}. \tag{63}$$

Then it is immediate to compute

$$\|\mathcal{J}_{\Phi_\infty}(\xi)\| := \|\frac{\partial \Phi_\infty}{\partial \xi_2} \wedge \dots \wedge \frac{\partial \Phi_\infty}{\partial \xi_n}\| = \sqrt{1 + \sum_{i=2}^m \left(\frac{\nu_i}{\nu_1}\right)^2} = \frac{1}{\nu_1}.$$

The preceeding computations yield

$$\begin{aligned} \lim_{r\to 0} r^{1-Q} |\partial E|_{\mathbb{G}}(U_d(0,r)) &= \int_{\Phi_\infty^{-1}(U_d(0,1)\cap T^g_{\mathbb{G}}S(0))} \frac{1}{\nu_1}\, d\mathcal{L}^{n-1} \\ &= \int_{U_d(0,1)\cap T^g_{\mathbb{G}}S(0)} d\mathcal{L}^{n-1} \\ &= \mathcal{L}^{n-1}\left(U_d(0,1) \cap T^g_{\mathbb{G}}S(0)\right). \end{aligned} \tag{64}$$

□

Corollary 3.7. *If S is a $\mathbb{G}$-regular hypersurface then the Hausdorff dimension of S, with respect to the cc-metric d_c or any other metric d comparable with it, is $Q-1$.*

3.2. Euclidean regular surfaces in $\mathbb{G}$.

Even if $\mathbb{G}$-regular surfaces are the natural regular surfaces inside a group $\mathbb{G}$, it may also be of some interest to study Euclidean C^1 surfaces in $\mathbb{R}^n = \mathbb{G}$.

Strictly speaking, an Euclidean regular surface S may be not $\mathbb{G}$-regular. Indeed, even if S is locally the zero set of a function $f \in \mathbf{C}^1(\mathbb{R}^n) \subset \mathbf{C}^1_{\mathbb{G}}(\mathbb{R}^n)$, the transversality condition $\nabla_{\mathbb{G}} f(x) \neq 0$, $\forall x \in S$ may fail to hold. Points of S where the transversality condition fails are usually called *characteristic points*. More precisely the characteristic set $C(S)$ of an Euclidean regular surface S inside a Carnot group is

$$C(S) = \{x \in S \,:\, H\mathbb{G}_x \subseteq T_S(x)\}$$

where $T_S(x)$ denotes the Euclidean tangent space to S at $x \in S$.

It follows from the non integrability of the vector fields $X_1, \dots, X_m$ (assumptions (15)), that $C(S)$ is *small* inside S. There are many results in this line, under various regularity hypotheses on the surfaces and using different surface measures (Euclidean versus intrinsic) to estimate the smallness. Balogh (see [5]) was the first one to prove that, in the Heisenberg groups, the intrinsic $(Q-1)$-Hausdorff measure of the characteristic set of an Euclidean C^1 surface vanishes. He obtained also many other related optimal estimates. Very recently, Balogh's estimate has been extended to step 2 Carnot groups (see [23]). Precisely we have

$$\mathcal{H}^{Q-1}_{\mathbb{G}}(C(S)) = 0 \tag{65}$$

if S is an Euclidean C^1 hypersurface in a Carnot group $\mathbb{G}$ of step 2. Since a C^1 hypersurface S in a Carnot group can be written as $S = C(S) \cup (S \setminus C(S))$ and $S \setminus C(S)$ is a $\mathbb{G}$-regular hypersurface, then, by combining (53) and (65) we get

$$|\partial E|_{\mathbb{G}} \llcorner (\Omega \setminus C(S)) = \mathfrak{s}\mathcal{S}^{Q-1}_{\mathbb{G}} \llcorner (S \cap \Omega).$$

On the other hand, the representation formula (14) for the $\mathbb{G}$-perimeter of $\mathcal{C}^1$ manifolds yields

$$|\partial E|_{\mathbb{G}} \llcorner \Omega = |C n_E| \mathcal{H}^{n-1} \llcorner (S \cap \Omega) = \mathfrak{s}\mathcal{S}^{Q-1}_{\mathbb{G}} \llcorner (S \cap \Omega) = |\partial E|_{\mathbb{G}} \llcorner (\Omega \setminus C(S)),$$

where n_E is the Euclidean outward normal vector field to S, since $Cn_E = 0$ on $C(S)$. Thus the following corollary holds

Corollary 3.8. *Let F be an open subset of $\mathbb{G}$ with boundary S. If S is an Euclidean C^1 hypersurface and $\mathbb{G}$ is step 2 group, then*

$$\begin{aligned} |\partial F|_{\mathbb{G}} \llcorner \Omega &= |Cn_E| \, \mathcal{H}^{n-1} \llcorner (S \cap \Omega) \\ &= \mathfrak{s} \, \mathcal{S}_{\mathbb{G}}^{Q-1} \llcorner (S \cap \Omega); \end{aligned} \tag{66}$$

here $\mathfrak{s}$ is given by Theorem 3.3 and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Euclidean Hausdorff measure on $\mathbb{G} \equiv \mathbb{R}^n$.

Remark 3.9. Recently, Magnani, (see [37]), extended (65) to a general Carnot group.

Remark 3.10. In order to make more evident the relationships among different integral rappresentations of the $\mathbb{G}$-perimeter, we observe the following: with the notations of Proposition 1.4 and Theorem 2.1, assume that $\partial F \cap \Omega$ is an Euclidean C^1 hypersurface with no characteristic points, formula (14) yields

$$|\partial F|_{\mathbb{G}} \llcorner \Omega = |Cn_F|_{\mathbb{R}^m} \, \mathcal{H}^{n-1} \llcorner (\partial F \cap \Omega). \tag{67}$$

On the other hand, if $\Omega \cap \partial F \subset \Phi(\mathrm{Int}\, I_\delta)$, (vi) of Theorem 2.1 reads as

$$|\partial F|_{\mathbb{G}} \llcorner \Omega = \Phi_\# \left(\frac{|\nabla_{\mathbb{G}} f|}{X_1 f} \circ \Phi \, \mathcal{L}^{n-1} \right). \tag{68}$$

Notice also that

$$|Cn_F|_{\mathbb{R}^m} = \frac{|C\nabla f|_{\mathbb{R}^m}}{|\nabla f|_{\mathbb{R}^n}} = \frac{|\nabla_{\mathbb{G}} f|}{|\nabla f|_{\mathbb{R}^n}},$$

and that, by the Euclidean area formula and the non trivial computations in the proof of Theorem 2.1,

$$\Phi_\# \left(\frac{|\nabla f|}{X_1 f} \circ \Phi \, \mathcal{L}^{n-1} \right) = \mathcal{H}^{n-1} \llcorner (\partial F \cap \Omega).$$

Thus, (67) and (68) are also formally equivalent, but the right hand side of (67) becomes meaningless if the Euclidean regularity of ∂F fails, unlike (68) that turns out to be an intrinsic generalization of (67) for $\mathbb{G}$-regular hypersurfaces.

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