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Prescribing a Higher Order Conformal Invariant on S^n

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1. Introduction.

An important problem in differential geometry is to construct conformal metrics on S^2 whose Gauss curvature equals a given positive function f. This problem is equivalent to finding a solution of the equation

$$-\Delta_0 w = f\, e^{2w} - 1,$$

where Δ_0 denotes the Laplace operator associated with the standard metric g_0 on S^2. J. Moser [22] proved that this equation has a solution if the function f satisfies the condition $f(x) = f(-x)$ for all $x \in S^2$. The general case was studied by S.-Y. A. Chang, M. Gursky, and P. Yang [10, 11, 14].

A. Bahri and J. M. Coron [4, 5] and R. Schoen and D. Zhang [24] constructed metrics with prescribed scalar curvature on S^3. J. F. Escobar and R. Schoen [18] studied the prescribed scalar curvature problem on manifolds which are not necessarily conformally equivalent to the standard sphere.

Our aim is to generalize these results to higher dimensions. Let g be a conformal metric on S^4. We denote by R the scalar curvature of g and by Ric the Ricci tensor of g. Moreover, we denote by Δ the Laplace operator with respect to the metric g. A natural conformal invariant in dimension four is

$$Q = -\frac{1}{6}\,(\Delta R - R^2 + 3\,|Ric|^2).$$

The quantity Q plays an important role in conformal geometry, see [7, 12, 15]. Indeed, the quantity Q enjoys similar properties as the Gauss curvature in dimension two. For a given positive function f on S^4, we want to construct a conformal metric g on S^4 such that

$$Q = 6f.$$

If we denote by g_0 the standard metric on S^4, then this problem is equivalent to the equation

$$\Delta_0^2 w - 2\Delta_0 w = 6\,(fe^{4w} - 1).$$

If the function f satisfies the condition $f(x) = f(-x)$ for all $x \in S^4$, then this equation has a solution. The solution can be constructed by means of an evolution equation, see [9].

More generally, we can consider the standard sphere S^n, where n is even. By the work of C. Fefferman and R. Graham [19, 20], there exists a conformally invariant self-adjoint operator with leading term $(-\Delta_0)^{\frac{n}{2}}$. On the standard sphere S^n, this operator is given by

$$P_0 = \prod_{k=1}^{\frac{n}{2}} (-\Delta_0 + (k-1)(n-k)),$$

see [8, 12]. We consider the equation

$$P_0 w = (n-1)!\,(f\,e^{nw} - 1)$$

for some positive function f on S^n. This is a semilinear elliptic equation of order n involving the critical Sobolev exponent. We assume that the function f satisfies the non-degeneracy condition

$$\nabla_0 f(p) \quad \Longrightarrow \quad \Delta_0 f(p) \neq 0.$$

Moreover, we identify the group of conformal transformations on S^n with the unit ball in $\mathbb{R}^{n+1}$. Moreover, we consider the map

$$H : B^{n+1} \to \mathbb{R}^{n+1}, \quad \sigma \mapsto \bigg(\int_{S^n} f \circ \sigma \, x_i \, dV_0 \bigg)_{1 \leq i \leq n+1}.$$

Then we have the following result:

Theorem 1.1. *Suppose that f satisfies the non-degeneracy condition and*

$$\deg(H, 0) \neq 0.$$

Then the equation

$$P_0 w = (n-1)!\,(f\,e^{nw} - 1)$$

has a solution.

As a consequence, we obtain:

Corollary 1.2. *Suppose that f satisfies the non-degeneracy condition and*

$$\sum_{\nabla_0 f(p)=0,\, \Delta_0 f(p)<0} (-1)^{\mathrm{ind}(f,p)} \neq 1.$$

Then the equation

$$P_0 w = (n-1)!\,(f\,e^{nw} - 1)$$

has a solution.

An index criterion similar to that in Corollary 1.2 was introduced by A. Bahri and J. M. Coron [5] in their work on the prescribed scalar curvature problem. Related results were established by Z. Djadli, A. Malchiodi and M. Ahmedou [16, 17].

In Section 2, we consider solutions of the equation

$$P_0 w = (n-1)!\,(f\,e^{nw} - 1)$$

satisfying the normalization condition

$$\int_{S^n} e^{nw}\, x_j\, dV_0 = 0$$

for $1 \leq j \leq n+1$. Using the estimates for the Paneitz operator from [12], we can show that the function w is bounded in H^n. If the function f is close to 1, we establish an estimate of the form

$$\|w\|_{H^n} \leq C\,\|f-1\|_{L^2}$$

for some constant C.

In Section 3, we show that the a-priori estimates remain valid even if the normalization condition is dropped. The proof relies on the Kazdan-Warner identity (see [12]) and the non-degeneracy condition for f.

In Section 4, we use a topological degree argument to show that the equation

$$P_0 w = (n-1)!\,(f\,e^{nw} - 1)$$

has a solution.

2. A-priori estimates for solutions satisfying a normalization condition.

Let f be a positive function on S^n, and let w be a function which satisfies the equation

$$P_0 w = (n-1)!\,(f\,e^{nw} - 1)$$

and the normalization condition

$$\int_{S^n} e^{nw}\, x_j\, dV_0 = 0$$

for $1 \leq j \leq n+1$. We begin with a simple estimate.

Lemma 2.1. *The function w satisfies*

$$\|w - \overline{w}\|_{W^{\frac{n}{2},p}} \leq C$$

for all $p < 2$.

Proof. The function $P_0 w$ satisfies

$$|P_0 w| \leq P_0 w + 2(n-1)!,$$

hence

$$⨍_{S^n} |P_0 w| \, dV_0 \leq 2(n-1)!.$$

Using Green's formula, we obtain

$$\|w - \overline{w}\|_{W^{\frac{n}{2},p}} \leq C$$

for all $p < 2$.

Using the normalization condition, it is possible to derive an improved Sobolev inequality for the function w. The proof follows ideas of T. Aubin [2, 3] and is included here for completeness.

Proposition 2.2. *The function w satisfies the inequality*

$$\log\left(\int_{S^n} e^{n(w-\overline{w})} \, dV_0\right) \leq ⨍_{S^n} \frac{n}{4(n-1)!} \, w \, P_0 w \, dV_0 + C.$$

Proof. We use the inequality

$$\begin{aligned}\log\left(\int_{S^n} e^{nu} \, dV_0\right) &\leq ⨍_{S^n} \frac{n}{2(n-1)!} \left((-\Delta_0)^{\frac{n}{4}} u\right)^2 dV_0 \\ &\quad + ⨍_{S^n} nu \, dV_0 + C\end{aligned}$$

(see [1, 6, 12]). Without loss of generality, we may assume that

$$\int_{S^n} e^{nw} \, dV_0 \leq C \int_{\{x_{n+1} \geq 2\delta\}} e^{nw} \, dV_0.$$

We first consider the case

$$\int_{\{x_{n+1} \geq \delta\}} \left((-\Delta_0)^{\frac{n}{4}} w\right)^2 dV_0 \leq \int_{\{x_{n+1} \leq \delta\}} \left((-\Delta_0)^{\frac{n}{4}} w\right)^2 dV_0.$$

This implies

$$\int_{\{x_{n+1}\geq\delta\}} \left((-\Delta_0)^{\frac{n}{4}} w\right)^2 dV_0 \leq \int_{S^n} \frac{1}{2}\left((-\Delta_0)^{\frac{n}{4}} w\right)^2 dV_0.$$

We choose a cut-off function η such that $\eta = 1$ for $x_{n+1} \geq 2\delta$ and $\eta = 0$ for $x_{n+1} \leq \delta$. For $u = \eta\,(w - \overline{w})$ we obtain

$$\log\left(\int_{S^n} e^{n\eta(w-\overline{w})}\, dV_0\right) \leq \fint_{S^n} \frac{n}{2(n-1)!}\left((-\Delta_0)^{\frac{n}{4}}(\eta\,(w-\overline{w}))\right)^2 dV_0 + \fint_{S^n} n\,\eta\,(w-\overline{w})\, dV_0 + C.$$

From this it follows that

$$\log\left(\int_{\{x_{n+1}\geq 2\delta\}} e^{n(w-\overline{w})}\, dV_0\right) \leq \fint_{S^n} \frac{n}{2(n-1)!}\,\eta^2\left((-\Delta_0)^{\frac{n}{4}} w\right)^2 dV_0 + C.$$

Therefore, we obtain

$$\log\left(\int_{S^n} e^{n(w-\overline{w})}\, dV_0\right) \leq \fint_{S^n} \frac{n}{4(n-1)!}\left((-\Delta_0)^{\frac{n}{4}} w\right)^2 dV_0 + C.$$

We now consider the case

$$\int_{\{x_{n+1}\leq\delta\}} \left((-\Delta_0)^{\frac{n}{4}} w\right)^2 dV_0 \leq \int_{\{x_{n+1}\geq\delta\}} \left((-\Delta_0)^{\frac{n}{4}} w\right)^2 dV_0.$$

This implies

$$\int_{\{x_{n+1}\leq\delta\}} \left((-\Delta_0)^{\frac{n}{4}} w\right)^2 dV_0 \leq \int_{S^n} \frac{1}{2}\left((-\Delta_0)^{\frac{n}{4}} w\right)^2 dV_0.$$

We choose a cut-off function η such that $\eta = 1$ for $x_{n+1} \leq 0$ and $\eta = 0$ for $x_{n+1} \geq \delta$. For $u = \eta\,(w - \overline{w})$ we obtain

$$\log\left(\int_{S^n} e^{n\eta(w-\overline{w})}\, dV_0\right) \leq \fint_{S^n} \frac{n}{2(n-1)!}\left((-\Delta_0)^{\frac{n}{4}}(\eta\,(w-\overline{w}))\right)^2 dV_0 + \fint_{S^n} n\,\eta\,(w-\overline{w})\, dV_0 + C.$$

From this it follows that

$$\log\left(\int_{\{x_{n+1}\leq 0\}} e^{n(w-\overline{w})}\, dV_0\right) \leq \fint_{S^n} \frac{n}{2(n-1)!}\,\eta^2\left((-\Delta_0)^{\frac{n}{4}} w\right)^2 dV_0 + C.$$

Using the inequality

$$\begin{aligned}\int_{S^n} e^{n(w-\overline{w})}\,dV_0 &\leq C \int_{\{x_{n+1}\geq 2\delta\}} e^{n(w-\overline{w})}\,dV_0 \\ &\leq C \int_{\{x_{n+1}\geq 2\delta\}} e^{n(w-\overline{w})}\,x_{n+1}\,dV_0 \\ &= -C \int_{\{x_{n+1}\leq 2\delta\}} e^{n(w-\overline{w})}\,x_{n+1}\,dV_0 \\ &\leq -C \int_{\{x_{n+1}\leq 0\}} e^{n(w-\overline{w})}\,x_{n+1}\,dV_0 \\ &\leq C \int_{\{x_{n+1}\leq 0\}} e^{n(w-\overline{w})}\,dV_0,\end{aligned}$$

we obtain

$$\log\bigg(\int_{S^n} e^{n(w-\overline{w})}\,dV_0\bigg) \leq \fint_{S^n} \frac{n}{4(n-1)!}\,\big((-\Delta_0)^{\frac{n}{4}} w\big)^2\,dV_0 + C.$$

This proves the assertion.

On the other hand, S.-Y. A. Chang and P. Yang [12] established the following estimate:

Proposition 2.3. *Assume that*

$$0 < m \leq f \leq M.$$

Then the function w satisfies

$$\fint_{S^n} \frac{n}{2(n-1)!}\,w\,P_0 w\,dV_0 - \log\bigg(\int_{S^n} e^{n(w-\overline{w})}\,dV_0\bigg) \leq C.$$

Combining these statements, we obtain:

Corollary 2.4. *Assume that*

$$0 < m \leq f \leq M.$$

Then the function w satisfies

$$\int_{S^n} w\,P_0 w\,dV_0 \leq C.$$

As a consequence, we obtain:

Proposition 2.5. *If the function f satisfies*

$$0 < m \leq f \leq M,$$

then the function w satisfies the estimate $\|w\|_{H^n} \leq C$.

Proof. It follows from Corollary 2.4 that

$$\|w - \overline{w}\|_{H^{\frac{n}{2}}} \leq C.$$

Using an inequality of N. Trudinger, we obtain

$$\int_{S^n} e^{n(w-\overline{w})} \, dV_0 \leq C.$$

Since

$$P_0 w = (n-1)! \, (f \, e^{nw} - 1),$$

we obtain

$$⨍_{S^n} f \, e^{nw} \, dV_0 = 1.$$

This implies

$$\frac{1}{M} \leq ⨍_{S^n} e^{nw} \, dV_0 \leq \frac{1}{m}.$$

From this it follows that

$$-C \leq \overline{w} \leq C.$$

Thus, we conclude that $\|w\|_{H^{\frac{n}{2}}} \leq C$, hence

$$\int_{S^n} e^{2nw} \, dV_0 \leq C$$

by Trudinger's inequality. From this it follows that

$$\int_{S^n} (P_0 w)^2 \, dV_0 \leq C.$$

Since $\overline{w}$ is bounded, the assertion follows.

In the remaining part of this section, we assume that the function f is close to 1.

Lemma 2.6. *For every $\varepsilon > 0$, there exists a real number $\delta > 0$ with the following property: If the function f satisfies*

$$0 < m \leq f \leq M$$

and

$$\|f - 1\|_{L^2} \leq \delta,$$

then the function w satisfies the estimate $\|w\|_{H^n} \leq \varepsilon$.

Proof. We consider a sequence of functions w_k satisfying

$$P_0 w_k = (n-1)!\,(f_k\,e^{nw_k} - 1)$$

and

$$\int_{S^n} e^{nw_k}\,x_j\,dV_0 = 0$$

for $1 \leq j \leq n+1$. We assume that

$$0 < m \leq f_k \leq M$$

and

$$\|f_k - 1\|_{L^2} \to 0.$$

By Proposition 2.5, the function satisfies the estimate $\|w_k\|_{H^n} \leq C$. Hence, by passing to a subsequence, we may assume that

$$\|w_k - w\|_{L^\infty} \to 0$$

for some function w. Then the function w satisfies

$$P_0 w = (n-1)!\,(e^{nw} - 1).$$

From this it follows that w is smooth. Moreover, it follows from the results in [13] that the metric $e^{2w} g_0$ agrees with the standard metric g_0 up to conformal transformations. Using the normalization condition

$$\int_{S^n} e^{nw}\,x_j\,dV_0 = 0$$

for $1 \leq j \leq n+1$, we conclude that $w = 0$. This implies

$$\|w_k\|_{L^\infty} \to 0.$$

Since

$$P_0 w_k = (n-1)!\,(f_k\,e^{nw_k} - 1),$$

it follows that

$$\|P_0 w_k\|_{L^2} \to 0.$$

Therefore, we obtain

$$\|w_k\|_{H^n} \to 0.$$

This proves the assertion.

Proposition 2.7. *Assume that the function f satisfies*

$$0 < m \leq f \leq M$$

and

$$\|f - 1\|_{L^2} \leq \delta.$$

Then the function w satisfies an estimate of the form

$$\|w\|_{H^n} \leq C\,\|f - 1\|_{L^2}.$$

Proof. The function w satisfies

$$P_0 w - n!\,w = (n-1)!\,(f-1)\,e^{nw} + (n-1)!\,(e^{nw} - nw - 1)$$

and

$$\int_{S^n} w\,x_j\,dV_0 = -\int_{S^n} \frac{1}{n}\,(e^{nw} - nw - 1)\,x_j\,dV_0$$

for $1 \leq j \leq n+1$. Using the Sobolev embedding theorem, we obtain

$$\|w\|_{L^\infty} \leq C\,\|w\|_{H^n} \leq C\,\varepsilon.$$

From this it follows that

$$\|P_0 w - n!\,w\|_{L^2} \leq C\,\|f - 1\|_{L^2} + C\,\varepsilon\,\|w\|_{L^2}$$

and

$$\left|\int_{S^n} w\,x_j\,dV_0\right| \leq C\,\varepsilon\,\|w\|_{L^2}$$

for $1 \leq j \leq n+1$. Thus, we conclude that

$$\|w\|_{H^n} \leq C\,\|f - 1\|_{L^2} + C\,\varepsilon\,\|w\|_{L^2},$$

hence

$$\|w\|_{H^n} \leq C\,\|f - 1\|_{L^2}.$$

This proves the assertion.

Proposition 2.8. *Let f be a function with*

$$\|f - 1\|_{L^2} \leq \delta.$$

Then there exists a unique pair $(w, \Lambda) \in H^n \times \mathbb{R}^{n+1}$ such that

$$P_0 w = (n-1)!\left(\left(f - \sum_{j=1}^{n+1} \Lambda_j\,x_j\right)e^{nw} - 1\right)$$

and

$$\int_{S^n} e^{nw}\, x_j\, dV_0 = 0$$

for $1 \leq j \leq n+1$ *and* $\|(w, \Lambda)\|_{H^n \times \mathbb{R}^{n+1}} \leq \varepsilon$.

Proof. Let

$$\mathcal{S} = \Big\{ w \in H^n : \int_{S^n} e^{nw}\, x_j\, dV_0 = 0 \quad \text{for } 1 \leq j \leq n+1 \Big\}.$$

We define a map

$$\Phi : \mathcal{S} \times \mathbb{R}^{n+1} \to L^2$$

by

$$\Phi(w, \Lambda) = e^{-nw}\, P_0 w + (n-1)!\, e^{-nw} + \sum_{j=1}^{n+1} (n-1)!\, \Lambda_j\, x_j.$$

We denote by

$$\Phi' : T\mathcal{S} \times \mathbb{R}^{n+1} \to L^2$$

the differential of Φ at the point $(0,0)$. We have

$$T\mathcal{S} = \Big\{ w \in H^n : \int_{S^n} w\, x_j\, dV_0 = 0 \quad \text{for } 1 \leq j \leq n+1 \Big\}$$

and

$$\Phi'(w, \Lambda) = P_0 w - n!\, w + \sum_{j=1}^{n+1} (n-1)!\, \Lambda_j\, x_j.$$

Therefore, the map Φ' is bijective. The implicite function theorem implies that Φ is a bijective map from a neighbourhood of $(0,0)$ in $\mathcal{S} \times \mathbb{R}^{n+1}$ to a neighbourhood of $(n-1)!$ in L^2. Since

$$\|f - 1\|_{L^2} \leq \delta,$$

there exists a pair $(w, \Lambda) \in \mathcal{S} \times \mathbb{R}^{n+1}$ such that

$$\Phi(w, \Lambda) = (n-1)!\, f$$

and

$$\|(w, \Lambda)\|_{H^n \times \mathbb{R}^{n+1}} \leq \varepsilon.$$

From this the assertion follows.

3. A-priori estimates for solutions in the absence of a normalization condition.

Let f be a fixed positive function on S^n. In this section, we establish the following result:

Proposition 3.1. *Let w be a solution of the equation*

$$P_0 w = (n-1)!\,\big((s\,f + 1 - s)\,e^{nw} - 1\big)$$

for some $0 < s \leq 1$. Then the function w satisfies the estimate $\|w\|_{H^n} \leq C$.

Proof. Assume that there exists a sequence of functions w_k satisfying

$$P_0 w_k = (n-1)!\,\big((s_k\,f + 1 - s_k)\,e^{nw_k} - 1\big)$$

for some $0 < s_k \leq 1$ and

$$\|w_k\|_{H^n} \to \infty.$$

We choose conformal transformations σ_k such that

$$\int_{S^n} e^{n\tilde{w}_k}\,x_j\,dV_0 = 0$$

for $1 \leq j \leq n+1$, where

$$e^{2\tilde{w}_k} g_0 = \sigma_k^*(e^{2w_k} g_0).$$

Then the functions $\tilde{w}_k$ satisfy the equation

$$P_0 \tilde{w}_k = (n-1)!\,\big((s_k\,f + 1 - s_k) \circ \sigma_k\,e^{n\tilde{w}_k} - 1\big).$$

Since f is a fixed positive function on S^n, we have

$$0 < m \leq (s_k\,f + 1 - s_k) \circ \sigma_k \leq M.$$

Hence, it follows from Proposition 2.5 that $\|\tilde{w}_k\|_{H^n} \leq C$. Since

$$\|w_k\|_{H^n} \to \infty,$$

we conclude that the sequence σ_k tends to infinity. This implies

$$\|(s_k\,f + 1 - s_k) \circ \sigma_k - e^{-nr_k}\|_{L^2} = o(1)$$

for a suitable constant r_k. Using Lemma 2.6, we obtain

$$\|\tilde{w}_k - r_k\|_{H^n} = o(1).$$

Moreover, the Kazdan-Warner identity (see [12]) implies that

$$\int_{S^n} \langle d(f \circ \sigma_k), dx_j \rangle \, e^{n(\tilde{w}_k - r)} \, dV_0 = 0$$

for $1 \leq j \leq n+1$. If we identify S^n with $\mathbb{R}^n \cup \{\infty\}$ via the stereographic projection, then we may assume that the conformal transformation σ_k is given by

$$\sigma_k(y) = \frac{1}{t_k} \, y$$

for a suitable sequence $t_k \to \infty$. The pull-back of the standard metric on S^n under the stereographic projection is given by

$$(g_0)_{ij} = \frac{4}{(1+|y|^2)^2} \, \delta_{ij}.$$

Moreover, we have

$$x_j = \frac{2y_j}{1+|y|^2}$$

for $1 \leq j \leq n$ and

$$x_{n+1} = -\frac{1-|y|^2}{1+|y|^2}.$$

This implies

$$dx_j = \frac{2}{1+|y|^2} \, dy_j - \sum_{i=1}^n \frac{4y_i y_j}{(1+|y|^2)^2} \, dy_i$$

for $1 \leq j \leq n$ and

$$dx_{n+1} = \sum_{i=1}^n \frac{4y_i}{(1+|y|^2)^2} \, dy_i.$$

Using the formula

$$f(y) = f(0) + \sum_{i=1}^n \alpha_i \, y_i + \frac{1}{2} \sum_{i,j=1}^n \beta_{ij} \, y_i \, y_j + o(|y|^2),$$

we obtain

$$(f \circ \sigma_k)(y) = f(0) + \frac{1}{t_k} \sum_{i=1}^n \alpha_i \, y_i + \frac{1}{2t_k^2} \sum_{i,j=1}^n \beta_{ij} \, y_i \, y_j + o\Big(\frac{|y|^2}{t_k^2}\Big),$$

hence

$$d(f \circ \sigma_k)(y) = \frac{1}{t_k} \sum_{i=1}^n \alpha_i \, dy_i + \frac{1}{t_k^2} \sum_{i,j=1}^n \beta_{ij} \, y_j \, dy_i + o\Big(\frac{|y|}{t_k^2}\Big).$$

From this it follows that

$$\begin{aligned}
0 &= \int_{S^n} \langle d(f \circ \sigma_k), dx_j \rangle \, e^{n(\tilde{w}_k - r_k)} \, dV_0 \\
&= \int_{S^n} \langle d(f \circ \sigma_k), dx_j \rangle \, dV_0 + o\Big(\frac{1}{t_k}\Big) \\
&= \frac{1}{t_k} \int_{\mathbb{R}^n} \frac{2^{n-1}\alpha_j}{(1+|y|^2)^{n-1}} \, dy_1 \cdots dy_n - \frac{1}{t_k} \sum_{i=1}^n \int_{\mathbb{R}^n} \frac{2^n \alpha_i y_i y_j}{(1+|y|^2)^n} \, dy_1 \cdots dy_n + o\Big(\frac{1}{t_k}\Big) \\
&= \frac{1}{t_k} \int_{\mathbb{R}^n} \frac{2^{n-1}\alpha_j}{(1+|y|^2)^{n-1}} \, dy_1 \cdots dy_n - \frac{1}{t_k} \int_{\mathbb{R}^n} \frac{2^n \alpha_j y_j^2}{(1+|y|^2)^n} \, dy_1 \cdots dy_n + o\Big(\frac{1}{t_k}\Big) \\
&= \frac{1}{t_k} \int_{\mathbb{R}^n} \frac{2^{n-1}\alpha_j}{(1+|y|^2)^{n-1}} \, dy_1 \cdots dy_n - \frac{1}{t_k} \int_{\mathbb{R}^n} \frac{2^n \alpha_j |y|^2}{n(1+|y|^2)^n} \, dy_1 \cdots dy_n + o\Big(\frac{1}{t_k}\Big) \\
&= \frac{1}{t_k} \int_{\mathbb{R}^n} \frac{2^{n-1} \, \alpha_j \, (n + (n-2)\, |y|^2)}{n(1+|y|^2)^n} \, dy_1 \cdots dy_n + o\Big(\frac{1}{t_k}\Big)
\end{aligned}$$

for $1 \leq j \leq n$. Thus, we conclude that $\alpha_j = o(1)$ for $1 \leq j \leq n$. From this it follows that

$$\|(s_k \, f + 1 - s_k) \circ \sigma_k - e^{-nr_k}\|_{L^2} \leq o\Big(\frac{1}{t_k}\Big),$$

where $e^{-nr_k} = s_k \, f(0) + 1 - s_k$. This implies

$$\|\tilde{w}_k - r_k\|_{H^n} \leq o\Big(\frac{1}{t_k}\Big).$$

Using this estimate, we obtain

$$\begin{aligned}
0 &= \int_{S^n} \langle d(f \circ \sigma_k), dx_{n+1} \rangle \, e^{n(\tilde{w}_k - r_k)} \, dV_0 \\
&= \int_{S^n} \langle d(f \circ \sigma_k), dx_{n+1} \rangle \, dV_0 + o\Big(\frac{1}{t_k^2}\Big) \\
&= \frac{1}{t_k^2} \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{2^n \beta_{ij} \, y_i \, y_j}{(1+|y|^2)^n} \, dy_1 \cdots dy_n + o\Big(\frac{1}{t_k^2}\Big) \\
&= \frac{1}{t_k^2} \sum_{i=1}^n \int_{\mathbb{R}^4} \frac{2^n \beta_{ii} |y|^2}{n(1+|y|^2)^n} \, dy_1 \cdots dy_n + o\Big(\frac{1}{t_k^2}\Big).
\end{aligned}$$

Thus, we conclude that

$$\sum_{i=1}^n \beta_{ii} = 0.$$

Therefore, the concentration point p satisfies

$$\nabla_0 f(p) = 0$$

and

$$\Delta_0 f(p) = 0.$$

This contradicts the non-degeneracy condition.

4. Existence results.

Let

$$\mathcal{M}_s = \Big\{ w \in H^n : \fint_{S^n} (s\,f + 1 - s)\, e^{nw}\, dV_0 = 1 \Big\}.$$

We define a map

$$\Psi_s : \mathcal{M}_s \to H^n$$

by

$$\Psi_s(w) = w - (n-1)!\, P_0^{-1}\big((s\,f + 1 - s)\, e^{nw} - 1\big).$$

We first show that the degree of Ψ_s is independent of s.

Proposition 4.1. *We have*

$$\deg(\Psi_1, 0) = \deg(\Psi_s, 0)$$

for all $0 < s \leq 1$.

Proof. It follows from Proposition 3.1 that the set

$$\big\{ (s, w) : 0 < s \leq 1,\, w \in \mathcal{M}_s \text{ and } \Psi_s(w) = 0 \big\}$$

is bounded in $\mathbb{R} \times H^n$. The assertion is now a consequence of the homotopy invariance of the degree (see [23]).

We now choose $0 < s \leq 1$ sufficiently small. By Proposition 2.8, for every conformal transformation σ, there exists a unique function $\tilde{w}_\sigma$ which satisfies

$$P_0 \tilde{w}_\sigma = (n-1)!\, \bigg(\bigg((s\,f + 1 - s) \circ \sigma - \sum_{j=1}^{n+1} \Lambda_{\sigma,j}\, x_j \bigg)\, e^{n\tilde{w}_\sigma} - 1 \bigg)$$

and

$$\int_{S^n} e^{n\tilde{w}_\sigma}\, x_j\, dV_0 = 0$$

for $1 \leq j \leq n+1$ and

$$\|(\tilde{w}_\sigma, \Lambda_\sigma)\|_{H^n \times \mathbb{R}^{n+1}} \leq \varepsilon.$$

Using Proposition 2.7, we obtain $\|\tilde{w}_\sigma\|_{H^n} \leq Cs$. We now define functions w_σ by

$$e^{2\tilde{w}_\sigma} g_0 = \sigma^*(e^{2w_\sigma} g_0).$$

Then the function w_σ satisfies the equation

$$P_0 w_\sigma = (n-1)! \bigg(\bigg((s\,f + 1 - s) - \sum_{j=1}^{n+1} \Lambda_{\sigma,j}\, x_j \circ \sigma^{-1} \bigg) e^{n w_\sigma} - 1 \bigg).$$

In the first step, we show that the zeroes of Ψ_s are in one-to-one correspondence with the zeroes of Λ.

Proposition 4.2. *A function w satisfies $\Psi_s(w) = 0$ if and only if there exists a conformal transformation σ such that $w = w_\sigma$ and $\Lambda_\sigma = 0$.*

Proof. Suppose that $w \in \mathcal{M}_s$ satisfies $\Psi_s(w) = 0$. Then the function w satisfies the equation

$$P_0 w = (n-1)! \big((s\,f + 1 - s)\, e^{nw} - 1 \big).$$

We choose a conformal transformation σ such that

$$\int_{S^n} e^{n\tilde{w}}\, x_j \, dV_0 = 0$$

for $1 \leq j \leq n+1$, where

$$e^{2\tilde{w}} g_0 = \sigma^*(e^{2w} g_0).$$

Then the function $\tilde{w}$ satisfies the equation

$$P_0 \tilde{w} = (n-1)! \big((s\,f + 1 - s) \circ \sigma\, e^{n\tilde{w}} - 1 \big).$$

If s is sufficiently small, then we have

$$\|(s\,f - s) \circ \sigma\|_{L^2} \leq \delta.$$

Using Proposition 2.6, we obtain

$$\|\tilde{w}\|_{H^n} \leq \varepsilon.$$

Hence, it follows from the uniqueness statement in Proposition 2.8 that $\tilde{w} = \tilde{w}_\sigma$ and $\Lambda_\sigma = 0$. Conversely, if σ is a conformal transformation satisfying $\Lambda_\sigma = 0$, then the function w_σ belongs to the space $\mathcal{M}_s$ and $\Psi_s(w_\sigma) = 0$.

Let σ be a conformal transformation satisfying $\Lambda_\sigma = 0$, and let Λ' be the differential of Λ at σ. Furthermore, we denote by Ψ_s' the differential of Ψ_s at w_σ. We want to compare the number of negative eigenvalues of Λ' and Ψ_s'.

To this end, we differentiate the identity

$$P_0 w_\tau = (n-1)!\,\bigg(\bigg((s\,f + 1 - s) - \sum_{j=1}^{n+1} \Lambda_{\tau,j}\, x_j \circ \tau^{-1}\bigg)\, e^{n w_\tau} - 1\bigg)$$

with respect to τ. This gives a collection of functions u_i such that

$$P_0 u_i = n!\,(s\,f + 1 - s)\, e^{n w_\sigma}\, u_i - (n-1)! \sum_{j=1}^{n+1} \Lambda_{i,j}'\, x_j \circ \sigma^{-1}\, e^{n w_\sigma}$$

for $1 \leq i \leq n+1$. By definition of w_σ, we have

$$\int_{S^n} e^{n w_\sigma}\, x_j \circ \sigma^{-1}\, dV_0 = \int_{S^n} e^{n \tilde{w}_\sigma}\, x_j\, dV_0 = 0$$

for $1 \leq j \leq n+1$. Let v_j be the solution of the linear equation

$$P_0 v_j = -x_j \circ \sigma^{-1}\, e^{n w_\sigma}.$$

Then we obtain the identity

$$P_0 u_i = n!\,(s\,f + 1 - s)\, e^{n w_\sigma}\, u_i + (n-1)! \sum_{j=1}^{n+1} \Lambda_{i,j}'\, x_j \circ \sigma^{-1}\, e^{n w_\sigma}.$$

Thus, we conclude that $u_i \in T\mathcal{M}_s$ and

$$\Psi_s'(u_i) = (n-1)! \sum_{j=1}^{n+1} \Lambda_{i,j}'\, v_j.$$

We now establish precise estimates for the functions u_i and v_j.

Lemma 4.3. *The function u_i satisfies the estimate*

$$\|u_i + x_i \circ \sigma^{-1}\|_{H^n} \leq Cs.$$

Proof. Since

$$\|(s\,f+1-s)\circ\tau-(s\,f+1-s)\circ\sigma\|_{L^2}\leq Cs\,\mathrm{dist}(\tau,\sigma),$$

it follows from the proof of Proposition 2.8 that

$$\|\tilde{w}_\tau-\tilde{w}_\sigma\|_{H^n}\leq Cs\,\mathrm{dist}(\tau,\sigma).$$

This implies

$$\|\tilde{w}_\tau\circ\tau^{-1}-\tilde{w}_\sigma\circ\sigma^{-1}\|_{H^n}\leq Cs\,\mathrm{dist}(\tau,\sigma).$$

Using the relations

$$\tilde{w}_\sigma\circ\sigma^{-1}=w_\sigma+\frac{1}{n}\log\det d\sigma\circ\sigma^{-1}$$

and

$$\tilde{w}_\tau\circ\tau^{-1}=w_\tau+\frac{1}{n}\log\det d\tau\circ\tau^{-1},$$

we obtain

$$\|w_\tau-w_\sigma+\frac{1}{n}\log\det d\tau\circ\tau^{-1}-\frac{1}{n}\log\det d\sigma\circ\sigma^{-1}\|_{H^n}\leq Cs\,\mathrm{dist}(\tau,\sigma),$$

hence

$$\|w_\tau-w_\sigma-\frac{1}{n}\log\det d(\tau^{-1}\circ\sigma)\circ\sigma^{-1}\|_{H^n}\leq Cs\,\mathrm{dist}(\tau,\sigma).$$

From this it follows that

$$\|u_i+x_i\circ\sigma^{-1}\|_{H^n}\leq Cs.$$

This proves the assertion.

Lemma 4.4. *The function v_j satisfies the estimate*

$$\|n!\,v_j+x_j\circ\sigma^{-1}\|_{H^n}\leq Cs.$$

Proof. Since $-\Delta_0 x_j=n\,x_j$, we obtain

$$P_0x_j=\prod_{k=1}^{\frac{n}{2}}(n+(k-1)(n-k))\,x_j=\prod_{k=1}^{\frac{n}{2}}k(n-k+1)\,x_j=n!\,x_j.$$

Using the conformal invariance of the Paneitz operator, we conclude that

$$P_0(x_j\circ\sigma^{-1})\,\det d\sigma\circ\sigma^{-1}=n!\,x_j\circ\sigma^{-1},$$

hence

$$P_0(x_j \circ \sigma^{-1})\, e^{n\tilde{w}_\sigma \circ \sigma^{-1}} = n!\, x_j \circ \sigma^{-1}\, e^{nw_\sigma}.$$

This implies

$$n!\, P_0 v_j = -P_0(x_j \circ \sigma^{-1})\, e^{n\tilde{w}_\sigma \circ \sigma^{-1}}.$$

Using the estimate

$$\|\tilde{w}_\sigma\|_{H^n} \leq Cs,$$

we obtain

$$\|n!\, v_j + x_j \circ \sigma^{-1}\|_{H^n} \leq C\, \|P_0(n!\, v_j + x_j \circ \sigma^{-1})\|_{L^2} \leq Cs.$$

This proves the assertion.

Proposition 4.5. *If s is sufficiently small, then the degree of Ψ_s coincides with the degree of Λ.*

Proof. By Lemma 4.3 and Lemma 4.4, the finite-dimensional approximations of Ψ_s' are of the form

$$\begin{pmatrix} \Lambda' E & \Lambda' F \\ X & Y \end{pmatrix}^T,$$

where

$$\|E - 1\| \leq Cs$$

and

$$\|F\| \leq Cs.$$

Using the identity

$$\begin{pmatrix} \Lambda' E & \Lambda' F \\ X & Y \end{pmatrix} = \begin{pmatrix} \Lambda' & 0 \\ XE^{-1} & Y - XE^{-1}F \end{pmatrix} \begin{pmatrix} E & F \\ 0 & 1 \end{pmatrix}$$

we obtain

$$\begin{aligned} \det \begin{pmatrix} \Lambda' E & \Lambda' F \\ X & Y \end{pmatrix} &= \det \begin{pmatrix} \Lambda' & 0 \\ XE^{-1} & Y - XE^{-1}F \end{pmatrix} \det \begin{pmatrix} E & F \\ 0 & 1 \end{pmatrix} \\ &= \det \Lambda' \, \det(Y - XE^{-1}F)\, \det E. \end{aligned}$$

Hence, if s is sufficiently small, then $\det \begin{pmatrix} \Lambda' E & \Lambda' F \\ X & Y \end{pmatrix}$ and $\det \Lambda'$ have the same sign. Thus, we conclude that

$$\deg(\Psi_s, 0) = \deg(\Lambda, 0)$$

if s is sufficiently small.

We now identify the the group of conformal transformations on S^n with the unit ball in $\mathbb{R}^{n+1}$. We consider the map

$$H : B^{n+1} \to \mathbb{R}^{n+1}, \quad \sigma \mapsto \left(\int_{S^n} f \circ \sigma \, x_i \, dV_0 \right)_{1 \leq i \leq n+1}.$$

Then we have the following result:

Proposition 4.6. *If s is sufficiently small, then the degree of Λ coincides with the degree of H.*

Proof. Using the Kazdan-Warner identity, we obtain

$$\int_{S^n} \left\langle d((s\, f + 1 - s) \circ \sigma) - \sum_{j=1}^{n+1} \Lambda_{\sigma,j} \, dx_j, dx_i \right\rangle e^{n\tilde{w}_\sigma} \, dV_0 = 0.$$

This implies

$$s \int_{S^n} \langle d(f \circ \sigma), dx_i \rangle \, e^{n\tilde{w}_\sigma} \, dV_0 = \sum_{j=1}^{n+1} \Lambda_{\sigma,j} \int_{S^n} \langle dx_j, dx_i \rangle \, e^{n\tilde{w}_\sigma} \, dV_0.$$

Therefore, the degree of Λ coincides with the degree of the map

$$G : B^{n+1} \to \mathbb{R}^{n+1}, \quad \sigma \mapsto \left(\int_{S^n} \langle d(f \circ \sigma), dx_i \rangle \, e^{n\tilde{w}_\sigma} \, dV_0 \right)_{1 \leq i \leq n+1}.$$

On the other hand,

$$\begin{aligned} |G(\sigma) - n\, H(\sigma)| &\leq \sum_{i=1}^{n+1} \left| \int_{S^n} \langle d(f \circ \sigma), dx_i \rangle \, e^{n\tilde{w}_\sigma} \, dV_0 - n \int_{S^n} f \circ \sigma \, x_i \, dV_0 \right| \\ &\leq \sum_{i=1}^{n+1} \left| \int_{S^n} \langle d(f \circ \sigma), dx_i \rangle \, (e^{n\tilde{w}_\sigma} - 1) \, dV_0 \right| \\ &\leq Cs. \end{aligned}$$

If s is sufficiently small, then G and H are homotopic, and therefore the degree of G agrees with the degree of H.

Combining these statements, we obtain

$$\deg(\Psi_1, 0) = \deg(H, 0).$$

By assumption, we have $\deg(H, 0) \neq 0$, hence $\deg(\Psi_1, 0) \neq 0$. Therefore, there exists a function $w \in \mathcal{M}_1$ such that $\Psi_1(w) = 0$. This implies

$$\fint_{S^n} f\, e^{nw}\, dV_0 = 1$$

and

$$w - (n-1)!\, P_0^{-1}(f\, e^{nw} - 1) = 0.$$

Thus, we conclude that

$$P_0 w = (n-1)!\, (f\, e^{nw} - 1).$$

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