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On the Instanton Complex of Holomorphic Morse Theory

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Consider a holomorphic torus action on a complex manifold which lifts to a holomorphic vector bundle. When the connected components of the fixed-point set form a partially ordered set, we construct, using sheaf-theoretical techniques, two spectral sequences that converges to the twisted Dolbeault cohomology groups and those with compact support, respectively. These spectral sequences are the holomorphic counterparts of the instanton complex in standard Morse theory. The results proved imply holomorphic Morse inequalities and fixed-point formulas on a possibly non-compact manifold. Finally, examples and applications are given.

1. Introduction.

Given a Morse function on a compact real manifold, the Morse inequalities bound the Betti numbers in terms of the information of critical points. However, the former can not be determined by the Morse inequalities alone unless the Morse function is perfect. If the Morse function satisfies the transversality condition [43], then there is a finite dimensional complex, called the Thom-Smale-Witten complex or the instanton complex [48], which computes the cohomology groups of the manifold. The instanton complex consists of vector spaces spanned by the critical points (when they are isolated), graded by their Morse indices. The coboundary operators come from counting (with orientation) the number of gradient paths between critical points whose Morse indices differ by one. The latter is related to the instanton tunneling effect in supersymmetric quantum mechanics [48].

Consider a complex manifold with a holomorphic group action and a holomorphic vector bundle over the manifold on which the group action lifts holomorphically. We want to determine the Dolbeault cohomology groups (twisted by the vector bundle) as representations of the group. When the manifold is compact, the fixed-point formula of Atiyah and Bott [2] (for isolated fixed points) and of Atiyah and Singer [3] computes the alternating sum of the characters on the cohomology groups. For holomorphic Morse theory, this (equivariant) index theorem is the counterpart of the Hopf (or Lefschetz) formula. When the manifold is compact and Kähler and the group is the circle group, Morse-type inequalities were obtained by Witten [49] using a holomorphic version of supersymmetric quantum mechanics. These (equivariant) holomorphic Morse inequalities put constraints on the sizes of Dolbeault cohomology groups but do not completely determine them. In [37], a heat kernel proof of these inequalities was given under the additional assumption that the fixed points are isolated. In [50], the result was generalized to cases with torus and non-Abelian group actions. Furthermore, it was shown that the Kähler assumption was necessary for holomorphic Morse inequalities [50], although not so for the fixed-point theorem. In [51], these inequalities were proved analytically for compact Kähler manifolds with possibly non-isolated fixed points.

In this paper, we construct the holomorphic counterpart of the instanton complex which computes the Dolbeault cohomology groups using the combinatorial data of the group action. At the same time, we investigate more closely the condition on complex manifolds for establishing a holomorphic Morse theory. Holomorphic Morse theory differs from ordinary Morse theory in a number of ways. If the circle group acts on a compact Kähler manifold in a Hamiltonian fashion, the moment map is a perfect Morse function whose critical points have even Morse indices only, which can not differ by one. Furthermore, Smale's transversality condition fails in general and the gradient paths are never isolated because of the circular symmetry. Consequently, the techniques for holomorphic Morse theory will be quite different from those for ordinary Morse theory.

We start with a complex manifold with a holomorphic action of a (complex) torus. The action of a non-compact 1-parameter subgroup is analogous to the gradient flow of a Morse function. Holomorphic actions of $\mathbb{C}^\times$ were studied extensively by Białynicki-Birula [6, 7] in the algebraic case and by Carrell and Sommese [15, 16, 17] in greater generalities. The extension of their results to higher rank tori is straightforward. An action is meromorphic if, roughly speaking, all such orbits start from and end at some points in the manifold, which must be fixed points of the torus. If so, then there is a relation on the connected components of the fixed-point set given by the direction of the flows. The central result of this paper is that if this relation is a partial ordering, then there are two (equivariant) spectral sequences converging (equivariantly) to the twisted Dolbeault cohomology groups and those with compact support, respectively. These spectral sequences will be constructed using sheaf-theoretic techniques from a filtration of the complex manifold determined by the group action. The spectral sequences, with the natural coboundary maps, are the couterparts in holomorphic Morse theory of the instanton complex in ordinary Morse theory. The information of the E_1-terms already implies the holomorphic Morse inequalities. But unlike ordinary Morse theory, the spectral sequences do not always degenerate at E_2. When the manifold is compact and Kähler, the partial order condition is automatically satisfied. Thus the results of [49, 37, 50, 51] are recovered. On the other hand, the example in [50] shows that without the partial order condition, the holomorphic Morse inequalities can fail. Therefore the partial order condition is fundamental to holomorphic Morse theory.

The rest of the paper is organized as follows. In section 2, we review and establish some facts about meromorphic torus actions on a compact or a suitably non-compact complex manifold. In section 3, we construct two spectral sequences converging to Dolbeault cohomology groups and those with compact support, respectively, under the partial order condition. In particular, we obtain holomorphic Morse inequalities and fixed-point formulas for a possibly non-compact manifold. We also study the condition under which the spectral sequences degenerate to cochain complexes. In section 4, the application to flag manifolds yields a geometric realization of the Bernstein-Gelfand-Gelfand resolution and its generalizations. We also study the Dolbeault cohomologies and geometric quantization on non-compact manifolds.

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^{\pm}$, $\mathbb{C}$ and $\mathbb{C}^{\times}$ denote the sets of non-negative integers, real numbers, positive (negative) real numbers, complex numbers and non-zero complex numbers, respectively.

2. Holomorphic torus actions.

We first recall from [50, 51] some notations of holomorphic torus actions but without making the compact or Kähler assumption.

Let T be a complex torus with Lie algebra $\mathfrak{t}$. Let $T_{\mathbb{R}}$ be the (real) maximal compact torus subgroup of T and $\mathfrak{t}_{\mathbb{R}} = \sqrt{-1}\,\mathrm{Lie}(T_{\mathbb{R}})$. Let ℓ be the integral lattice in $\mathfrak{t}_{\mathbb{R}}$, and $\ell^* \subset \mathfrak{t}_{\mathbb{R}}^*$, the dual lattice. If $T = \mathbb{C}^{\times}$, the multiplicative group of non-zero complex numbers, then $T_{\mathbb{R}} = S^1$, $\mathfrak{t}_{\mathbb{R}} = \mathbb{R}$, and $\ell = \mathbb{Z}$. In general, for any $v \in \ell - \{0\}$, there is an embedding $j_v \colon \mathbb{C}^{\times} \longrightarrow T$ whose image $\mathbb{C}^{\times}_v$ is a $\mathbb{C}^{\times}$-subgroup of T.

The ring of formal characters of T is $\mathbb{Z}[\ell^*] = \{q = \sum_{\xi\in\ell^*} q_\xi e^\xi \,|\, q_\xi \in \mathbb{Z}\}$. The *support* of $q \in \mathbb{Z}[\ell^*]$ is $\operatorname{supp} q = \{\xi \in \ell^* \,|\, q_\xi \neq 0\}$. We say that $q \geq 0$ if $q_\xi \geq 0$ for all $\xi \in \ell^*$. Consider a representation R of T. If every weight $\xi \in \ell^*$ of R has a finite multiplicity r_ξ, then the character $\operatorname{char} R = \sum_{\xi\in\ell^*} r_\xi e^\xi \in$

$\mathbb{Z}[\ell^*]$ is well-defined. Let $\operatorname{supp} R = \operatorname{supp}(\operatorname{char} R)$. As in [37, 50], we write

$$\frac{e^{\eta}}{1-e^{\xi}} \stackrel{\text{def.}}{=} \sum_{k=0}^{\infty} e^{\eta+k\xi}, \qquad \xi,\eta \in \ell^*. \tag{2.1}$$

We emphasize here that the left-hand side is a notation for the formal series in $\mathbb{Z}[\ell^*]$ on the right-hand side. More generally, if R is a finite dimensional representation of T, we can write

$$\operatorname{char}{}^{-1}\!\det(1-R) = \sum_{k=0}^{\infty} \operatorname{char} S^k(R) = \operatorname{char} S(R), \tag{2.2}$$

where ${}^{-1}\!\det(1-R)$ is a notation for $\oplus_{k=0}^{\infty} S^k(R)$.

Let X be a complex manifold of dimension n. Suppose T acts holomorphically and effectively on X. The fixed-point set X^T of T in X, if non-empty, is a complex submanifold of X. Let F be the set of connected components of X^T. Then $X^T = \bigcup_{\alpha\in F} X^T_\alpha$, where X^T_α is the component labeled by $\alpha \in F$. Let $n_\alpha = \dim_{\mathbb{C}} X^T_\alpha$. Let $N_\alpha \longrightarrow X^T_\alpha$ be the (holomorphic) normal bundle of X^T_α in X. T acts on N_α preserving the base X^T_α pointwise. The weights of the isotropy representation on the normal fiber remain constant within any connected component. Let $\lambda_{\alpha,k} \in \ell^* - \{0\} \subset \mathfrak{t}^*_{\mathbb{R}}$ $(1 \le k \le n-n_\alpha)$ be the isotropy weights on N_α. The hyperplanes $(\lambda_{\alpha,k})^\perp \subset \mathfrak{t}_{\mathbb{R}}$ cut $\mathfrak{t}_{\mathbb{R}}$ into open polyhedral cones called *action chambers* [42]. Choose an action chamber C. Let $\lambda^C_{\alpha,k} = \pm\lambda_{\alpha,k}$, with the sign chosen so that $\lambda^C_{\alpha,k} \in C^*$. (Here C^* is the dual cone in $\mathfrak{t}^*_{\mathbb{R}}$ defined by $C^* = \{\xi \in \mathfrak{t}^*_{\mathbb{R}} \mid \langle \xi, C\rangle > 0\}$.) We define ν^C_α as the number of weights $\lambda_{\alpha,k} \in C^*$. Let N^C_α be the direct sum of the sub-bundles corresponding to the weights $\lambda_{\alpha,k} \in C^*$. Then $N_\alpha = N^C_\alpha \oplus N^{-C}_\alpha$. ν^C_α is the rank of the holomorphic vector bundle N^C_α; that of N^{-C}_α is $\nu^{-C}_\alpha = n - n_\alpha - \nu^C_\alpha$, which is called the *polarizing index* of X^T_α with respect to C.

In subsection 2.1, we will consider holomorphic torus actions on compact manifolds. This will be a straightforward generalization of the work [6, 7, 15, 16, 17] from rank 1 to higher ranks. A non-compact setting will be studied in subsection 2.2.

2.1. Meromorphic torus actions on compact manifolds.

Throughout this subsection, X is a compact, connected complex manifold with an effective holomorphic action of the torus T. Then F is a finite set and each component X^T_α $(\alpha \in F)$ is compact.

Definition 2.1. A holomorphic T-action on X is *meromorphic* if for any $x \in X$ and any $v \in \ell - \{0\}$, the limit $\pi^v(x) = \lim_{u \longrightarrow 0} j_v(u)x$ exists.

If $T = \mathbb{C}^\times$, the action is meromorphic if and only if for any $x \in X$, the limits $\pi^+(x) = \lim_{u \longrightarrow 0} ux$ and $\pi^-(x) = \lim_{u \longrightarrow \infty} ux$ exist. In this case, the holomorphic map $\mathbb{C}^\times \times X \longrightarrow X$ can be extended to a meromorphic map $\mathbb{CP}^1 \times X \longrightarrow X$.

Suppose T acts holomorphically on X and Y and U is a (not necessarily T-invariant) open subset of X. We say that a map $f\colon U \longrightarrow Y$ is *locally T-equivariant* if $f(gx) = gf(x)$ for $g \in T$, $x \in U$ whenever $gx \in U$.

Proposition 2.2. *If the T-action on X is meromorphic, then*
1. for any $v \in \ell - \bigcup_{\alpha \in F, 1 \le k \le n-n_\alpha} (\lambda_{\alpha,k})^\perp$, i.e., for any v in the lattice but not annihilated by any of the isotropy weights, the fixed-point set of $\mathbb{C}_v^\times$ coincides with X^T;
2. for any $x \in X$ and action chamber C, the limit $\pi^v(x)$ for $v \in \ell \cap C$ depends only on C and not on the choice of v.

Proof. 1. Let X' be a connected component of the fixed-point set of $\mathbb{C}_v^\times$. Then $X' \cap X^T$ is a closed subset of X'. For any $x \in X' \cap X^T$, let X_α^T be the component of X^T that contains x. Since the T-action is effective and $\lambda_{\alpha,k}(v) \ne 0$ for any $0 \le k \le n - n_\alpha$, we have $\dim X_\alpha^T = \dim X'$. Therefore $X' \cap X^T$ is also an open subset of X'. Finally, choose $v_1, \dots, v_{r-1} \in \ell$ ($r = \dim_{\mathbb{C}} T$) such that $\{v, v_1, \dots, v_{r-1}\}$ is a basis of $\mathfrak{t}_{\mathbb{R}}$. (In the non-compact setting of subsection 2.2, we need to choose all $v_i \in \ell \cap C$ for an action chamber C.) Pick any $x' \in X'$. Since the T-action is meromorphic, the iterated limit $x = \pi^{v_1}\pi^{v_2} \cdots \pi^{v_{r-1}}(x')$ exists. It is clear that $x \in X' \cap X^T$. So $X' \cap X^T \ne \emptyset$. Consequently, $X' \cap X^T = X' = X_\alpha^T$.
2. From part 1, we have $y = \pi^v(x) \in X_\alpha^T$ for some $\alpha \in F$. By [15, Proposition I], there is a $T_{\mathbb{R}}$-invariant neighborhood W_y of y in N_α and a locally T-equivariant holomorphic embedding $\psi_y\colon W_y \longrightarrow X$. Let $X_y^v = (\pi^v)^{-1}(X_\alpha^T)$. Then from the linear T-action on N_α, we get $X_y^v \cap \psi_y(W_y) = \psi_y(N_\alpha^C \cap W_y)$. Hence $X_y^v = T\,\psi_y(N_\alpha^C \cap W_y)$; this depends only on C and not on the choice of v. □

We denote $\pi^v(x)$ by $\pi^C(x)$ when $v \in \ell \cap C$.

Remark 2.3. 1. If X is a compact, connected Kähler manifold and $X^T \ne \emptyset$, then the $T_{\mathbb{R}}$-action is Hamiltonian [23]. Let $\mu\colon X \longrightarrow \mathfrak{t}_{\mathbb{R}}^*$ be a moment map. For $v \in \ell - \{0\}$, the 1-parameter subgroup $\{j_v(e^t) \,|\, t \in \mathbb{R}\}$ generates the gradient flows of $\langle \mu, v \rangle$, along which its value strictly decreases. Therefore

the limit $\pi^v(x)$ for any $x \in X$ exists. Hence the T-action is meromorphic.
2. A holomorphic action on X may not be meromorphic even if X is compact and Kähler. For example, let $\mathbb{Z}$ act on $\mathbb{C} - \{0\}$ by $k\colon z \mapsto 2^k z$ ($k \in \mathbb{Z}$, $z \in \mathbb{C} - \{0\}$) and let $X = (\mathbb{C} - \{0\})/\mathbb{Z}$ be the quotient. Then the standard multiplication of $\mathbb{C}^\times$ on $\mathbb{C} - \{0\}$ induces a holomorphic action on X which has no fixed points and hence is not meromorphic.

In order to capture the topology of X by the fixed-point information, it is necessary to assume that the T-action is meromorphic. If so, then X has a cell decomposition according to the connected components of X^T that π^C maps to.

Definition 2.4. Suppose the T-action on X is meromorphic. Set $X_\alpha^C = (\pi^C)^{-1}(X_\alpha^T)$. Then

$$X = \bigcup_{\alpha \in F} X_\alpha^C \tag{2.3}$$

is called the *Białynicki-Birula decomposition* with respect to C.

Consider the case $T = \mathbb{C}^\times$. If the $\mathbb{C}^\times$-action is meromorphic, set $X_\alpha^\pm = (\pi^\pm)^{-1}(X_\alpha^T)$. The decompositions $X = \bigcup_{\alpha \in F} X_\alpha^\pm$ are called the *plus- (minus-) decompositions*, respectively [6, 7, 15, 16, 17].

The cells X_α^C are T-invariant. If the transversality condition (of Smale [43]) is satisfied, then the decomposition (2.3) is a stratification in the sense that the closure of each cell is a union of cells [7, Theorem 5]. In general, this is not true even when X is Kähler. An example is the Hirzebruch surface (the blow-up of $\mathbb{CP}^2$ at one point) [7, Example 1].

Definition 2.5. For $\alpha, \beta \in F$, we write $\alpha \longrightarrow \beta$ if there is $x \in X$ such that $\pi^C(x) \in X_\alpha^T$ and $\pi^{-C}(x) \in X_\beta^T$. We write $\alpha \prec \beta$ if either $\alpha = \beta$ or there is a *chain* from α to β, i.e., a finite sequence $\alpha_0 = \alpha, a_1, \ldots, \alpha_{r-1}, \alpha_r = \beta$ in F such that $\alpha_{i-1} \longrightarrow \alpha_i$ for all $1 \le i \le r$ ($r > 0$). Such a chain is called a *quasicycle* of length r if $\alpha = \beta$.

Obviously, the relation $\prec$ on F depends on the choice of C. Results on meromorphic $\mathbb{C}^\times$-actions generalize straightforwardly to meromorphic T-actions.

Remark 2.6. Under the assumptions of this subsection, the following statements are equivalent [17]:
1. $(F, \prec)$ is a partially ordered set;
2. There is no quasicycle in $(F, \prec)$;

3. There is a strictly decreasing function on $(F, \prec)$, i.e., a function $f\colon F \longrightarrow \mathbb{R}$ satisfying $f(\alpha) > f(\beta)$ if $\alpha \prec \beta$ and $\alpha \neq \beta$.

Consequently, $(F, \prec)$ is a partially ordered set if one of the following is true [7, 16, 17]:

1. X is Kähler;
2. $\nu_\alpha^C > \nu_\beta^C$ if $\alpha \prec \beta$ and $\alpha \neq \beta$;
3. The Białynicki-Birula decomposition is a stratification.

In each of the above cases, the moment map (projected along some direction in C), $\nu_\cdot^C$, and $\dim_{\mathbb{C}} X_\cdot^C$, respectively, provides a strictly decreasing function on $(F, \prec)$.

Example 2.7. Jurkiewicz [30] constructed a smooth compact toric 3-manifold with a meromorphic T^3-action that has 22 isolated fixed points. Choosing an appropriate action chamber, there is a quasicycle of length 6 [30]. Therefore $(F, \prec)$ is not a partially ordered set. In [50, § 4], it is shown that there exists a T^3-equivariant holomorphic line bundle such that the (strong) holomorphic Morse inequalities fail. This shows that the holomorphic Morse inequalities are not valid on an arbitrary complex manifold [50], in contrast with the fixed-point theorems in [2, 3]. In section 3, we construct the analog of the instanton complex in holomorphic Morse theory when $(F, \prec)$ is a partially ordered set. The existence of such a construction implies the holomorphic Morse inequalities. The partial order condition is weaker than the Kähler condition.

Definition 2.8. Suppose X has a meromorphic T-action. The Białynicki-Birula decomposition with respect to C is *filterable* if there is a descending sequence of closed T-invariant subvarieties

$$X = Z_0 \supset Z_1 \supset \cdots \supset Z_m \supset Z_{m+1} = \emptyset \tag{2.4}$$

such that for all $0 \leq p \leq m$, $Z_p - Z_{p+1} = \bigcup_{\alpha \in F_p} X_\alpha^C$ for a subset $F_p \subset F$ such that neither $\alpha \prec \beta$ nor $\beta \prec \alpha$ if $\alpha \neq \beta \in F_p$.

Notice that we allow $Z_p - Z_{p+1}$ to be a union of cells labeled by elements in F unrelated by $\prec$. In [7, Definition 2], $Z_p - Z_{p+1}$ is required to be a single cell. Since $\overline{X_\alpha^C} \bigcap X_\beta^C \neq \emptyset$ implies $\alpha \prec \beta$ [17, Lemma 1], the two notions are equivalent. Notice that the function $\alpha \mapsto p(\alpha)$ where $\alpha \in F_{p(\alpha)}$ is strictly increasing on $(F, \prec)$.

Alternatively, (2.4) can be written as

$$X = V_0 \supset V_1 \supset \cdots \supset V_m \supset V_{m+1} = \emptyset, \tag{2.5}$$

where $V_p = X - Z_{m+1-p}$ $(0 \le q \le m+1)$ are open sets in X such that $V_p - V_{p+1} = Z_{m-p} - Z_{m-p+1} = \bigcup_{\alpha \in F_{m-p}} X_\alpha^C$ for $0 \le p \le m$.

Proposition 2.9. ([17]) *Consider a meromorphic T-action on X. Then the Białynicki-Birula decomposition* (2.3) *is filterable if and only if $(F, \prec)$ is a partially ordered set. If so, then*
1. the projection $\pi^C \colon X_\alpha^C \longrightarrow X_\alpha^T$ is a T-equivariant holomorphic fibration and the fiber $(\pi^C)^{-1}(x)$ over any $x \in X_\alpha^T$ is T-equivariantly isomorphic to $(N_\alpha^C)_x$;
2. there is a T-equivariant isomorphism $TX_\alpha^C|_{X_\alpha^T} \cong N_\alpha^C \oplus TX_\alpha^T$ of holomorphic vector bundles over X_α^T;
3. the closure $\overline{X_\alpha^C}$ in X is analytic and contains X_α^C as a Zariski open set. Consequently, X_α^C is locally closed in X.

Proof. If $T = \mathbb{C}^\times$, the necessary and sufficient condition for (2.3) to be filterable was proved in [17]. As in [17], properties 1 and 2 follow from the arguments of [15] and property 3 follows from the arguments in [16, § IIb] where the Kähler assumption was not made. The generalization to a higher rank torus T is straightforward. □

The three properties of Proposition 2.9 were shown to be valid when X is a Kähler manifold [15, 16, 24, 33] or a complete smooth algebraic variety [6, 7, 32], prior to the work of [17]. Without any of these assumptions, one or more of the properties in Proposition 2.9 could fail [44].

Example 2.7 was originally constructed to provide a non-filterable Białynicki-Birula decomposition [30].

Remark 2.10. The restriction of π^{-C} to $X_\alpha^C - X_\alpha^T$ may be discontinuous and the image $\pi^{-C}(X_\alpha^C - X_\alpha^T)$ may fall into more than one connected components of X^T. For example, let $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ with the diagonal $\mathbb{C}^\times$-action. Then $X^T = \{0, \infty\} \times \{0, \infty\}$ and $X^+_{(0,0)} = \mathbb{C} \times \mathbb{C}$. We have $\pi^-(\{0\} \times (\mathbb{C} - \{0\})) = (0, \infty)$, $\pi^-((\mathbb{C} - \{0\}) \times \{0\}) = (\infty, 0)$, and $\pi^-((\mathbb{C} - \{0\}) \times (\mathbb{C} - \{0\})) = (\infty, \infty)$. The reason is that the holomorphic embedding $X_\alpha^C \longrightarrow X$ extends only meromorphically at infinity [16, Lemma 2], where it can be discontinuous.

Notice that despite of part 2 of Proposition 2.9, a tubular neighborhood of X_α^T in X_α^C can not be identified holomorphically with that in N_α^C in general [15]. There is an infinite series of obstruction to this [26, 20]. However, the identification is possible locally on X_α^T. Consider a holomorphic vector bundle E over X on which the T-action lifts holomorphically. For future applications, we also put E into a standard local form.

Lemma 2.11. *For any $x \in X^T_\alpha$, there is a neighborhood U_x of x in X^T_α, a $T_\mathbb{R}$-invariant open set W^C_x in N_α containing $N^C_\alpha|_{U_x}$ as a closed subset, and a locally T-equivariant holomorphic embedding $\psi_x \colon W^C_x \longrightarrow X$ such that $\psi_x(N^C_\alpha|_{U_x}) = (\pi^C)^{-1}(U_x) \subset X^C_\alpha$. Moreover, ψ_x can be lifted to a locally T-equivariant isomorphism $\tilde{\psi}_x \colon W^C_x \times E_x \longrightarrow E|_{\psi_x(W^C_x)}$ of holomorphic vector bundles.*

Proof. As in the proof of [15, Proposition I], there is a neighborhood U_x of x in X^T_α, a $T_\mathbb{R}$-invariant open set W_x in N_α containing U_x, and a T-equivariant holomorphic embedding $\psi_x \colon W_x \longrightarrow X$ such that $\psi_x(N^C_\alpha \cap W_x) \subset X^C_\alpha$. Pick any $v \in \ell \cap C$. Let $W^C_x = \bigcup_{t \geq 0} j_v(e^t)\, W_x$. W^C_x is a $T_\mathbb{R}$-invariant open set in N_α. Moreover, for any $y \in N^C_\alpha|_{U_x}$, we have $\pi^C(y) \in U_x$, hence there exists $t \geq 0$ such that $j_v(e^{-t})y \in W_x$, i.e., $y \in W^C_x$. So W^C_x contains $N^C_\alpha|_{U_x}$. We extend ψ_x from W_x to W^C_x by $\psi_x(j_v(e^t)y) = j_v(e^t)\psi_x(y)$ for $y \in W_x$ and $t \geq 0$. Clearly, the extension is well-defined, locally T-equivariant and holomorphic. Next, there is a holomorphic isomorphism $\tilde{\psi}_x \colon W_x \times E_x \longrightarrow E|_{W_x}$ of vector bundles, perhaps on a smaller neighborhood W_x. By [15, Lemma I], $\tilde{\psi}_x$ can be made $T_\mathbb{R}$-equivariant (hence T-equivariant). We extend $\tilde{\psi}_x$ to $W^C_x \times E_x$ by $\tilde{\psi}_x(j_v(e^t)y, \xi) = j_v(e^t)\tilde{\psi}_x(y, j_v(e^{-t})\xi)$ for $y \in W_x$, $\xi \in E_x$ and $t \geq 0$. The extension is again well-defined and is a T-equivariant holomorphic isomorphism of vector bundles. □

2.2. A non-compact setting.

In this subsection, we consider a class of non-compact complex manifolds with holomorphic torus actions. We hope that this class is broad enough to include many interesting examples.

Let X be a (possibly non-compact) complex manifold with an effective holomorphic action of the torus T.

Definition 2.12. Let C be an action chamber. The T-action on X is C-*meromorphic* if
1. for any $x \in X$, $v \in \ell \cap C$, the limit $\pi^v(x)$ exists;
2. the set F of connected components of X^T is finite and each component X^T_α ($\alpha \in F$) is compact.

The simplest example is $X = \mathbb{C}$ with the standard multiplication by $\mathbb{C}^\times$. The action is plus-meromorphic. The plus-decomposition is $X = X^+_0$ (a single 2-cell); there is no minus-decomposition.

Remark 2.13. Consider a C-meromorphic T-action on X.
1. By Proposition 2.2, which applies to the non-compact setting here, the limit $\pi^v(x)$ $(x \in X)$ does not depend on the choice of $v \in \ell \cap C$ and is therefore denoted by $\pi^C(x)$. Moreover $\pi^C(x) \in X^T$. X^T has a finite number of connected components, and we have the Białynicki-Birula decomposition (2.3) with respect to C. For $x \in X$, the limit $\pi^{-C}(x)$ may not exist in X.
2. As in the compact situation, there is a relation $\prec$ on F. If $(F, \prec)$ is a partially ordered set, then the properties of Proposition 2.9 for X are satisfied. In particular, $\overline{X^C_\alpha}$ is a closed subvariety in X that contains (and may be equal to) X^C_α as a Zariski open set. Furthermore, the Białynicki-Birula decomposition of X with respect to C is filterable and we have filtrations of X by closed subsets (2.4) and by open subsets (2.5).
3. If E is a holomorphic vector bundle over X on which the T-action lifts holomorphically, Lemma 2.11 also holds.

Assumption 2.14. There exists an action chamber C such that the T-action on X is C-meromorphic and the set $(F, \prec)$ is partially ordered.

In section 3, we will establish holomorphic Morse theory on a (possibly non-compact) complex manifold with a holomorphic T-action satisfying Assumption 2.14. An immediate way of obtaining such non-compact manifolds is as follows. We start with a compact complex manifold (or orbifold) $\tilde{X}$ that has a meromorphic T-action. Suppose the Białynicki-Birula decomposition of $\tilde{X}$ with respect to an action chamber $\tilde{C}$ is filterable and is filtered by the closed sets

$$\tilde{X} = \tilde{Z}_0 \supset \tilde{Z}_1 \supset \cdots \supset \tilde{Z}_{\tilde{m}} \supset \tilde{Z}_{\tilde{m}+1} = \emptyset. \tag{2.6}$$

Pick any m such that $0 \le m \le \tilde{m} - 1$ and let $X = \tilde{X} - \tilde{Z}_{m+1}$. T acts holomorphically on X. Let C be the action chamber of the T-action on X that contains $\tilde{C}$. Then the T-action on X is C-meromorphic. Each connected component of X^T is that of $\tilde{X}^T$. Moreover the Białynicki-Birula decomposition of X with respect to C has a filtration (2.4) by closed subsets $Z_p = \tilde{Z}_p - \tilde{Z}_{m+1}$ $(0 \le p \le m+1)$ of X. The simple example $X = \mathbb{C}$ falls into this category, with $\tilde{X} = \mathbb{CP}^1$.

More interestingly, the non-compact setting here is a complex analog of the symplectic setting considered in [41, 42], which we now recall. Let (X, ω) be a (possibly non-compact) symplectic manifold with a Hamiltonian action of the compact torus $T_\mathbb{R}$, with a moment map $\mu\colon X \longrightarrow \mathfrak{t}^*_\mathbb{R}$. The fixed-point set X^T of the torus $T_{\mathbb{R}}$ is a symplectic submanifold of X. Let F be the set of connected components of X^T.

Assumption 2.15. ([42, Assumption 1.3]) There is $v \in \mathfrak{t}_{\mathbb{R}}$ such that $\langle \mu, v\rangle \colon X \longrightarrow \mathbb{R}$ is proper and not surjective. F is a (non-empty) finite set.

If in addition (X, ω) is Kähler and the $T_{\mathbb{R}}$-action preserves the complex structure on X, then there is an infinitesimal holomorphic T-action on X. However this action does not always exponentiate even with Assumption 2.15. A simple example is the hyperbolic disc $\{z, |z| < 1\}$ with the Kähler form $\omega = \frac{\sqrt{-1}}{2}\frac{dz\wedge d\bar{z}}{(1-|z|^2)^2}$ and the standard complex structure. The moment map $\mu(z) = -^{-1}2(1-|z|^2)$ is a proper function. But since its gradient flows are incomplete, the rotation on the disc can not be extended to a holomorphic $\mathbb{C}^\times$-action.

Proposition 2.16. *Let (X, ω) be a Kähler manifold with a holomorphic T-action. If the $T_{\mathbb{R}}$-action is Hamiltonian and satisfies Assumption* 2.15*, then*
1. the T-action satisfies Assumption 2.14*;*
2. there is a compact Kähler orbifold with a meromorphic T-action that contains X as a T-invariant Zariski open set.

Proof. 1. By [42, Proposition 1.6], there is an action chamber C such that for any $v \in C$, the function $\langle \mu, v\rangle$ on X is proper and bounded from above. If $v \in \ell \cap C$, then the limit $\pi^v(x)$ exists for any $x \in X$. Thus the T-action is C-meromorphic. For $v \in \ell \cap C$, the function $\langle \mu, v\rangle$ is strictly decreasing on $(F, \prec)$. Properness of $\langle \mu, v\rangle$ also implies that each connected component of X^T is compact.
2. Since F is finite, there is $a \in \mathbb{R}$ such that $\langle \mu(X^T), v\rangle > a$. We construct a symplectic cut $X_{\geq a}$ [36]. Let $\mathbb{C}^\times$ act on $X \times \mathbb{C}$ by $u \colon (x, z) \mapsto (j_v(u)x, uz)$. The action of $S^1 \subset \mathbb{C}^\times$ on $X \times \mathbb{C}$ is Hamiltonian with a moment map $\tilde{\mu}(x, z) = \langle \mu(x), v\rangle - a - {}^{-1}2|z|^2$. $\tilde{\mu}$ is a proper function on $X \times \mathbb{C}$ and 0 is a regular value. The symplectic quotient $X_{\geq a} = \tilde{\mu}^{-1}(0)/S^1$ is a compact symplectic orbifold with a Hamiltonian $T_{\mathbb{R}}$-action. Since X is Kähler, $X_{\geq a} = (X \times \mathbb{C})^{\mathrm{s}}/\mathbb{C}^\times$ holomorphically and is also Kähler [27]. Here $(X \times \mathbb{C})^{\mathrm{s}} = \{(x, z) \in X \times \mathbb{C} \,|\, \mathbb{C}^\times(x, z) \cap \tilde{\mu}^{-1}(0) \neq \emptyset\}$ is the stable subset of $X \times \mathbb{C}$. There is a holomorphic T-action on $X_{\geq a}$ defined by $g \colon (x, z) \mapsto (gx, z)$, $g \in T$. The action is meromorphic since $X_{\geq a}$ is compact and Kähler. We want to construct a T-equivariant holomorphic embedding $X \longrightarrow X_{\geq a}$. Clearly, $\tilde{\mu}(u(x, 1)) = \langle \mu(j_v(u)x), v\rangle - a - {}^{-1}2|u|^2$, where $u \in \mathbb{C}^\times$ and $x \in X$. For any $x \in X$, since $\pi^v(x) \in X^T$, $\lim_{u\longrightarrow 0} \tilde{\mu}(u(x, 1)) = \langle \mu(\pi^v(x)), v\rangle - a > 0$. On the

other hand, since $\langle \mu, v\rangle$ is bounded from above, $\lim_{u\longrightarrow\infty}\tilde{\mu}(u(x,1)) = -\infty$. Therefore there is $u \in \mathbb{C}^\times$ such that $\tilde{\mu}(u(x,1)) = 0$. Hence $X \times \{1\} \subset (X \times \mathbb{C})^{\mathrm{s}}$. The composition $X \longrightarrow X \times \{1\} \subset (X \times \mathbb{C})^{\mathrm{s}} \longrightarrow X_{\geq a}$ of the inclusion and the quotient is a T-equivariant holomorphic embedding. The image is $X_{>a} = (X \times (\mathbb{C} - \{0\}))/\mathbb{C}^\times$. Since $X_{\geq a} - X_{>a} = (X \times \{0\})^{\mathrm{s}}/\mathbb{C}^\times = \mu^{-1}(a)/S^1$ is a complex subvariety of $X_{\geq a}$, X is embedded as a Zariski open set. □

3. Equivariant spectral sequences in holomorphic Morse theory.

We consider an effective holomorphic T-action on a (possibly non-compact) complex manifold X. F is the set of connected components of the fixed-point set X^T. Throughout this section, we make Assumption 2.14. Then the Białynicki-Birula decomposition is filterable, with descending sequences of closed sets (2.4) and open sets (2.5) in X. Let E be a holomorphic vector bundle over X on which the T-action lifts holomorphically. We want to determine the Dolbeault cohomology groups $H^*_{\mathrm{c}}(X, \mathcal{O}(E))$ (with compact support) and $H^*(X, \mathcal{O}(E))$ as representations of T.

Given a representation R of T, let R_ξ be the subspace of weight $\xi \in \ell^* \subset \mathfrak{t}^*_{\mathbb{R}}$. Given an open subset $U \subset X$, there is an infinitesimal T-action on $\mathcal{O}(E)(U)$. It is easy to see that $\mathcal{O}(E)_\xi(U) = (\mathcal{O}(E)(U))_\xi$ defines a sheaf $\mathcal{O}(E)_\xi$ for each $\xi \in \ell^*$. Moreover, $H^*_{\mathrm{c}}(X, \mathcal{O}(E)_\xi) = H^*_{\mathrm{c}}(X, \mathcal{O}(E))_\xi$ and $H^*(X, \mathcal{O}(E)_\xi) = H^*(X, \mathcal{O}(E))_\xi$.

Definition 3.1. A (cohomological) spectral sequence $\{E_r^{pq}, d_r^{pq}\}$ is *T-equivariant* if the spaces E_r^{pq} are representations of T and the coboundary maps $d_r^{pq}\colon E_r^{pq} \longrightarrow E_r^{p+r,q-r+1}$ are T-equivariant. The spectral sequence *converges T-equivariantly* to the representations H^* if the spaces E_∞^{pq} are the graded components of H^* as representations of T.

3.1. Spectral sequence for cohomologies with compact support.

In this subsection, we construct a spectral sequence converging to the Dolbeault cohomology groups $H^*_{\mathrm{c}}(X, \mathcal{O}(E))$ with compact support.

Recall that if $A \subset X$ is a locally closed subset, then for any sheaf $\mathcal{F}$ on X, there is a unique sheaf on X, denoted by $\mathcal{F}_A$, such that the restrictions $\mathcal{F}_A|_A = \mathcal{F}|_A$ and $\mathcal{F}|_{X-A} = 0$. Moreover, $\mathcal{F}_A$ exists for any sheaf $\mathcal{F}$ only if A is locally closed [25, Théorème II.2.9.1]. Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^*(\mathcal{F})$ be the canonical resolution of $\mathcal{F}$ [25, § II.4.3]. It is easy to see that $0 \longrightarrow \mathcal{F}_A \longrightarrow \mathcal{C}^*(\mathcal{F})_A$ is a flabby resolution of $\mathcal{F}_A$. Finally, if A is an open subset, then $\mathcal{F}_A$

is a subsheaf of $\mathcal{F}$.

Lemma 3.2. *Under Assumption* 2.14*, there is a spectral sequence with*

$$E_1^{pq} = H_{\mathrm{c}}^{p+q}(X, \mathcal{F}_{V_p - V_{p+1}}) \tag{3.1}$$

that converges to $H_{\mathrm{c}}^*(X, \mathcal{F})$*.*

Proof. From (2.5), we have a filtration of $\mathcal{F}$ by subsheaves

$$\mathcal{F} = \mathcal{F}_{V_0} \supset \mathcal{F}_{V_1} \supset \cdots \supset \mathcal{F}_{V_m} \supset \mathcal{F}_{V_{m+1}} = 0 \tag{3.2}$$

and hence a filtration of the cochain complex $\Gamma_{\mathrm{c}}(\mathcal{C}^*(\mathcal{F}))$ by

$$\begin{aligned} \Gamma_{\mathrm{c}}(\mathcal{C}^*(\mathcal{F})) = \Gamma_{\mathrm{c}}(\mathcal{C}^*(\mathcal{F})_{V_0}) \supset \Gamma_{\mathrm{c}}(\mathcal{C}^*(\mathcal{F})_{V_1}) \supset \cdots \\ \supset \Gamma_{\mathrm{c}}(\mathcal{C}^*(\mathcal{F})_{V_m}) \supset \Gamma_{\mathrm{c}}(\mathcal{C}^*(\mathcal{F})_{V_{m+1}}) = 0. \end{aligned} \tag{3.3}$$

This induces a spectral sequence that converges to $H^*(\Gamma_{\mathrm{c}}(\mathcal{C}^*(\mathcal{F}))) = H_{\mathrm{c}}^*(X, \mathcal{F})$, with

$$E_0^{pq} = \Gamma_{\mathrm{c}}(\mathcal{C}^{p+q}(\mathcal{F})_{V_p}) / \Gamma_{\mathrm{c}}(\mathcal{C}^{p+q}(\mathcal{F})_{V_{p+1}}) = \Gamma_{\mathrm{c}}(\mathcal{C}^{p+q}(\mathcal{F})_{V_p - V_{p+1}}). \tag{3.4}$$

Since the maps $d_0^{pq} \colon E_0^{pq} \longrightarrow E_0^{p,q+1}$ are induced by the resolution, we get

$$E_1^{pq} = H^{p+q}(\Gamma_{\mathrm{c}}(\mathcal{C}^*(\mathcal{F})_{V_p - V_{p+1}})) = H_{\mathrm{c}}^{p+q}(X, \mathcal{F}_{V_p - V_{p+1}}). \tag{3.5}$$

□

Lemma 3.3.

$$H_{\mathrm{c}}^*(X, \mathcal{F}_{V_p - V_{p+1}}) = \bigoplus_{\alpha \in F_{m-p}} H_{\mathrm{c}}^*(X_\alpha^C, \mathcal{F}|_{X_\alpha^C}). \tag{3.6}$$

Proof. Since $\overline{X_\alpha^C} \cap X_\beta^C = \emptyset$ for any $\alpha \neq \beta \in F_{m-p}$, we have $\mathcal{F}_{V_p - V_{p+1}} = \bigoplus_{\alpha \in F_{m-p}} \mathcal{F}_{X_\alpha^C}$ and hence

$$H_{\mathrm{c}}^*(X, \mathcal{F}_{V_p - V_{p+1}}) = \bigoplus_{\alpha \in F_{m-p}} H_{\mathrm{c}}^*(X, \mathcal{F}_{X_\alpha^C}). \tag{3.7}$$

The support of $\mathcal{F}_{X_\alpha^C}$ is contained in the closed subvariety $\overline{X_\alpha^C}$. Therefore we have [25, Théorème II.4.9.1]

$$H_{\mathrm{c}}^*(X, \mathcal{F}_{X_\alpha^C}) = H_{\mathrm{c}}^*(\overline{X_\alpha^C}, \mathcal{F}_{X_\alpha^C}). \tag{3.8}$$

Since $\overline{X_\alpha^C} - X_\alpha^C$ is a closed subset in $\overline{X_\alpha^C}$ and $\mathcal{F}_{X_\alpha^C}|_{\overline{X_\alpha^C}-X_\alpha^C} = 0$, we deduce from [25, Théorème II.4.10.1] that

$$H_c^*(X_\alpha^C, \mathcal{F}|_{X_\alpha^C}) = H_c^*(\overline{X_\alpha^C}, \mathcal{F}_{X_\alpha^C}). \tag{3.9}$$

The result follows from (3.7), (3.8) and (3.9). □

Recall that $\pi^C\colon X_\alpha^C \longrightarrow X_\alpha^T$ is a holomorphic fibration with fiber $\mathbb{C}^{\nu_\alpha^C}$. The sheaf $\mathcal{F}|_{X_\alpha^C}$ is on the total space X_α^C. To calculate the right hand side of (3.6), we need another spectral sequence.

We consider a general fibration $\pi\colon Y \longrightarrow B$ over a compact base B with possibly non-compact fibers. The cohomology groups with compact support are $H_c^q(Y, \mathcal{F}) = H^q(\Gamma_\Phi(Y, \mathcal{C}^*(\mathcal{F})))$ $(q \geq 0)$, where Φ is the family of supports that consists of the compact subsets of Y. Let $\mathcal{A}$, $\mathcal{L}^*$ be the sheaves on B defined by the presheaves $\mathcal{A}(U) = \Gamma_{\Phi\cap\pi^{-1}(U)}(\pi^{-1}(U), \mathcal{F})$, $\mathcal{L}^*(U) = \Gamma_{\Phi\cap\pi^{-1}(U)}(\pi^{-1}(U), \mathcal{C}^*(\mathcal{F}))$, respectively, where U is any open subset of B. Then $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{L}^*$ is a differential sheaf in the sence of [25, § II.4.1]. Let $\mathcal{H}_c^q(Y, \mathcal{F})$ $(q \geq 0)$ be the sheaves on B defined by the presheaves $\mathcal{H}_c^q(Y, \mathcal{F})(U) = H^q(\mathcal{L}^*(U))$, for any open subset $U \subset B$.

Lemma 3.4. *1. At $b \in B$, the stalk of $\mathcal{H}_c^q(Y, \mathcal{F})$ for any $q \geq 0$ is*

$$\mathcal{H}_c^q(Y, \mathcal{F})_b \cong H_c^q(Y_b, \mathcal{F}|_{Y_b}). \tag{3.10}$$

2. There is a spectral sequence with

$$E_2^{pq} = H^p(B, \mathcal{H}_c^q(Y, \mathcal{F})) \tag{3.11}$$

that converges to $H_c^*(Y, \mathcal{F})$.

Proof. 1. This the analog of [25, Remarque II.4.17.1] for cohomologies with compact support. First, $\mathcal{H}_c^q(Y, \mathcal{F})_b = \varinjlim_{U\ni b} \mathcal{H}_c^q(Y, \mathcal{F})(U) = \varinjlim_{U\ni b} H^q_{\Phi\cap\pi^{-1}(U)}(\pi^{-1}(U), \mathcal{F})$. By [25, Théorème II.3.3.1], any section $s \in \Gamma(Y_b, \mathcal{C}^*(\mathcal{F})|_{Y_b})$ can be extended to a neighborhood of Y_b in Y. If $\operatorname{supp} s \in \Phi \cap Y_b$, then the neighborhood can be chosen as $\pi^{-1}(U)$ for some open set $U \subset B$. Therefore $\varinjlim_{U\ni b} H^q_{\Phi\cap\pi^{-1}(U)}(\pi^{-1}(U), \mathcal{F}) = \varinjlim_{V\supset Y_b} H^q_{\Phi\cap V}(V, \mathcal{F})$. Following the proof of [25, Théorème II.4.11.1], we get $\varinjlim_{V\supset Y_b} H^q_{\Phi\cap V}(V, \mathcal{F}) = H_c^q(Y_b, \mathcal{F}|_{Y_b})$.

2. It is clear that $\mathcal{L}^*$ are flabby sheaves. By [25, Théorème II.4.6.1], associated to the differential sheaf $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{L}^*$ there is a spectral sequence with (3.11) that converges to $H^*(\Gamma(\mathcal{L}^*))$. By the definition of $\mathcal{L}^*$, $\Gamma(B, \mathcal{L}^*) = \Gamma_c(Y, \mathcal{C}^*(\mathcal{F}))$. The result follows. □

Lemma 3.5. *Let R be a representation of T with $\dim_{\mathbb{C}} R = n$ and let A, $A^{\perp}$ be T-invariant subspaces of R such that $\dim_{\mathbb{C}} A = \nu$ and $R = A \oplus A^{\perp}$. Let E_0 be a representation of T and E, the trivial holomorphic vector bundle over R with fiber E_0. Then, for any $\xi \in \ell^*$,*

$$\begin{aligned} H^q_{\mathrm{c}}(A, \mathcal{O}(E)|_A)_\xi &= H^q_{\mathrm{c}}(R, \mathcal{O}(E)_A)_\xi \\ &= \begin{cases} (S(A^{\perp *}) \otimes S(A) \otimes \wedge^{\nu}(A) \otimes E_0)_\xi, & \text{if } q = \nu, \\ 0, & \text{if } q \neq \nu. \end{cases} \end{aligned} \tag{3.12}$$

Proof. It suffices to prove the case when $E_0 = \mathbb{C}$ is a trivial representation. If $A = \{0\}$, then

$$H^q_{\mathrm{c}}(R, \mathcal{O}_{\{0\}})_\xi = H^q(R, \mathcal{O}_{\{0\}})_\xi = \begin{cases} S(R^*)_\xi, & \text{if } q = 0, \\ 0, & \text{if } q \neq 0. \end{cases} \tag{3.13}$$

If $A = R$, then (see [34] for an analytic version)

$$H^q_{\mathrm{c}}(R, \mathcal{O})_\xi = \begin{cases} (S(R) \otimes \wedge^n(R))_\xi, & \text{if } q = n, \\ 0, & \text{if } q \neq n. \end{cases} \tag{3.14}$$

The general case is a consequence of the Künneth formula. □

Lemma 3.6.

$$\begin{aligned} &H^q_{\mathrm{c}}(X^C_\alpha, \mathcal{O}(E)|_{X^C_\alpha})_\xi \\ &= H^{q-\nu^C_\alpha}(X^T_\alpha, \mathcal{O}(S((N^{-C}_\alpha)^*) \otimes S(N^C_\alpha) \otimes \wedge^{\nu^C_\alpha}(N^C_\alpha) \otimes E|_{X^T_\alpha}))_\xi \end{aligned} \tag{3.15}$$

for any $\xi \in \ell^*$.

Proof. Consider the holomorphic fibration $\pi^C \colon X^C_\alpha \longrightarrow X^T_\alpha$ with fiber $\mathbb{C}^{\nu^C_\alpha}$ and the sheaf $\mathcal{F} = \mathcal{O}(E)_\xi|_{X^C_\alpha}$ on X^C_α. For any $x \in X^T_\alpha$, we want to find the stalk $\mathcal{H}^q_{\mathrm{c}}(X^C_\alpha, \mathcal{F})_x$, which depends only on an open neighborhood of $(\pi^C)^{-1}(x) \subset X^C_\alpha$ in X. By Lemma 2.11, we can replace $X^C_\alpha \subset X$ by $N^C_\alpha|_{U_x} \subset N_\alpha|_{U_x}$ and E by a trivial vector bundle with fiber E_x. Moreover there is a T-equivariant isomorphism $(N_\alpha, N^C_\alpha)|_{U_x} \cong U_x \times (N_x, N^C_x)$. By Lemma 3.4.1 and Lemma 3.5,

$$\begin{aligned} &\mathcal{H}^q_{\mathrm{c}}(X^C_\alpha, \mathcal{F})_x \\ &= H^q(N^C_x, \mathcal{O}(W^C_x, E_x)_\xi|_{N^C_x}) \\ &= \begin{cases} \mathcal{O}((S((N^{-C}_\alpha)^*) \otimes S(N^C_\alpha) \otimes \wedge^{\nu^C_\alpha}(N^C_\alpha) \otimes E|_{X^T_\alpha})_\xi)_x, & \text{if } q = \nu^C_\alpha, \\ 0, & \text{if } q \neq \nu^C_\alpha. \end{cases} \end{aligned} \tag{3.16}$$

So the spectral sequence of Lemma 3.4.2 degenerates at E_2 and the result follows. $\square$

Though the bundle $S((N_\alpha^{-C})^*) \otimes S(N_\alpha^C) \otimes \wedge^{\nu_\alpha^C}(N_\alpha^C) \otimes E|_{X_\alpha^T}$ over X_α^T is infinite dimensional, its sub-bundle of any given weight is of finite rank. Therefore each weight has a finite multiplicity in the cohomology groups (3.15), and their formal characters in $\mathbb{Z}[\ell^*]$ exist.

Theorem 3.7. *Let X be a complex manifold with an effective holomorphic T-action satisfying Assumption* 2.14. *Let E be a holomorphic vector bundle over X on which the T-action lifts holomorphically. Then*
1. there is a T-equivariant spectral sequence converging T-equivariantly to $H_c^*(X, \mathcal{O}(E))$ with

$$E_1^{pq} = \bigoplus_{\alpha \in F_{m-p}} H^{p+q-\nu_\alpha^C}(X_\alpha^T, \mathcal{O}(S((N_\alpha^{-C})^*) \otimes S(N_\alpha^C) \otimes \wedge^{\nu_\alpha^C}(N_\alpha^C) \otimes E|_{X_\alpha^T})); \tag{3.17}$$

2. there is a character valued polynomial $Q_c^C(t) \geq 0$ such that

$$\begin{aligned}\sum_{\alpha \in F} t^{\nu_\alpha^C} \sum_{q=0}^{n_\alpha} t^q \operatorname{char} H^q(X_\alpha^T, \mathcal{O}(S((N_\alpha^{-C})^*) \otimes S(N_\alpha^C) \otimes \wedge^{\nu_\alpha^C}(N_\alpha^C) \otimes E|_{X_\alpha^T})) \\ = \sum_{q=0}^{n} t^q \operatorname{char} H_c^q(X, \mathcal{O}(E)) + (1+t) Q_c^C(t);\end{aligned} \tag{3.18}$$

3.

$$\begin{aligned}&\sum_{q=0}^{n} (-1)^q \operatorname{char} H_c^q(X, \mathcal{O}(E)) \\ &= \sum_{\alpha \in F} (-1)^{\nu_\alpha^C} \int_{X_\alpha^T} \operatorname{ch}_T \left(\frac{E|_{X_\alpha^T} \otimes \det(N_\alpha^C)}{\det(1 - (N_\alpha^{-C})^*) \otimes \det(1 - N_\alpha^C)} \right) \operatorname{td}(X_\alpha^T),\end{aligned} \tag{3.19}$$

where ch_T and td stand for the equivariant Chern character and the Todd class, respectively.

Proof. 1. The result follows from Lemma 3.2, Lemma 3.3 with $\mathcal{F} = \mathcal{O}(E)_\xi$ and Lemma 3.6 for all $\xi \in \ell^*$.
2. Since E_{r+1}^{pq} is the cohomology of (E_r^{pq}, d_r^{pq}), we have

$$\sum_{p,q} t^{p+q} \operatorname{char} E_r^{pq} = \sum_{p,q} t^{p+q} \operatorname{char} E_{r+1}^{pq} + (1+t) Q_r(t) \tag{3.20}$$

for a character valued polynomial $Q_r(t) \geq 0$. Using (3.20) recursively, we get (3.18) with $Q^C_{\mathrm{c}}(t) = \sum_{r\geq 1} Q_r(t) \geq 0$.
3. By setting $t = -1$ in (3.18) and using

$$\begin{aligned}&\sum_{q=0}^{n_\alpha}(-1)^q \operatorname{char} H^q(X^T_\alpha, \mathcal{O}(S((N^{-C}_\alpha)^*) \otimes S(N^C_\alpha) \otimes \wedge^{\nu^C_\alpha}(N^C_\alpha) \otimes E|_{X^T_\alpha}))\\ &= \int_{X^T_\alpha} \mathrm{ch}_T \left(\frac{E|_{X^T_\alpha} \otimes \det(N^C_\alpha)}{\det(1-(N^{-C}_\alpha)^*) \otimes \det(1-N^C_\alpha)}\right) \mathrm{td}(X^T_\alpha),\end{aligned} \tag{3.21}$$

we obtain (3.19). See [51, Remark 2.3.2]. □

Corollary 3.8. *If in addition X^T is discrete (and is identified with F), then*
1. there is a T-equivariant spectral sequence converging T-equivariantly to $H^*_{\mathrm{c}}(X, \mathcal{O}(E))$ with

$$E_1^{pq} = \bigoplus_{x\in F_{m-p},\, \nu^C_x = p+q} S((N^{-C}_x)^*) \otimes S(N^C_x) \otimes \wedge^{\nu^C_x}(N^C_x) \otimes E_x. \tag{3.22}$$

2. there is a character valued polynomial $Q^C_{\mathrm{c}}(t) \geq 0$ such that

$$\begin{aligned}&\sum_{x\in F} t^{\nu^C_x} \operatorname{char} E_x \prod_{\lambda_{x,k}\in C^*} \frac{e^{\lambda_{x,k}}}{1-e^{\lambda_{x,k}}} \prod^{-1}_{\lambda_{x,k}\in -C^*} 1 - e^{-\lambda_{x,k}}\\ &= \sum_{q=0}^{n} t^q \operatorname{char} H^q_{\mathrm{c}}(X, \mathcal{O}(E)) + (1+t) Q^C_{\mathrm{c}}(t);\end{aligned} \tag{3.23}$$

3.

$$\begin{aligned}&\sum_{q=0}^{n}(-1)^q \operatorname{char} H^q_{\mathrm{c}}(X, \mathcal{O}(E))\\ &= \sum_{x\in F}(-1)^{\nu^C_\alpha} \operatorname{char} E_x \prod_{\lambda_{x,k}\in C^*} \frac{e^{\lambda_{x,k}}}{1-e^{\lambda_{x,k}}} \prod^{-1}_{\lambda_{x,k}\in -C^*} 1 - e^{-\lambda_{x,k}}.\end{aligned} \tag{3.24}$$

Remark 3.9. 1. If X is compact, then $H^*_{\mathrm{c}}(X, \mathcal{O}(E)) = H^*(X, \mathcal{O}(E))$, and the right-hand sides of (3.19) and (3.24) are often written as

$$\sum_{\alpha\in F} \int_{X^T_\alpha} \mathrm{ch}_T \left(\frac{E|_{X^T_\alpha}}{\det(1-N^*_\alpha)}\right) \mathrm{td}(X^T_\alpha) \qquad \text{and} \qquad \sum_{x\in F} \frac{\operatorname{char} E_x}{\prod_{k=1}^n (1-e^{-\lambda_{x,k}})}, \tag{3.25}$$

respectively. In this case, parts 3 of Theorem 3.7 and Corollary 3.8 are the fixed-point theorems of [3, 2], which do not require $\prec$ to be a partial ordering. Here X can be non-compact. We obtain a fixed-point theorem for Dolbeault cohomology groups with compact support under the partial order condition. When X is compact and Kähler, parts 2 are the results of [49, 37, 50, 51]. Parts 1 strengthen these results under a weaker condition, namely, Assumption 2.14. In particular, all the weights of T in $H^*_{\mathrm{c}}(X, \mathcal{O}(E))$ are of finite multiplicity. It would be interesting to have an independent analytic proof of the results in parts 2 when X is a non-compact Kähler manifold satisfying Assumption 2.15. They are the discrete versions of [42, Theorem 3.2].
2. The coboundary maps $\{d_r^{pq}\}$ in the spectral sequence in Theorem 3.7 or Corollary 3.8 are the holomorphic counterparts of the instanton tunneling operators in [48]. Through this spectral sequence, the cohomology groups $H^*_{\mathrm{c}}(X, \mathcal{O}(E))$ are determined by the combinatorial data of the T-action on X. However unlike the real case, the spectral sequence of holomorphic Morse theory does not always degenerate at E_2. A sufficient condition for degeneracy at E_2 is

$$E_1^{pq} = 0 \quad \text{for all} \quad q \neq 0. \tag{3.26}$$

If so, then the spectral sequence reduces to a cochain complex $\{E_1^{*0}, d_1^{*0}\}$, whose cohomology is $E_2^{*0} = H^*(X, \mathcal{O}(E))$. This would be exactly like the Thom-Smale-Witten complex [48]. For example, if $X^T = F$ is discrete, $m = n$ in (2.5), and $F_p = \{x \in F \,|\, \nu_x^{-C} = p\}$ for $0 \leq p \leq n$, then (3.26) is satisfied.

3.2. Spectral sequence with local cohomology groups.

In this subsection, we construct an alternative spectral sequence converging to the Dolbeault cohomology groups $H^*(X, \mathcal{O}(E))$.

For any locally closed subset $A \subset X$, let Γ_A be the functor which associates every sheaf $\mathcal{F}$ an Abelian group $\Gamma_A(\mathcal{F}) = \{s \in \Gamma(\mathcal{F}) \,|\, \operatorname{supp} s \subset A\}$. Recall that the *local cohomology groups* H_A^q ($q \geq 0$) are the derived functors of Γ_A, i.e., $H_A^q(X, \mathcal{F}) = H^q(\Gamma_A(\mathcal{C}^*(\mathcal{F})))$. The *sheaves of local cohomology* $\mathcal{H}_A^q(\mathcal{F})$ with supports in A are the sheaves associated to the presheaves $U \mapsto H^q_{U\cap A}(U, \mathcal{F})$, where U is any open subset of X. (We refer the reader to [4, chap. II] and [31, § 7-10] for details.) For any closed subset A' of A, let $\Gamma_{A/A'}(\mathcal{F}) = \Gamma_A(\mathcal{F})/\Gamma_{A'}(\mathcal{F})$. If $\mathcal{F}$ is flabby, then $\Gamma_{A/A'}(\mathcal{F}) = \Gamma_{A-A'}(\mathcal{F})$ [31, Lemma 7.3]. The derived functors of $\Gamma_{A/A'}$ are denoted by $H^q_{A/A'}$. We have $H^q_{A/A'}(\mathcal{F}) = H^q_{A-A'}(\mathcal{F})$ for any sheaf $\mathcal{F}$. Let $\mathcal{H}^q_{A/A'}(\mathcal{F})$ be the sheaves associated to the presheaves $U \mapsto H^q_{U\cap A/U\cap A'}(U, \mathcal{F})$, where U is any open subset of X.

If A is a T-invariant locally closed subset of X, then the local cohomology groups $H^q_A(X,\mathcal{O}(E))$ $(q \geq 0)$ are representations of T. Furthermore, $H^q_A(X,\mathcal{O}(E)_\xi) = H^q_A(X,\mathcal{O}(E))_\xi$ for any $\xi \in \ell^*$.

Lemma 3.10. *Under Assumption* 2.14, *there is a spectral sequence with*

$$E_1^{pq} = H^{p+q}_{Z_p - Z_{p+1}}(X,\mathcal{F}) = \bigoplus_{\alpha \in F_p} H^{p+q}_{X^C_\alpha}(X,\mathcal{F}) \tag{3.27}$$

that converges to $H^*(X,\mathcal{F})$.

Proof. From (2.4), we have a filtration of the cochain complex

$$\begin{aligned} \Gamma(\mathcal{C}^*(\mathcal{F})) = \Gamma_{Z_0}(\mathcal{C}^*(\mathcal{F})) \supset \Gamma_{Z_1}(\mathcal{C}^*(\mathcal{F})) \supset \cdots \\ \supset \Gamma_{Z_m}(\mathcal{C}^*(\mathcal{F})) \supset \Gamma_{Z_{m+1}}(\mathcal{C}^*(\mathcal{F})) = 0. \end{aligned} \tag{3.28}$$

This induces a spectral sequence that converging to $H^*(X,\mathcal{F})$ with

$$E_0^{pq} = \Gamma_{Z_p}(\mathcal{C}^{p+q}(\mathcal{F}))/\Gamma_{Z_{p+1}}(\mathcal{C}^{p+q}(\mathcal{F})) = \Gamma_{Z_p - Z_{p+1}}(\mathcal{C}^{p+q}(\mathcal{F})). \tag{3.29}$$

Therefore

$$E_1^{pq} = H^{p+q}_{Z_p - Z_{p+1}}(\mathcal{C}^*(\mathcal{F})) = H^{p+q}_{Z_p - Z_{p+1}}(X,\mathcal{F}). \tag{3.30}$$

(See for example [52, Theorem 1.1]; the proof is included here for completeness.) Since $Z_p - Z_{p+1} = \bigcup_{\alpha \in F_p} X^C_\alpha$ and $\overline{X^C_\alpha} \cap X^C_\beta = \emptyset$ for $\alpha \neq \beta \in F_p$, we have $\Gamma_{Z_p - Z_{p+1}}(\mathcal{C}^*(\mathcal{F})) = \bigoplus_{\alpha \in F_p} \Gamma_{X^C_\alpha}(\mathcal{C}^*(\mathcal{F}))$. Hence

$$H^*_{Z_p - Z_{p+1}}(X,\mathcal{F}) = \bigoplus_{\alpha \in F_p} H^*_{X^C_\alpha}(X,\mathcal{F}). \tag{3.31}$$

□

Similar to the study of cohomology with compact support, we consider a general fibration $\pi\colon Y \longrightarrow B$. Suppose for the time being that X is any topological space containing Y as a locally closed subset and that $\mathcal{F}$ is any sheaf on X. We want to compute the local cohomology groups $H^q_Y(X,\mathcal{F})$ $(q \geq 0)$. Let $\mathcal{A}$, $\mathcal{L}^*$ be the sheaves on B defined by the presheaves $\mathcal{A}(U) = \Gamma_{\pi^{-1}(U)}(X,\mathcal{F})$, $\mathcal{L}^*(U) = \Gamma_{\pi^{-1}(U)}(X,\mathcal{C}^*(\mathcal{F}))$, respectively, where U is any open subset of B. Then $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{L}^*$ is a differential sheaf in the sence of [25, § II.4.1]. Let $\mathcal{H}^q_Y(X,\mathcal{F})$ $(q \geq 0)$ be the sheaves on B defined by the presheaves $\mathcal{H}^q_Y(X,\mathcal{F})(U) = H^q(\mathcal{L}^*(U))$, for any open subset $U \subset B$.

Lemma 3.11. *1. At $b \in B$, the stalk of $\mathcal{H}^q_Y(X, \mathcal{F})$ for any $q \geq 0$ is*

$$\mathcal{H}^q_Y(X, \mathcal{F})_b \cong \varinjlim_{U \ni b} H^q_{\pi^{-1}(U)}(X, \mathcal{F}), \tag{3.32}$$

where the limit is taken over the open sets $U \subset B$ containing b.
2. There is a spectral sequence with

$$E_2^{pq} = H^p(B, \mathcal{H}^q_Y(X, \mathcal{F})) \tag{3.33}$$

that converges to $H^*_Y(X, \mathcal{F})$.

Proof. 1. This is because

$$\mathcal{H}^q_Y(X, \mathcal{F})_b = \varinjlim_{U \ni b} H^q(\mathcal{L}^*(U)) = \varinjlim_{U \ni b} H^q_{\pi^{-1}(U)}(X, \mathcal{F}).$$

2. It is clear that $\mathcal{L}^*$ are flabby sheaves and that $\Gamma(B, \mathcal{L}^*) = \Gamma_Y(X, \mathcal{C}^*(\mathcal{F}))$. The rest of the proof is identical to that of Lemma 3.4.2. □

Notice that $\mathcal{H}^q_Y(X, \mathcal{F})_b$ is different from $H^q_{Y_b}(X, \mathcal{F})$. In fact, $\mathcal{H}^q_Y(X, \mathcal{F}) = \pi_*(\mathcal{H}^q_Y(\mathcal{F})|_Y)$.

Lemma 3.12. *Under the conditions of Lemma 3.5, we have, for any $\xi \in \ell^*$,*

$$H^q_A(R, \mathcal{O}(E))_\xi = \begin{cases} (S(A^*) \otimes S(A^\perp) \otimes \wedge^{n-\nu}(A^\perp) \otimes E_0)_\xi, & \text{if } q = n - \nu, \\ 0, & \text{if } q \neq n - \nu. \end{cases} \tag{3.34}$$

Proof. As in the proof of Lemma 3.5, the general result follows from

$$H^q_{\{0\}}(R, \mathcal{O})_\xi = \begin{cases} (S(R) \otimes \wedge^n(R))_\xi, & \text{if } q = n, \\ 0, & \text{if } q \neq n \end{cases} \tag{3.35}$$

and

$$H^q(R, \mathcal{O})_\xi = \begin{cases} S(R^*)_\xi, & \text{if } q = 0, \\ 0, & \text{if } q \neq 0 \end{cases} \tag{3.36}$$

by the Künneth formula. See also [31, Proposition 11.9(e)]. □

Lemma 3.13.

$$\begin{aligned} H^q_{X^C_\alpha}(X, \mathcal{O}(E))_\xi = H^{q+\nu^C_\alpha+n_\alpha-n}(X^T_\alpha, \mathcal{O}(S((N^C_\alpha)^*) \otimes S(N^{-C}_\alpha) \\ \otimes \wedge^{n-n_\alpha-\nu^C_\alpha}(N^{-C}_\alpha) \otimes E|_{X^T_\alpha}))_\xi \end{aligned} \tag{3.37}$$

for any $\xi \in \ell^*$.

Proof. Let $\mathcal{F} = \mathcal{O}(E)_\xi$. Consider the fibration $\pi^C\colon X_\alpha^C \longrightarrow X_\alpha^T$. For any $x \in X_\alpha^T$, we want to find the stalk $\mathcal{H}^q_{X_\alpha^C}(X, \mathcal{F})_x$, which by excision [4, § II.1, Lemma 1.1] depends only on an open neighborhood of $(\pi^C)^{-1}(x) \subset X_\alpha^C$ in X. By Lemma 2.11, we can replace $X_\alpha^C \subset X$ by $N_\alpha^C|_{U_x} \subset N_\alpha|_{U_x}$ and E by a trivial vector bundle with fiber E_x. Moreover there is a T-equivariant isomorphism $(N_\alpha, N_\alpha^C)|_{U_x} \cong U_x \times (N_x, N_x^C)$. By Lemma 3.11.1 and Lemma 3.12,

$$\mathcal{H}^q_{X_\alpha^C}(X,\mathcal{F})_x = \varinjlim_{U \ni x} H^q_{(\pi^C)^{-1}(U)}(X,\mathcal{F}) = \varinjlim_{U\ni x} H^q_{N_\alpha^C|_U}(N_\alpha|_U, \mathcal{O}(N_\alpha|_U, E_x)_\xi)$$
$$= \begin{cases} \mathcal{O}((S((N_\alpha^C)^*)\otimes S(N_\alpha^{-C})\otimes \wedge^{n-n_\alpha-\nu_\alpha^C}(N_\alpha^{-C})\otimes E|_{X_\alpha^T})_\xi)_x, & \text{if } q = n-n_\alpha-\nu_\alpha^C, \\ 0, & \text{if } q \neq n-n_\alpha-\nu_\alpha^C. \end{cases} \tag{3.38}$$

In the above limit, U can be any open subset of X_α^T such that $x \in U \subset U_x$. So the spectral sequence of Lemma 3.11.2 degenerates at E_2 and the result follows. □

Theorem 3.14. *Under the conditions of Theorem 3.7,*
1. there is a T-equivariant spectral sequence converging T-equivariantly to $H^*(X, \mathcal{O}(E))$ with

$$E_1^{pq} = \bigoplus_{\alpha\in F_p} H^{p+q+\nu_\alpha^C+n_\alpha-n}(X_\alpha^T, \mathcal{O}(S((N_\alpha^C)^*)\otimes S(N_\alpha^{-C}) \otimes \wedge^{n-n_\alpha-\nu_\alpha^C}(N_\alpha^{-C})\otimes E|_{X_\alpha^T})); \tag{3.39}$$

2. there is a character valued polynomial $Q^C(t) \geq 0$ such that

$$\sum_{\alpha\in F} t^{n-n_\alpha-\nu_\alpha^C} \sum_{q=0}^{n_\alpha} t^q \operatorname{char} H^q(X_\alpha^T, \mathcal{O}(S((N_\alpha^C)^*)\otimes S(N_\alpha^{-C}) \otimes \wedge^{n-n_\alpha-\nu_\alpha^C}(N_\alpha^{-C})\otimes E|_{X_\alpha^T})) = \sum_{q=0}^{n} t^q \operatorname{char} H^q(X,\mathcal{O}(E)) + (1+t)Q^C(t); \tag{3.40}$$

3.

$$\sum_{q=0}^{n}(-1)^q \operatorname{char} H^q(X,\mathcal{O}(E)) = \sum_{\alpha\in F}(-1)^{n-n_\alpha-\nu_\alpha^C}\int_{X_\alpha^T}\operatorname{ch}_T\left(\frac{E|_{X_\alpha^T}\otimes\det(N_\alpha^{-C})}{\det(1-(N_\alpha^C)^*)\otimes\det(1-N_\alpha^{-C})}\right)\operatorname{td}(X_\alpha^T). \tag{3.41}$$

Proof. Part 1 follows from Lemma 3.10 with $\mathcal{F} = \mathcal{O}(E)_\xi$ and Lemma 3.13 for all $\xi \in \ell^*$. Parts 2 and 3 are proved in the same way as in Theorem 3.7. □

Corollary 3.15. *Under the conditions of Corollary* 3.8,
1. there is a T*-equivariant spectral sequence converging* T*-equivariantly to* $H^*(X, \mathcal{O}(E))$ *with*

$$E_1^{pq} = \bigoplus_{x \in F_p,\, \nu_x^C = n-p-q} S((N_x^C)^*) \otimes S(N_x^{-C}) \otimes \wedge^{n-\nu_x^C}(N_x^{-C}) \otimes E_x. \tag{3.42}$$

2. there is a character valued polynomial $Q^C(t) \geq 0$ *such that*

$$\begin{aligned} &\sum_{x \in F} t^{n-\nu_x^C} \operatorname{char}(E_x) \prod_{\lambda_{x,k} \in C^*}^{-1} 1 - e^{-\lambda_{x,k}} \prod_{\lambda_{x,k} \in -C^*} \frac{e^{\lambda_{x,k}}}{1 - e^{\lambda_{x,k}}} \\ &= \sum_{q=0}^{n} t^q \operatorname{char} H^q(X, \mathcal{O}(E)) + (1+t) Q^C(t); \end{aligned} \tag{3.43}$$

3.

$$\begin{aligned} &\sum_{q=0}^{n} (-1)^q \operatorname{char} H^q(X, \mathcal{O}(E)) \\ &= \sum_{x \in F} (-1)^{n-\nu_\alpha^C} \operatorname{char}(E_x) \prod_{\lambda_{x,k} \in C^*}^{-1} 1 - e^{-\lambda_{x,k}} \prod_{\lambda_{x,k} \in -C^*} \frac{e^{\lambda_{x,k}}}{1 - e^{\lambda_{x,k}}}. \end{aligned} \tag{3.44}$$

Remark 3.16. 1. The same observations in Remark 3.9.1 apply to Theorem 3.14 and Corollary 3.15. In particular, all the weights of T in $H^*(X, \mathcal{O}(E))$ are also of finite multiplicities. When X is non-compact, the Dolbeault cohomology groups are different from those with compact support. Therefore the results of Theorem 3.7 and Theorem 3.14 are distinct. Again, it would be interesting to have an independent analytic proof of parts 2 of Theorem 3.14 and Corollary 3.15 when X is a non-compact Kähler manifold satisfying Assumption 2.15. When X is compact, Theorem 3.7 is identical to Theorem 3.14 with an opposite action chamber. The two theorems are also related to each other by Serre duality. The local models in Lemma 3.5 and Lemma 3.12 are also dual to each other.
2. Remark 3.9.2 applies here as well. In particular, the complex $\{E_1^{*0}, d_1^{*0}\}$, i.e.,

$$\begin{aligned} 0 &\longrightarrow \mathit{\Gamma}(X, \mathcal{F}) \longrightarrow H^0_{Z_0 - Z_1}(X, \mathcal{F}) \longrightarrow H^1_{Z_1 - Z_2}(X, \mathcal{F}) \longrightarrow \cdots \\ &\longrightarrow H^m_{Z_m}(X, \mathcal{F}) \longrightarrow 0, \end{aligned} \tag{3.45}$$

is called the *global Grothendieck-Cousin complex* [28, 31]. If condition (3.26) is satisfied, then the complex (3.45) computes the cohomology groups $H^*(X, \mathcal{O}(E))$. Again a sufficient condition for (3.26) is that $X^T = F$ is discrete, $m = n$ in (2.4), and $F_p = \{x \in F \mid \nu_x^{-C} = p\}$ for $0 \le p \le n$. In [31, § 10], a few other sufficient conditions were found. If $\mathcal{H}^q_{Z_p/Z_{p+1}}(\mathcal{F}) = 0$ for all $q \neq p$, then the complex of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{H}^0_{Z_0/Z_1}(\mathcal{F}) \longrightarrow \mathcal{H}^1_{Z_1/Z_2}(\mathcal{F}) \longrightarrow \cdots \longrightarrow \mathcal{H}^m_{Z_m}(\mathcal{F}) \longrightarrow 0, \quad (3.46)$$

called the *local Grothendieck-Cousin complex*, is a resolution of $\mathcal{F}$ (see for example [31, Theorem 8.7] or [10, Lemma 1.2]). In this case, the sheaf $\mathcal{F}$ is called *locally Cohen-Macaulay* with respect to the filtration (2.4). The global Grothendieck-Cousin complex (3.45), which computes the cohomology groups $H^*(X, \mathcal{F})$, is obtained from (3.46) by applying the functor $\Gamma(X, \cdot\,)$.

4. Examples and Applications.

4.1. Flag manifolds and generalized Bernstein-Gelfand-Gelfand resolutions.

The spectral sequence for the cohomology of a flag manifold leads to the geometric realization of the Bernstein-Gelfand-Gelfand [5] and related resolutions. Though our approach does not provide new insights to the latter, it is an important example of the instanton complex in the holomorphic setting.

Let G be a complex semi-simple Lie group and T, a maximal torus of G. Let $\mathfrak{g}$, $\mathfrak{t}$ be the Lie algebras of G, T, respectively. Let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the root space decomposition, where $\Delta \subset \ell^* - \{0\} \subset \mathfrak{t}^*_{\mathbb{R}}$ is the root system of the pair $(\mathfrak{g}, \mathfrak{t})$ and $\mathfrak{g}_\alpha = \mathbb{C} e_\alpha$ $(\alpha \in \Delta)$. Let Δ_+ be a set of positive roots and let $\Delta_- = -\Delta_+$. Let $\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$. Let B be the Borel subgroup corresponding to the Borel subalgebra $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}_+$. Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{t})$. Denote by w_0 the element in W of maximal length $l(w_0) = |\Delta_+|$.

Recall that the Verma module of highest weight λ is the $U(\mathfrak{g})$-module $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C} v_\lambda$, where $\mathbb{C} v_\lambda$ is the 1-dimensional $U(\mathfrak{b})$-module defined by $\lambda \in \mathfrak{t}^*_{\mathbb{R}}$. M_λ is free over $U(\mathfrak{n}_-)$. As a $U(\mathfrak{t})$-module, M_λ is determined by $\operatorname{char} M_\lambda = \frac{e^\lambda}{\prod_{\alpha \in \Delta_+}(1 - e^{-\alpha})}$. When λ is a dominant weight, let R_λ be the (finite dimensional) irreducible module of highest weight λ. We have a resolution of R_λ by Verma modules

$$\begin{aligned} 0 \longrightarrow M_{w_0\lambda-2\rho} \longrightarrow \bigoplus_{l(w)=|\Delta_+|-1} M_{w(\lambda+\rho)-\rho} \longrightarrow \cdots \\ \longrightarrow \bigoplus_{l(w)=1} M_{w(\lambda+\rho)-\rho} \longrightarrow M_\lambda \longrightarrow R_\lambda \longrightarrow 0, \end{aligned} \tag{4.1}$$

where $\rho = {}^{-1}2\sum_{\alpha\in\Delta_+}\alpha$. This is called the Bernstein-Gelfand-Gelfand resolution [5] of R_λ.

For any $w \in W$, put $\mathfrak{n}_\pm^w = w\mathfrak{n}_\pm w^{-1}$. The *twisted Verma module* M_λ^w is a $U(\mathfrak{g})$-module of highest weight λ that is free over $U(\mathfrak{n}_+^w \cap \mathfrak{n}_-)$ and co-free over $U(\mathfrak{n}_+^w \cap \mathfrak{n}_+)$ [21]. In particular, $M_\lambda^1 \cong M_\lambda^*$ and $M_\lambda^{w_0} \cong M_\lambda$; the $U(\mathfrak{g})$-module structure of the dual M_λ^* is given by [13, §2.3]

$$\langle x\xi, v\rangle = -\langle \xi, \tau(x)v\rangle \quad \text{for } x \in \mathfrak{g},\ \xi \in M_\lambda^*,\ v \in M_\lambda, \tag{4.2}$$

where τ is an automorphism of $\mathfrak{g}$ such that $\tau(h) = -h$ ($h \in \mathfrak{t}$) and $\tau(e_\alpha) = e_{-\alpha}$ ($\alpha \in \Delta$). If M_λ is irreducible, then $M_\lambda^w \cong M_\lambda$ for any $w \in W$. As $U(\mathfrak{t})$-modules, we always have $\operatorname{char} M_\lambda^w = \operatorname{char} M_\lambda$.

We consider the non-degenerate flag manifold $X = G/B^-$, where B^- is the Borel subgroup opposite to B. The maximal torus T acts meromorphically on X. The fixed-point set is $X^T = \{wB^- \mid w \in W\}$. The isotropy weights at wB^- are $w\alpha$ ($\alpha \in \Delta_+$). The action chambers in $\mathfrak{t}$ are the Weyl chambers. Denote the positive Weyl chamber by "+" and the opposite chamber by "−". Then the polarizing index of $wB^- \in X^T$ is $\nu_w^- = |\Delta_- \cap w\Delta_+| = l(w)$ for any $w \in W$. The Białynicki-Birula decomposition is precisely the Bruhat decomposition $X = \bigcup_{w\in W} X_w^+$, where $X_w^+ = BwB^-/B^-$ ($w \in W$) are the Bruhat cells [1]. These cells are also the B-orbits in X. Moreover, the relation $\prec$ on $F \cong W$ is the Chevalley-Bruhat order [18], which is a partial ordering. Consequently, the Białynicki-Birula decomposition is filterable, and we have the filtration (2.4), where $m = |\Delta_+| = \dim_{\mathbb{C}} X$. The closed sets $Z_p = \bigcup_{l(w)\geq p} X_w^+$ ($0 \leq p \leq |\Delta_+|$) are the Schubert varieties. Since $Z_p - Z_{p+1} = \bigcup_{l(w)=p} X_w^+$ ($0 \leq p \leq |\Delta_+|$) and $\nu_w^- = l(w)$, the cohomology groups $H^*(X, \mathcal{F})$ with coefficients in any sheaf $\mathcal{F}$ can be computed by the (global) Grothendieck-Cousin complex (3.45), which becomes

$$\begin{aligned} 0 \longrightarrow H^0_{X_1^+}(X,\mathcal{F}) \longrightarrow \bigoplus_{l(w)=1} H^1_{X_w^+}(X,\mathcal{F}) \longrightarrow \cdots \\ \longrightarrow \bigoplus_{l(w)=|\Delta_+|-1} H^{|\Delta_+|-1}_{X_w^+}(X,\mathcal{F}) \longrightarrow H^{|\Delta_+|}_{X_{w_0}^+}(X,\mathcal{F}) \longrightarrow 0. \end{aligned} \tag{4.3}$$

Given any integral weight $\lambda \in \ell^*$, we have a holomorphic line bundle $L_\lambda = G \times_{B^-} \mathbb{C}v_\lambda$ over X, where $\mathbb{C}v_\lambda$ is the 1-dimensional holomorphic representation of B^- defined by λ. The weight of T on the fiber $(L_\lambda)_{wB^-}$ $(w \in W)$ is $w\lambda$. Set $\mathcal{F}_\lambda = \mathcal{O}(L_\lambda)$. Then from subsection 3.2, we have for any $w \in W$,

$$\begin{aligned} \operatorname{char} H^{l(w)}_{X_w^+}(X, \mathcal{F}_\lambda) &= e^{w\lambda} \prod_{\alpha \in \Delta_+ \cap w^{-1}\Delta_+}^{-1} 1 - e^{-w\alpha} \prod_{\alpha \in \Delta_+ \cap w^{-1}\Delta_-} \frac{e^{w\alpha}}{1 - e^{w\alpha}} \\ &= \frac{e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta_+}(1 - e^{-\alpha})}. \end{aligned} \tag{4.4}$$

So as representations of T, $H^{l(w)}_{X_w^+}(X, \mathcal{F}_\lambda)$ is the same as the Verma module $M_{w(\lambda+\rho)-\rho}$. In fact the above local cohomology groups are $U(\mathfrak{g})$-modules. This is because the canonical resolution $\mathcal{C}^*(\mathcal{F}_\lambda)$ of $\mathcal{F}_\lambda$, on which the Lie algebra $\mathfrak{g}$ acts, is $U(\mathfrak{g})$-equivariant. (Notice however that a representation of $\mathfrak{g}$ on an infinite dimensional space may not exponentiate to that of G.) Therefore the Grothendieck-Cousin complex (4.3) is $U(\mathfrak{g})$-equivariant. Moreover, we have $H^{l(w)}_{X_w^+}(X, \mathcal{F}_\lambda) \cong M^w_{w(\lambda+\rho)-\rho}$ as $U(\mathfrak{g})$-modules [21, §2.2]. So (4.3) becomes

$$0 \longrightarrow M^*_\lambda \longrightarrow \bigoplus_{l(w)=1} M^w_{w(\lambda+\rho)-\rho} \longrightarrow \cdots \longrightarrow \bigoplus_{l(w)=|\Delta_+|-1} M^w_{w(\lambda+\rho)-\rho} \longrightarrow M_{w_0\lambda-2\rho} \longrightarrow 0, \tag{4.5}$$

whose cohomology groups $H^*(X, \mathcal{F}_\lambda)$ are given by [8]. If $\lambda + \rho$ is regular, then

$$H^q(X, \mathcal{F}_\lambda) = \begin{cases} R_{w_\lambda(\lambda+\rho)-\rho}, & \text{if } q = l(w_\lambda), \\ 0, & \text{if } q \neq l(w_\lambda), \end{cases} \tag{4.6}$$

where w_λ is the unique element in W such that $w_\lambda(\lambda + \rho) - \rho$ is a dominant weight. The complex (4.5) is called the *generalized Bernstein-Gelfand-Gelfand resolution* of $R_{w_\lambda(\lambda+\rho)-\rho}$ [21, §2.3]. When λ is a dominant weight, (4.5) is the dual of the Bernstein-Gelfand-Gelfand resolution (4.1) for R_λ [31, 12, 13]. When $w_0\lambda - 2\rho$ is dominant, (4.5) is the Bernstein-Gelfand-Gelfand resolution for $R_{w_0\lambda-2\rho}$.

Remark 4.1. 1. Lepowsky [35] found a Bernstein-Gelfand-Gelfand-type resolution of any finite dimensional irreducible $U(\mathfrak{g})$-module by the generalized Verma modules, which are induced from representations of a parabolic subgroup $P \subset G$. In [40], a geometric realization of this resolution was constructed using the local cohomology of the P-orbits in G/B^- (rather than the B-orbits in G/P^-). Let H be the Levi subgroup of P, and $\mathfrak{h}$, its

Lie algebra. Let Δ_H be the root system of the pair $(\mathfrak{h}, \mathfrak{t})$, and W_H, the corresponding Weyl group. Then $X = G/B^-$ decomposes into its P-orbits according to

$$X = \bigcup_{w' \in W/W_H} P w' B^- / B^-, \tag{4.7}$$

where $W/W_H = \{W_H w \,|\, w \in W\}$ (see for example [47, §1.2]). H is the centralizer of a torus subgroup $T' \subset T$, whose Lie algebra $\mathfrak{t}' \subset \mathfrak{t}$. Consider the (meromorphic) T'-action on X. The fixed-point set $X^{T'} = \bigcup_{w' \in W/W_H} H w' B^- / B^-$. Choose the action chamber $C' \subset \mathfrak{t}'$ such that $\langle \alpha, C' \rangle > 0$ for all $\alpha \in \Delta_+ - \Delta_H \cap \Delta_+$. The the Białynicki-Birula decomposition of X with respect to C' is precisely (4.7). Therefore (3.45) gives the geometric realizations of Lepowsky's resolution and similar generalizations.
2. More interestingly, the (holomorphic) instanton complex can be used to study the cohomology groups of vector bundles over spherical varieties. (See [11] for an extension of the Borel-Weil theorem.)

4.2. Cohomology and geometric quantization of non-compact manifolds.

In section 3, we obtained equivariant holomorphic Morse inequalities and equivariant index theorems for non-compact complex manifolds under Assumption 2.14. In this subsection, we apply them to establish some results on the cohomology groups and on geometric quantization.

Let X be a (possibly non-compact) complex manifold of dimension n with a holomorphic T-action satisfying Assumption 2.14. Let $H^{pq}(X) = H^q(M, \mathcal{O}(\wedge^p TX))$, $H^{pq}_{\mathrm{c}}(X) = H^q_{\mathrm{c}}(M, \mathcal{O}(\wedge^p TX))$ $(p, q = 0, 1, \ldots, n)$ be the Dolbeault cohomology groups of X and those with compact support, respectively. Let $P(X; s, t) = \sum_{p,q=0}^n s^p t^q \operatorname{char} H^{pq}(X)$, $P_{\mathrm{c}}(X; s, t) = \sum_{p,q=0}^n s^p t^q \operatorname{char} H^{pq}_{\mathrm{c}}(X)$, the character-valued Poincaré-Hodge polynomials. If the cohomology groups are finite dimensional, then $h^{pq}(X) = \dim_{\mathbb{C}} H^{pq}(X)$, $h^{pq}_{\mathrm{c}}(X) = \dim_{\mathbb{C}} H^{pq}_{\mathrm{c}}(X)$ are the Hodge numbers of X and $p(X; s, t) = \sum_{p,q=0}^n s^p t^q h^{pq}(X)$, $p_{\mathrm{c}}(X; s, t) = \sum_{p,q=0}^n s^p t^q h^{pq}_{\mathrm{c}}(X)$, the (usual) Poincaré-Hodge polynomials. Notice that if X is a non-compact Kähler manifold, the Hodge numbers or the Poincaré-Hodge polynomials do not necessarily satisfy the symmetry relations valid for compact manifolds. For example let $X = \mathbb{C}$. Then $H^{01}(X) = H^{10}_{\mathrm{c}}(X) = 0$ whereas $H^{10}(X)$ and $H^{01}_{\mathrm{c}}(X)$ are infinite dimensional.

Proposition 4.2. *Under Assumption* 2.14, *1.* $\operatorname{supp} H_{\rm c}^{pq}(X) \subset \overline{C^*} \cap \ell^*$ *for all* C *such that the* T*-action is* C*-meromorphic. Moreover, for any such* C*, there is a polynomial* $q_{\rm c}^C(s,t) \geq 0$ *such that*

$$\sum_{\alpha \in F} (st)^{\nu_\alpha^C} p_{\rm c}(X_\alpha^T; s,t) = \sum_{p,q=1}^{n} s^p t^q \dim_{\mathbb{C}} H_{\rm c}^{pq}(X)^T + (1+t) q_{\rm c}^C(s,t); \quad (4.8)$$

2. $\operatorname{supp} H^{pq}(X) \subset -\overline{C^*} \cap \ell^*$ *for all* C *such that the* T*-action is* C*-meromorphic. Moreover, for any such* C*, there is a polynomial* $q^C(s,t) \geq 0$ *such that*

$$\sum_{\alpha \in F} (st)^{n-n_\alpha-\nu_\alpha^C} p(X_\alpha^T; s,t) = \sum_{p,q=1}^{n} s^p t^q \dim_{\mathbb{C}} H^{pq}(X)^T + (1+t) q^C(s,t). \quad (4.9)$$

Proof. The results follow from (3.18) and (3.40) as in the proof of [51, Theorem 4.1]. □

Remark 4.3. 1. If in addition there is an action chamber C such that the T-action is both C-meromorphic and $(-C)$-meromorphic, then the cohomology groups $H_{\rm c}^{pq}(X)$ and $H^{pq}(X)$ are trivial representations of T. This is true when X is compact [51, Theorem 4.1.1, Remark 4.2.1] but not so in general. For example, let $X = \mathbb{C}$ with the standard multiplication by $\mathbb{C}^\times$, which is plus-meromorphic. Then $\operatorname{supp} H^{00}(X) = -\mathbb{N}$ and $\operatorname{supp} H_{\rm c}^{01}(X) = \mathbb{N} - \{0\}$.
2. As in [51, Corollary 4.5], we conclude from Proposition 4.2 that if $|p-q| > \max_{\alpha \in F} n_\alpha$, then $H_{\rm c}^{pq}(X)^T = H^{pq}(X)^T = 0$. In particular, if all the fixed points are isolated, then $H_{\rm c}^{pq}(X)^T = H^{pq}(X)^T = 0$ when $p \neq q$. The result [14] for the full cohomology groups does not hold in our non-compact setting. In the above example with $X = \mathbb{C}$, $H_{\rm c}^{01}(X) \neq 0$ although the only fixed point 0 is isolated.

We now consider geometric quantization on a Kähler manifold X with a holomorphic $\mathbb{C}^\times$-action satisfying Assumption 2.15. Recall that a prequantum line bundle L on (X, ω) is a holomorphic line bundle whose curvature is $\frac{\omega}{\sqrt{-1}}$. Suppose such an L exists and the $\mathbb{C}^\times$-action lifts to a holomorphic action on L.

Definition 4.4. The *quantization* of (X, ω) is the virtual vector space

$$H(X) = \bigoplus_{q=0}^{n} (-1)^q H^q(X, \mathcal{O}(L)). \quad (4.10)$$

Applying Theorem 3.14.1 to the pre-quantum line bundle, we obtain

$$H(X) = \bigoplus_{p,q} (-1)^{p+q} E_1^{pq} \tag{4.11}$$

as virtual representations of $\mathbb{C}^\times$, where the spaces E_1^{pq} are given by (3.17).

Without loss of generality, we assume that the moment map μ is bounded from above. Then the $\mathbb{C}^\times$-action is plus-meromorphic. Suppose 0 is a regular value of μ. For simplicity, we assume that the S^1-action on $\mu^{-1}(0)$ is free. Then the symplectic quotient $X_0 = \mu^{-1}(0)/S^1 = X^{\mathrm{s}}/\mathbb{C}^\times$ is a smooth Kähler manifold. We construct the symplectic cuts $(X_\pm, \omega_\pm)$ as the symplectic quotients of the S^1-action on $X \times \mathbb{C}$, where the weights on $\mathbb{C}$ are ± 1, respectively [36]. The two cuts are Kähler manifolds with holomorphic $\mathbb{C}^\times$-actions. X_+ is compact and X_- satisfies Assumption 2.15. The sets of connected components of $X_\pm^{\mathbb{C}^\times}$ are $F_\pm = \{0\} \cup \{\alpha \in F \mid \mu(X_\alpha^C) \in \mathbb{R}^\pm\}$, respectively, and $X_{\pm,0}^{\mathbb{C}^\times} \cong X_0$, $X_{\pm,\alpha}^{\mathbb{C}^\times} \cong X_\alpha^{\mathbb{C}^\times}$ as complex manifolds [51, Lemma 4.6], which we now identify. Let $N_0 \longrightarrow X_0$ be the holomorphic line bundle associate to the circle bundle $\mu^{-1}(0) \longrightarrow X_0$. Then $\mathbb{C}^\times$ acts on the fibers of N_0 with weight 1. The holomorphic normal bundles of X_0 in $X_\pm$ are isomorphic to $N_0^{\pm 1}$, respectively. Since the action of $\mathbb{C}^\times$ lifts to L, the pre-quantum line bundles $L_0 \longrightarrow X_0$ and $L_\pm \longrightarrow X_\pm$ exist. We have the isomorphisms $L_\pm|_{X_0} \cong L_0$ and $L_\pm|_{X_\pm - X_0} \cong L|_{\mu^{-1}(\mathbb{R}^\pm)}$ (see for example [51, Lemma 4.9]).

Proposition 4.5. *Under the above assumptions, we have*
1. a gluing formula under symplectic cutting

$$\operatorname{char} H(X) = \operatorname{char} H(X_+) + \operatorname{char} H(X_-) - \dim_{\mathbb{C}} H(X_0); \tag{4.12}$$

2. that quantization commutes with reduction, i.e.,

$$\dim_{\mathbb{C}} H(X)^{\mathbb{C}^\times} = \dim_{\mathbb{C}} H(X_0). \tag{4.13}$$

Proof. The results follow from (3.41) using the same techniques as in [19]. □

Remark 4.6. 1. We can define $H_{\mathrm{c}}(X) = \bigoplus_{q=0}^n (-1)^q H_{\mathrm{c}}^q(X, \mathcal{O}(L))$ as the counterpart of (4.10) with compact support. It satisfies a similar gluing formula

$$\operatorname{char} H_{\mathrm{c}}(X) = \operatorname{char} H(X_+) + \operatorname{char} H_{\mathrm{c}}(X_-) - \dim_{\mathbb{C}} H(X_0). \tag{4.14}$$

However $\dim_{\mathbb{C}} H_{\mathrm{c}}(X)^{\mathbb{C}^\times} \neq \dim_{\mathbb{C}} H(X_0)$ in general. For example, take $X = \mathbb{C}$ and choose the moment map $\mu(z) = -1 - {}^{-1}2|z|^2$. Then $\dim_{\mathbb{C}} H_{\mathrm{c}}(X)^{\mathbb{C}^\times} = 1$

but $X_0 = \emptyset$.
2. When (X, ω) is symplectic, the individual cohomology groups in (4.10) do not make sense, but $H(X)$ can be defined as the index of a Spin^c-Dirac operator. In fact, (4.12) and (4.13) were proved for compact symplectic manifolds in [19]. (4.13) is the S^1-case of a conjecture by Guillemin and Sternberg [27]. See [29, 38, 39, 45, 46] for higher rank torus and non-Abelian group actions. For non-compact symplectic manifolds satisfying Assumption 2.15, the validity of (4.12) and (4.13) remains open.

Remark 4.7. In ordinary Morse theory, the underlying real manifold is (the bosonic part of) the configuration space of a supersymmetric system [48]. In holomorphic Morse theory, the complex manifold X, if it is Kähler, can be interpreted as the phase space of a bosonic system; this interpretation is adopted in Definition 4.4. The spectral sequence in Theorem 3.14.1 or Corollary 3.15.1 that converges to the quantum Hilbert space (4.10) is a finite dimensional model of the BRST approach in conformal field theory [22]. In [21, 9], the case of flag manifolds (see subsection 4.2) was considered. Here we show that the analogy works for any quantizable Kähler manifold with a Hamiltonian S^1-action satisfying Assumption 2.15. The extension of the present work to infinite dimensional settings could have significant implications.

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