Evolution of radial graphs in hyperbolic space by their mean curvature

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We consider the evolution of a surface $\mathbf{F} : M^n \mapsto \mathcal{H}^{n+1}$ in hyperbolic space by mean curvature flow. That is, we study the one parameter family $\mathbf{F}_t = \mathbf{F}(.,t)$ of immersions with corresponding images $M_t = \mathbf{F}_t(M^n)$ such that

$$\begin{aligned} \tfrac{\partial}{\partial t}\mathbf{F}(p,t) &= \mathbf{H}(p,t), \quad p \in M^n \\ \mathbf{F}(p,0) &= \mathbf{F}_0(p) \end{aligned} \tag{1}$$

where $\mathbf{H}(p,t)$ is the mean curvature vector of the hypersurface M_t at $\mathbf{F}(p,t)$ in hyperbolic space.

Mean curvature flow was first studied by Brakke [Bra78] in the context of geometric measure theory. Later, smooth compact surfaces evolving in Euclidean space were investigated in [Hui84] and [Hui90], and on arbitrary ambient manifolds in [Hui86]. The study of the evolution of complete graphs in $\mathcal{R}^{n+1}$ was also studied in [EH89], the results being improved in [EH91].

Minimal and constant mean curvature surfaces in hyperbolic space have also been studied by many authors with Anderson [And82] (see also [And83]) first showing that given any immersed closed submanifold M^{n-1} in the boundary at infinity there exists a complete area minimising current with M^{n-1} as its asymptotic boundary. Hardt and Lin [HL87] then considered the boundary regularity at infinity of these currents after which Lin [Lin89] showed that the regularity of a particular type of minimal graph is the same as that of its asymptotic boundary. During this period first Do Carmo and Lawson [DCL83] and then Levitt and Rosenberg [LR85] classified, under certain assumptions, constant mean curvature surfaces in hyperbolic space.

We will consider only initial surfaces $\mathbf{F}_0(M^n)$ which in the upper half space model of hyperbolic space, $\mathcal{H}^{n+1}$, can be written as entire Euclidean

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[1]This article is published posthumously by Philip Unterberger who passed away on 8 October 2002. In 1998, Phil completed his PhD in Mathematics at the University of Melbourne. This paper is part of his doctoral thesis. Afterwards Phil held a position as a Senior Analyst with the National Australia Bank. His courage and love of life were truly inspirational. He is sadly missed by his wife, daughter, family and all who knew him.

radial graphs above the hyperplane $S^n_+ = S^n_+(1)$, the Euclidean upper hemi-sphere of radius one about the origin. That is $\mathbf{F}_0(M^n) \subset \mathcal{R}^n \times (0, \infty)$ and $\langle \nu, x \rangle > 0$ where ν is an outward normal to $\mathbf{F}_0(M^n)$. If h is the log of the Euclidean radial height of M_t above S^n_+ and $H^{\mathcal{H}}$ is the mean curvature of the surface as seen from hyperbolic space, then we will see in section 3 that (1) is equivalent up to tangential diffeomorphisms to

$$\frac{\partial}{\partial t} h = -x_{n+1} \langle \nu, \frac{x}{|x|} \rangle^{-1} H^{\mathcal{H}}. \tag{2}$$

The term $\langle \nu, \frac{x}{|x|} \rangle^{-1}$ accounts for the fact that we are flowing in the normal direction rather than in the direction of x. This implies that provided we stay in the class of solutions with finite curvature, the asymptotic boundary of a flowing surface is fixed. The geometrical nature of this problem, that of an evolving radial graph in hyperbolic space, suggests its study by calculations on the hypersurface as seen from hyperbolic space.

Our main result, for $B^n \subset \mathcal{R}^{n+1}$ an open ball, is

Theorem 0.1. *For $M_0 = F_0(B^n)$ a locally Lipschitz continuous entire radial graph over $S^n_+ \subset \mathcal{H}^{n+1}$ the initial value problem (1) has a smooth solution $M_t = F_t(B^n)$ for all time $t > 0$. Moreover each M_t is an entire radial graph over S^n_+.*

In particular we do not require any initial bounds on either the radial height (defined to be $\log |x|$) or gradient. Thus a horosphere for example could be taken as our initial surface and an initial surface can be found whose asymptotic boundary is any given star shaped manifold. If these boundaries are compact and have non-negative mean curvature and we obtain a limit surface M_∞ which is minimal then by a uniqueness result of Lin [Lin89] the surface M_∞ is one of the type studied by Lin [Lin89]. If we also include the condition that the geodesic height of our initial surface over S^n_+ is bounded we obtain the following convergence result:

Theorem 0.2. *If M_0 has bounded hyperbolic height over S^n_+ then under mean curvature flow M_t converges in C^∞ to S^n_+.*

Remarks.
1. From Lin [Lin89] any radial graph with asymptotic boundary ∂M which is a minimal surface, is the unique area minimising surface with asymptotic boundary ∂M. This can also be shown using a calibration method [Mor87, pg 72].

2. There exists a geodesic graph, a surface which intersects each geodesic perpendicular to S^n_+ once, which is rotationally symmetric about the x_{n+1} axis, and which does not remain a graph under mean curvature flow. In fact using a 'clearing out' lemma it can be shown that a neck of this graph pinches under the flow [Unt98].

3. A curve which is a geodesic graph (in $\mathcal{H}^2$) remains a graph throughout its flow.

I'd like to thank my supervisor Prof. Klaus Ecker for his enthusiasm, patience and support, without which this work would not have been possible. My thanks also to my fellow students, Miles, Nick, and Denis, for many useful and interesting discussions.

1. Hyperbolic space.

We begin this section by defining some notation and noting some of the relevant facts about hyperbolic space and the model of it that we will use:

The $(n+1)-$dimensional upper half space model of hyperbolic space is the space $\mathcal{R}^n \times (0, \infty)$ endowed with the metric $g_{ij}(x) = \frac{1}{x^2_{n+1}} \delta_{ij}$, and we will denote by $\partial \mathcal{H}^{n+1} = (\mathcal{R}^n \times \{0\}) \cup \{\infty\}$ the boundary of its standard compactification. The Riemannian curvature tensor and Ricci and scalar curvatures are given by

$$R^{\mathcal{H}}_{ijkl} = \frac{1}{x^4_{n+1}} (\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}), \ R^{\mathcal{H}}_{ij} = -n\frac{1}{x^2_{n+1}}\delta_{ij} \text{ and } R^{\mathcal{H}} = -n(n+1). \quad (3)$$

Expressions in hyperbolic space will, where necessary, be denoted by the subscript or superscript $\mathcal{H}$. We will identify points in $T_p\mathcal{R}^{n+1}$ and $T_p\mathcal{H}^{n+1}$ so for ν a unit vector in $T\mathcal{R}^{n+1}$, $\nu_{\mathcal{H}} = x_{n+1}\nu$ is a unit vector in $T\mathcal{H}^{n+1}$.

Letting $f_\lambda(x) = (1+\lambda)x$ we see $\langle f_\lambda(x), f_\lambda(y) \rangle_{\mathcal{H}} = \langle x, y \rangle_{\mathcal{H}}$ which makes f_λ an isometry for all λ. Differentiating f_λ with respect to λ implies that the vector field x is a Killing vector field. Integral curves of this vector field will be called radial lines, and we define a radial graph to be any hypersurface which intersects each radial line at most once. For a point $p \in M_t$ we define the height and gradient of our graph to be

$$h(\frac{p}{|p|}) = log|x(p)| \qquad \text{and}$$

$$v = \langle \nu, \omega \rangle^{-1} = \langle \nu_{\mathcal{H}}, \omega_{\mathcal{H}} \rangle^{-1}_{\mathcal{H}} = (1 + |\nabla h|^2)^{\frac{1}{2}}$$

respectively where ν is the normal to M_t and $\omega = \frac{x}{|x|}$ (We note that the height function used is the one induced by the Killing vector field x.) . By

using

$$\nabla_X^{\mathcal{H}} Y = \nabla_X Y + \langle X, Y \rangle_{\mathcal{H}} \overset{\mathcal{H}}{e_{n+1}} - \langle X, \overset{\mathcal{H}}{e_{n+1}} \rangle_{\mathcal{H}} Y - \langle Y, \overset{\mathcal{H}}{e_{n+1}} \rangle_{\mathcal{H}} X \qquad (4)$$

(see [GKM75, pg90]) to obtain

$$
\begin{aligned}
\mathrm{div}_{\mathcal{H}} X &= \langle \nabla_{\overset{\mathcal{H}}{e_i}}^{\mathcal{H}} X, \overset{\mathcal{H}}{e_i} \rangle_{\mathcal{H}} \\
&= \langle \nabla_{e_i} X + \langle X, e_i \rangle_{\mathcal{H}} \overset{\mathcal{H}}{e_{n+1}} - \langle e_i, \overset{\mathcal{H}}{e_{n+1}} \rangle_{\mathcal{H}} X - \langle X, \overset{\mathcal{H}}{e_{n+1}} \rangle_{\mathcal{H}} e_i, e_i \rangle \quad (5) \\
&= \mathrm{div} X - \langle \langle X, \overset{\mathcal{H}}{e_{n+1}} \rangle_{\mathcal{H}} e_i, e_i \rangle \\
&= \mathrm{div} X - \tfrac{n+1}{2} \langle X, \nabla x_{n+1}^2 \rangle_{\mathcal{H}},
\end{aligned}
$$

and so

$$H^{\mathcal{H}} = x_{n+1} H^{\mathcal{R}} - n\langle \nu_{\mathcal{R}}, e_{n+1} \rangle, \qquad (6)$$

we are also able to calculate the mean curvature of many graphs quite simply. Finally we also need that the distance between two points, see [BP92], is given by

$$d^{\mathcal{H}}(p, q) = \cosh^{-1}\left(\frac{d^{\mathcal{R}}(p, q)^2}{2 p_{n+1} q_{n+1}} + 1 \right). \qquad (7)$$

We are now able to consider a number of examples of surfaces in hyperbolic space and their evolution under mean curvature flow in some detail. Apart from being interesting in their own right they can be used as barriers to other flowing surfaces and motivate many of the methods used in the remainder of the paper.

The Hyperplane $\mathbf{M_0} = \{\mathbf{x} \,|\, |\mathbf{x}| = \mathbf{c}\}$.
    These appear as Euclidean hemispheres with centres on $\partial \mathcal{H}^{n+1}$ (in particular S_+^n) and are radial graphs of constant height, gradient of 1, and zero mean curvature. Therefore

$$M_t = \{x \,|\, |x| = c\}.$$

They are totally geodesic and we consider our graphs to be graphs over $S_+^n(1)$.

The Horosphere $\mathbf{M_0} = \{\mathbf{x} \,|\, \mathbf{x_{n+1}} = \mathbf{c}\}$.
    This surface is an entire radial graph of unbounded height and gradient. It has constant curvature $-n$ and evolves by translation in the e_{n+1} direction with

$$M_t = \{x \,|\, x_{n+1} = ce^{nt}\}.$$

The Sphere $\mathbf{M_0} = \{\mathbf{x} \,|\, \mathbf{d}^{\mathcal{H}}(\mathbf{p}, \mathbf{x}) = \mathbf{r}\}$.

This sphere of radius r with centre p is not a radial graph. It is a compact surface of constant curvature $n \coth r$ and looks like a Euclidean sphere of radius $x_{n+1}(p) \sinh r$. It contracts by homothety to a point with

$$M_t = \{x \mid \cosh d^{\mathcal{H}}(p, x) = e^{-nt} \cosh r\}.$$

If we surround any compact minimal surface in $\mathcal{H}^{n+1}$ by a sphere and then flow both surfaces they will touch at some time. By the compact maximum principle this cannot occur and so there don't exist any compact minimal surfaces in $\mathcal{H}^{n+1}$.

The Hypersphere $M_0 = \{x \mid x_1^2 + ... + x_n^2 + (x_{n+1} - \sinh a_0)^2 = \cosh^2 a_0\}$.

These are level sets of the signed distance a from $S_+^n(1)$, and look like the intersections of Euclidean spheres with the upper half space. They are geodesic graphs of constant curvature $n \tanh a$, remain hyperspheres under (1) with

$$M_t = \{x \mid x_1^2 + ... + x_n^2 + (x_{n+1} - \sinh a_t)^2 = \cosh^2 a_t\}$$

where $\sinh a_t = \sinh a_0 e^{-nt}$, and collapse to $S_+^n(1)$ in infinite time. So by (2) and a compact maximum principle any surface which evolves for all time and initially lies between two hyperspheres must converge to $S_+^n(1)$. We note that $\sinh a = \frac{|x|^2 - 1}{2x_{n+1}}$ for later reference.

The Cylinder $M_0 = \{x \mid \frac{|x|}{x_{n+1}} = c\}$.

This surface is found by rotating a radial line about the x_{n+1}-axis. It has distance r to the x_{n+1}-axis where $\cosh r = \frac{|x|}{x_{n+1}}$, is not a radial graph, and has constant curvature $(n-1) \coth r + \tanh r$. It collapses to the x_{n+1} axis in finite time with

$$M_t = \{x \mid \frac{|x|}{x_{n+1}} = \sqrt{\frac{1}{n}(1 + (nc^2 - 1)e^{-2nt})}\}$$

and will be employed in defining cut-off functions for use in interior estimates. In the existance proof for solutions of (1) we will also be required to consider the evolution of our surfaces restricted to various cylinders.

2. Heat Equations.

Using methods similar to those in [EH91] we establish interior estimates for the height and all its derivatives. In determining an interior gradient estimate however, we are required to consider a combination of gradient and height which leads to an estimate which increases in time. In order to

understand how the geometry of M_t changes under (1) it will be necessary to calculate $\left(\frac{\partial}{\partial t} - \Delta_M\right)$ of many geometric quantities where Δ_M is the Laplace-Beltrami operator on M_t.

Proposition 2.1. *Any function* $f : M_t \mapsto \mathcal{R}$ *on* $M_t \subset \mathcal{H}^{n+1}$ *satisfies*

$$\left(\tfrac{\partial}{\partial t} - \Delta_M\right) f = -x_{n+1}^2 (\Delta f - \langle \nabla_\nu \nabla f, \nu \rangle)$$
$$+ x_{n+1}((n-2)\langle \nabla f, e_{n+1} \rangle + 2 \langle \nabla f, \nu \rangle \langle \nu, e_{n+1} \rangle)$$

Proof. For any function f and vector field X

$$\operatorname{div}_{\mathcal{H}} f X = f \operatorname{div}_{\mathcal{H}} X + X(f),$$
$$\text{and} \qquad \nabla^{\mathcal{H}} f = x_{n+1}^2 \nabla f.$$

Making use of (4) and (5) we then have

$$\left(\tfrac{\partial}{\partial t} - \Delta_M\right) f = -\operatorname{div}_M \nabla^{\mathcal{H}} f$$
$$= -\operatorname{div}_{\mathcal{H}} \nabla^{\mathcal{H}} f + \langle \nabla_\nu^{\mathcal{H}} \nabla^{\mathcal{H}} f, \nu \rangle$$
$$= -x_{n+1}^2 \operatorname{div}_{\mathcal{H}} \nabla f - \nabla f(x_{n+1}^2) + x_{n+1}^2 \langle \nabla_\nu^{\mathcal{H}} \nabla f, \nu \rangle + \langle \nabla f, \nu \rangle \nu(x_{n+1}^2)$$
$$= -x_{n+1}^2 \Delta f + \tfrac{n+1}{2} \langle \nabla f, \nabla x_{n+1}^2 \rangle - \nabla f(x_{n+1}^2)$$
$$+ x_{n+1}^2 \langle \nabla_\nu \nabla f + \langle \nu, \nabla f \rangle_{\mathcal{H}} e_{n+1}^{\mathcal{H}} - \langle \nabla f, e_{n+1}^{\mathcal{H}} \rangle_{\mathcal{H}} \nu$$
$$- \langle \nu, e_{n+1}^{\mathcal{H}} \rangle_{\mathcal{H}} \nabla f, \nu \rangle + \langle \nabla f, \nu \rangle \langle \nabla x_{n+1}^2, \nu \rangle$$
$$= -x_{n+1}^2 (\Delta f - \langle \nabla_\nu \nabla f, \nu \rangle)$$
$$+ x_{n+1}((n-2)\langle \nabla f, e_{n+1} \rangle + 2 \langle \nabla f, \nu \rangle \langle \nu, e_{n+1} \rangle)$$

$\square$

Using this result we now calculate the heat operator of a number of expressions suggested in part by the examples at the end of the first section.

Lemma 2.2. *For a radial graph* M_t *evolving by (1) we have*

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) |x|^2 = -4|x|^2 \langle \omega, e_{n+1} \rangle (\langle \omega, e_{n+1} \rangle - \langle \nu, \omega \rangle \langle \nu, e_{n+1} \rangle), \qquad (8)$$

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) x_{n+1} = (n-2)x_{n+1} + 2x_{n+1} \langle \nu, e_{n+1} \rangle^2, \qquad (9)$$

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) \sinh a = -n \sinh a, \text{ and} \qquad (10)$$

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) \cosh r = \frac{1}{\cosh r}(1 - \langle \nu, \omega \rangle^2) - n \cosh r. \qquad (11)$$

where a *is the signed distance from* $S_+^n(1)$ *and* r *is the distance to the* x_{n+1}-*axis.*

Proof. For the proof of (8) we use proposition 2.1 and note

$$\nabla x_{n+1} = e_{n+1},$$
$$\nabla_\nu \nabla x_{n+1} = 0 \quad \text{and}$$
$$\Delta x_{n+1} = 0.$$

To prove (9) we proceed similarly noting

$$\nabla |x|^2 = 2x,$$
$$\nabla_\nu \nabla |x|^2 = 2\nu \quad \text{and}$$
$$\Delta |x|^2 = 2(n+1).$$

For (10) we use the identity $\sinh a = \frac{|x|^2 - 1}{2x_{n+1}}$ and calculate

$$\nabla \frac{|x|^2 - 1}{2x_{n+1}} = \frac{x}{x_{n+1}} - \frac{|x|^2 - 1}{2x_{n+1}^2} e_{n+1},$$

$$\nabla_\nu \nabla \frac{|x|^2 - 1}{2x_{n+1}} = \frac{1}{x_{n+1}}\nu - \frac{1}{x_{n+1}^2}\langle \nu, e_{n+1}\rangle x - \frac{1}{x_{n+1}^2}\langle x, \nu\rangle e_{n+1}$$
$$+ \frac{|x|^2 - 1}{x_{n+1}^3}\langle e_{n+1}, \nu\rangle e_{n+1}, \quad \text{and}$$

$$\Delta \frac{|x|^2 - 1}{2x_{n+1}} = \frac{n-1}{x_{n+1}} + \frac{|x|^2 - 1}{x_{n+1}^3}.$$

Identity (11) is found noting that $\cosh r = \frac{|x|}{x_{n+1}}$ (see the cylinder example) and using

$$\nabla \frac{|x|}{x_{n+1}} = \frac{x}{x_{n+1}|x|} - \frac{|x|}{x_{n+1}^2} e_{n+1},$$

$$\nabla_\nu \nabla \frac{|x|}{x_{n+1}} = -\frac{1}{x_{n+1}|x|}\langle \nu, \omega\rangle\omega - \frac{1}{x_{n+1}^2}\langle \nu, e_{n+1}\rangle\omega + \frac{1}{x_{n+1}|x|}\nu$$
$$+ \frac{2|x|}{x_{n+1}^3}\langle \nu, e_{n+1}\rangle e_{n+1} - \frac{1}{x_{n+1}^2}\langle \nu, \omega\rangle e_{n+1} \quad \text{and}$$

$$\Delta \frac{|x|}{x_{n+1}} = \frac{n-2}{x_{n+1}|x|} + \frac{2|x|}{x_{n+1}^3}.$$

$\square$

To bound the gradient $\langle \nu, \omega\rangle^{-1}$ we need to consider the heat operator of the inner product of the normal and some vector $\vec{u}$ in the direction of the Killing vector field x. As there exist examples where the gradient increases during the flow (see [Unt98]) the heat operator of the gradient is not in itself useful. We start by considering $\vec{u}$ to be the Killing vector field x itself.

Proposition 2.3. *For a radial graph M_t evolving by mean curvature flow in hyperbolic space we have*

$$\left(\frac{\partial}{\partial t} - \Delta_M\right)\langle \nu_\mathcal{H}, x\rangle_\mathcal{H} = (|A|^2 - n)\langle \nu_\mathcal{H}, x\rangle_\mathcal{H}. \tag{12}$$

Proof. We combine the well known formula

$$\Delta_M \langle \nu_\mathcal{H}, x\rangle_\mathcal{H} = \langle \nabla^M H^\mathcal{H}, x\rangle_\mathcal{H} - \langle \nu_\mathcal{H}, x\rangle_\mathcal{H} R^\mathcal{H}_{\nu_\mathcal{H}\nu_\mathcal{H}} - \langle \nu_\mathcal{H}, x\rangle_\mathcal{H}|A|^2 - x(H)$$

(see [Bar84]) where $x(H)$ is the rate with which the mean curvature changes as the manifold is translated along the vector field x, with

$$
\begin{aligned}
\frac{\partial}{\partial t}\langle \nu_\mathcal{H}, x\rangle_\mathcal{H} &= -H^\mathcal{H}\nu_\mathcal{H}(\langle \nu_\mathcal{H}, x\rangle_\mathcal{H}) \\
&= \langle \nabla^M H^\mathcal{H}, x\rangle_\mathcal{H} + H^\mathcal{H}\langle \nu_\mathcal{H}, \nabla^\mathcal{H}_{\nu_\mathcal{H}} x\rangle_\mathcal{H}
\end{aligned}
$$

to derive

$$\left(\frac{\partial}{\partial t} - \Delta_M\right)\langle \nu_\mathcal{H}, x\rangle_\mathcal{H} = \langle \nu_\mathcal{H}, x\rangle_\mathcal{H}(|A|^2 + R^\mathcal{H}_{\nu_\mathcal{H}\nu_\mathcal{H}}) + x(H^\mathcal{H}) - H^\mathcal{H}\langle \nabla^\mathcal{H}_{\nu_\mathcal{H}} x, \nu_\mathcal{H}\rangle_\mathcal{H}. \tag{13}$$

Since $R^\mathcal{H}_{\nu_\mathcal{H}\nu_\mathcal{H}} = -n$ by (3) and $x(H^\mathcal{H}) = 0$ as x is a Killing vector field, we only need to show the last term is 0. Using (4) we calculate

$$
\begin{aligned}
\langle \nabla^\mathcal{H}_{\nu_\mathcal{H}} x, \nu_\mathcal{H}\rangle_\mathcal{H} &= \langle \nabla^\mathcal{H}_\nu x, \nu\rangle \\
&= \langle \nabla_\nu x, \nu\rangle + \frac{1}{x_{n+1}}(\langle \nu, x\rangle\langle e_{n+1}, \nu\rangle - \langle \nu, e_{n+1}\rangle\langle x, \nu\rangle \\
&\quad - \langle x, e_{n+1}\rangle\langle \nu, \nu\rangle) \\
&= 0
\end{aligned}
$$

since $\langle \nabla_\nu x, \nu\rangle = \langle \nu, \nu\rangle$. $\qquad\qquad\square$

We now look at the heat operator of $\langle \nu, x\rangle$ which is the product of $\langle \nu_\mathcal{H}, x\rangle_\mathcal{H}$ and the inverse of the square root of the conformal factor.

Corollary 2.4. *Under the conditions as in the previous proposition*

$$\left(\frac{\partial}{\partial t} - \Delta_M\right)\langle \nu, x\rangle = \langle \nu, x\rangle|A|^2 - 2\langle \nabla^M\langle \nu, x\rangle, x_{n+1}e_{n+1}\rangle_\mathcal{H}.$$

Proof. Combining (9) and (12) implies,

$$
\begin{aligned}
\left(\tfrac{\partial}{\partial t} - \Delta_M\right)\langle \nu, x\rangle &= x_{n+1}\left(\tfrac{\partial}{\partial t} - \Delta_M\right)\langle \nu_{\mathcal{H}}, x\rangle_{\mathcal{H}} + \langle \nu_{\mathcal{H}}, x\rangle_{\mathcal{H}}\left(\tfrac{\partial}{\partial t} - \Delta_M\right)x_{n+1} \\
&\quad -2\langle \overset{\mathcal{H}}{\nabla} x_{n+1}, \overset{\mathcal{H}}{\nabla}\langle \nu_{\mathcal{H}}, x\rangle_{\mathcal{H}}\rangle_{\mathcal{H}} \\
&= (|A|^2 - n)\langle \nu, x\rangle + (n-2)\langle \nu, x\rangle + 2\langle \nu, x\rangle\langle \nu, e_{n+1}\rangle^2 \\
&\quad -2\langle \overset{M}{\nabla}\langle \nu, x\rangle - \langle \nu_{\mathcal{H}}, x\rangle_{\mathcal{H}}\overset{M}{\nabla} x_{n+1}, \tfrac{1}{x_{n+1}}\overset{\mathcal{H}}{\nabla} x_{n+1}\rangle_{\mathcal{H}} \\
&= \langle \nu, x\rangle(|A|^2 - 2 + 2\langle \nu, e_{n+1}\rangle^2) - 2\langle \overset{M}{\nabla}\langle \nu, x\rangle, x_{n+1}e_{n+1}\rangle_{\mathcal{H}} \\
&\quad +2\langle \nu, x\rangle(1 - \langle \nu, e_{n+1}\rangle^2) \\
&= \langle \nu, x\rangle|A|^2 - 2\langle \overset{M}{\nabla}\langle \nu, x\rangle, x_{n+1}e_{n+1}\rangle_{\mathcal{H}}.
\end{aligned}
$$

$\square$

We can now use these results to obtain interior estimates.

Lemma 2.5. *Let $M_t \subset \mathcal{H}^{n+1}$ be a radial graph evolving by mean curvature flow and $\eta = \cosh R - e^{nt}\cosh r$ where r is the distance from the x_{n+1} axis. Then*

$$
\left(\frac{\partial}{\partial t} - \Delta_M\right)\eta^3\langle \nu, x\rangle^{-1} \le 2\eta^3\langle \nu, x\rangle^{-1}
$$

on the support of $\overset{+}{\eta}$ (the positive part of η).

Proof. From (11) we obtain $\left(\tfrac{\partial}{\partial t} - \Delta_M\right)\eta \le 0$ and $\left(\tfrac{\partial}{\partial t} - \Delta_M\right)\eta^3 \le -6\eta|\overset{M}{\nabla}\eta|^2$. Corollary 2.4 then gives

$$
\begin{aligned}
\left(\tfrac{\partial}{\partial t} - \Delta_M\right)\langle \nu, x\rangle^{-1} &= -\langle \nu, x\rangle^{-1}|A|^2 - 2\langle \overset{M}{\nabla}\langle \nu, x\rangle^{-1}, \overset{\mathcal{H}}{e}_{n+1}\rangle_{\mathcal{H}} \\
&\quad -2\langle \nu, x\rangle^{-3}|\overset{M}{\nabla}\langle \nu, x\rangle|^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left(\tfrac{\partial}{\partial t} - \Delta_M\right)\eta^3\langle \nu, x\rangle^{-1} &\le -\eta^3\langle \nu, x\rangle^{-1}|A|^2 - 2\eta^3\langle \overset{M}{\nabla}\langle \nu, x\rangle^{-1}, \overset{\mathcal{H}}{e}_{n+1}\rangle_{\mathcal{H}} \\
&\quad -2\eta^3\langle \nu, x\rangle^{-3}|\overset{M}{\nabla}\langle \nu, x\rangle|^2 - 6\eta\langle \nu, x\rangle^{-1}|\overset{M}{\nabla}\eta|^2 \\
&\quad +6\eta^2\langle \nu, x\rangle^{-2}\langle \overset{M}{\nabla}\eta, \overset{M}{\nabla}\langle \nu, x\rangle\rangle_{\mathcal{H}}.
\end{aligned}
$$

The result follows by making the substitutions

$$
-2\eta^3\langle \overset{M}{\nabla}\langle \nu, x\rangle^{-1}, \overset{\mathcal{H}}{e}_{n+1}\rangle_{\mathcal{H}} \le \frac{1}{2}\eta^3\langle \nu, x\rangle^{-3}|\overset{M}{\nabla}\langle \nu, x\rangle|^2 + 2\eta^3\langle \nu, x\rangle^{-1}
$$

and

$$
6\eta^2\langle \nu, x\rangle^{-2}\langle \overset{M}{\nabla}\eta, \overset{M}{\nabla}\langle \nu, x\rangle\rangle_{\mathcal{H}} \le 6\eta\langle \nu, x\rangle^{-1}|\overset{M}{\nabla}\eta|^2 + \frac{3}{2}\eta^3\langle \nu, x\rangle^{-3}|\overset{M}{\nabla}\langle \nu, x\rangle|^2.
$$

$\square$

Lemma 2.6. *Let $\omega = \frac{x}{|x|}$ and $R > 0$ be such that $\{x \in M_t \,|\, e^{nt}\cosh r \leq \cosh R\}$ is a compact radial graph for all $t \in [0, T]$. Then for any $t \in [0, T]$ and $0 \leq \theta < 1$ we have the estimate*

$$\sup_{\{x \in M_t \,|\, e^{nt}\cosh r \leq \theta\cosh R\}} \langle \nu, \omega \rangle^{-1} \leq e^{2T + h_{max} - h_{min}}(1 - \theta)^{-3} \sup_{\{x \in M_0 \,|\, r \leq R\}} \langle \nu, \omega \rangle^{-1}$$

where h_{max} and h_{min} are the supremum and infimum respectively of the radial height in $\{x \in M_t \,|\, r \leq R\}$.

Proof.   The previous Lemma with ([Ham89]) imply for almost every $t \in (0, T)$

$$\frac{\partial}{\partial t} \max_{M_t} \eta^3 \langle \nu, x \rangle^{-1} \leq 2 \max_{M_t} \eta^3 \langle \nu, x \rangle^{-1}.$$

Integrating from 0 to T then gives

$$\max_{M_T} \eta^3 \langle \nu, x \rangle^{-1} \leq e^{2T} \max_{M_0} \eta^3 \langle \nu, x \rangle^{-1}$$

and using $h_{min} \leq log|x| \leq h_{max}$ we have

$$\max_{M_T} \eta^3 \langle \nu, \omega \rangle^{-1} \leq e^{2T + h_{max} - h_{min}} \max_{M_0} \eta^3 \langle \nu, \omega \rangle^{-1}.$$

Since in $\{x \in M_t \,|\, e^{nt}\cosh r \leq \theta\cosh R\}$, $\eta^3 \geq (1 - \theta)^3 \cosh^3 R$, we finally obtain by letting T be any $t \in [0, T]$

$$\sup_{\{x \in M_t \,|\, e^{nt}\cosh r \leq \theta\cosh R\}} \langle \nu, \omega \rangle^{-1} \leq e^{2T + h_{max} - h_{min}}(1 - \theta)^{-3} \sup_{\{x \in M_0 \,|\, r \leq R\}} \langle \nu, \omega \rangle^{-1}$$

$\square$

Proposition 2.7. *For $M_t \subset \mathcal{H}^{n+1}$ evolving by mean curvature flow*

$i \qquad \left(\frac{\partial}{\partial t} - \Delta_M\right)|A|^2 = -2|\nabla A|^2 + 2|A|^2(|A|^2 + n) - 4H^2,$

$ii \quad \left(\frac{\partial}{\partial t} - \Delta_M\right)|\nabla^{Mm} A|^2 \leq -2|\nabla^{Mm+1} A|^2$
$$\qquad\qquad\qquad + C(n, m)(\textstyle\sum_{i+j+k=m} |\nabla^{Mi} A||\nabla^{Mj} A||\nabla^{Mk} A||\nabla^{Mm} A|$$
$$\qquad\qquad\qquad + |\nabla^{Mm} A|^2).$$

Proof.   The first formula was derived in [Hui86] for general manifolds. Direct substitution gives the result in the case of $\mathcal{H}^{n+1}$. From this the second is derived as in [Hui84] and [Ham82, Ch13]. $\square$

Lemma 2.8. *Let $M_t \subset \mathcal{H}^{n+1}$ evolve by (1) and define*

$$c_T = \frac{1}{2\sup_{[0,T)} \sup_{M_t} \langle \nu, x \rangle^{-2}}, \quad \text{and} \quad \varphi = \varphi(\langle \nu, x \rangle^{-2}) = \frac{\langle \nu, x \rangle^{-2}}{1 - c_T \langle \nu, x \rangle^{-2}}.$$

Then

$$\left(\frac{\partial}{\partial t} - \Delta_M \right) |A|^2 \varphi \;\leq\; -2 c_T |A|^4 \varphi^2 + \left(\frac{c(n)}{c_T} - c_T \varphi' |\nabla^M \langle \nu, x \rangle^{-1}|^2 \right) |A|^2 \varphi$$
$$- \varphi^{-1} \langle \nabla^M \varphi, \nabla^M (|A|^2 \varphi) \rangle_\mathcal{H}.$$

Proof. From Corollary 2.4 we derive

$$\left(\frac{\partial}{\partial t} - \Delta_M \right) \langle \nu, x \rangle^{-2} \;=\; -2 \langle \nu, x \rangle^{-2} |A|^2 + 4 \langle \nu, x \rangle^{-3} \langle \nabla^M \langle \nu, x \rangle, e_{n+1}^\mathcal{H} \rangle_\mathcal{H}$$
$$- 6 \langle \nu, x \rangle^{-4} |\nabla^M \langle \nu, x \rangle|^2.$$

Following the steps in [EH91, Theorem 3.1] yields the result.   □

Lemma 2.9. *Let $R > 0$ be such that $\{x \in M_t \mid r \leq R\}$ is a compact radial graph for any $t \in [0, T]$. Then for any $t \in [0, T]$ and $0 \leq \theta < 1$ we have the estimate*

$$\sup_{\{x \in M_t \mid \cosh r \leq \theta \cosh R\}} |A|^2 \leq c(n)(1-\theta)^{-2}\left(1 + \frac{1}{t}\right) \sup_{\{x \in M_s \mid r \leq R, s \in [0,t]\}} v^4.$$

Proof. In this proof we restrict ourselves to the region $\{x \mid r < R\}$ and utilize the methods of [EH91]. Replacing $|A|^2 \varphi$ by g in the previous lemma leads to

$$\left(\frac{\partial}{\partial t} - \Delta_M \right) g \;=\; -2 c_T g^2 + \left(\frac{c(n)}{c_T} - c_T \varphi' |\nabla^M \langle \nu, x \rangle^{-2}|^2 \right) g$$
$$- 2\varphi \langle \nu, x \rangle^3 \langle \nabla^M \langle \nu, x \rangle^{-1}, \nabla^M g \rangle_\mathcal{H},$$

and for $\eta = (\cosh R - \cosh r)^2$ we have using (11)

$$\left(\frac{\partial}{\partial t} - \Delta_M \right) \eta \leq 2n \cosh^2 R - 2|\nabla^M \cosh r|^2$$

and $|\nabla^M \cosh r| \leq \sinh r$. Now

$$
\begin{aligned}
-2\varphi\langle\nu,x\rangle^3\langle\nabla^M\langle\nu,x\rangle^{-1},\nabla^M g\rangle_{\mathcal{H}}\eta \;\leq\; & -2\varphi\langle\nu,x\rangle^3\langle\nabla^M\langle\nu,x\rangle^{-1},\nabla^M g\eta\rangle_{\mathcal{H}} \\
& +c_T\varphi'|\nabla^M\langle\nu,x\rangle^{-1}|^2 g\eta \\
& +\frac{\varphi^2}{c_T\varphi'}\langle\nu,x\rangle^6 g\frac{(\eta')^2}{\eta}|\nabla^M\cosh r|^2 \\
=\; & -2\varphi\langle\nu,x\rangle^3\langle\nabla^M\langle\nu,x\rangle^{-1},\nabla^M g\eta\rangle_{\mathcal{H}} \\
& +c_T\varphi'|\nabla^M\langle\nu,x\rangle^{-1}|^2 g\eta \\
& +\frac{4}{c_T}\langle\nu,x\rangle^2 g|\nabla^M\cosh r|^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta_M\right)g\eta \;=\; & -2c_T g^2\eta+(\frac{c(n)}{c_T}-c_T\varphi'|\nabla^M\langle\nu,x\rangle^{-2}|^2)g\eta \\
& -2\varphi\langle\nu,x\rangle^3\langle\nabla^M\langle\nu,x\rangle^{-1},\nabla^M g\rangle_{\mathcal{H}}\eta \\
& +2ng\cosh^2 R+6|\nabla^M\cosh r|^2 g \\
& -2\eta^{-1}\langle\nabla^M(g\eta),\nabla^{\mathcal{H}}\eta\rangle_{\mathcal{H}} \\
\leq\; & -2c_T g^2\eta+\frac{c(n)}{c_T}g\eta \\
& -2\langle\eta^{-1}\nabla^M\eta+\varphi\langle\nu,x\rangle^3\nabla^M\langle\nu,x\rangle^{-1},\nabla^{\mathcal{H}}g\eta\rangle_{\mathcal{H}} \\
& +2ng\cosh^2 R+(6+\frac{4}{c_T}\langle\nu,x\rangle^2)6|\nabla^M\cosh r|^2 g \\
\leq\; & -2c_T g^2\eta+\frac{c(n)}{c_T}g\eta \\
& -2\langle\eta^{-1}\nabla^M\eta+\varphi\langle\nu,x\rangle^3\nabla^M\langle\nu,x\rangle^{-1},\nabla^{\mathcal{H}}g\eta\rangle_{\mathcal{H}} \\
& +c(n)(1+\frac{\langle\nu,x\rangle^2}{c_T})g\cosh^2 R,
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta_M\right)g\eta t \;\leq\; & -2c_T g^2\eta t+(\frac{c(n)}{c_T}t+1)g\eta \\
& -2\langle\eta^{-1}\nabla^M\eta+\varphi\langle\nu,x\rangle^3\nabla^M\langle\nu,x\rangle^{-1},\nabla^{\mathcal{H}}g\eta t\rangle_{\mathcal{H}} \\
& +c(n)(1+\frac{\langle\nu,x\rangle^2}{c_T})gt\cosh^2 R.
\end{aligned}
$$

At a point where $\sup_{0 \le t \le T} \sup_{x \in M_t \,|\, r \le R} tg\eta \ne 0$ is attained for $t_0 > 0$ we compute

$$2c_T g^2 \eta t \le (\frac{c(n)}{c_T}t + 1)g\eta + c(n)(1 + \frac{\langle \nu, x \rangle^2}{c_T}) \cosh^2 Rgt.$$

Dividing by $2c_T g$ we compute

$$g\eta T \le \frac{1}{2c_T}(\frac{c(n)}{c_T}T + 1)\eta + \frac{c(n)}{2c_T}(1 + \frac{1}{c_T}) \cosh^2 RT,$$

and since $\eta \le \cosh^2 R$ we have

$$
\begin{aligned}
g\eta T &\le \frac{1}{2c_T}(\frac{c(n)}{c_T}T + 1) \cosh^2 R + \frac{c(n)}{c_T}(1 + \frac{1}{c_T}) \cosh^2 RT \\
&\le \frac{c(n)}{c_T^2}(1 + T) \cosh^2 R
\end{aligned}
$$

in the set $\{x \in M_t \,|\, r \le R\}$.
As $\eta \ge (1-\theta)^2 \cosh^2 R$ in $\{x \in M_t \,|\, \cosh r \le \theta \cosh R\}$ we obtain the required result by replacing T by any $t \in [0, T]$. $\qquad \square$

Lemma 2.10. *Let $R > 0$ be such that $\{x \in M_t \,|\, r \le R\}$ is a compact radial graph for any $t \in [0, T]$. Then for any $t \in [0, T]$, $m \ge 0$, and $0 \le \theta < 1$ we have the estimate*

$$\sup_{\{x \in M_t \,|\, \cosh r \le \theta \cosh R\}} |\nabla^{Mm} A|^2 \le c_m (1 - \theta)^{-2}(1 + \frac{1}{t})^{m+1}$$

where $c_m = c_m(n, m, \sup_{\{x \in M_s \,|\, r \le R, s \in [0,t]\}} v)$.

Proof. Throughout this proof we restrict ourselves to the region $\{x \,|\, r < R\}$. Lemma 2.9 implies that the above holds for $m = 0$. Suppose now that for $R > 0$ and $\psi(t) = \frac{t}{t+1}$

$$\sup_{\{x \in M_t \,|\, r \le R, t \in [0,T]\}} \psi^k |\nabla^{Mk-1} A|^2 \le c_{k-1}$$

has been established for $0 \le k \le m$. We then want to estimate $\psi^{m+1} |\nabla^{Mm} A|^2$ in $\{x \in M_t \,|\, \cosh r \le \theta \cosh R, t \in [0, T]\}$.
Making use of proposition 2.7 we adopt the same method as in [EH91] with

$f = \psi^{m+1}|\nabla^{Mm}A|^2(7c_{m-1}+1+\psi^m|\nabla^{Mm-1}A|^2)$ and $c = (n, m, c_0, ..., c_{m-1})$ to obtain

$$\left(\frac{\partial}{\partial t} - \Delta_M\right)f \leq -\psi^{-1}(\delta f^2 - c)$$

where δ has the same dependence as c. Using η as in lemma 2.9 we have

$$\left(\frac{\partial}{\partial t} - \Delta_M\right)\eta f \leq -\psi^{-1}\delta\eta f^2 + \psi^{-1}c\eta + 2nf\cosh^2 R$$
$$-2|\nabla^M\cosh r|^2 f - 2\langle\nabla^M\eta, \nabla^{\mathcal{H}}f\rangle_{\mathcal{H}}$$

and

$$-2\langle\nabla^M\eta, \nabla^{\mathcal{H}}f\rangle_{\mathcal{H}} = -4\eta^{-\frac{1}{2}}\langle\nabla^M\eta^{\frac{1}{2}}, \nabla^{\mathcal{H}}(\eta f)\rangle_{\mathcal{H}} + 8|\nabla^M\cosh r|^2 f$$

which imply

$$\left(\frac{\partial}{\partial t} - \Delta_M\right)\eta f \leq -\psi^{-1}\delta\eta f^2 + \psi^{-1}c\eta + 2nf\cosh^2 R + 6|\nabla^M\cosh r|^2 f$$
$$-4\eta^{-\frac{1}{2}}\langle\nabla^M\eta^{\frac{1}{2}}, \nabla^{\mathcal{H}}(\eta f)\rangle_{\mathcal{H}}.$$

Since $\psi(0) = 0$ we have $\eta f = 0$ at time $t = 0$. At the point where, for $t_0 > 0$, $m(T) = \sup_{0 < t_0 \leq T} \sup_{\{x \in M_t \mid \cosh r \leq \cosh R\}} \eta f$ is attained we have

$$\begin{aligned}
\delta\eta f^2 &\leq c\eta + 6\psi|\nabla^M\eta^{\frac{1}{2}}|^2 f + 2n\psi f\cosh^2 R \\
&\leq c\eta + 6|\nabla^M\eta^{\frac{1}{2}}|^2 f + 2nf\cosh^2 R \\
&\leq c\cosh^2 R + 6f\cosh^2 R + 2nf\cosh^2 R.
\end{aligned}$$

On $\{x \mid \cosh r \leq \theta\cosh R\}$, the inequality $\eta \geq (1-\theta)^2\cosh^2 R$ holds and so

$$\begin{aligned}
(1-\theta)^2\delta f^2 &\leq c + (6+2n)f \\
&\leq c(1-\theta)^{-2} + (1-\theta)^2\frac{\delta}{2}f^2,
\end{aligned}$$

which implies $\quad f \leq c(1-\theta)^2.$

Thus

$$\sup_{\{x \in M_t \mid \cosh r \leq \theta\cosh R, t \in [0,T]\}} \psi^{m+1}|\nabla^{Mm}A|^2 \leq c_m(1-\theta)^{-2}$$

and $\quad\displaystyle\sup_{\{x \in M_t \mid \cosh r \leq \theta\cosh R, t \in [0,T]\}} |\nabla^{Mm}A|^2 \leq c_m(1-\theta)^{-2}(1+\frac{1}{t})^{m+1}.$

□

To find solutions of (1) we are first required to restrict the problem to obtaining solutions over a compact subset of S_+^n.

Define $\Omega_\varepsilon = \{S_+^n \,|\, x_{n+1} \geq \varepsilon\}$ and $\Sigma_R = \{\Omega_\varepsilon \,|\, d(x) \leq R\}$ where here $d(x) = dist^E(x, \partial\Omega_\varepsilon)$ is the Euclidean distance as measured on Ω_ε from x to $\partial\Omega_\varepsilon$. We now find an upper barrier for the flow over Ω_ε. That is we show there exists a surface which lies over Σ_R, above the initial surface and has positive mean curvature. This prevents the gradient of the evolving surface over Ω_ε from increasing on $\partial\Omega_\varepsilon$.

Proposition 2.11. *Let $h = \log |x| \in C^2(\Omega_\varepsilon)$ be the height of the section of the initial surface, $\mathbf{F}_0(M^n)$, which lies over Ω_ε. Then*

$$\delta^+ = h(x) + \frac{1}{c}\log(1 + \beta d(x)).$$

for appropriate $c(n,h)$ and $\beta(n,h)$ is an upper barrier for mean curvature flow in $\mathcal{H}^{n+1}$ over Ω_ε.

Proof. [Giu84]. □

3. Existence of solutions.

We reformulate the problem to simplify the analysis. Firstly note that the system

$$\left(\frac{\partial}{\partial t}\mathbf{F}(p,t)\right)^\perp = \mathbf{H}(p,t)$$

is, up to tangential diffeomorphisms, equivalent to (1). Using a coordinate chart in which the first n coordinates $\vec{\theta}(p,t)$ represent a point on S_+^n and the $(n+1)^{st}$ coordinate the Euclidean radial height over S_+^n we have $\mathbf{F}(p,t) = (\vec{\theta}(p,t), e^{h(\vec{\theta}(p,t),t)})$ and $\nu_\mathcal{H} = \frac{x_{n+1}(-\nabla h, 1)}{\sqrt{1+|\nabla h|^2}}$ so that

$$\langle \frac{\partial}{\partial t}\mathbf{F}, \nu_\mathcal{H}\rangle_\mathcal{H} = \frac{1}{x_{n+1}\sqrt{1 + |\nabla h|^2}}\frac{\partial}{\partial t}h = -H^\mathcal{H}$$

and

$$\frac{\partial}{\partial t}h = -x_{n+1}\langle \nu, \omega\rangle^{-1}H^\mathcal{H}.$$

Using $H^{\mathcal{H}} = div_M \nu_{\mathcal{H}}$ we then obtain

$$
\frac{\partial}{\partial t} h = x_{n+1}^2 \left(\Delta^s h - g^{ik} g^{jl} \frac{\overset{s}{\nabla}_i h \overset{s}{\nabla}_j h}{1 + |\overset{s}{\nabla} h|^2} \overset{s}{\nabla}_k \overset{s}{\nabla}_l h \right) - n x_{n+1} \langle \overset{s}{\nabla} h, e_{n+1} \rangle
$$
$$
- \frac{x_{n+1}^2 |\overset{s}{\nabla} h|^2}{(1 + |\overset{s}{\nabla} h|^2)} \text{ on } S_+^n
$$
$$
h(\vec{\theta}, 0) = h_0(\vec{\theta})
$$

where Δ^s, $\overset{s}{\nabla}$, and g^{ij}, are the Laplacian, gradient and metric coefficients on S_+^n with the metric induced from $\mathcal{R}^{n+1}$. By projecting down onto $B_1^n(0) \subset \mathcal{R}^n$, defining $s(y) = h((y, x_{n+1}))$ for $y \in B_1^n(0)$ we arrive at

$$
\frac{\partial}{\partial t} s = A(x, s, Ds, D^2 s) \text{ on } B_1^n(0)
$$
$$
s(y, 0) = s_0(y)
$$

where

$$
A(x, s, Ds, D^2 s) = (1 - |y|^2) a^{ij} D_i D_j s
$$
$$
+ (1 - |y|^2) \left(n + \frac{1}{1 + |Ds|^2 - (Ds.y)^2} \right) Ds.y
$$
$$
+ (1 - |y|^2) \frac{|Ds|^2 - (Ds.y)^2}{1 + |Ds|^2 - (Ds.y)^2}
$$

and

$$
a^{ij} = \delta^{ij} - y^i y^j - \frac{(Ds - (Ds.y)y)^i (Ds - (Ds.y)y)^j}{1 + |Ds|^2 - (Ds.y)^2}.
$$

The coefficient matrix a^{ij} has :

1. For $y \parallel Ds$ an eigenvalue $\frac{1 - |y|^2}{1 + (1 - |y|^2)|Ds|^2}$ with multiplicity 1 and an eigenvalue 1 with multiplicity $n - 1$.

2. For $y \perp Ds$ eigenvalues $1 - |y|^2$ and $\frac{1}{1 + |Ds|^2}$ with multiplicity 1 and an eigenvalue 1 with multiplicity $n - 2$, and

3. Otherwise two eigenvalues

$$
\frac{(1 - |y|^2)(1 + |Ds|^2) + 1 \pm \sqrt{(1 - (1 - |y|^2)(1 + |Ds|^2))^2 + 4(1 - |y|^2)(Ds.y)^2}}{2(1 + |Ds|^2 - (Ds.y)^2)}
$$

with multiplicity 1 and an eigenvalue 1 with multiplicity $n - 2$.

Clearly as $|y| \to 1$ or $|Ds| \to \infty$ some eigenvalues may become zero and thus our equation may not be uniformly parabolic. This is one of the main problems in proving existence of a solution of the flow.

Theorem 3.1. *Suppose $F_0(M_\varepsilon^n)$ is a smooth radial graph over $\Omega_\varepsilon = \{S_+^n \,|\, x_{n+1} \geq \varepsilon\}$. Then the initial value problem*

$$\begin{aligned}
\tfrac{\partial}{\partial t}\mathbf{F}(p,t) &= \mathbf{H}(p,t), & p &\in M_\varepsilon^n \\
\mathbf{F}(p,0) &= \mathbf{F}_0(p), & p &\in M_\varepsilon^n \\
\mathbf{F}(p,t) &= \mathbf{F}_0(p), & p &\in \partial M_\varepsilon^n
\end{aligned} \tag{14}$$

has a smooth radial graph solution for all time.

Proof. The result follows if we have a smooth solution for all time to the problem

$$\begin{aligned}
\tfrac{\partial}{\partial t}s &= A(x,s,Ds,D^2 s), & y &\in B^n_{\sqrt{1-\varepsilon^2}}(0) \\
s(y,0) &= s_0(y), & y &\in B^n_{\sqrt{1-\varepsilon^2}}(0) \\
s(y,t) &= s_0(y), & y &\in \partial B^n_{\sqrt{1-\varepsilon^2}}(0).
\end{aligned} \tag{15}$$

As $|y| \leq \sqrt{1-\varepsilon^2}$, this system is uniformly parabolic (see eigenvalues on previous page) provided $|Ds| < \infty$. Given these the results of Ladyzhenskaya et al. [LSU68] apply and yield a smooth solution of (15). Since $|Ds|^2 \leq \frac{|\nabla h|^2}{\varepsilon^2} = \frac{v^2-1}{\varepsilon^2}$ our result follows provided $v^{-1} = \langle \nu, \omega \rangle$ is bounded away form zero. Employing Corollary 2.4 and a compact maximum principle shows $\langle \nu, x \rangle$ takes its minimum on the parabolic boundary $B^n_{\sqrt{1-\varepsilon^2}}(0) \times 0 \cup \partial B^n_{\sqrt{1-\varepsilon^2}}(0) \times (0,\infty)$. Initial bounds and the barriers (Proposition 2.11) imply $\langle \nu, x \rangle$ is bounded here. The barrier $S_+^n(e^{h_{max}})$ is used to bound $|x|$ above, and so $\langle \nu, \omega \rangle$ is bounded away from zero completing the proof. $\square$

Theorem 3.2. *Let $M_0 = F_0(B^n)$ be a locally Lipschitz continuous entire radial graph over $S_+^n \subset \mathcal{H}^{n+1}$. Then the initial value problem (1) has a smooth solution $M_t = F_t(B^n)$ for all time $t > 0$. Moreover, each M_t is an entire graph over S_+^n.*

Proof. We first assume M_0 is smooth. For the solid cylinder $C_\varepsilon = \{x \,|\, \frac{|x|}{x_{n+1}} \leq \frac{1}{\varepsilon}\}$, let M_t^ε be the solution of (14) with $M_\varepsilon^n = \mathbf{F}_0^{-1}(M_0 \cap C_\varepsilon)$ which by Theorem 3.1 exists for all time, and let h_t^ε and s_t^ε be the radial height of M_t^ε over S_+^n and its projection onto a ball in $\mathcal{R}^n$ respectively.

Denote now by B_R^n the restriction of the hyperbolic ball of radius R and center $(\vec{0}, 1)$ to S_+^n. Then choose R_0 so that $B_{4R_0}^n \subset\subset \Omega_\varepsilon$. For $h^m = \sup_{\{x \in M_0 \,|\, r < 4R_0\}} h_0^\varepsilon$ we find (see "The Sphere" example)

$$M_0^\varepsilon \cap B_{4R_0}^{n+1}(p^+) = \emptyset$$

where $p^+ = (\vec{0}, \frac{1+e^{h_m}}{\cosh 4R_0 - \sinh 4R_0})$. Furthermore

$$\partial M_t^\varepsilon \cap B_{4R_0}^{n+1}(p^+) = \emptyset.$$

Using a compact maximum principle (see Lemma 3.2 of [Hui86]) and "The Sphere" example we infer

$$M_t^\varepsilon \cap B_{2R_0}^{n+1}(p^+) = \emptyset$$

for $t \in [0, t_0]$ where $t_0 = \frac{1}{n} \log \frac{\cosh 4R_0}{\cosh 2R_0}$. Using a similar argument with $B_{4R_0}^{n+1}(p^-)$ where $p^- = (\vec{0}, \frac{e^{h_m}}{2\cosh R + \sinh R})$ and $h_m = \inf_{\{x \in M_0 \mid r < 4R_0\}} h_0^\varepsilon$ we conclude that the height function satisfies

$$\sup_{B_{2R_0}^n \times [0,t_0]} |s_t^\varepsilon| < c_0,$$

where c_0 depends only on n, R_0 and $\sup_{B_{4R_0}^n \times \{0\}} |h_0^\varepsilon|$.
Applying the gradient estimate of Lemma 2.6 we obtain

$$\sup_{B_{\frac{3}{2}R_0}^n \times [0,t_1]} |Ds_t^\varepsilon| \le c_1$$

where $t_1 < t_0$ and $c_1 = c_1(n, R_0, c_{02}, \sup_{B_{2R_0}^n \times \{0\}} |Ds_t^\varepsilon|)$. We note that as $R_0 \to \infty$ both t_0 and t_1 may approach ∞. From Lemma 2.10 we then conclude for any integer $m \ge 0$

$$\sup_{B_{R_0}^n \times [0,t_1]} |D^m s_t^\varepsilon| \le c_m$$

where $c_m = c_m(m, n, R_0, c_1)$.
Since R_0 and t_1 are arbitrary we can select a sequence of solutions $(s_t^{\varepsilon_i})$ for $\varepsilon_i \to 0$ $(\varepsilon_i < \varepsilon)$ such that $s_t^{\varepsilon_i} \overset{i \to \infty}{\to} s$ in C^∞ uniformly on compact subsets of $M^n \times (0, \infty)$. This establishes the existence of a family of entire graphs $M_t = \text{graph } h_t$ solving (1) where $h \in C^\infty(S_+^n \times (0, \infty))$. As the second and higher order derivative estimates for h on each compact subset of S_+^n depend only on the initial gradient on a slightly larger subset, an approximation argument yields a smooth solution of (1) also for locally Lipschitz initial data. $\square$

Theorem 3.3. *If M_0 has bounded gradient and hyperbolic height over S_+^n then under mean curvature flow M_t converges in C^∞ to S_+^n.*

Proof. As M_0 has bounded hyperbolic height it lies between two hyperspheres (see "The Hypersphere" example). In view of (2), the solution M_t does not move at the boundary and therefore by the compact maximum principle remains between the evolving barriers. Since both barriers evolve to S_+^n our solution converges uniformly to S_+^n in height.

Allowing $R \to \infty$ in lemma 2.6 we infer that v remains bounded on some small time interval $[0, T]$ and hence by letting $R \to \infty$ in the interior estimate for $|A|^2$, lemma 2.9, that $|A|^2$ is bounded on some small time interval $[t_0, T]$ for $t_0 > 0$. We may thus use the non compact maximum principle [EH91, Theorem 4.3] which with the equation for the heat operator of $\langle \nu, x \rangle^{-1}$, Corollary 2.4, imply that $\langle \nu, x \rangle^{-1}$ is bounded above on $[t_0, T]$ by its maximum value at time t_0. It follows that $\langle \nu, x \rangle^{-1}$ is bounded above by its initial maximum for all time. Our height bounds then imply that v is bounded independent of time. From the interior estimate, lemma 2.10, with $R \to \infty$ we then obtain bounds for $|\nabla^{Mm} A|^2$ for $m \geq 0$ independent of time. Thus height and all its derivatives are bounded independent of time. Employing the Theorem of Arzela-Ascoli the result follows. $\square$

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